Vers la classification des variétés de caractères sauvages

Towards the classification of wild character varieties

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Hear and attend and listen; for this befell and behappened and became and was, O my Best Beloved, when the Tame animals were wild. The Dog was wild, and the Horse was wild, and the Cow was wild, and the Sheep was wild, and the Pig was wild—as wild as wild could be—and they walked in the Wet Wild Woods by their wild lones. But the wildest of all the wild animals was the Cat. He walked by himself, and all places were alike to him.

R. Kipling.
The Cat that Walked by Himself

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Contents

Chapter 1

Introduction

Notre résultat principal est la construction d'un diagramme associé à toute connexion algébrique sur un fibré vectoriel sur un ouvert de Zariski de la sphère de Riemann, ainsi que d'une preuve que ce diagramme est invariant sous l'action du groupe des automorphismes symplectiques de l'algèbre de Weyl, comprenant la transformation de Fourier-Laplace. En général, ceci change le rang des fibrés et le nombre de pôles. En particulier, l'invariance du diagramme entraîne l'égalité des dimensions des espaces de modules correspondants.

1.1 Contexte et motivations

1.1.1 Variétés de caractères (sauvages) et systèmes intégrables

La motivation principale pour développer une théorie des "diagrammes de Dynkin" associés à des espaces de modules de connexions est la perpective d'une classification des espaces de Hodge non-abéliens ([22] Defn. 7). Ce sont des variétés hyperkähleriennes (complètes) associées à des surfaces de Riemann compactes, avec des données de singularité spécifiées à certains points marqués. Leur construction générale [10] étend la construction [59, 47] dans le cas compact (sans singularités), ainsi que [95, 66, 77] dans le cas modéré (avec des pôles simples). La théorie de Hodge non-abélienne et la correspondence de Riemann-Hilbert entraînent le fait remarquable que ces espaces de modules analytiques constituent des espaces de modules pour trois types d'objets algébriques ou topologiques. Le premier côté est le côté des fibrés de Higgs (irréguliers), appelé le côté de Dolbeault. Le second est le côté des connexions (irrégulières), connu comme celui de de Rham. Enfin, le troisième côté, de nature topologique, est celui des représentations du groupe fondamental, appelé le côté de Betti, ainsi que sa généralisation aux représentations de Stokes.

La correspondence de Hodge non-abélienne [10, 90] (reposant sur les travaux de Hitchin, Donaldson, Corlette et Simpson [59, 42, 33, 96] dans le cas compact, ainsi que Simpson, Biquard, Konno, Nakajima [95, 9, 66, 77] dans le cas modéré) implique que les espaces de modules de Dolbeault $\mathcal{M}_{Dol}$ et de de Rham $\mathcal{M}_{dR}$ sont difféomorphes. Ils possèdent une structure hyperkählerienne, provenant du fait que ces espaces de modules sont isomorphes aux espaces de modules $\mathfrak{M}$ des solutions des équations de Yang-Mills autoduales en deux dimensions (souvent appelées équations de Hitchin). Au sein de la sphère des structures complexes de la variété hyperkählerienne $\mathfrak{M}$, il y a deux choix préférés qui correspondent aux espaces $\mathcal{M}_{Dol}$ et $\mathcal{M}_{dR}$ respectivement.

Par ailleurs, la correspondance de Riemann-Hilbert [38] et son extension au cas irrégulier [69] entraînent que l'espace de modules de de Rham $\mathcal{M}_{dR}$ et l'espace de modules de Betti $\mathcal{M}_B$ sont difféomorphes. Cela mène à la situation générale suivante:

$$\begin{tikzcd}[column sep=large]
 & & \mathcal{M}_{Dol} \\
\begin{array}{c}(\Sigma,\mathbf{a},\Theta,\mathcal{C})\\ \text{surface de Riemann sauvage}\\ \text{et classes de conjugaison}\end{array} \arrow[r] & \begin{array}{c}\mathfrak{M}\\ \text{espace de Hodge non-abélien}\end{array} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{M}_{dR} \\
 & & \mathcal{M}_B
\end{tikzcd}$$

où les trois applications à droite sont des difféomorphismes. L'espace de Hodge non-abélien $\mathfrak{M}$ dépend du choix d'une courbe algébrique complexe lisse de base Σ, avec des points marqués $\mathbf{a}$ et des données de singularité $(\Theta, \boldsymbol{\mathcal{C}})$ qui encodent le type de singularités de la connexion.

Plus précisément, la classe irrégulière Θ encode le rang des fibrés vectoriels et les facteurs exponentiels intervenant dans les sections horizontales de la connexion au voisinage de chaque singularité, tandis que $\boldsymbol{\mathcal{C}}$ est la collection des classes de conjugaison des monodromies formelles aux points singuliers (voir §3.1 pour les détails). D'un point de vue de théorie des déformations, il est naturel de voir le triplet $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ comme une généralisation d'une surface de Riemann avec des points marqués; il s'agit d'une *surface de Riemann sauvage* au sens de [21, 25].

Le point de vue le plus simple est celui de Betti, et c'est celui que nous allons adopter dans cette thèse. Les espaces de modules de Betti sont des variétés algébriques symplectiques de dimension finie, et peuvent être décrites par une présentation explicite. Lorsque la courbe est compacte, de telle sorte qu'il n'y a pas de singularités, les espaces de modules de Betti sont les variétés de caractères habituelles, c'est-à-dire les espaces de modules de représentations du groupe fondamental de la surface:

$$\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G \cong \{A_1, B_1, \ldots, A_g, B_g \in G, [A_1, B_1] \cdots [A_g, B_g] = 1\}/G,$$

où g est le genre de Σ, et $G = GL_n(\mathbb{C})$, n correspondant au rang de la connexion, et $[A, B] = ABA^{-1}B^{-1}$ désigne le commutateur multiplicatif. Le cas où la courbe de base n'est pas compacte, par exemple lorsque $\Sigma = \mathbb{P}^1 \smallsetminus \{a_1, \ldots, a_m\}$ est un ouvert de Zariski de la sphère de Riemann, correspond à des connexions ayant des singularités. Lorsque toutes les singularités sont *régulières*, il découle de la correspondance de Riemann-Hilbert qu'une connexion à singularités régulières peut être reconstruite à partir de ses monodromies autour de ses points singuliers, de telle sorte que l'espace de modules de Betti est donné par

$$\mathcal{M}_B \cong \{(M_1, \ldots, M_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r,\ M_r \ldots M_1 = 1\}/G,$$

où $\mathcal{C}_1, \ldots, \mathcal{C}_r \subset G$ sont des classes de conjugaison dans G.

Lorsque $r = 3$ et que les matrices sont de taille 2, la situation correspond à l'équation hypergéométrique de Gauss, et l'espace de modules est réduit à un point. L'exemple le plus simple où l'espace de modules est de dimension strictement positive, apparaît pour $r = 4$ et des matrices de taille 2: l'espace de modules est de dimension complexe 2 et la situation correspond à l'équation de Heun (déformée) apparaissant dans la représentation standard de l'équation de Painlevé VI.

Cependant, lorsque les singularités sont irrégulières, la représentation de monodromie ne suffit plus à reconstruire la connexion. À cause du *phénomène de Stokes*, il y a des données topologiques supplémentaires à extraire: la raison principale est que les développements asymptotiques des sections horizontales des solutions de la connexion sont seulement valables dans certains secteurs angulaires autour d'une singularité irrégulière, et peuvent sauter de façon discontinue lorsqu'on traverse certaines directions particulières.

Une présentation explicite de l'espace de module de Betti peut néanmoins encore être donnée [21, 25]. Elle fait intervenir des matrices de Stokes qui rendent compte du passage d'un secteur au suivant. Si de nouveau la courbe de base est $\Sigma = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$, cette présentation explicite est de la forme

$$\mathcal{M}_B \cong \{(\mathbf{C}, h, \mathbf{S}) \mid (C_1^{-1} h_1 S_{k_1}^{(1)} \dots S_1^{(1)} C_1) \dots (C_m^{-1} h_m S_{k_m}^{(m)} \dots S_1^{(m)} C_m) = 1\}/G \times \boldsymbol{H},$$

où $C_i \in G$, h_i^{-1} est dans la classe de conjugaison $\mathcal{C}_i$ de la monodromie formelle en a_i, $\boldsymbol{H} = H_1 \times \dots \times H_m$, avec H_j un sous-groupe réductif de G, et les *matrices de Stokes* $S_i^{(j)}$ appartiennent à certains sous-groupes unipotents de G, les sous-groupes de Stokes (voir §3.2 pour les détails). De telles présentations sont connues depuis les travaux de Birkhoff [11] au début du XXième siècle. Ces espaces de modules de Betti irréguliers sont connus aujourd'hui sous le nom de variétés de caractères sauvages (voir [21, 25] où elles sont construites comme variétés algébriques symplectiques, parallèlement aux constructions analytiques précédentes [13, 10], et étendant les premières constructions algébriques [101, 16, 18] à des cas plus généraux). On obtient de cette manière une vaste famille d'espaces de modules associés à une surface de Riemann munie de données de singularités.

Il s'avère que, quand Σ est une sphère de Riemann, il y a de nombreux exemples de surfaces de Riemann sauvages différentes, correspondant à des connexions avec des rangs différents, des nombres de singularités différents et des singularités d'ordre différents, donnant lieu à des espaces de modules isomorphes. En particulier, un espace de modules provenant d'une connexion à singularités régulières peut être isomorphe à un espace de modules provenant d'un espace de modules à singularités irrégulières.

On peut dès lors voir les surfaces de Riemann sauvages donnant des espaces de modules isomorphes comme des "réalisations" ou "représentations" différentes de la même variété algébrique affine symplectique $\mathcal{M}_B$. Cela soulève naturellement la question suivante [23], qui est la motivation principale de ce travail :

Question Peut-on décrire toutes les représentations d'une même variété de caractères sauvage (abstraite) $\mathcal{M}_B$? Étant données deux surfaces de Riemann sauvages, existe-t-il un moyen direct de savoir si elles correspondent à des espaces de modules isomorphes ?

Une motivation supplémentaire pour s'intéresser à cette question provient du fait que les variétés de caractères sauvages ont des liens étroits avec de nombreuses équations différentielles non-linéaires, et systèmes intégrables. Du côté de Dolbeault, les espaces de modules de fibrés de Higgs méromorphes sont des systèmes intégrables algébriques [27, 71], correspondant aux systèmes de Hitchin dans le cas des surfaces de genre supérieur sans pôles, ainsi que de nombreux systèmes intégrables classiques en genre zéro avec des pôles, le plus souvent irréguliers [1, 2, 8, 40, 51, 88], sont obtenus de cette façon.

Par ailleurs, la théorie des déformations isomonodromiques engendre des équations différentielles non-linéaires à partir des espaces de de Rham. L'idée de base est que les surfaces de Riemann sauvages viennent en famille: on peut faire varier le module de Σ et les positions des singularités, mais aussi les paramètres définissant la classe irrégulière Θ.

Si $\underline{\boldsymbol{\Sigma}} \to \mathbb{B}$ est une famille de surfaces de Riemann sauvages sur une base $\mathbb{B}$ de déformations admissibles, on obtient une famille $\boldsymbol{\mathcal{M}} \to \mathbb{B}$ d'espaces de modules sur la base $\mathbb{B}$, où la fibre au dessus de $b \in \mathbb{B}$ est l'espace de modules $\mathcal{M}_{dR}(\boldsymbol{\Sigma}_b)$, où $\boldsymbol{\Sigma}_b$ désigne la surface de Riemann sauvage au-dessus de b. Lorsque b varie dans $\mathbb{B}$, faire varier les connexions dans $\mathcal{M}_{dR}(\boldsymbol{\Sigma}_b)$ de telle sorte que les monodromies généralisées restent constantes fournit une connexion plate d'Ehresmann (complète) sur $\boldsymbol{\mathcal{M}}$, la connexion d'isomonodromie [13]. Si on l'écrit de façon explicite en coordonnées, la connexion d'isomonodromie correspond à un système d'équations différentielles non linéaires. Du point de vue de Betti, cela revient au fait que les variétés de caractères sauvages forment un "système local de variétés" au-dessus de $\mathbb{B}$ [21].

De nombreuses équations différentielles non linéaires bien connues apparaissent de cette façon, notamment les équations de Painlevé [93, 50, 62, 60], qui correspondent à des espaces de modules de dimension 2, ainsi que bien d'autres équations différentielles étudiées par les physiciens (par exemple [61]). Une représentation de $\mathcal{M}_B$ correspond à un choix de représentation de Lax (ou "linéarisation") du système d'équations différentielles non-linéaires.

Dans l'étude des sytèmes intégrables, il arrive très souvent qu'un système donné possède plusieurs représentations de Lax. Des transformations comme les transformations de Bäcklund permettent de passer d'une représentation de Lax à une autre. Ainsi, on peut voir la question de la classification des variétés de caractères sauvages comme recouvrant celle de la classification des sytèmes intégrables provenant des sytèmes de Hitchin méromorphes et des déformations isomonodromiques, la question de trouver toutes les réalisations d'un espace de modules donné revenant à trouver toutes les représentations de Lax du système intégrable correspondant.

Les espaces de modules de fibrés de Higgs méromorphes apparaissent aussi dans divers contextes liés à la physique des hautes énergies: par exemple Seiberg-Witten [94] ont relié des théories de super Yang-Mills $N = 2$ à des systèmes intégrables algébriques, et la majorité des exemples sont reliés à des fibrés de Higgs méromorphes [41, 27, 71, 72, 4]. Des extension récentes [48, 49, 79, 102] de cette histoire retrouvent de façon conjecturale des cas supplémentaires des variétés hyperkähler complètes construites dans [10], comme leurs branches de Coulomb compactifiées sur un cercle. D'autres exemples incluent les monopôles périodiques [32, 31] via la transformation de Nahm, L'approche de Witten pour Langlands géométrique [100] et les solutions de cordes complexes ("voidons") des équations de Yang-Mills autoduales dans [29].

1.1.2 Liens avec les carquois, graphes et diagrammes

Il s'avère que, de façon remarquable, dans certains cas il semble y avoir une manière élégante de paramétriser les données déterminant un espace de module de connexions en termes d'un graphe (c'est-à-dire d'un carquois doublé), et ces liens fournissent un éclairage sur la question de la classification.

Rappelons qu'un graphe fini, d'ensemble de sommets N (sans boucles) détermine une matrice de Cartan (Kac-Moody) par

$$C_{ij} = 2 - B_{ij}$$

où $i, j \in N$ et $B_{ij} = B_{ji}$ est le nombre d'arêtes entre les sommets i, j.

La question principale est de trouver une application des données de singularités vers de telles données de Cartan :

Question Étant donnée une connexion méromorphe sur la sphère de Riemann, peut-on définir de façon uniforme un graphe Γ et un vecteur dimension $\mathbf{d} \in \mathbb{N}^N$ tels que la variété de caractère sauvage $\mathcal{M}_B$ déterminée par (E, ∇) a pour dimension

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

où $(\ ,\ )$ est la forme bilinéaire symétrique déterminée par la matrice de Cartan de Γ ?

Avant d'énoncer nos résultats principaux, commençons par rappeler brièvement les choses déjà connues.

Les groupes de Weyl affines d'Okamoto Les exemples les plus simples d'espaces de Hodge non-abéliens sont les surfaces H3 [23], c'est-à-dire ceux de dimension 2 (analogues non-compacts des surfaces K3). Certaines de ces surfaces sont reliées aux équations de Painlevé : l'espace de module de de Rham est isomorphe aux "espaces de conditions initiales" de l'équation de Painlevé

numéro	ordre des pôles	groupe de Weyl affine	nombre de paramètres
VI	$1+1+1+1$	D_4	4
V	$2+1+1$	A_3	3
IV	$3+1$	A_2	2
III	$2+2$	B_2	2
II	4	A_1	1
I	$\tilde{4}$	-	0

Figure 1.1: Représentations standard et symétries des équations de Painlevé, cf. [86].

correspondante, construit explicitement par Okamoto [82, 84, 83, 85, 80]. Okamoto a montré que chacune des six équations de Painlevé admet un groupe de Weyl affine de symétries. Par ailleurs, chaque équation de Painlevé est un système d'isomonodromie pour une connexion linéaire sur la sphère de Riemann, de plusieurs manières différentes. Des choix de telles connexions linéaires ont été trouvés par R. Fuchs [46] et R. Garnier [50], puis réécrits sous forme matricielle par Schlesinger [93] and Jimbo-Miwa [61].

Les données de singularité de ces représentations de Lax standard pour chaque équation de Painlevé sont résumées dans le tableau 1.1 (elles correspondent toutes à des connexions sur le fibré vectoriel trivial de rang 2 sur la sphère de Riemann).

Cela fournit une application de certains exemples de données de singularités vers des groupes de Weyl affines. L'application cherchée entre données de singularité et données de Cartan devrait étendre cette application (en notant que toute matrice de Cartan détermine un group de Weyl).

Observons que $\dim(\mathcal{M}_B) = 2$ entraîne $(\mathbf{d}, \mathbf{d}) = 0$ c'est-à-dire que $\mathbf{d}$ est isotrope, comme c'est le cas pour toute racine imaginaire d'une algèbre de Kac-Moody affine. Les vecteurs dimension correspondant aux surfaces H3 vont en effet être des racines imaginaires minimales.

À noter qu'Okamoto définit aussi un diagramme de Dynkin affine *différent* pour chaque équation de Painlevé, le "diagramme d'Okamoto" (ils sont listés dans [86]). Par exemple le diagramme d'Okamoto de Painlevé II is $\widehat{E}_7$ tandis que le groupe de Weyl affine de symétrie est de type $\widehat{A}_1$. On ne sait pas s'il est possible de généraliser les diagrammes d'Okamoto à des espaces de modules de dimension supérieure.

Modularité des variétés de carquois L'étape suivante consiste à identifier les variétés de carquois de Nakajima de certains graphes à des espaces de modules de connexions. L'histoire fait intervenir les *espaces de de Rham additifs* $\mathcal{M}^*$ definis dans [13], qui sont les espaces de modules de connexions méromorphes sur des fibrés holomorphes *triviaux* sur la sphère de Riemann. L'application de Riemann-Hilbert-Birkhoff envoyant une connexion vers ses données de monodromies généralisées induit un difféomorphisme symplectique (voir [13, §6])

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_B.$$

entre l'espace de module additif et un ouvert dense de l'espace de Betti tout entier.

Les variétés de carquois de Nakajima [76] sont des objets importants en théorie des représentations. Étant donné un graphe Γ (par graphe nous désignons un carquois doublé), d'ensemble de sommets N, $\mathbf{d} \in \mathbb{Z}^N$ un vecteur dimension, et $\boldsymbol{\lambda} = (\lambda_i)_{i\in N} \in \mathbb{C}^N$, la variété de carquois de Nakajima $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ est une variété algébrique symplectique, de dimension $2 - (\mathbf{d}, \mathbf{d})$, où $(\cdot, \cdot)$ est la forme bilinéaire symétrique sur $\mathbb{Z}^N$ définie par la matrice de Cartan du graphe (voir chapitre 3 pour les détails).

Dans plusieurs cas, il est possible d'associer un graphe Γ ainsi qu'un vecteur dimension $\mathbf{d}$ et des labels $\boldsymbol{\lambda}$ a une connexion (E, ∇), tels que l'espace de module additif $\mathcal{M}^*$ est isomorphe à la variété de carquois $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$.

Le premier ingrédient permettant de relier les connexions méromorphes aux carquois est une construction qui associe à une classe de conjugaison $\mathcal{C} \subset GL_n(\mathbb{C})$ un carquois linéaire [67, 76], ainsi qu'un vecteur dimension et un vecteur de labels. La construction dépend du choix d'un *marquage* de la classe de conjugaison (voir ch. 3). Dans le cas des connexions à singularités régulières, les données formelles sont entièrement déterminées par les classes de conjugaison des monodromies: les données de singularité consistent simplement en une liste $\boldsymbol{\mathcal{C}}$ de m classes de conjugaison dans $G = GL_n(\mathbb{C})$. des travaux de Crawley-Boevey et ses collaborateurs associent à la connexion [34, 35, 36] un graphe étoilé, où les "jambes" de l'étoile sont les carquois encodant les classes de conjugaison des monodromies. En rang 2 avec quatre pôles, cela donne le graphe de type affine D_4, qui coïncide avec le groupe de Weyl affine de symétries de Painlevé VI.

Il s'avère que dans ce cas l'espace de de Rham additif $\mathcal{M}^*(\boldsymbol{\mathcal{C}})$ est isomorphe à la variété de carquois de Nakajima $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ définie par le graphe. Ce résultat a permis à Crawley-Boevey de résoudre la version additive du problème de Deligne–Simpson, qui demande pour quelles données formelles l'espace de module additif est non vide. Dans ce cadre, les réflexions de Weyl simple associées au carquois sont liées à l'action de la convolution moyenne sur les connexions, et la réponse au problème de Deligne-Simpson additif est formulée en termes du système de racines défini par le carquois. Par ailleurs, l'espace de module $\mathcal{M}_B$ entier peut être vu comme une variété de carquois multiplicative [36].

Ce tableau peut être étendu au cas des connexions avec une singularité irrégulière, potentiellement avec d'autres pôles simples. Cette généralisation a d'abord été entreprise par Boalch [17, 20], en se restreignant au cas simplement lacé où la singularité irrégulière n'est pas ramifiée et est un pôle d'ordre inférieur ou égal à 3. Dans ce cadre, il est encore possible de définir un triplet $(\Gamma, \mathbf{d}, \boldsymbol{\lambda})$ associé à une connexion (E, ∇). Les graphes qui apparaissent sont une généralisation des graphes étoilés, les graphes supernova. Ils sont constitués d'un cœur, qui est un graphe complet k-parti, auquel viennent se coller des jambes. Les sommets du cœur correspondent aux *facteurs exponentiels* $q_i \in z\mathbb{C}[z]$ de la connexion à la singularité irrégulière à l'infini, qui sont essentiellement les valeurs propres de la partie singulière à l'infini de la connexion. Plus précisément, la connexion peut être amenée à une forme normale

$$\nabla_0 = d - \left(dQ + \frac{\Lambda}{z} dz \right),$$

où Q est une matrice diagonale par blocs de la forme

$$Q = \mathrm{diag}(q_1 1_{n_1}, \ldots, q_k 1_{n_k}),$$

(et Λ est une matrice constante).

À ces sommets il faut coller des jambes encodant comme auparavant les classes de conjugaison des composantes de la monodromie formelle associées à chaque facteur exponentiel, ainsi que les classes de conjugaison des monodromies aux singularités régulières. Si $q_1, \ldots, q_k$ sont les facteurs exponentiels à l'infini, le nombre d'arêtes dans le cœur du digramme entre q_i et q_j est donné par

$$B_{ij} := \deg(q_i - q_j) - 1, \tag{1.1.1}$$

où $\deg(q_i - q_j)$ est le degré de la différence $q_i - q_j$ comme polynôme en z. La définition du graphe a été généralisée dans [17, Appendix C] au cas des connexions avec une singularité irrégulière non-ramifiée d'ordre quelconque. Dans ces cas, nous disposons encore d'un "théorème de modularité des carquois":

Theorem 1.1.1. *Étant donnée une connexion méromorphe ∇ sur un fibré vectoriel trivial E sur la sphère de Riemann $\mathbb{P}^1 = \mathbb{C} \cup \infty$, ayant seulement des pôles simples sur $\mathbb{C}$ et une singularité (irrégulière) non-ramifiée à ∞, on peut définir de façon uniforme un graphe Γ et un vecteur dimension $\mathbf{d} \in \mathbb{N}^N$ tels que l'espace de de Rham additif $\mathcal{M}^*$ of (E, ∇) est isomorphe à la variété* *de carquois de Nakajima $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ pour un vecteur de labels $\boldsymbol{\lambda}$. En particulier, la variété de caractères sauvage $\mathcal{M}_B$ déterminée par (E, ∇) a pour dimension*

$$\dim(\mathcal{M}_B) = \dim(\mathcal{M}^*) = 2 - (\mathbf{d}, \mathbf{d})$$

où $(\ ,\ )$ est la forme bilinéaire symétrique déterminée par la matrice de Cartan $C = 2 - B$ de Γ.

Ce résultat a été conjecturé dans [17, appendix C], prouvé dans le cas “simplement lacé” dans [17, 20], et en général dans [57]. Ce résultat permet de nouveau de donner une solution au problème de Deligne-Simpson additif en termes du système de racines associé au carquois. L’espace de modules entier peut encore être vu comme une sorte de variété de carquois généralisée [22].

Diagrammes de Boalch-Yamakawa Cependant, cette stratégie ne fonctionne pas dans le cas général, puisque tous les espaces de modules $\mathcal{M}^*$ ne sont pas isomorphes à des variétés de carquois. Par exemple, on sait que l’espace de modules $\mathcal{M}^*$ n’est pas isomorphe à une variété de carquois de Nakajima dans le cas de 2 pôles d’ordre 2 [17, p. 3].

Récemment, Boach et Yamakawa [26] ont compris comment voir la façon dont le graphe apparaît depuis le point de vue de Betti, en termes des présentations explicites des variétés de caractères sauvages. Ils ont observé que cette reformulation a encore un sens pour les connexions avec un type arbitraire de singularité irrégulière à l’infini, ainsi que des pôles simples à distance finie. En général, ce qu’on obtient n’est plus un graphe proprement dit, mais un objet plus général qu’ils nomment un “diagramme”.

Definition 1.1.2. Un diagramme (au sens de [26]) est la donnée d’un ensemble N de sommets ainsi que d’un entier $B_{ij} = B_{ji} \in \mathbb{Z}$ pour chaque $i, j \in N$, tels que B_{ii} est pair pour tout $i \in N$.

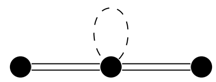
Ainsi un graphe correspond à un cas particulier d’un diagramme pour lequel. $B_{ij} \geq 0$ et $B_{ii} = 0$. Le résultat principal de [26] est le suivant :

Theorem 1.1.3. *Étant donnée une connexion algébrique (E, ∇) sur un ouvert de Zariski de la droite affine, avec uniquement des pôles simples sur $\mathbb{C}$ et un type de singularité quelconque à ∞, on peut définir de manière uniforme un diagramme Γ ainsi qu’un vecteur dimension $\mathbf{d} \in \mathbb{N}^N$ tels que la variété de caractères sauvage $\mathcal{M}_B$ déterminée par (E, ∇) est de dimension*

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

où $(\ ,\ )$ est la forme bilinéaire symétrique déterminée par la matrice de Cartan $C = 2 - B$ of Γ.

Un exemple de base considéré dans [26] est constitué par la troisième équation de Painlevé, en utilisant la représentation de Lax connue sous le nom de Painlevé V dégénérée. Le diagramme et la matrice de Cartan sont :



$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

où la ligne en tirets indique une boucle de multiplicité négative.

Le groupe de Weyl défini par cette matrice de Cartan est isomorphe au groupe de Weyl de type affine B_2 (le groupe de symétrie d’Okamoto de Painlevé III), puisque la matrice de Cartan normalisée (non-symétrique) est la transposée de celle de type affine B_2.

Pour énoncer la définition du diagramme, commençons par rappeler brièvement la description des données de singularité $(\Theta, \boldsymbol{\mathcal{C}})$ (voir ch. 3 pour plus de détails) Chaque point $a \in \mathbb{P}^1$ détermine une collection $\mathcal{I}_a$ de cercles $\langle q \rangle$. En bref, chaque cercle $\langle q \rangle$ est la surface de Riemann d'un germe de fonction de la forme

$$q = \sum_{i=0}^{s} a_i z_a^{-\alpha_i/\beta}, \tag{1.1.2}$$

sur un secteur ouvert en a, où $a_i \in \mathbb{C}$, $a_i \neq 0$ et $\alpha_i, \beta \geq 1$ sont des entiers, dans une coordonnée locale z_a au voisinage de la singularité $a \in \mathbb{P}^1$. À chaque cercle $\langle q \rangle, q = \sum a_i z_a^{-\alpha_i/\beta}$ sont associés plusieurs entiers comme suit :

- $\mathrm{slope}(q)$ est le plus grand rationnel α_i/β présent dans q,
- L'ordre de ramification $\mathrm{ram}(q)$ est le plus petit entier possible $\beta \geq 1$ dans q,
- L'irrégularité $\mathrm{Irr}(q) \in \mathbb{N}$ est le produit $\mathrm{ram}(q)\mathrm{slope}(q)$.

Par définition, une *classe irrégulière* Θ en a est un choix d'un nombre fini de cercles $\langle q_i \rangle \subset \mathcal{I}_a$, chacun ayant une multiplicité $n_i \geq 1$. Une classe irrégulière peut être écrite comme une somme formelle :

$$\Theta = n_1\langle q_1 \rangle + \cdots + n_m \langle q_m \rangle.$$

Toute connexion algébrique sur un ouvert de Zariski de la droite affine détermine une classe irrégulière : essentiellement les sections horizontales de la connexion au voisinage de a sont des combinaisons linéaires de termes faisant intervenir les exponentielles $e^{q(z)}$. L'irrégularité d'une classe irrégulière $\Theta = \sum n_i \langle q_i \rangle$ est $\mathrm{Irr}(\Theta) = \sum n_i \mathrm{Irr}(q_i)$, et le rang d'une classe irrégulière est donnée par $\sum n_i \,\mathrm{ram}(q_i)$. Chaque paire de cercles $\langle q_1 \rangle, \langle q_2 \rangle$ en a détermine une classe irrégulière $\mathrm{Hom}(\langle q_1 \rangle, \langle q_2 \rangle)$ de rang égal au produit $\mathrm{ram}(q_1)\,\mathrm{ram}(q_2)$.

Des données formelles $(\Theta, \boldsymbol{\mathcal{C}})$ en a consistent en une classe irrégulière $\Theta = \sum n_i \langle q_i \rangle$, ainsi que des classes de conjugaison $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, pour chaque cercle de Θ. Il s'agit des "données de monodromie formelles" déterminées par la connexion (voir §3.1). Deux connexions sont formellement isomorphes en a si et seulement si elles ont les mêmes données formelles en a.

La définition du diagramme dans [26] est la suivante : soit (E, ∇) ne connexion sur la droite affine, avec seulement un pôle en ∞. Soient $(\Theta, \boldsymbol{\mathcal{C}})$ les données formelles de la connexion à l'infini. Notons N_c l'ensemble des cercles présents dans Θ. Le diagramme est construit en collant ensemble un cœur Γ_c et des jambes. Le cœur a pour ensemble de sommets N_c, et il y a une jambe pour chaque sommet du cœur (déterminée comme avant par la classe de conjugaison $\mathcal{C}_i$). Les multiplicités des arêtes B_{ij} du cœur sont définies comme suit :

Definition 1.1.4 ([26]). Notons $\Theta = \sum n_i \langle q_i \rangle$. For any $i, j \in N_c$:

- Si $i \neq j$ alors
$$B_{ij} = A_{ij} - \beta_i \beta_j, \tag{1.1.3}$$
où $A_{ij} = \mathrm{Irr}(\mathrm{Hom}(\langle q_i \rangle, \langle q_j \rangle))$ et $\beta_i = \mathrm{ram}(q_i)$,
- Si $i = j$ alors
$$B_{ii} = A_{ii} - \beta_i^2 + 1. \tag{1.1.4}$$

Cette définition peut aussi s'étendre [26, §2.1] au cas des connexions ayant aussi des pôles simples sur $\mathbb{C}$. Le but de cette thèse est d'étendre la définition au cas des connexions avec un nombre quelconque de pôles de type quelconque. Ainsi, par exemple, la définition de [26]§2.1 s'applique au cas de Painlevé III en utilisant une représentation de Lax alternative (avec deux pôles simples et un pôle à l'infini de classe irrégulière $\langle z^{1/2} \rangle$) mais elle ne s'applique pas à la re-sprésentation standard, qui a deux pôles d'ordre deux. Nous allons voir que le même diagramme apparaît directement à partir de la représentation standard de Painlevé III en utilisant notre définition générale.

1.2 Résultats principaux

Notre premier résultat principal est la définition d'un diagramme dans le cas général des connexions avec un nombre arbitraire de pôles de type quelconque sur la sphère de Riemann :

Theorem 1.2.1. *Étant donnée une connexion algébrique quelconque (E, ∇) sur un ouvert de Zariski de la droite affine $\mathbb{C} = \mathbb{P}^1 \setminus \infty$, il est possible de définir de façon uniforme un diagramme Γ et un vecteur dimension $\mathbf{d} \in \mathbb{N}^N$ tels que la variété de caractères sauvage $\mathcal{M}_B$ déterminée par (E, ∇) soit de dimension*

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

où $(\,,\,)$ est la forme bilinéaire symétrique définie par la matrice de Cartan $C = 2 - B$ de Γ.

Le diagramme $\Gamma(E, \nabla) = \Gamma(\Theta, \boldsymbol{\mathcal{C}})$ que nous associons à (E, ∇) dépend seulement des données formelles $(\Theta, \boldsymbol{\mathcal{C}})$ à tous les points singuliers. Il a la structure suivante : il est constitué d'un cœur Γ_c auquel on vient coller des jambes encodant certaines classes de conjugaison. Soit $\mathcal{I} = \bigcup_{a \in \mathbb{P}^1} \mathcal{I}_a$ l'union disjointe de tous les cercles à tous les points de la sphère de Riemann. Notons $\pi : \mathcal{I} \to \mathbb{P}^1$ l'application envoyant un cercle sur le point auquel il correspond. Comme précédemment, le type formel d'une connexion définit un entier n_i et une classe de conjugaison $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ pour chaque cercle $i \subset \mathcal{I}$, de telle sorte que $n_i = 0$ pour tous les cercles sauf un nombre fini en chaque point singulier.

On modifie alors les données formelles de la façon suivante : si i est un cercle modéré à distance finie ($\pi(i) \in \mathbb{C}$ et $q_i = 0$) alors on remplace n_i par l'entier

$$m_i = \operatorname{rank}(A - 1)$$

où $A \in \mathcal{C}_i$. Ainsi, $A = 1 + vu$ pour une application linéaire surjective $u : \mathbb{C}^{n_i} \to \mathbb{C}^{m_i}$ et une application linéaire injective $v : \mathbb{C}^{m_i} \to \mathbb{C}^{n_i}$. On remplace alors $\mathcal{C}_i$ par la classe de conjugaison $\breve{\mathcal{C}}_i$ de $(1 + uv)$ dans $GL_{m_i}(\mathbb{C})$. Voir 4.1 pour plus de détails; cette classe est appelée le *fils* de $\mathcal{C}_i$ dans [22, appendix] Si $\pi(i) = \infty$ ou si i n'est pas modéré ($q_i \neq 0$), on ne fait pas de modification: $m_i = n_i$ et $\breve{\mathcal{C}}_i = \mathcal{C}_i$. Cela définit des données formelles modifiées $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$, où $\breve{\Theta}$ est la collection des *cercles actifs*, c'est-à-dire les cercles de multiplicité $m_i \geq 1$. Les sommets du cœur N_c de notre diagramme sont donnés par cette collection de cercles :

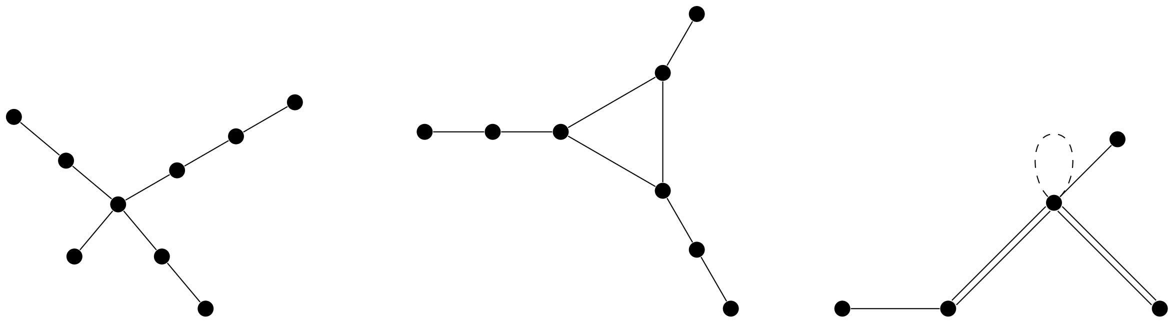
$$N_c = \{\text{circles } i \subset \mathcal{I} \mid m_i \geq 1\}.$$

On colle ensuite la jambe déterminée par la classe de conjugaison $\breve{\mathcal{C}}_i$ sur le noeud $i \in N_c$ comme précédemment. Il reste seulement à définir les entiers $B_{ij} = B_{ji}$ pour $i, j \in N_c$:

Definition 1.2.2. Soient $i, j \in N_c$ des cercles actifs, et notons $a_i = \pi(i), a_j = \pi(j) \in \mathbb{C} \cup \infty$, de telle sorte que $i = \langle q_i \rangle$, $j = \langle q_j \rangle$ soient des cercles en a_i, a_j respectivement. Notons $\alpha_i = \operatorname{Irr}(q_i), \beta_i = \operatorname{ram}(q_i)$ et de même pour j. Si $a_i = a_j$, désignons par $B^{\infty}_{i,j}$ le nombre déterminé par les expressions (2.1.3) and (2.1.4), appliqué aux cercles i, j.

1. Si $a_i = a_j = \infty$ alors $B_{ij} = B^{\infty}_{ij}$.
2. Si $a_i = \infty$ alors $a_j \neq \infty$ then $B_{ij} = B_{ji} = \beta_i(\alpha_j + \beta_j)$.
3. Si $a_i \neq \infty, a_j \neq \infty$ et $a_i \neq a_j$ alors $B_{ij} = B_{ji} = 0$.
4. Si $a_i = a_j \neq \infty$ alors $B_{ij} = B_{ji} = B^{\infty}_{ij} - \alpha_i\beta_j - \alpha_j\beta_i$.

Nous allons prouver de façon directe que ceci définit bien un diagramme dans 4.3, indépendemment de la preuve esquissée dans [26] dans le cas particulier d'une seule singularité irrégulière. Nous allons prouver le thm. 1.2.1 donnant la dimension dans 5.



(a) Singularités régulières (b) cas simplement lacé (c) Cas général

Figure 1.2: Exemples de diagrammes associés à des connexions

L'idée de base menant à cette définition est la suivante : on cherche à se ramener à la situation de [26] en appliquant la transformation de Fourier-Laplace. Celle-ci joue déjà un rôle essentiel dans les constructions précédentes, par exemple elle joue un rôle dans les différentes lectures du graphe dans [17, 20], et permet souvent de passer d'une représentation de Lax à une autre pour les équations de Painlevé. Le transformation de Fourier-Laplace induit une transformation de Fourier formelle au niveau des données formelles [12, 68]. Plus explicitement, la formule de phase stationnaire [70, 44, 91] relie les données formelles globales d'une connexion à celle de sa transformation de Fourier-Laplace. Une propriété remarquable est que la transformation de Fourier-Laplace agit sur les données formelles de manière non locale. Nous utilisons cette propriété pour envoyer tous les cercles actifs à l'infini et nous ramener au cadre de [26]. Le point non trivial est de s'assurer que le diagram obtenu de cette manière ne dépend pas des choix effectués en cours de route.

Esquissons les idées principales de façon un peu plus précise. Les connexions sur des ouverts de Zariski de la droite affine sont étroitement reliées aux modules sur l'algèbre de Weyl $A_1 := \mathbb{C}[z]\langle\partial_z\rangle$. Or, il existe une action naturelle de $SL_2(\mathbb{C})$ on sur l'algèbre de Weyl: une matrice

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

de $SL_2(\mathbb{C})$ induit un automorphisme de l'algèbre de Weyl, par

$$(z, \partial_z) \mapsto (az + b\partial_z, cz + d\partial_z).$$

Parmi ces transformations, la transformation de Fourier-Laplace, correspondant à la matrice

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a une importance particulière.

La formule de phase stationnaire [44, 91] entraîne que ces transformations symplectiques induisent une action de $SL_2(\mathbb{C})$ sur les données de singularité modifiées $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ des connexions. Le résultat principal de cette thèse, garantissant que le diagramme est indépendant des choix, est le suivant :

Theorem 1.2.3. *Le diagramme est invariant par les transformations symlectiques : si $A \in SL_2(\mathbb{C})$, alors on a*

$$\Gamma(A \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})) = \Gamma(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}).$$

Le diagramme constitue donc un invariant sous l'action de $SL_2(\mathbb{C})$. Par conséquent, un même diagramme va pouvoir être lu de différentes manières comme provenant de connexions avec différentes données formelles, reliées par une transformation symplectique.

Un ingrédient important dans la preuve du théorème est une formule explicite pour le nombre d'arêtes et de boucles comme fonction des degrés des termes des facteurs exponentiels, qui a aussi son intérêt propre :

Theorem 1.2.4. *Soient $\langle q\rangle, \langle q'\rangle$ deux facteurs exponentiels distincts à l'infini, et notons*

$$q = \sum_{j=0}^{p} b_j z_\infty^{-\alpha_j/\beta}, \qquad q' = \sum_{j=0}^{p'} b'_j z_\infty^{-\alpha'_j/\beta'}$$

avec $b_j \neq 0$, $b'_j \neq 0$. Leurs ordres de ramification sont respectivement β, β' et leurs pentes sont α/β, α'/β', avec $\alpha := \alpha_0$, $\alpha' := \alpha'_0$. Soit r l'entier telle que les pentes de leurs parties différentes (voir §4.3.2 pour la définition) sont α_r/β et α'_r/β'.

- *Supposons que $\alpha_r/\beta \geq \alpha'_r/\beta'$. Alors le nombre d'arêtes entre $I = \langle q\rangle$ et $I' = \langle q'\rangle$ est*

$$\begin{aligned} B_{I,I'} =& (\beta' - (\alpha'_0, \beta'))\alpha_0 + ((\alpha'_0, \beta') - (\alpha'_0, \alpha'_1, \beta'))\alpha_1 + \cdots + ((\alpha'_0, \dots, \alpha'_{r-2}, \beta') - (\alpha'_0, \dots, \alpha'_{r-1}, \beta'))\alpha_{r-1} \\ &+ (\alpha'_0, \dots, \alpha'_{r-1}, \beta')\alpha_r - \beta\beta'. \end{aligned}$$

- *En particulier, si q et q' n'ont pas de partie commune et $\alpha/\beta \geq \alpha'/\beta'$, alors*

$$B_{I,I'} = \beta'(\alpha - \beta).$$

Ici, la notation $(\cdot, \dots, \cdot)$ désigne le plus grapnd multiple commun d'un n-uplet d'entiers. On a une formule similaire pour le nombre de boucles à un cercle $\langle q\rangle$ (voir §4.3.2).

Plusieurs autres propriétés des carquois dans les cas précédents restent vraies dans le cas plus général. On a encore différentes représentations d'un même diagramme, correspondant à des connexions de rangs différents et de nombres de singularités différents. Par ailleurs, les réflexions de Weyl simples définies à partir de la matrice de Cartan du diagramme, pour certains sommets, peuvent être interprétées comme provenant d'opérations sur les connexions. Il y a en effet plusieurs opérations que l'on peut appliquer aux connexions : les transformations $SL_2(\mathbb{C})$, mais aussi des transformations de Möbius, et l'opération consistant à prendre le produit tensoriel avec une connexion de rang 1. Dans les cas particuliers étudiés dans [17, 20], les éléments du groupe de Weyl associé au diagramme sont la contrepartie, du côté du diagramme, de l'action d'une combinaison de telles opérations sur les connexions correspondantes. Ce phénomène s'étend partiellement au cas général : il existe une classe de cercles (que nous appelons *cercles simples*) tels que les réflexions de Weyl par rapport à ces sommets proviennent d'une combinaison de ces opérations sur les connexions (voir §7.3).

Theorem 1.2.5. *Soient $(\breve{\Theta}, \breve{\mathcal{C}})$ les données formelles modifiées d'une connexion (E, ∇), et $(\Gamma, \mathbf{d}, \mathbf{q})$ le triplet constitué du diagramme (complet), du vecteur dimension et du vecteur de labels multiplicatifs associés. Soit I l'ensemble des sommets de Γ et $i \in I$. Supposons que i n'est pas dans le cœur, ou correspond à un cercle simple. Il existe une combinaison de twists et de transformations $SL_2(\mathbb{C})$ tels que le triplet associé à $\Phi \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ soit*

$$(\Gamma, s_i(\mathbf{d}), r_i(\mathbf{q})),$$

où s_i et r_i sont les réflexions simples par rapport à i, agissant sur $\mathbb{Z}^I$ et $(\mathbb{C}^*)^I$ respectivement.

Applications aux équations de Painlevé Notre construction des diagrammes nous permet d'obtenir un diagramme pour la troisième équation de Painlevé, dont la représentation de Lax standard possède deux singularités irrégulières. Nous retrouvons de cette manière le diagramme de [26], obtenu à partir d'une représentation de Lax alternative. Cette autre représentation est maintenant interprétée comme une représentation différente du même diagramme. Nous obtenons aussi des diagrammes pour les versions "dégénérées" de la troisième équation de Painlevé, dont les singularités irrégulières sont ramifiées.

Il y a des analogues des équations de Painlevé en dimension supérieure, correspondant à des variétés de caractères sauvages de dimension supérieure. En particulier, les systèmes d'isomonodromie en dimension 4 ont été étudiés de façon approfondie par l'école japonaise d'équations différentielles. L'article [65] donne ainsi une liste de représentations de systèmes d'isomonodromie de dimension 4, qui sont obtenus par dégénération de connexions fuchsiennes. Dans de nombreux cas, ces systèmes ont plusieurs représentations correspondant à des lectures différentes du même graphe supernova. Nos diagrammes nous permettent de généraliser cette observation aux cas restants qui n'entrent pas dans le cadre des graphes supernova, de telle sorte que nous avons :

Theorem 1.2.6. *Dans la liste de [81] de représentations des équations de Painlevé (en dimension 2), ainsi que dans la liste de [65] de systèmes d'isomonodromie en dimension 4, toutes les représentations mentionnées pour un même système d'isomonodromie correspondent à des représentations différentes d'un même diagramme.*

Par ailleurs de la même manière que pour les autres équations de Painlevé dont les représentations standard ont seulement une singularité irrégulière, les symétries d'Okamoto de Painlevé III ont une interprétation modulaire en termes d'opérations sur les connexions.(voir §7.5).

Theorem 1.2.7. *Les symétries d'Okamoto des paramètres de l'équation de Painlevé III admettent une réalisation géométrique en termes d'opérations sur les connexions.*

Une partie de ces opérations correspond aux réflexions de Weyl simples des diagrammes associés à ces représentations de Lax. Une spécificité du cas de Painlevé III est qu'il est nécessaire de passer d'un diagramme à un autre (et pas juste à des représentations différentes du même diagramme) pour rendre compte de toutes les symétries.

Orbites sous les opérations sur les connexions Enfin, nous nous intéressons à quelques questions reliées à l'action sur les connexions d'applications successives de transformations $SL_2(\mathbb{C})$, de twists et de transformations de Möbius, que nous appelons opérations de base. Une orbite sous l'action du groupe engendré par ces opérations va toujours contenir des connexions de rang arbitrairement grand (voir par exemple [20, §11.3] pour un exemple explicite).

L'extension au case irrégulier, due à Arinkin [6], de l'algorithme de convolution moyenne de Katz [64] montre que toute connexion rigide irréductible peut être amenée à la connexion triviale de rang 1 par application répétée de telles opérations. Dans le cas non rigide, c'est donc une question intéressante de savoir si toute connexion peut être amenée de façon similaire à une connexion *minimale*, en un sens à préciser.

En utilisant notre formule explicite pour les nombres de boucles pour un diagramme à un sommet, associé à un facteur exponentiel, nous sommes en mesure de formuler quelques résultats préliminaires dans cette direction. Un cas simple à considérer est en effet celui des connexions avec seulement une singularité à l'infini, et un seul cercle actif. Dans ce cas, il existe un algorithme de simplification explicite, qui amène tout facteur exponentiel à un facteur minimal. Pour de petits nombres de boucles, nous obtenons que le diagramme classifie les facteurs exponentiels à applications successives de twists et de la transformation de Fourier-Laplace près : si le nombre de boucles est 0, 1 ou 2, il y a seulement une classe de facteurs exponentiels.

Theorem 1.2.8. *Soit $\langle q \rangle_\infty$ un facteur exponentiel à l'infini et k le nombre de boucles du diagramme associé à $\langle q \rangle$.*

- $k = 0$ *si et seulement si il existe une combinaison de twists et de transformation de Fourier, fournie explicitement par l'algorithme, envoyant $\langle q \rangle_\infty$ sur le cercle modéré $\langle 0 \rangle_\infty$.*
- $k = 1$ *si et seulement si l'algorithme appliqué à $\langle q \rangle_\infty$ termine à un facteur exponentiel avec les mêmes niveaux que $\langle z^{5/3} \rangle_\infty$.*
- $k = 2$ *si et seulement si l'algorithme appliqué à $\langle q \rangle_\infty$ termine à un facteur exponentiel avec les mêmes niveaux que $\langle z^{7/5} \rangle_\infty$.*

Chapter 2

Introduction

Our main result is the construction of a diagram associated to any algebraic connection on a vector bundle on a Zariski open subset of the Riemann sphere, together with a proof that the diagram is invariant under the action of the group of symplectic automorphisms of the Weyl algebra, including the Fourier–Laplace transform. In general this changes the rank of the bundles and the number of poles. As an application, the invariance of the diagram implies the dimensions of the corresponding moduli spaces are the same.

2.1 Context and motivations

2.1.1 (Wild) character varieties and integrable systems

The main motivation for developing a theory of "Dynkin diagrams" for moduli spaces of connections is the perspective of the classification of non-abelian Hodge spaces ([22] Defn. 7). These are (complete) hyperkähler manifolds attached to compact Riemann surfaces with fixed singularity data at some marked points. Their general construction [10] extends the construction [59, 47] in the compact case (no singularities), and [95, 66, 77] in the tame case (simple poles).

Wild non-abelian Hodge theory and the Riemann–Hilbert–Birkhoff correspondence imply the remarkable fact that these analytic moduli spaces are moduli spaces for three different kinds of algebraic or topological objects. The first side is the side of (irregular) Higgs bundles, the so-called Dolbeault side. The second one is the side of (irregular) connections, known as the de Rham side. Finally, the third one is the topological side of representations of the fundamental group, known as the Betti side, and its generalization to Stokes representations. The wild non-abelian Hodge correspondence on curves [10, 90] (building on Hitchin, Donaldson, Corlette and Simpson [59, 42, 33, 96] in the compact case, and Simpson, Biquard, Konno, Nakajima [95, 9, 66, 77] in the tame case) implies that the Dolbeault moduli space $\mathcal{M}_{Dol}$ and the de Rham moduli spaces $\mathcal{M}_{dR}$ are diffeomorphic. They carry a hyperkähler structure, due to the fact that these moduli spaces are isomorphic to moduli spaces $\mathfrak{M}$ of solutions of the 2d self-dual Yang–Mills equations (often called the Hitchin equations). Within the sphere of complex structures on the hyperkähler manifold $\mathfrak{M}$ there are two preferred choices, where $\mathfrak{M}$ becomes $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ respectively. On the other hand, the Riemann–Hilbert correspondence [38], and its extension to the irregular case [69] imply that the de Rham moduli space $\mathcal{M}_{dR}$ and the Betti moduli space $\mathcal{M}_B$ are diffeomorphic. This leads to the following general picture:

$$\begin{tikzcd}
 & & \mathcal{M}_{Dol} \\
\begin{array}{c}(\Sigma, \mathbf{a}, \Theta, \mathcal{C}) \\ \text{wild Riemann surface} \\ \text{and conjugacy classes}\end{array} \arrow[r] & \begin{array}{c}\mathfrak{M} \\ \text{non-abelian Hodge space}\end{array} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{M}_{dR} \\
 & & \mathcal{M}_B
\end{tikzcd}$$

where the three maps on the right are diffeomorphisms. The nonabelian Hodge space $\mathfrak{M}$ depends on the choice of the base smooth complex algebraic curve Σ, together with some marked points $\mathbf{a}$, and singularity data $(\Theta, \mathcal{C})$ at the marked points encoding the type of singularities of the connection. The irregular class Θ encodes the rank of the vector bundles and the exponential factors appearing in horizontal sections of the connection around each singularity, whereas $\mathcal{C}$ denotes the conjugacy classes of the formal monodromies at the singular points (see §3.1 for details). Deformation theory implies it is natural to view the triple $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ as a generalization of a surface with marked points; it is a *wild Riemann surface* in the sense of [21, 25].

The simplest approach is from the Betti side, and we will focus on that in this thesis. The Betti moduli spaces are affine finite-dimensional algebraic symplectic varieties, and can be described using an explicit presentation. When the curve is compact so that there are no singularities, the Betti moduli spaces are the usual character varieties i.e. moduli spaces of representations of the fundamental group of the surface.

$$\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G \cong \{A_1, B_1, \dots, A_g, B_g \in G, [A_1, B_1] \cdots [A_g, B_g] = 1\}/G,$$

where g is the genus of Σ, and $G = GL_n(\mathbb{C})$, n corresponding to the rank of the connection, and $[A, B] = ABA^{-1}B^{-1}$ denotes the multiplicative commutator. The case when the base curve Σ is non-compact e.g. if $\Sigma = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ is a Zariski open subset of the Riemann sphere corresponds to connections with singularities. When all singularities are *regular*, it follows from the Riemann-Hilbert correspondence that a regular singular connection can be reconstructed from its monodromies around the singular points, and the Betti moduli space is given by

$$\mathcal{M}_B \cong \{(M_1, \dots, M_r) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_r,\ M_r \dots M_1 = 1\}/G,$$

where $\mathcal{C}_1, \dots, \mathcal{C}_r \subset G$ are conjugacy classes in G. When $r = 3$ and the matrices have rank 2, this situation corresponds to the well-known Gauss hypergeometric equation and the moduli space is a point. The simplest example of positive dimension is when $r = 4$ and the matrices have rank 2, the moduli space then has complex dimension 2 and this situation corresponds to the (deformed) Heun equation appearing in the standard representation of the Painlevé VI equation.

When the singularities are irregular however, the monodromy representation is no longer enough to reconstruct the connection. Because of the *Stokes phenomenon* there is more topological data to be extracted: the basic reason is that the asymptotic expansions of the horizontal sections of the connection are only valid in some sectors around an irregular singularity and may jump when crossing some particular directions. An explicit presentation of the Betti moduli space can still be given [21, 25], involving Stokes matrices accounting for passing from one sector to the next. If again the base curve is $\Sigma = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$, this explicit presentation is of the form

$$\mathcal{M}_B \cong \{(\mathbf{C}, h, \mathbf{S}) \mid (C_1^{-1} h_1 S_{k_1}^{(1)} \dots S_1^{(1)} C_1) \dots (C_m^{-1} h_m S_{k_m}^{(m)} \dots S_1^{(m)} C_m) = 1\}/G \times \boldsymbol{H},$$

where $C_i \in G$, h_i^{-1} lies in the conjugacy class $\mathcal{C}_i$ of the formal monodromy at a_i, $\boldsymbol{H} = H_1 \times \cdots \times H_m$, with H_j a reductive subgroup of G, and the *Stokes matrices* $S_i^{(j)}$ belong to some unipotent subgroups of G, the Stokes subgroups (see §3.2 for details). Such kinds of presentations are known since the work of Birkhoff [11] at the beginning of the 20th century. These Betti moduli spaces in the irregular case are nowadays referred to as wild character varieties (see [21, 25] where they were constructed as symplectic algebraic varieties, parallel to the earlier analytic constructions [13, 10] and extending the first algebraic constructions [101, 16, 18] in more generic settings). We obtain in this way a large family of moduli spaces associated to a Riemann surface together with singularity data.

It turns out that, when Σ is a Riemann sphere, there are many examples of different wild Riemann surfaces, corresponding to connections with different ranks, different numbers of singularities and order of singularities, giving rise to isomorphic moduli spaces. In particular, a Betti moduli space coming from a connection with regular singularities may be isomorphic to one associated to a connection with irregular singularities. We may view the wild Riemann surfaces yielding isomorphic moduli spaces as determining different "realizations", or "representations" of the same abstract affine symplectic variety $\mathcal{M}_B$. This naturally leads to the following far-reaching question [23], which is the core motivation for this work:

Question Can we describe all representations of a given (abstract) wild character variety $\mathcal{M}_B$? Given two wild Riemann surfaces, is there a direct way to know whether they give rise to isomorphic moduli spaces?

A further motivation for being interested in this question is that the wild character varieties are closely related to many important nonlinear differential equations and integrable systems. On the Dolbeault side, moduli spaces of meromorphic Higgs bundles are algebraic integrable systems [27, 71], specializing to the Hitchin systems [58] in the higher genus case with no poles, and many classical integrable systems in the genus zero case with poles, usually irregular [1, 2, 8, 40, 51, 88], arise in this way. On the other hand, the theory of isomonodromic deformations generates nonlinear differential equations from the de Rham moduli spaces. The basic idea is that wild Riemann surfaces come in families, e.g. we can vary the modulus of Σ and the positions of the singularities, but also the parameters defining the irregular class Θ. If $\underline{\boldsymbol{\Sigma}} \to \mathbb{B}$ is a family of wild Riemann surfaces over a base space $\mathbb{B}$ of admissible deformations, we get a family $\boldsymbol{\mathcal{M}} \to \mathbb{B}$ of moduli spaces over the base $\mathbb{B}$ where the fiber above $b \in \mathbb{B}$ is the moduli space $\mathcal{M}_{dR}(\boldsymbol{\Sigma}_b)$, with $\boldsymbol{\Sigma}_b$ being the wild Riemann surface over b. When b varies in $\mathbb{B}$ varying the connections in $\mathcal{M}_{dR}(\boldsymbol{\Sigma}_b)$ in such a way that the generalized monodromy remains constant yields a (complete) flat Ehresmann connection on $\boldsymbol{\mathcal{M}}$, the isomonodromy connection [13]. When written explicitly in coordinates, the isomonodromy connection corresponds to a system of nonlinear differential equations. From the Betti viewpoint this corresponds to the fact that the wild character varieties form a "local system of varieties" over $\mathbb{B}$ [21].

Many well-known non-linear differential equations arise in this way, most notably the Painlevé equations [93, 50, 62, 60], which correspond to two dimensional moduli spaces, as well as many other differential equations studied by physicists (e.g. [61]). A representation of $\mathcal{M}_B$ corresponds to a choice of Lax representation (or "linearization") of the system of nonlinear differential equations.

In the study of integrable systems, it is very often the case that a given system possesses several Lax representations. Some transformations, such as the Bäcklund transformations, allow to pass from one Lax representation to another. In this way, one may view the question of the classification of wild character varieties as encompassing the problem of the classification of integrable systems arising from meromorphic Hitchin systems and isomonodromy, with the question of finding all possible realisations of one given moduli space amounting to finding the Lax representations giving rise to the associated integrable system or isomonodromy system.

Moduli of meromorphic Higgs bundles also appear in several contexts related to high energy physics: e.g. Seiberg-Witten [94] related some four dimensional $\mathcal{N} = 2$ super Yang-Mills theories to algebraic completely integrable systems, and most examples turn out to be related to irregular meromorphic Higgs bundles [41, 27, 71, 72, 4]. Recent extensions [48, 49, 79, 102] of this story conjecturally recover more of the new complete hyperkahler manifolds constructed in [10], as their "Coulomb branches compactified on a circle". Other instances are the relation to periodic monopoles [32, 31] via the Nahm transform, Witten's approach to wild geometric Langlands [100] and the complex string solutions ("voidons") of the 2d self-dual Yang-Mills equations in [29].

2.1.2 Link with quivers, graphs, diagrams

It turns out that, remarkably, in some cases there seems to be a nice way to parameterize the data determining a moduli space of connections in terms of a graph (i.e. a doubled quiver), and this link sheds light on the question of the classification.

Recall that a finite graph with nodes N (and no edge loops) determines a symmetric (Kac-Moody) Cartan matrix C with

$$C_{ij} = 2 - B_{ij}$$

where $i, j \in N$ and $B_{ij} = B_{ji}$ is the number of edges of the graph between the nodes i, j.

The basic question is to find a map from singularity data to Cartan data:

Question Given a meromorphic connection (E, ∇) on the Riemann sphere, is there a uniform way to define a graph Γ and a dimension vector $\mathbf{d} \in \mathbb{N}^N$ such that the wild character variety $\mathcal{M}_B$ determined by (E, ∇) has dimension

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

where $(\ ,\ )$ is the symmetric bilinear form determined by the Cartan matrix of Γ?

Before stating our main result we will review what is already known.

Okamoto's affine Weyl symmetry groups The simplest examples of nonabelian Hodge spaces are the so-called H3 surfaces, i.e. those of complex dimension two (non-compact analogues of the K3 surfaces). Some of these surfaces are related to Painlevé equations: the de Rham moduli space is isomorphic to the "space of initial conditions" of the Painlevé equation, constructed explicitly by Okamoto [82, 84, 83, 85, 80].

Okamoto showed that each of the six Painlevé equations admits affine Weyl symmetry groups. On the other hand each Painlevé equation is the isomonodromy system of a linear connection on the Riemann sphere, in many different ways. Choices of such linear connections were found by R. Fuchs [46] and R. Garnier [50] and rewritten in matrix form by Schlesinger [93] and Jimbo-Miwa [61]. The singularity data of these "standard Lax representations" of each Painlevé equation is summarised in the table 1.1 (they all correspond to connections on the trivial rank two vector bundle on the Riemann sphere).

This gives a map from some examples of singularity data to affine Weyl groups. The desired map from singularity data to Cartan data, should specialize to give this map (noting that any Cartan matrix determines a Weyl group). Observe that $\dim(\mathcal{M}_B) = 2$ implies $(\mathbf{d}, \mathbf{d}) = 0$ i.e. that $\mathbf{d}$ is null, as is the case for any imaginary root of an affine Kac-Moody algebra: the dimension vectors corresponding to H3 surfaces will indeed be minimal imaginary roots.

Beware that Okamoto also defined a *different* affine Dynkin diagram for each Painlevé equation, the "Okamoto diagram" (listed in [86]). For example the Okamoto diagram of Painlevé II

equation number	pole orders	affine Weyl group symmetry	number of parameters
VI	$1+1+1+1$	D_4	4
V	$2+1+1$	A_3	3
IV	$3+1$	A_2	2
III	$2+2$	B_2	2
II	4	A_1	1
I	$\tilde{4}$	-	0

Figure 2.1: Standard representations and Okamoto symmetries of the Painlevé equations, from [86].

is $\widehat{E}_7$ whereas the affine Weyl symmetry group is of type $\widehat{A}_1$. It is not known how to generalize the Okamoto diagrams to higher dimensional moduli spaces.

Modularity of quiver varieties The next step involves identifying the Nakajima quiver varieties of certain graphs with moduli spaces of connections.

The story involves *additive de Rham moduli spaces* $\mathcal{M}^*$ defined in [13], which are moduli spaces of meromorphic connections on *trivial* holomorphic vector bundles on the Riemann sphere. The Riemann-Hilbert-Birkhoff map sending a connection to its generalized monodromy data induces a symplectic diffeomorphism (see [13, §6])

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_B.$$

between the additive moduli space and a dense open part of the full Betti moduli space.

Nakajima quiver varieties [76] are important objects in representation theory. Given Γ a graph (by graph we mean a doubled quiver) with set of vertices N, $\mathbf{d} \in \mathbb{Z}^N$ a dimension vector, and $\boldsymbol{\lambda} = (\lambda_i)_{i\in N} \in \mathbb{C}^N$, the Nakajima quiver variety $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ is an algebraic symplectic variety, with dimension $2 - (\mathbf{d}, \mathbf{d})$, where $(\cdot, \cdot)$ is the symmetric bilinear form on $\mathbb{Z}^N$ defined from the Cartan matrix of the graph (see chapter 3 for details).

In several cases, it is possible to associate a graph Γ together with a dimension vector $\mathbf{d}$ and labels $\boldsymbol{\lambda}$ to an algebraic connection (E, ∇), such that the additive moduli space $\mathcal{M}^*$ is isomorphic to the quiver variety $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$.

The first ingredient allowing to relate meromorphic connections to quivers is a construction which associates to a conjugacy class $\mathcal{C} \subset GL_n(\mathbb{C})$ a linear quiver [67, 76, 35], together with a dimension vector and labels. The construction depends on a choice of *marking* of the conjugacy class.

In the case of connections with regular singularities, the formal data are entirely determined by the conjugacy classes of the monodromies: the singularity data is just a list $\boldsymbol{\mathcal{C}}$ of m conjugacy classes of $G = GL_n(\mathbb{C})$. Works of Crawley-Boevey and collaborators associate to the connection [34, 35, 36] a star-shaped graph, with the "legs" of the star being the quivers encoding the conjugacy classes of the monodromies. For rank two with four punctures this yields the affine D_4 graph, coinciding with the affine Weyl symmetry group of Painlevé VI.

It turns out that in this case the additive de Rham moduli space $\mathcal{M}^*(\boldsymbol{\mathcal{C}})$ is isomorphic to the Nakajima quiver variety $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ defined by the graph. This result allowed Crawley-Boevey to solve the additive version of the Deligne-Simpson problem, asking for which formal data the addtive moduli space is non-empty. In this setup, simple Weyl reflections associated to the quiver are related to the action of the middle convolution of connections, and the answer to the additive Deligne-Simpson problem is formulated in terms of the root system defined by the quiver. Furthermore, the full moduli space $\mathcal{M}_B$ can be seen as a multiplicative quiver variety [36].

This picture can be extended to the case of irregular connections. This was first done by Boalch in [17, 20], focussing on the simply laced case where the irregular singularity is unramified and is a pole of order at most 3. In this setting, it is still possible to define a triple $(\Gamma, \mathbf{d}, \boldsymbol{\lambda})$ associated to a connection (E, ∇). The quivers that appear are a generalization of star-shaped graphs, the supernova graphs. They consist of a core graph $\Gamma_c(E, \nabla)$, which is a complete k-partite graph. Its vertices correspond to the *exponential factors* $q_i \in z\mathbb{C}[z]$ of the connection at the irregular singularity at infinity, which are basically the eigenvalues of the singular part at infinity of the connection. More precisely the connection can be brought to a formal normal form

$$\nabla_0 = d - \left(dQ + \frac{\Lambda}{z} dz \right),$$

with Q a diagonal matrix of the form

$$Q = \mathrm{diag}(q_1 1_{n_1}, \ldots, q_k 1_{n_k}),$$

(and Λ is a constant matrix). To these vertices are glued legs encoding the conjugacy classes of the pieces of formal monodromy associated to each exponential factor, as well as the conjugacy classes of the monodromies at the regular singularities. If $q_1, \ldots, q_k$ are the exponential factors at infinity, the number of edges in the core diagram between q_i and q_j is given by

$$B_{ij} := \deg(q_i - q_j) - 1, \tag{2.1.1}$$

where $\deg(q_i - q_j)$ is the degree of the difference $q_i - q_j$ as a polynomial in z. The definition of the graph was generalized [17, Appendix C] to connections with an unramified irregular singularity with arbitrary order. In these setting, we still have a "quiver modularity theorem":

Theorem 2.1.1. *Given a meromorphic connection ∇ on a trivial vector bundle E on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$, with only tame poles on $\mathbb{C}$ and an unramified (irregular) singularity at ∞, there is a uniform way to define a graph Γ and a dimension vector $\mathbf{d} \in \mathbb{N}^N$ such that the additive de Rham space $\mathcal{M}^*$ of (E, ∇) is isomorphic to the Nakajima quiver variety $\mathcal{N}_\Gamma(\mathbf{d}, \boldsymbol{\lambda})$ for some parameters $\boldsymbol{\lambda}$. In particular the wild character variety $\mathcal{M}_B$ determined by (E, ∇) has dimension*

$$\dim(\mathcal{M}_B) = \dim(\mathcal{M}^*) = 2 - (\mathbf{d}, \mathbf{d})$$

where $(\ ,\ )$ is the symmetric bilinear form determined by the Cartan matrix $C = 2 - B$ of Γ.

This was conjectured in [17, appendix C] and shown in the "simply laced" case where the irregular singularity is a pole of order at most 3 in [17, 20], and in general in [57]. This result allows again to give a solution of the additive Deligne-Simpson problem in terms of the root system associated to the quiver. The full moduli space may still be considered as a kind of generalized multiplicative quiver variety [22].

Diagrams of Boalch-Yamakawa However, this strategy does not work in general since not all of the additive moduli spaces $\mathcal{M}^*$ are isomorphic to quiver varieties. For example it is known that the additive moduli space $\mathcal{M}^*$ is not isomorphic to a Nakajima quiver variety in the case of 2 poles of order 2 [17, p. 3].

Recently Boalch and Yamakawa [26] understood how to see how the graph in 2) appears from the Betti point of view, in terms of the explicit presentations of the wild character varieties. Moreover they observed that this reformulation works for connections with any type of irregular singularity at ∞, together with any number of tame poles at finite distance. In general the output is no longer a graph, but a generalized object that they called a "diagram".

Definition 2.1.2. A diagram (in the sense of [26]) is a set N of nodes together with the choice of an integer $B_{ij} = B_{ji} \in \mathbb{Z}$ for each $i, j \in N$, such that B_{ii} is even for each $i \in N$.

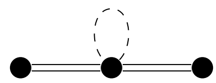
Thus a graph is the special case of a diagram when every $B_{ij} \geq 0$ and $B_{ii} = 0$. The main statement of [26] is that

Theorem 2.1.3. *Given an algebraic connection (E, ∇) on a Zariski open subset of the affine line, with only tame poles on $\mathbb{C}$ and any type of singularity at ∞, there a uniform way to define a diagram Γ and a dimension vector $\mathbf{d} \in \mathbb{N}^N$ such that the wild character variety $\mathcal{M}_B$ determined by (E, ∇) has dimension*

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

where $(\ ,\ )$ is the symmetric bilinear form determined by the Cartan matrix $C = 2 - B$ of Γ.

A basic example considered in [26] is for the third Painlevé equation, using the Lax representation known as degenerate Painlevé V. The diagram and the Cartan matrix are:



$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

where the dashed line indicates a loop of negative multiplicity. The Weyl group of this Cartan matrix is isomorphic to affine B_2 (the Okamoto symmetry group of Painlevé III), since the normalized (non-symmetric) Cartan matrix is the transpose of affine B_2.

To recall the definition of the diagram, we first review briefly the singularity data $(\Theta, \boldsymbol{\mathcal{C}})$ (see chapter 3 for more details). Each point $a \in \mathbb{P}^1$ determines a collection $\mathcal{I}_a$ of circles $\langle q \rangle$. In brief each circle $\langle q \rangle$ is the Riemann surface of a germ of a function of the form

$$q = \sum_{i=0}^{s} a_i z_a^{-\alpha_i/\beta}, \tag{2.1.2}$$

on an open sector at a, where $a_i \in \mathbb{C}$, $a_i \neq 0$ and $\alpha_i, \beta \geq 1$ are integers, in some local coordinate z_a at the singularity $a \in \mathbb{P}^1$. Some numbers attached to each circle $\langle q \rangle, q = \sum a_i z_a^{-\alpha_i/\beta}$ are as follows:

- slope(q) is the largest rational number α_i/β present in q,
- The ramification degree ram(q) is the lowest possible integer $\beta \geq 1$ in q,
- The irregularity Irr$(q) \in \mathbb{N}$ is the product ram(q)slope(q).

By definition an *irregular class* Θ at a is the choice of a finite number of circles $\langle q_i \rangle \subset \mathcal{I}_a$, each with an integer multiplicity $n_i \geq 1$. An irregular class can be written as a formal sum:

$$\Theta = n_1\langle q_1 \rangle + \cdots + n_m\langle q_m \rangle.$$

Any algebraic connection on a Zariski open subset of the affine line determines an irregular class at each singularity; in brief the horizontal sections of the connection in the vicinity of a are linear combinations of terms featuring the exponentials $e^{q(z)}$. The irregularity of any irregular class $\Theta = \sum n_i \langle q_i \rangle$ is $\mathrm{Irr}(\Theta) = \sum n_i \mathrm{Irr}(q_i)$, and the rank of an irregular class is $\sum n_i \,\mathrm{ram}(q_i)$. Any circles $\langle q_1 \rangle, \langle q_2 \rangle$ at a determine an irregular class $\mathrm{Hom}(\langle q_1 \rangle, \langle q_2 \rangle)$ of rank equal to the product $\mathrm{ram}(q_1)\,\mathrm{ram}(q_2)$.

The singularity data $(\Theta, \boldsymbol{\mathcal{C}})$ at a consists of an irregular class $\Theta = \sum n_i \langle q_i \rangle$ together with conjugacy classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, for each circle in Θ. They are the "formal monodromy classes" determined by the connection (see §3.1). Two connections are formally isomorphic at a if and only if they have the same singularity data at a.

The definition of the diagram in [26] is as follows: Suppose we have a connection (E, ∇) on the Riemann sphere with just one pole, at ∞. Let $(\Theta, \mathcal{C})$ be the singularity data of the connection at ∞. Let N_c be the set of circles present in Θ. The diagram is built by gluing together a core diagram Γ_c and legs. The core diagram has nodes N_c and there is one leg for each node of the core diagram (determined by the conjugacy class $\mathcal{C}_i$, as above). The edge multiplicities B_{ij} for the core diagram are as follows:

Definition 2.1.4 ([26])**.** Write $\Theta = \sum n_i \langle q_i \rangle$. For any $i, j \in N_c$:

- If $i \neq j$ then
$$B_{ij} = A_{ij} - \beta_i \beta_j, \tag{2.1.3}$$
where $A_{ij} = \text{Irr}(\text{Hom}(\langle q_i \rangle, \langle q_j \rangle)$ and $\beta_i = \text{ram}(q_i)$,

- If $i = j$ then
$$B_{ii} = A_{ii} - \beta_i^2 + 1. \tag{2.1.4}$$

This definition was extended in [26] §2.1 to the case of connections with further tame poles on $\mathbb{C}$. The aim of this thesis is to extend the definition to the case of connections with any number of poles of any type. For example the definition of [26] §2.1 covered the case of Painlevé III using an alternative Lax representation (with two tame poles and a pole at ∞ with irregular class $\langle z^{1/2} \rangle$) but it did not cover the standard representation, with two poles of order two. We will see how the same diagram appears directly from the standard representation of Painlevé III using our general definition.

2.2 Main results

Our first main result is to define a diagram in the general case of connections with any number of poles of any type on the Riemann sphere:

Theorem 2.2.1. *Given any algebraic connection (E, ∇) on a Zariski open subset of the affine line $\mathbb{C} = \mathbb{P}^1 \setminus \infty$, there is a uniform way to define a diagram Γ and a dimension vector $\mathbf{d} \in \mathbb{N}^N$ such that the wild character variety $\mathcal{M}_B$ determined by (E, ∇) has dimension*

$$\dim(\mathcal{M}_B) = 2 - (\mathbf{d}, \mathbf{d})$$

where $(\ ,\ )$ is the symmetric bilinear form determined by the Cartan matrix $C = 2 - B$ of Γ.

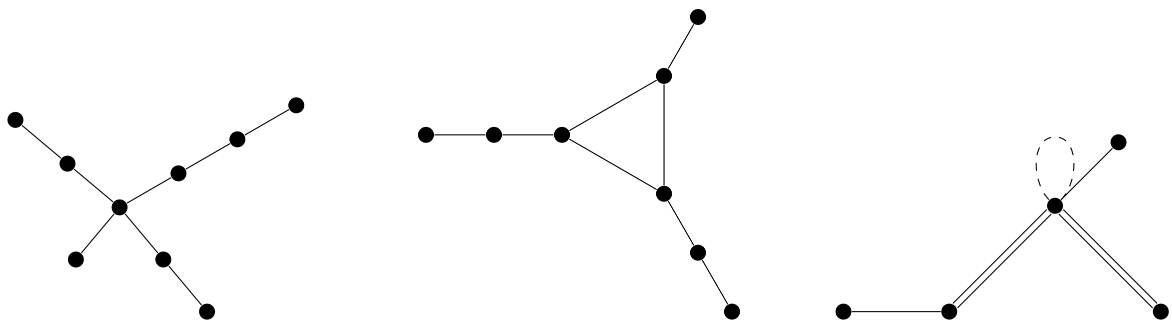
The diagram $\Gamma(E, \nabla) = \Gamma(\Theta, \mathcal{C})$ that we associate to the connection (E, ∇) only depends the singularity data $(\Theta, \mathcal{C})$ at all of the singular points. It has the following structure: it consists of a core diagram Γ_c to which are then glued legs encoding certain conjugacy classes.

Let $\mathcal{I} = \bigcup_{a \in \mathbb{P}^1} \mathcal{I}_a$ be the disjoint union of all the circles at all the points of the Riemann sphere. There is a map $\pi : \mathcal{I} \to \mathbb{P}^1$ taking a circle to the point it lies over. As before the formal type of the connection determines an integer n_i and a conjugacy class $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ for each circle $i \subset \mathcal{I}$, so that $n_i = 0$ for all but a finite number of circles at each singular point.

We then modify this formal data as follows: if i is a tame circle at finite distance ($\pi(i) \in \mathbb{C}$ and $q_i = 0$) then replace n_i by the integer

$$m_i = \text{rank}(A - 1)$$

where $A \in \mathcal{C}_i$. Thus $A = 1 + vu$ for a surjective map $u : \mathbb{C}^{n_i} \to \mathbb{C}^{m_i}$ and an injective map $v : \mathbb{C}^{m_i} \to \mathbb{C}^{n_i}$. Then replace $\mathcal{C}_i$ by the conjugacy class $\breve{\mathcal{C}}_i$ of $(1 + uv)$ in $GL_{m_i}(\mathbb{C})$. See section 4.1 for more details; this class is called the *child* of $\mathcal{C}_i$ in [22, appendix] If $\pi(i) = \infty$ or if i is not tame ($q_i \neq 0$) then we do no modification: $m_i = n_i$ and $\breve{\mathcal{C}}_i = \mathcal{C}_i$. This defines modified formal



(a) Regular singular case (b) simply laced case (c) General case

Figure 2.2: Examples of diagrams associated to connections

data $(\breve{\Theta}, \breve{\mathcal{C}})$, where $\breve{\Theta}$ is the collection of circles with multiplicities $m_i \geq 1$. The core nodes N_c of our diagram are given by this collection of circles:

$$N_c = \{\text{circles } i \subset \mathcal{I} \mid m_i \geq 1\}.$$

We glue the leg determined by the conjugacy class $\breve{\mathcal{C}}_i$ onto the node $i \in N_c$ as before. The remaining step to define the diagram is to define the integers $B_{ij} = B_{ji}$ for $i, j \in N_c$:

Definition 2.2.2. Suppose $i, j \in N_c$ are active circles and write $a_i = \pi(i), a_j = \pi(j) \in \mathbb{C} \cup \infty$, so that $i = \langle q_i \rangle$, $j = \langle q_j \rangle$ are circles at a_i, a_j respectively. Let $\alpha_i = \mathrm{Irr}(q_i), \beta_i = \mathrm{ram}(q_i)$ and similarly for j. If $a_i = a_j$ let us denote by $B^{\infty}_{i,j}$ the number determined by the expressions (2.1.3) and (2.1.4), applied to the circles i, j.

1. If $a_i = a_j = \infty$ then $B_{ij} = B^{\infty}_{ij}$.
2. If $a_i = \infty$ and $a_j \neq \infty$ then $B_{ij} = B_{ji} = \beta_i(\alpha_j + \beta_j)$.
3. If $a_i \neq \infty, a_j \neq \infty$ and $a_i \neq a_j$ then $B_{ij} = B_{ji} = 0$.
4. If $a_i = a_j \neq \infty$ then $B_{ij} = B_{ji} = B^{\infty}_{ij} - \alpha_i\beta_j - \alpha_j\beta_i$.

We will prove directly that this does indeed define a diagram in §4.3, independently of the proof suggested in [26] for the special case of one irregular singularity. We will prove Thm. 2.2.1 computing the dimension in chapter 5.

The basic idea leading to this definition goes as follows: we want to reduce to the situation of [26] by applying the Fourier-Laplace transform. The Fourier-Laplace transform already plays an important role in previous constructions, e.g. it is underlying the different readings of the quiver in [17, 20], and often allows to pass from one Lax representation to another for the Painlevé equations. The Fourier-Laplace transform induces a formal Fourier transform at the level of the formal data [12, 68]. More explicity, the stationary phase formula [70, 44, 91] relates the global formal data of a connection to those of its Fourier-Laplace transform. A remarkable fact is that the Fourier-Laplace transform acts on the formal data in a non-local way. We use this property to send all active circles to infinity and recover the setting of [26]. The non-trivial point is to ensure that the diagram we obtain is independent of the choices made along the way.

Let us sketch the main ideas a little more precisely. Connections on Zariski open subsets of the affine line are closely related to modules over the Weyl algebra $A_1 := \mathbb{C}[z]\langle\partial_z\rangle$. There is a natural action of the group $SL_2(\mathbb{C})$ on the Weyl algebra: a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbb{C})$ induces an automorphism of the Weyl algebra by

$$(z, \partial_z) \mapsto (az + b\partial_z, cz + d\partial_z).$$

Of particular significance among those transformations is the Fourier-Laplace transform, corresponding to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The stationary phase formula [44, 91] implies that those symplectic transformations induce an action of $SL_2(\mathbb{C})$ on modified singularity data $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ of connections. The main result of this work, ensuring the diagram is independent of the choices, leading to our definition, is the following:

Theorem 2.2.3. *The diagram is invariant under symplectic transformations: if $A \in SL_2(\mathbb{C})$, then*

$$\Gamma(A \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})) = \Gamma(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}).$$

The diagram is thus an invariant under the action on $SL_2(\mathbb{C})$. As a consequence, a given diagram can be read in different ways, as coming from connections with different formal data related by a symplectic transformation.

An important ingredient in the proof of this theorem is an explicit formula for the number of edges or loops as a function of the degrees of the terms of the exponential factors, which is of independent interest:

Theorem 2.2.4. *Let $\langle q \rangle, \langle q' \rangle$ two distinct exponential factors at infinity, and write:*

$$q = \sum_{j=0}^{p} b_j z_\infty^{-\alpha_j/\beta}, \qquad q' = \sum_{j=0}^{p'} b'_j z_\infty^{-\alpha'_j/\beta'}$$

with $b_j \neq 0$, $b'_j \neq 0$. Their ramification orders are respectively β, β' and their slopes are α/β, α'/β', with $\alpha := \alpha_0$, $\alpha' := \alpha'_0$. Let r the integer such that the slopes of their different parts (see §4.3.2 for the definition) are α_r/β and α'_r/β'.

- *Assume that and $\alpha_r/\beta \geq \alpha'_r/\beta'$. Then the number of edges between $I = \langle q \rangle$ and $I' = \langle q' \rangle$ is*

$$\begin{aligned} B_{I,I'} =& (\beta' - (\alpha'_0, \beta'))\alpha_0 + ((\alpha'_0, \beta') - (\alpha'_0, \alpha'_1, \beta'))\alpha_1 + \cdots + ((\alpha'_0, \dots, \alpha'_{r-2}, \beta') - (\alpha'_0, \dots, \alpha'_{r-1}, \beta'))\alpha_{r-1} \\ &+ (\alpha'_0, \dots, \alpha'_{r-1}, \beta')\alpha_r - \beta\beta'. \end{aligned}$$

- *In particular, if q and q' have no common parts and $\alpha/\beta \geq \alpha'/\beta'$, then*

$$B_{I,I'} = \beta'(\alpha - \beta).$$

Here, the notation $(\cdot, \dots, \cdot)$ refers to the greatest common divisor of a tuple of integers. There is a similar formula for the number of loops at a circle $\langle q \rangle$ (see §4.3.2).

Several other properties of the quivers in the previous known cases remain true in the more general setting. There are still several representations of the same diagram, generalizing the different readings of the simply laced case, corresponding to different formal data associated to connections with different ranks and different numbers of singularities.

Furthermore, the simple Weyl reflections defined from the Cartan matrix of the diagram with respect to some vertices can be interpreted as coming from operations on connections. There are indeed several operations that one can apply to connections: $SL_2(\mathbb{C})$ transformations, Möbius transformations, and the operation of tensoring with a rank one connection. In the particular cases considered in [17, 20], elements of the Weyl group associated to the diagram are the counterpart on the diagram side of the action of some combination of such operations. This picture partially extends to the general case: there is a class of vertices (corresponding to what we call *simple circles*) such that simple Weyl reflections at those vertices come from some combination of these operations on connections (see §7.3).

Theorem 2.2.5. *Let $(\breve{\Theta}, \breve{\mathcal{C}})$ be modified formal data of a connection (E, ∇), and $(\Gamma, \mathbf{d}, \mathbf{q})$ the triple consisting of the associated (full) diagram, dimension vector, and vector of multiplicative labels. Let I the set of vertices of Γ and $i \in I$. Assume that i is not in the core, or corresponds to a simple circle. There is a combination of twists and $SL_2(\mathbb{C})$ transformations such that the triple associated to $\Phi \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ is*

$$(\Gamma, s_i(\mathbf{d}), r_i(\mathbf{q})),$$

where s_i and r_i are the i-th simple Weyl reflections acting on $\mathbb{Z}^I$ and $(\mathbb{C}^*)^I$ respectively.

Applications to Painlevé equations Our construction of diagrams allows us to obtain a diagram for the third Painlevé equation, whose standard Lax representation features two irregular singularities. We recover in this way the diagram given in [26], which was obtained there from one alternative Lax representation having one ramified irregular singularity along with two simple poles. This alternative Lax representation is simply interpreted in our framework as a different representation of the same diagram. We also obtain diagrams for the "degenerate" versions of the third Painlevé equation, for which the irregular singularities are ramified.

There are higher dimensional analogues of Painlevé equations, corresponding to higher dimensional wild character varieties. In particular, 4-dimensional isomonodromy systems have been extensively studied by the Japanese school of differential equations. The article [65] gives a list of representations of 4-dimensional isomonodromy systems, where they are obtained as degenerations of Fuchsian connections. In many cases, they have several Lax representations corresponding to several representations of the same supernova graph. Our diagrams allow us to generalize this fact to the remaining cases which do not fit in the supernova framework, so that we have:

Theorem 2.2.6. *In the list of [81] of representations of (two-dimensional) Painlevé equations, and in the list of [65] 4-dimensional isomonodromy systems, all different representations mentioned there for a given isomonodromy system correspond to different representations of the corresponding diagram.*

As for the other Painlevé equations involving only one irregular singularity, the Okamoto symmetries of Painlevé III have a modular interpretation in terms of operations on connections (see §7.5).

Theorem 2.2.7. *The Okamoto symmetries of the Painlevé III equation admit a geometrical realization in terms of operations on connections.*

Part of these operations correspond to simple Weyl reflections of the diagrams associated to its Lax representations. The novelty of the Painlevé III case is that we need to pass from one diagram to another (and not just between different representations of the same diagram) to make sense of all the symmetries.

Orbits under operations of connections We finally investigate a few questions related to the action on connections of $SL_2(\mathbb{C})$ transformations, twists, and Möbius transformations. An orbit under the action of the group generated by these operations will always contain connections with arbitrarily large ranks. (see e.g. [20, §11.3] for an explicit example). Arinkin's extension

[6] of Katz' middle convolution algorithm [64] shows that any irreducible rigid connection can be brought to the trivial rank one connection by repeated application of such operations. In the non-rigid case, it is thus an interesting question to ask whether any connection can be brought in a similar way to a connection which is "minimal" in some sense.

Using our explicit formula for the number of edges/loops in the diagram with one vertex associated to one exponential factor, we are able to obtain a couple of preliminary results in this direction. A simple case to consider is the case of connections with just one singularity at infinity, with only one active circle. There is an explicit simplification algorithm bringing any exponential factor to a "minimal" one. For small number of loops, we obtain that the diagram classifies the exponential factors up to twists and Fourier transform: when then number of loops is equal to 0,1 or 2, there is only one class of minimal exponential factors.

Theorem 2.2.8. *Let $\langle q\rangle_\infty$ an exponential factor at infinity and k the number of loops in the diagram associated to $\langle q\rangle$.*

- $k = 0$ *if and only if there exists a combination of twists and applications of the Fourier transform, explicitly provided by the algorithm, sending $\langle q\rangle_\infty$ to the tame circle $\langle 0\rangle_\infty$.*
- $k = 1$ *if and only if the algorithm applied to $\langle q\rangle_\infty$ terminates at an exponential factor with the same levels as $\langle z^{5/3}\rangle_\infty$.*
- $k = 2$ *if and only if the algorithm applied to $\langle q\rangle_\infty$ terminates at an exponential factor with the same levels as $\langle z^{7/5}\rangle_\infty$.*

Chapter 3

Irregular connections, wild character varieties and quivers

In this chapter, we review the main facts about the formal classification of irregular connections, the Stokes phenomenon and the quasi-Hamiltonian description of the wild character varieties. We also recall the known results relating wild character varieties to quivers, part of which we wish to generalize.

3.1 Formal classification of irregular connections

3.1.1 Formal normal form

Let us recall the definition of an algebraic connection (in the one-dimensional case):

Definition 3.1.1. Let Σ a smooth complex algebraic curve. An algebraic connection on Σ is a pair (E, ∇) such that

1. E is an algebraic vector bundle on Σ (i.e. a locally free $\mathcal{O}_\Sigma$-module).
2. $\nabla : E \to E \otimes \Omega^1$ satisfying the Leibniz rule: $\nabla(fs) = f\nabla s + dfs$, for $f \in \mathcal{O}_\Sigma$ and s a section of E.

We have the analogous definitions in the smooth and analytic categories, using the appropriate sheaves of functions and one-forms. In this work, we will be interested in algebraic connections on a Zariski open subsets of the affine line.

In a local trivialization of E above an open subset $U \subset \Sigma$, a connection takes an explicit form

$$\nabla = d - A,$$

where A is a matrix-valued one form. A change of local trivialization is determined by a local gauge transformation, that is a map $g : U \to GL_n(\mathbb{C})$. Under this gauge transformation, the matrix A is transformed into

$$g \cdot A = gAg^{-1} + dgg^{-1}.$$

The problem of formal classification of connections deals with connections on the formal punctured disk.

Definition 3.1.2. Let $K := \mathbb{C}((z))$ the field of formal Laurent series in the variable z. A connection on the formal punctured disk is a pair (E, ∇) where E is a K-vector space and $\nabla : E \to Edz$ a $\mathbb{C}$-linear map satisfying the Leibniz rule.

It turns out that, using formal gauge transformations and possibly passing to a finite cover, i.e. passing to a variable t such that $z = t^r$ for some integer r, it is always possible to transform a connection on the formal punctured disk to a block diagonal normal form. This is the content of the classical Fabry-Hukuhara-Turritin-Levelt theorem:

Theorem 3.1.3 (see e.g. [43, 99, 70, 91])**.** *Let (E, ∇) a connection on the formal punctured disk given by $\nabla = d - Adz$ with $A \in M_n(\mathbb{C}((z)))$. There exists an integer r, a formal gauge transformation $g \in GL_n(\mathbb{C}((t)))$ with $t^r = z$ sending ∇ to a connection of the form*

$$\nabla_0 = d - \left(dQ + \frac{\Lambda}{z}dz\right),$$

with Q a diagonal matrix of the form

$$Q = \mathrm{diag}(q_1 1_{n_1}, \dots, q_k 1_{n_k}),$$

where $q_i \in t^{-1}\mathbb{C}[t^{-1}]$ for $i = 1, \dots, k$ are distinct polynomials in t^{-1} and Λ a block diagonal matrix

$$\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_k),$$

with $\Lambda_i \in M_{n_i}(\mathbb{C})$ a constant matrix.

∇_0 is a formal normal form of the connection. The polynomials $q_1, \dots, q_k$ are the *exponential factors* of the connection, and Λ is its exponent of formal monodromy. Indeed, a fundamental solution of ∇_0 is given by

$$e^Q z^\Lambda,$$

so its coefficients are linear combinations of terms involving the exponentials e^{q_i}, and when going around the singularity at $z = 0$ the fundamental solution gets multiplied by $e^{2i\pi\Lambda}$. This is the monodromy of ∇_0, and is often called the "formal monodromy" of ∇.

Definition 3.1.4. (E, ∇) is *regular singular* if $Q = 0$, that is if it has no exponential factors. Otherwise, (E, ∇) is *irregular*.

3.1.2 The exponential local system $\mathcal{I}$

In this work, it will be more convenient for us to use a different formulation of the formal classification in terms of local systems, that we review now. This is Deligne's approach to the formal classification, which was given an intrinsic formulation in [24] (see also [25]).

The idea is to see the exponential factors as sections of a local system on the circle of directions around the singularity. Let Σ a Riemann surface and $a \in \Sigma$. Let $\pi : \widehat{\Sigma} \to \Sigma$ the real oriented blow-up at a of Σ. The preimage $\partial := \pi^{-1}(a)$ is a circle whose points correspond to the directions around a. An open subset of ∂ corresponds to a germ of sector at a.

Let z be a local coordinate at a. The exponential local system $\mathcal{I}$ is a local system of sets (that is a covering space) on ∂ whose sections are germs of holomorphic functions on sectors of the form

$$\sum_i a_i z^{-k_i},$$

where $k_i \in \mathbb{Q}_{>0}$, and $a_i \in \mathbb{C}$. The connected component of such a local section is a finite order cover of the circle ∂. More precisely, let r the smallest integer such that the expression $q = \sum_i a_i z^{-k_i}$ is a polynomial in $z^{-1/r}$. The corresponding holomorphic function is multivalued, and becomes single-valued when passing to a finite cover $t^r = z$. Therefore, the corresponding connected component, which we denote by $\langle q \rangle$, is a r-sheeted cover of ∂. As a topological space, it is homeomorphic to a circle, and $\mathcal{I}$ is thus a disjoint union of (an infinite number of) circles. The exponential local system can also be defined in an intrinsic way without using an explicit local coordinate (see e.g. [25] Rmk. 2).

Notice that there are several polynomials in $z^{-1/r}$ giving rise to the same connected component $I = \langle q \rangle$. They correspond to the Galois orbit of q, under the Galois group of $I \to \partial$ which is isomorphic to $\mathbb{Z}/r\mathbb{Z}$. Explicitly, if we write $q = \sum_{j=1}^{s} a_j z^{-j/r}$, the polynomials p such that $\langle q \rangle = \langle p \rangle$ are the

$$q_i = \sum_{i=1}^{s} a_j \omega^j z^{-j/r}, \quad i \in \mathbb{Z}/r\mathbb{Z},$$

where $\omega = e^{-\frac{2i\pi}{r}}$. We call r the *ramification order* of the circle $I = \langle q \rangle$, and denote it by $\mathrm{ram}(q)$. T. The degree of q as a polynomial in $z^{-1/r}$ is its irregularity, which we denote by $\mathrm{Irr}(q)$. he slope of q is the quotient $\frac{s}{r}$. If $r = 1$ we say that the circle $\langle q \rangle$ is unramified. We call $\langle 0 \rangle$ the *tame circle*.

If $d \in \partial$, we denote by $\mathcal{I}_d$ the fibre of $\mathcal{I}$ over the direction d. Taking d as a basis point of ∂, the monodromy of $\mathcal{I}$ is the automorphism

$$\rho : \mathcal{I}_d \to \mathcal{I}_d$$

of the fibre $\mathcal{I}_d$ obtained when going around ∂.

3.1.3 Local systems on $\mathcal{I}$

Definition 3.1.5. An $\mathcal{I}$-graded local system on ∂ is a local system $V^0 \to \partial$ of finite dimensional vector spaces, together with a locally constant grading of each fibre I_d, with $d \in \partial$ by the set $\mathcal{I}_d$, where $\mathcal{I}_d$ is the fibre of $\mathcal{I} \to \partial$ above d. This means that for each $d \in \partial$ there is a direct sum decomposition

$$V_d^0 = \bigoplus_{i \in \mathcal{I}_d} V_d^0(i).$$

Let us set $n_i := \dim V_d^0(i)$ for $i \in \mathcal{I}_d$. Since V^0 is finite dimensional, there is only a finite number of elements $i \in \mathcal{I}_d$ for which n_i is nonzero. Furthermore, since the grading is locally constant, n_i only depends on the connected component I of $\mathcal{I}$ containing i, and we set $n_I := n_i$.

Definition 3.1.6. An irregular class is a function $\Theta : \pi_0(\mathcal{I}) \to \mathbb{N}$ with compact support.

The dimensions n_i of the graded pieces thus define an irregular class $\Theta(V^0)$ associated to V^0. We define the *active circles* to be the connected components $I \in \pi_0(\mathcal{I})$ such that $n_I \neq 0$.

Let us again fix a basis direction $d \in \partial$ and denote also by $\rho \in \mathrm{Aut}(V_d)$ the monodromy of V:

$$\rho : V_d^0 \to V_d^0.$$

Since the grading is locally constant, the monodromy of V has to be compatible with the monodromy of $\mathcal{I}$, that is

$$\rho(V_d^0(i)) = V_d^0(\rho_i),$$

for any $i \in \mathcal{I}_d$.

More explicitly, let $I = \langle q \rangle$ be a circle in $\mathcal{I}$ with ramification r. Fix $d \in \partial$ and set $I_d =: \{i_0, \dots, i_{r-1}\}$ such that $\rho(i_k) = i_{k+1}$ for $i = 0, \dots, r-1$ (with $i_r := i_0$). The monodromy of the piece $V_{I,d}^0 := \bigoplus_{k=0}^{r-1} V_d^0(i_k)$ has the form

$$\rho_I = \begin{pmatrix} 0 & \dots & 0 & \rho_{r-1,0} \\ \rho_{0,1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \rho_{r-2,r-1} & 0 \end{pmatrix} \in \mathrm{Aut}(V_I^0) \tag{3.1.1}$$

with $\rho_{i_k,i_{k+1}} : V_d^0(i_k) \to V_d^0(i_{k+1})$.

The group $\mathrm{GrAut}(V^0)$ of graded automorphisms of V^0 is isomorphic to

$$\mathrm{GrAut}(V^0) = \prod_{i \in \mathcal{I}_d} GL(V_d^0(i)).$$

An $\mathcal{I}$-graded local system is entirely determined by the data of its irregular class and the isomorphism class of its monodromy under graded automorphisms.

This language enables us to give a more geometric formulation of the formal classification, due to Deligne.

Theorem 3.1.7 (see [69]). *The category of connections on the formal punctured disk is equivalent to the category of $\mathcal{I}$-graded local systems.*

In this description the active circles of the local system $V^0 \to \mathcal{I}$ correspond to the exponential factors of the connection, and the monodromy of V^0 is related to the exponent of formal monodromy. In particular, a connection is regular if and only if its only active circle is the tame circle $\langle 0 \rangle$.

It is possible to view an $\mathcal{I}$-graded local system $V^0 \to \partial$ as a local system on $\mathcal{I}$, such that the following diagram commutes:

$$\begin{tikzcd} V^0 \arrow[r] \arrow[dr] & \mathcal{I} \arrow[d, "\pi"] \\ & \partial \end{tikzcd}$$

If $i \in \mathcal{I}$, the fibre of V^0 over i is $V_i^0 := V_d^0(i)$, where $d = \pi(i) \in \partial$. If I is a circle of ramification r, when going once around I we go r times around ∂. As a consequence, the monodromy $\hat{\rho}_I$ of the piece V_I^0, now seen as a local system over I, is related to the monodromy of V_I^0 seen as a I-graded local system by

$$\hat{\rho}_I = \rho_{0,r-1} \circ \rho_{r-1,r-2} \circ \cdots \circ \rho_{1,0} : V^0(i) \to V^0(i),$$

keeping the previous notations, with $i_0 = i$.

Theorem 3.1.8. *The category of connections on the formal punctured disk is equivalent to the category of local systems on $\mathcal{I}$ with compact support.*

3.1.4 Global formal data

We now return to the global situation. Let Σ be a Riemann surface. If $a \in \Sigma$, we define as before an exponential local system $\mathcal{I}_a \to \partial_a$, where ∂_a is the circles of directions at a. Let $\mathbf{a} = \{a_1, \ldots, a_m\} \subset \Sigma$ and (E, ∇) be an algebraic connection on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$. Let $\pi : \widehat{\Sigma} \to \Sigma$ the real oriented blowup of Σ at $\mathbf{a}$. We have $\pi^{-1}(\mathbf{a}) = (\partial_{a_1}, \ldots, \partial_{a_m})$. The formal classification associates to (E, ∇) at each singularity a_i a local system $V_{a_i}^0 \to \mathcal{I}_{a_i}$. If $a \in \Sigma$ is not a singularity, we may define $V_a^0 \to \mathcal{I}_a$ to be the trivial local system with all pieces having zero rank. This yields a local system $V^0 \to \mathcal{I}$ on the union of the exponential local systems

$$\mathcal{I} := \sqcup_{a \in \Sigma} \mathcal{I}_a.$$

(In the previous paragraph, we had dropped the subscript in $\mathcal{I}_a$ to simplify the notations, from now on $\mathcal{I}$ refers to the global exponential local system).

Definition 3.1.9. $V^0 \to \mathcal{I}$ is the *formal local system* associated to the connection (E, ∇).

The formal local system $V^0 \to \mathcal{I}$ defines a *global irregular class* $\Theta : \pi_0(\mathcal{I}) \to \mathbb{N}$. We denote by I the set $\Theta^{-1}(\mathbb{N}_{>0})$ of active circles.

3.2 The Stokes phenomenon

The Stokes phenomenon was discovered by Stokes while studying asymptotic expansions for solutions of the Airy equation. The basic idea is the following. The Hukuhara-Turritin-Level theorem answers the question of the formal classification of connections on the punctured disk, but this does not answer the question of the *analytic* classification. If (E, ∇) a connection on the punctured disk. It can be brought to its formal normal form using a formal gauge transformation. When the connection is regular, it turns out that the formal gauge transformation is actually convergent, and defines an analytic gauge transformation. In this case, the formal and analytic classifications coincide. However, when the connection is irregular, this is not the case any more. One can find an analytic gauge transformation which is asymptotic to the formal one, but this only works in limited sectors around the singularity. To characterize the analytic isomorphism class of the connection, one needs further data, the so-called *Stokes data*, to encode how to pass from one sector to another.

The basic reason for which formal gauge transformations can only be resummed on sectors around the singularity is that the asymptotic behaviour of the exponential factors e^q strongly depend on the direction along which one goes towards the singularity. As a consequence, the relative order of dominance of the different exponential factors changes as one goes around the singularity. In this paragraph, we review the description of Stokes data, focussing, among the several approaches, on the description in terms of Stokes local systems, which is the one leading to explicit presentations of the wild character varieties. We refer the reader to [24] for a detailed discussion of the different points of views on Stokes data.

3.2.1 Singular directions, Stokes arrows and Stokes groups

Let (E, ∇) a connection on the punctured disk and $V^0 \to \mathcal{I}_0$ the corresponding formal local system, with irregular class Θ. Let $I \subset \mathcal{I}_0$ the finite subcover of ∂ consisting of the active circles of V^0.

Definition 3.2.1. Let $d \in \partial$. We define a partial order $\prec_d$ on the I_d on the following way. Let $i, j \in I_d$, and let q_i, q_j be the corresponding germs of holomorphic functions. We say that $i \prec_d j$ if and only if the exponential $e^{q_i - q_j}$ decays the fastest on the direction d when z tends to 0. This happens if the leading term of the difference $(q_i - q_j)(z)$ is in $\mathbb{R}_{<0}$ when z is in the direction d.

In this case, we will say there is a *Stokes arrow* from j to i.

Definition 3.2.2. Let $d \in \partial$. If there exists $i, j \in I_d$ such that $i \prec_d j$, then d is a *singular direction*, or *anti-Stokes direction* for (E, ∇). We denote by $\mathbb{A} \subset \partial$ the set of singular directions.

We can give an alternative description of singular directions, using the notion of $\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)$ already mentioned in the introduction.

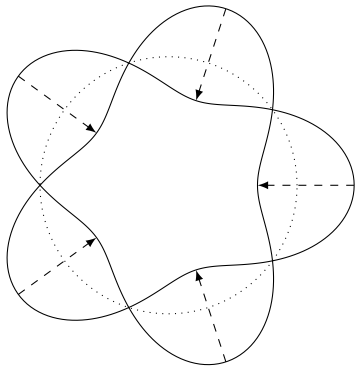
Definition 3.2.3. Let $\langle q \rangle, \langle q' \rangle \in \pi_0(\mathcal{I})$ two circles (not necessarily distinct). We define $\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)$ to be the subcover of $\mathcal{I}$ whose local sections on a small sector U are of the form

$$q_U - q'_U,$$

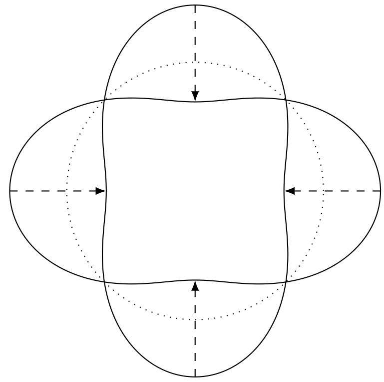
with q_U a local section of $\langle q \rangle$, and q'_U a local section of q'_U.

Because of ramification, the cover $\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)$ can in general have several connected components, possibly with different slopes and ramification orders. For example we have $\mathrm{Hom}(\langle z^{1/2} \rangle, \langle z^{1/2} \rangle) = \langle 0 \rangle \cup \langle 2z^{1/2} \rangle$.

Definition 3.2.4. Let $\langle q \rangle \in \pi_0(\mathcal{I})$ a circle. A point of maximal decay of $\langle q \rangle$ is a point q of $\langle q \rangle$ such that e^q has the fastest decay in the direction $\pi(q) \in \partial$.



(a) One active circle $\langle z^{5/2}\rangle$



(b) Two active circles: $\langle z^2\rangle$ and $\langle -z^2\rangle$

Figure 3.1: Stokes diagrams for two examples of irregular classes. The diagram represents the growth rate of the exponential factors as a function of the direction around the singularity. The dotted circle separates when the exponential factor is growing or decreasing. The Stokes arrows are also represented.

All circles except the tame circle $\langle 0\rangle$ have points of maximal decay: the number of points of maximal decay of a circle $\langle q\rangle$ is equal to the irregularity $\mathrm{Irr}(q)$.

If $\langle q_1\rangle,\dots,\langle q_s\rangle$ are the active circles of the irregular class Θ, the Stokes arrows correspond to the points of maximal decay of the covers $\mathrm{Hom}(\langle q_i\rangle,\langle q_j\rangle)$, $i,j=1,\dots,s$. The singular directions are the images by π of these points of maximal decay.

Definition 3.2.5. Let $d\in\partial$. We say that d is a Stokes direction for (E,∇) is there exists $i,j\in I_d$ such that the leading term of the difference $(q_i-q_j)(z)$ is in $i\mathbb{R}$ when z tends to 0 in the direction d. The Stokes directions are the directions where the dominance ordering of the exponential factors change.

When the exponential factors have just one level, that is consist of only one monomial, the relative dominance order of the exponential factors can be visualized on the *Stokes diagram* of the connection, which is obtained by plotting $|e^q|$ as a function of $d\in\partial$ for each active circle $\langle q\rangle$. The Stokes directions are the directions where some of the strands of the diagram cross.

Definition 3.2.6. Let $d\in\mathbb{A}$ a singular direction. The Stokes group associated to d is the unipotent subgroup $\mathbb{S}\mathrm{to}_d\subset GL(V_d^0)$ whose Lie algebra is given by

$$sto_d := \bigoplus_{i\prec_d j}\mathrm{Hom}(V_d^0(j),V_d^0(i))$$

3.2.2 Stokes local systems

The way to pass from one sector to the next can be axiomatized by the notion of *Stokes local systems* [25, 24], that we now describe. The idea is to define a new Riemann surface by adding a tangential puncture at each singular direction. A Stokes local system is then a local system on this new surface, such that the monodromies around the tangential punctures have to belong to the corresponding Stokes groups.

More precisely, we define the new Riemann surface $\widetilde{\Sigma}(\Theta)$ as follows. We define the halos $\mathbb{H}\subset\widehat{\Sigma}$ as a tubular neighbourhood of ∂. $\mathbb{H}$ is a union $\mathbb{H}=\mathbb{H}_1\cup\cdots\cup\mathbb{H}_m$ of annuli at each singularity. One of the boundary of each annulus $\mathbb{H}_i$ is the circle ∂_i. Let ∂_i' be the other boundary of $\mathbb{H}_i$, and $\partial'=\partial_1'\cup\dots\partial_m'$. Let $e:\partial\to\partial'$ an homeomorphism preserving the order.

Let Θ be a global irregular class corresponding to Σ°. It defines a set $\mathbb{A}\subset\partial$ of (global) singular directions. We define the new Riemann surface as

$$\widetilde{\Sigma}(\Theta) := \widehat{\Sigma}\setminus e(\mathbb{A}),$$

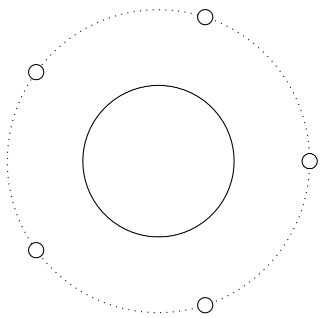


Figure 3.2: Local picture: the halo and the tangential punctures at a singulartity.

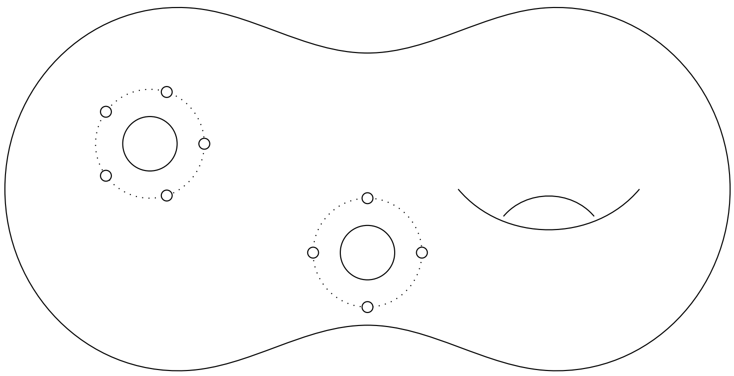


Figure 3.3: Global picture of a Stokes local system: at each singularity there is a halo with tangential punctures.

that is for each singular direction $d \in \mathbb{A}$ we remove from $\widehat{\Sigma}$ the corresponding *tangential puncture* $e(d)$. Let us denote by γ_d the small positive loop in $\tilde{\Sigma}$ starting from d, going around the tangential puncture $e(d)$ and going back to d.

Definition 3.2.7. A Stokes local system is a pair $(\mathbb{V}, \Theta)$ where Θ is a global irregular class and $\mathbb{V}$ is a local system of vector spaces on $\widetilde{\Sigma}(\Theta)$ equipped with an I-grading over the halo $\mathbb{H}$ (of dimension Θ), such that the monodromy $\mathbb{S}_d := \rho(\gamma_d)$ is in $\mathbb{S}\mathrm{to}_d \subset GL(V_d)$ for each $d \in \mathbb{A}$.

The Stokes local system yield a topological description of the category of algebraic connections on Σ°.

Theorem 3.2.8. *The category of algebraic connections on Σ° is equivalent to the category of Stokes local systems $(\mathbb{V}, \Theta)$ with singularities on $\mathbf{a}$.*

3.3 Wild character varieties

The topological description of meromorphic connections in terms of Stokes local systems gives an explicit presentation of the Betti moduli spaces, the wild character varieties. In this paragraph, we review these presentations as well has their symplectic structures.

3.3.1 Quasi-Hamiltonian geometry

The wild character varieties are obtained as reductions of some quasi-Hamiltonian spaces. We thus recall here the basic facts about complex quasi-Hamiltonian geometry needed to describe their quasi-Hamiltonian structure, following mostly [21, 25]. The basic idea [3] is that quasi-Hamiltonian geometry is a multiplicative version of usual Hamiltonian geometry. Whereas for a Hamiltonian G-space the moment map takes its values in the (dual of the) Lie algebra of the

Lie algebra of the group G, for a quasi-Hamiltonian G-space it takes its values in the group G itself. This idea first appeared for the circle group [73] then for general compact Lie groups [3], before being translated to the complex algebraic setting in [16] to construct the generic wild character varieties, and then extended in [21] to the general case.

Let G a connected complex reductive group, and $\mathfrak{g}$ its Lie algebra (we will only need here the case $G = GL_n(\mathbb{C})$). We fix an invariant symmetric nondegenerate bilinear form $(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$. Let $\theta, \bar{\theta} \in \Omega^1(M, \mathfrak{g}$ be the Maurer-Cartan forms on G. In the case we $G = GL_n(\mathbb{C})$ we are interested in, G is a matrix group and we have $\theta = g^{-1}dg$, $\bar{\theta} = dgg^{-1}$. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Omega^1(M, \mathfrak{g})$ are $\mathfrak{g}$-valued one-forms, define $(\mathcal{ABC}) := (\mathcal{A}, [\mathcal{B}, \mathcal{C}])/2 \in \Omega^3(M)$ (this is invariant under permutation of $\mathcal{A}, \mathcal{B}, \mathcal{C}$). The canonical bi-invariant three-form on M is $\frac{1}{6}(\theta^3)$. We denote by $gXg^{-1} := \mathrm{Ad}_g(X)$ the adjoint action of G on $\mathfrak{g}$, with $g \in G$ and $X \in \mathfrak{g}$. If $X \in \mathfrak{g}$, the fundamental vector field v_X associated to X is defined by $(v_X)_m := -\frac{d}{dt}(e^{tX}\cdot m)_{|t=0}$.

Definition 3.3.1. A quasi-Hamiltonian G-space is a complex manifold M together with an action of G on M, an equivariant map $\mu : M \to G$ (where the action of G on itself is the conjugation), and a holomorphic 2-form $\omega \in \Omega^2(M)$, such that

1. $d\omega = \mu^*(\theta^3)/6$, with $\theta^3/6$ the canonical three-form on G.
2. For any X in $\mathfrak{g}$, $\omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X) \in \Omega^1(M)$.
3. For each point $m \in M$, $\mathrm{Ker}\,\omega_m \cap \mathrm{Ker}\,d\mu = \{0\}$.

Example 3.3.2. Let $G = GL_n(\mathbb{C})$ and $\mathcal{C} \subset G$ be a conjugacy class. Then $\mathcal{C}$ has a structure of quasi-Hamiltonian G-space, with the action of G by conjugation, the moment map simply given by the inclusion $\mathcal{C} \subset G$, and the two-form ω such that

$$\omega_g(v_X, v_Y) = \frac{1}{2}((X, gYg^{-1}) - (Y, gXg^{-1}))$$

In particular, if G is abelian or trivial, M is a symplectic manifold.

Example 3.3.3 (The double)**.** The space $G \times G$ is a quasi-Hamiltonian $G \times G$-space, with the action of $G \times G$ given by $(g, k) \cdot (C, h) = (kCg^{-1}, khk^{-1})$, with moment map

$$\mu(C, h) = (C^{-1}hC, h^{-1}) \in G \times G.$$

The two form is given by

$$2\omega = (\bar{\gamma}, \mathrm{Ad}_h\,\bar{\gamma}) + (\bar{\gamma}, \bar{\eta} + \eta),$$

where $\bar{\gamma} = C^*(\theta)$, $\eta = h^*(\theta)$, and $\bar{\eta} = h^*(\bar{\theta})$, where the notation $C^*(\theta)$ denotes the pullback under the map $(C, h) \mapsto C$ from $G \times G$ to G of the form θ, and similarly for the other forms.

The quasi-Hamiltonian version of Hamiltonian reduction can be stated as follows.

Theorem 3.3.4. *Let M a quasi-Hamiltonian $G \times H$-space, with moment map $(\mu_G, \mu_H) : M \to G \times H$. Assume that the quotient of $\mu_G^{-1}(1_G)$ by G of the inverse image of the identity in G under the first component of the moment map is a nonsingular manifold. Then the restriction of the two-form to $\mu_G^{-1}(1)$ induces a symplectic form on the quotient*

$$M /\!\!/ G := \mu^{-1}(1)/G$$

that makes it a quasi-Hamiltonian H-space.

In particular, if H is abelian or trivial, the quasi-Hamiltonian quotient is a symplectic manifold.

An important construction with quasi-Hamiltonian spaces is the fusion product, which we now define.

Theorem 3.3.5. *Let M be a quasi-Hamiltonian $G \times G \times H$-space with moment map $\mu = (\mu_1, \mu_2, \mu_3) : M \mapsto G \times G \times H$. Let us consider the action of $G \times H$ on M defined via the diagonal embedding $(g, h) \mapsto (g, g, h)$. This makes M a quasi-Hamiltonian $G \times H$ space, with two-form*

$$\widetilde{\omega} = \omega - \frac{1}{2}(\mu_1^* \theta, \mu_2^* \bar{\theta}),$$

and with moment map

$$\tilde{\mu} = (\mu_1 \cdot \mu_2) : M \to G \times H.$$

Definition 3.3.6. Let M_1 a $G \times H_1$-space and M_2 a $G \times H_2$-space. Their *fusion product*

$$M_1 \circledast M_2$$

is the quasi-Hamiltonian $G \times H_1 \times H_2$-space obtained from the quasi-Hamiltonian $G \times G \times H_1 \times H_2$-space $M_1 \times M_2$ by applying the previous theorem to the two factors of G.

In particular, this operation can be used to define the quasi-Hamiltonian reduction of a $G \times H$ space taken at a conjugacy class $\mathcal{C} \in G$ rather that at the value 1, by setting

$$M /\!\!/_{\mathcal{C}} G = \mu_G^{-1}(\mathcal{C})/G \cong (M \circledast \mathcal{C}') /\!\!/ G,$$

where $\mathcal{C}'$ denotes the inverse conjugacy class and we take the reduction of the fusion product at the value 1_G.

Example 3.3.7. The internally fused double $\mathbb{D}$ is obtained by taking the fusion of the two copies of G in the double $\mathbf{D} = G \times G$. It is a quasi-Hamilonian G-space, with G acting as $g \cdot (a, b) = (gag^{-1}, gbg^{-1})$. The moment map is given by the commutator

$$\mu(a, b) = aba^{-1}b^{-1},$$

and the two-form by

$$2\omega = -(a^*\theta, b^*\bar{\theta}) - (a^*\bar{\theta}, b^*\theta) - ((ab)^*\theta, (a^{-1}b^{-1})^*\bar{\theta}).$$

As we shall see, $\mathbb{D}$ will be a building block in the quasi-Hamiltonian description of wild character varieties for surfaces with nonzero genus.

3.3.2 Twisted quasi-Hamiltonian spaces

Because of the presence of ramification in the exponential factors, we need to consider a further extension: twisted quasi-Hamiltonian spaces [25].

As before, let G be a connected complex reductive group. The ingredient we add to the untwisted case is that we consider $\phi \in \mathrm{Aut}(G)$ an automorphism of G, that we call the *twist*. We fix a symmetric nondegenerate complex bilinear form $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$ of G, which is invariant under the adjoint action of G as well as under ϕ. Let $\Gamma \subset \mathrm{Aut}(G)$ be the subgroup generated by ϕ, and $\widetilde{G}(\phi) := G \ltimes \Gamma \subset G \ltimes \mathrm{Aut}(G)$. We define

$$G(\phi) := \{(g, \phi) |\ g \in G\} \subset \widetilde{\Gamma}(\phi).$$

The natural left and right actions of G on $G(\phi)$ are free and transitive: this implies that $G(\phi)$ is a G-bitorsor. Indeed, if $(g_1, g_2) \in G \times G$, we have $g_1 \cdot (g, \phi) \cdot g_2 = (g_1 g \phi(g_2), \phi)$. In particular, we have now the ϕ-conjugation action.

Definition 3.3.8. A twisted quasi-Hamiltonian G-space is a complex manifold M equipped with an action of G, with an invariant holomorphic two-form ω, and a G-equivariant moment map $\mu : M \to G(\phi)$ to a twist of G, where G acts on $G(\phi)$ by the twisted conjugation action.

A twisted quasi-Hamiltonian G-space is thus close to a quasi-Hamiltonian $\widetilde{G}$-space, the difference being that we restrict to the action of the identity component of $\widetilde{G}$.

The fusion operation carries over to the twisted case. If M_1, M_2 are twisted quasi-Hamiltonian G-spaces, with moment maps $\mu_1 : M_1 \to G(\phi_1)$, $\mu_2 : M_2 \to G(\phi_2)$, their fusion $M_1 \circledast M_2$ is a twisted quasi-Hamiltonian G-space with moment map $\mu : M_1 \circledast M_2 \to G(\phi_1\phi_2)$.

3.3.3 Quasi-Hamiltonian construction of wild character varieties

In the same way as moduli spaces of local systems are character varieties, moduli spaces of Stokes local systems give rise to the wild character varieties.

Keeping previous notations, let $(\mathbb{V}, \Theta)$ a Stokes local system with singularities on $\mathbf{a} \subset \Sigma$. Let us choose a base point $b_i \in \partial_{a_i}$ on the circle of directions around each a_i. Let us also fix a framing of $\mathbb{V}$ at d_i, that is an isomorphism of vector spaces

$$\mathbb{F}_i \cong \mathbb{V}_{b_i},$$

where

$$\mathbb{F}_i = \mathbb{C}^{\Theta_{a_i}} := \bigoplus_{j \in (\mathcal{I}_{a_i})_{b_i}} \mathbb{C}^{\Theta(j)}.$$

is graded by $(\mathcal{I}_{a_i})_{b_i}$, such that each piece $\mathbb{F}(j)$ has dimension $\Theta(j)$.

Let $\Pi := \Pi_1(\widetilde{\Sigma}, \mathbf{b})$, where $\mathbf{b} := \{b_1, \ldots, b_m\}$ the fundamental groupoid of $\widetilde{\Sigma}$ with base points b_i.

Let $\mathrm{Hom}(\Pi, G)$ be the set of isomorphism classes of representations of this groupoid into G. A framed local system determines via its monodromy an element in $\mathrm{Hom}(\Pi, G)$. Actually, because of the conditions in the definition of a Stokes local system. The representation associated to a framed Stokes local system lives in a subset $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of *Stokes representations*, that we will now describe.

Let us view each boundary circle ∂_i as a loop based in b_i. Let $V_i^0 \to \partial_{a_i}$ be the $\mathcal{I}_{a_i}$ graded local system on ∂_{a_i} with irregular class Θ_{a_i} associated to $\mathbb{V}$. Let us denote by $I_1, \ldots, I_k$ the active circles corresponding to Θ_{a_i}. In the direct sum decomposition $(V^0)_{b_i} = \bigoplus_{l=1}^{k} (V^0_{I_l})_{b_i}$, with $(V^0_{I_l})_{b_i} = \bigoplus_{j \in (I_l)_{b_i}} V^0(j)$, its monodromy (taking b_i as basis point) is of the form

$$\rho(\partial_{a_i}) = \begin{pmatrix} \rho_{I_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \rho_{I_k} \end{pmatrix} \subset GL(V^0_{b_i}) \cong GL(\mathbb{F}_i),$$

with each ρ_{I_l} of the form (3.1.1). The set of such matrices (seen as elements in $GL(\mathbb{F}_i)$ is a twist of the group

$$H_i := \mathrm{GrAut}(\mathbb{F}_i) = \prod_{j \in (\mathcal{I}_{a_i})_{b_i}} GL_{\Theta(j)}(\mathbb{C}),$$

that we denote by $H(\partial_{a_i})$.

If $d \in \mathbb{A}_i$ is a singular direction at a_i, let $\lambda_d \subset \partial_i$ an arc from b_i to d. By parallel translation along λ_d we may identify the Stokes group $\mathbb{S}\mathrm{to}_d \subset GL(\mathbb{V}_d)$ to a subgroup of $GL(\mathbb{F}_i) = GL_n(\mathbb{C})$. Let us also define

$$\hat{\gamma}_d := \lambda_d^{-1} \circ \gamma_d \circ \lambda_d \in \Pi$$

to be the simple loop around the tangential puncture $e(d)$ based on b_i where γ_d designates as before the simple loop around $e(d)$ based on d. From the definition of a Stokes local system we immediately get that the monodromy $\rho(\hat{\gamma}_d)$ of $\mathbb{V}$ around $\hat{\gamma}_d$ belongs to $\mathbb{S}\mathrm{to}_d$. In summary:

Lemma 3.3.9. *Let $\rho \in \mathrm{Hom}(\Pi, G)$ the representation associated to a Stokes local system $(\mathbb{V}, \Theta)$. Then ρ satisfies the following conditions:*

1. *$\rho(\partial_{a_i}) \in H(\partial_{a_i})$ for each boundary circle ∂_i.*
2. *$\rho(\hat{\gamma}_d) \in \mathbb{S}\mathrm{to}_d$ for any singular direction $d \in \mathbb{A}$.*

Definition 3.3.10. A representation $\rho \in \mathrm{Hom}(\Pi, G)$ is a *Stokes representations* if it satisfies these two conditions.

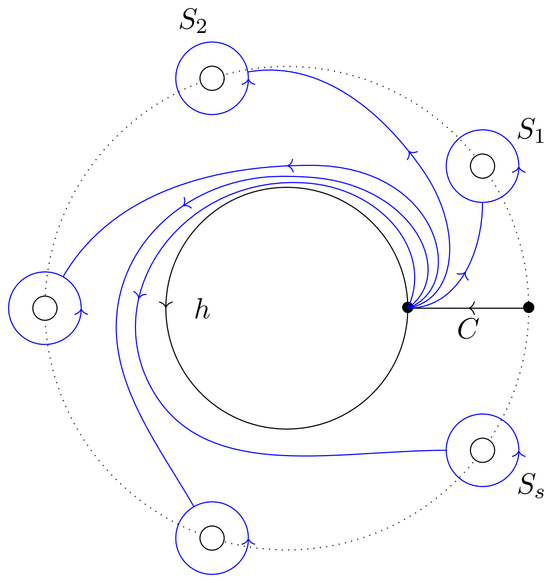


Figure 3.4: The paths for expressing the monodromies of a Stokes local system.

The set $\widetilde{M}_B(\Sigma, \Theta)$ of isomorphism classes framed local systems with irregular class Θ is in bijection with the space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of Stokes representations. There is a natural action of the group $\mathbf{H} := H_1 \times \cdots \times H_m$ on $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$, which amounts to changing the framings.

By choosing a basis of paths in Π, we get an explicit presentation of $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$, which we now describe. $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is obtained as the fusion of building blocks $\mathcal{A}(V^0_{a_i})$ associated to each singularity, which are quasi-Hamiltonian spaces.

Assuming for a while that there is only one singularity $a \in \Sigma$ to simplify notations and drop the subscripts i, we define

$$\mathcal{A}(V^0) := G \times H(\partial) \times \mathbb{S}\mathrm{to},$$

where $\mathbb{S}\mathrm{to} = \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d \subset GL_n(\mathbb{C})$ is the product of Stokes groups. Elements of $\mathcal{A}(V^0)$ will be denoted by $(C, h, \mathbf{S})$, with $\mathbf{S} = (S_1, \dots, S_s)$, $d_1, \dots, d_k$ being the successive singular directions at a (going in the positive direction from b). There is a natural action of $G \times H$ on $\mathcal{A}(V^0)$, given by

$$(g, k) \cdot (C, h, \mathbf{S}) = (kCg^{-1}, khk^{-1}, k\mathbf{S}k^{-1}). \tag{3.3.1}$$

We define an algebraic two-form on $\mathcal{A}(V^0)$ by the formula

$$2\omega = (\bar{\gamma}, \mathrm{Ad}_b(\bar{\gamma})) + (\bar{\gamma}, \bar{\beta}) + (\bar{\gamma}_s, \bar{\eta}) - \sum_{j=1}^{s} (\gamma_j, \gamma_{j-1}), \tag{3.3.2}$$

where $\gamma_j := c_j^*(\theta)$, $\bar{\gamma}_j := c_j^*(\bar{\theta})$, $\eta := h^*(\theta_H)$, $\beta := b^*(\bar{\theta})$, with $\theta, \bar{\theta}$ the Maurer-Cartan one-forms on $\widetilde{G}$, $\theta_H, \bar{\theta}_H$ the Maurer-Cartan forms on $\widetilde{H}$, $c_j : \mathcal{A}(V^0) \to G$ the application given by $c_j = S_j \dots S_1$ for $j = 1, \dots, s$ and $c_0 := C$, and $b : \mathcal{A}(V^0) \to G$ defined by $b = hS_s \dots S_1$. Let us denote by $\bar{\partial}$ the inverse of ∂ in Π that is going around the circle ∂ in the negative direction, and let $H(\bar{\partial})$ denote the corresponding twist of H.

Theorem 3.3.11. *$\mathcal{A}(V^0_{a_i})$ is a twisted quasi-Hamiltonian space, with moment map*

$$\mu_i(C, h, \mathbf{S}) = (C^{-1}hS_s \dots S_1, h^{-1}) \in G \times H(\bar{\partial}). \tag{3.3.3}$$

Going back to the general case with several singularities, the space of Stokes representations is now obtained as the quasi-Hamiltonian fusion of all $\mathcal{A}(V^0_{a_i})$, together with g copies of the double, where g is the genus of Σ.

Theorem 3.3.12 ([25])**.** *The space* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ *is a smooth affine complex algebraic variety, and is a quasi-Hamiltonian* $\mathbf{H}$*-space, with moment map given by*

$$\begin{aligned}\mu : \mathrm{Hom}_{\mathbb{S}}(\Pi, G) &\to \mathbf{H}(\bar{\partial}) \\ \rho &\mapsto \{\rho(\bar{\partial}_{a_i})\},\end{aligned}$$

where $\mathbf{H}(\bar{\partial}) := \prod_{i=1}^m H(\bar{\partial}_{a_i})$*, with* $\bar{\partial}_{a_i}$ *referring as above to the loop around* ∂_{a_i} *in the negative sense.*

Explicitly, given a choice of framing, we have

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \cong \left(\mathbb{D}^{\circledast g} \circledast \mathcal{A}(V^0_{a_1}) \circledast \cdots \circledast \mathcal{A}(V^0_{a_m})\right) /\!\!/ G. \tag{3.3.4}$$

with $\mathcal{A}(V^0_{a_i}) := G \times H(\partial_{a_i}) \times \mathbb{S}\mathrm{to}_i$, where $\mathbb{S}\mathrm{to}_i = \prod_{d \in \mathbb{A}_i} \mathbb{S}\mathrm{to}_d$ and the quasi-Hamiltonian structure described above.

Let us denote by $(A_j, B_j, C_i, h_i, \mathbf{S}^i, j = 1, \ldots, g,\ i = 1, \ldots, m)$ an element of the fusion product $\mathbb{D}^{\circledast g} \circledast \mathcal{A}(V^0_{a_1}) \circledast \cdots \circledast \mathcal{A}(V^0_{a_m})$. The G-component of the moment map is given by

$$\mu_G = \prod_{j=1}^{g} [A_j, B_j] \prod_{i=1}^{m} (C_i^{-1} h_i S^i_{s_i} \ldots S^i_1 C_i), \tag{3.3.5}$$

where $[A, B] := ABA^{-1}B^{-1}$ is the multiplicative commutator. This generalizes the presentations of usual character varieties.

Finally, the wild character varieties are obtained by taking the quasi-Hamiltonian reduction of $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ by $\mathbf{H}$. We define a twisted conjugacy class $\mathcal{C}_i \in H_i(\partial_{a_i})$ to be an orbit of $H_i(\partial_{a_i})$ under the twisted conjugation action of H_i. A global twisted conjugacy class is an element $\mathcal{C} := \prod_{i=1}^m \mathcal{C}_i \in \mathbf{H}(\partial)$. Let $\mathcal{M}_B(\Sigma, \Theta, \boldsymbol{\mathcal{C}})$ the affine variety associated to the ring of $\mathbf{H}$-invariant functions on $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$.

Theorem 3.3.13 ([25])**.** *The wild character variety* $\mathcal{M}_B(\Sigma, \Theta)$ *is an algebraic Poisson variety. Its symplectic leaves, the symplectic twisted wild character character varieties are the multiplicative symplectic quotients*

$$\mathcal{M}_B(\Sigma, \Theta, \mathcal{C}) = \mathrm{Hom}_{\mathbb{S}}(\Pi, G) /\!\!/ \mathbf{H} = \mu^{-1}(\mathcal{C})/\mathbf{H}.$$

If (E, ∇) is an algebraic connection on $\Sigma^\circ \subset \Sigma$, it determines an irregular class $\Theta : \pi_0(\mathcal{I}) \to \mathbb{N}$ and a formal local system $V^0 \to \mathcal{I}$ with irregular class Θ. At each singularity a_i, the isomorphism class of the inverse of the monodromy of $V^0_{a_i} \to \partial_{a_i}$ is a twisted conjugacy class $\mathcal{C}_i \subset H_i(\bar{\partial}_i)$. Let $\mathcal{C} := \prod_{i=1}^m \mathcal{C}_i \in \mathbf{H}(\bar{\partial})$. The connection (E, ∇) therefore determines a symplectic wild character variety $\mathcal{M}_B(E, \nabla) = \mathcal{M}_B(\Sigma, \Theta, \mathcal{C})$.

From the explicit presentation, it is straightforward to obtain a formula for the dimension of the wild character variety when it is nonempty. One has

$$\dim \mathcal{M}_B(\Sigma, \Theta, \mathcal{C}) = \sum_{i=1}^{m} \left(\dim \mathcal{A}(V^0_{a_i}) + \dim \mathcal{C}_i - 2 \dim H(\partial_{a_i})\right) + 2 \dim Z(G),$$

with

$$\dim \mathcal{A}(V^0_{a_i}) = \dim G + \dim H(\partial_{a_i}) + \dim \prod_{d \in \mathbb{A}_{a_i}} \mathbb{S}\mathrm{to}_d\,.$$

Example 3.3.14. Let us give a simple explicit example. We consider a rank 2 connection with just one singularity at infinity, with one active circle $\langle z^{5/2} \rangle$ with multiplicity 1. There are 5 singular directions. The twist $H(\partial) \subset GL_2(\mathbb{C})$ is the set of invertible matrices of the form

$$h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

and the Stokes matrices are of the form

$$S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 1 & s_5 \\ 0 & 1 \end{pmatrix}.$$

We have

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) = \{h, S_1, S_2, S_3, S_4, S_5 \mid hS_5S_4S_3S_2S_1 = 1\}.$$

3.4 Relation to quivers

In this paragraph, we review some constructions and results relating moduli spaces of meromorphic connections on the affine line to quivers, to sketch the picture we would like to extend to more general irregular classes.

Until know we have been focussing on the description of the Betti moduli spaces as wild character varieties. However, the link with quiver appears more directly on the De Rham side, for moduli spaces of meromorphic connections[1] on trivial bundles on $\mathbb{A}^1$. When restricting to trivial bundles, we get an open part

$$\mathcal{M}^* \subset \mathcal{M}_{dR}$$

of the full moduli space. In some cases, the space $\mathcal{M}^*$ turns out to be isomorphic to a Nakajima quiver variety. In turn, the full Betti moduli space may be seen as a multiplicative quiver variety.

3.4.1 Quiver varieties

Let us recall very briefly what is a quiver variety. A quiver $\mathcal{Q}$ is an oriented graph, with set of nodes I, and set of edges that we also denote by $\mathcal{Q}$. For each oriented edge $e \in \mathcal{Q}$, let $h(e)$ and $t(e)$ denote the head and tail of e. A representation of $\mathcal{Q}$ is the data an I-graded vector space $V = \bigoplus_{i \in I} V_i$, together with the data of a linear map $\phi_e : V_{h(e)} \to V_{t(e)}$ for each edge $e \in \mathcal{Q}$. The vector space of representations of a quiver $\mathcal{Q}$ on V is

$$\mathrm{Rep}(\mathcal{Q}, V) = \bigoplus_{e \in \mathcal{Q}} \mathrm{Hom}(V_{h(e)}, V_{t(e)}).$$

Now, let Γ be an unoriented graph with set of nodes I. Let $\bar{\Gamma}$ be the oriented quiver whose set of edges is the set of oriented edges of Γ, an oriented edge being an edge of Γ together with one choice of orientation among the two possible. Given an I-graded vector space V we define

$$\mathrm{Rep}(\Gamma, V) := \bigoplus_{e \in \bar{\Gamma}} \mathrm{Hom}(V_{h(e)}, V_{t(e)}).$$

This space has a natural symplectic structure. Indeed, let $\widehat{\Gamma}$ an oriented graph obtained from Γ by choosing an orientation for each edge of Γ among the two possible ones. We have $\bar{\Gamma} = \widehat{\Gamma} \cup \widehat{\Gamma}^*$, where $\widehat{\Gamma}^*$ denotes the oriented graph obtained from $\widehat{\Gamma}$ by inverting the orientation of all edges. This means that

$$\mathrm{Rep}(\Gamma, V) := \bigoplus_{e \in \widehat{\Gamma}} \mathrm{Hom}(V_{h(e)}, V_{t(e)}) \oplus \mathrm{Hom}(V_{t(e)}, V_{h(e)}),$$

which yields a natural identification $\mathrm{Rep}(\Gamma, V) = T^*(\mathrm{Rep}(\widehat{\Gamma}, V))$ of $\mathrm{Rep}(\Gamma, V)$ with the cotangent space of $\mathrm{Rep}(\widehat{\Gamma}, V))$. This endows $\mathrm{Rep}(\Gamma, V)$ with a symplectic structure.

[1]There is a notion of meromorphic connections on a compact algebraic curve with poles on a effective divisor D, slightly different from the notion of algebraic connections on the corresponding open curve, see the review in [19] and the discussion of "good/very good meromorphic connections" in [23].

There is on the space $\mathrm{Rep}(\Gamma, V)$ a natural action of the group

$$G := \prod_{i \in I} GL(V_i),$$

such that $g = (g_i)_{i\in I} \in G$ sends a linear map $\phi : V_i \to V_j$ to $g_j \phi g_i^{-1}$. This action is actually Hamiltonian, with moment map

$$\mu(\phi) = \sum_{e \in \hat{\Gamma}} \phi_e \circ \phi_{e^*} - \phi_{e^*} \circ \phi_e. \tag{3.4.1}$$

Taking the sum of the traces of the components of μ yields zero, and the subspace of $\prod i \in I \,\mathrm{End}(V_i)$ on which this sum of traces is zero can be naturally identified with the (dual of the) Lie algebra of G, and so μ is indeed a moment map for G. The Nakajima quiver variety associated to the quiver Γ is obtained by taking the Hamiltonian reduction of $\mathrm{Rep}(\Gamma, V)$ by G at some central value $\lambda = (\lambda_i \,\mathrm{Id}_{V_i})_{i\in I} \in \mathrm{Rep}(\Gamma, V)$:

$$\mathcal{N}_\Gamma(V, \lambda) := \mathrm{Rep}(\Gamma, V) /\!\!/_\lambda G = \mu^{-1}(\lambda)/G.$$

A notion of multiplicative quiver variety $\mathcal{N}_\Gamma^m(\mathbf{d}, \lambda)$ has been introduced [103], following work [36] on the multiplicative preprojective algebra. This amounts to replace the (additive) moment map by the following "multiplicative" group-valued moment map

$$\widetilde{\mu}(\phi) = \prod_{e \in \widehat{\Gamma}} (1 + \phi_e \circ \phi_{e^*})(1 + \phi_{e^*} \circ \phi_e)^{-1},$$

(where we fix an order for taking the products), and taking the quasi-Hamiltonian reduction at a central value $\mathbf{q} = (q_i \,\mathrm{Id}_{V_i})_{i\in I}$.

3.4.2 Regular connections, conjugacy classes and legs

Let us first consider connections on $\mathbb{A}^1$ with regular singularities. The formal local system $V^0 \to \mathcal{I}$ of a connection with regular singularities on $\mathbf{a} = (a_1, \dots, a_m)$ is determined by the collection $\boldsymbol{\mathcal{C}}$ of conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_m \subset GL_n(\mathbb{C})$ of the formal monodromies. The corresponding character variety is the quasi-Hamiltonian reduction.

$$\mathcal{M}_B(V^0) = (\mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m) /\!\!/ G.$$

The additive analogue on the de Rham side corresponds to the moduli space of Fuchsian systems of the form

$$\nabla = d - \left(\frac{A_1}{z - a_1} - \cdots - \frac{A_m}{z - a_m}\right) dz,$$

on the affine line, which is equivalent to a connection on a trivial bundle $V \to \mathbb{A}^1$, meromorphic on $\mathbf{a}$ with regular singularities. The moduli space has the structure of a usual symplectic quotient

$$\mathcal{M}^* = (\mathcal{O}_1 \times \cdots \times \mathcal{O}_m) /\!\!/ G.$$

To relate those moduli spaces to quivers, the main ingredient is a construction which associates to a coadjoint orbit or a conjugacy class in $GL_n(\mathbb{C})$ a leg-shaped quiver. Since we are most interested in the Betti moduli spaces, we state the multiplicative version involving conjugacy classes.

Definition 3.4.1. Let $\mathcal{C} \subset GL_n(\mathbb{C})$ a conjugacy class. A *marking* of $\mathcal{C}$ is a finite ordered set $(\xi_1, \dots, \xi_w)$ of nonzero complex number such that $\prod_{i=1}^w (M - \xi_i) = 0$ for any $M \in \mathcal{C}$.

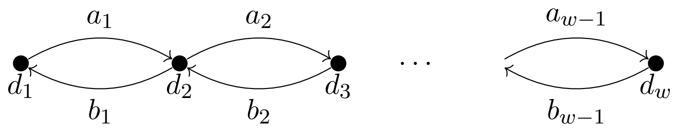


Figure 3.5: The leg-shaped quiver associated to a conjugacy class with a choice of marking.

In other words, this amounts to choose a monic annihilating polynomial $P \in \mathbb{C}[x]$ of $\mathcal{C}$, i.e. such that $P(M) = 0$ for any $M \in \mathcal{C}$, and an ordering of its roots. We say that a marking is *minimal* if w is minimal, that is if P is the minimal polynomial of any $M \in \mathcal{C}$. A marking will be called *special* if the first eigenvalue is $\xi_1 = 1$.

Given a marking $(\xi_1, \dots, \xi_w)$ of $\mathcal{C}$, we define

$$q_i := \frac{\xi_i}{\xi_{i-1}} \in \mathbb{C}^*,$$

for $i = 2, \dots, w$, and $q_1 := \xi_1$, as well as integers

$$d_i := \operatorname{rank}(M - \xi_1) \dots (M - \xi_{i-1}) \in \mathbb{N},$$

and $d_1 := n$.

The numbers q_i and the dimensions d_i are enough to determine the class q. This can be formulated in terms of a quiver. Let us consider the Dynkin quiver $\mathcal{Q}$ of type A_w with w vertices, with dimension vector $\mathbf{d} = (d_1, \dots, d_w)$.

We have the following result relating the multiplicative quiver variety associated to Q to the conjugacy class $\mathcal{C}$.

Theorem 3.4.2. *Let $\{(a_i, b_i)\}$ a representation of this leg-shaped quiver (see figure) such that a_i is injective and b_i is surjective for $i = 1, \dots, w-1$, and such that we have the following equalities*

$$M = q_1(1+a_1b_1),\ (1+a_1b_1) = q_2(1+a_2b_2),\ \dots,\ 1+a_{w-2}b_{w-2} = q_{w-1}(1+a_{w-1}b_{w-1}),\ 1+a_{w-1}b_{w-1} = q_w.$$

Then the matrix M belongs to the conjugacy class $\mathcal{C}$.

The relations in the statement of the theorem correspond to having the moment map of the multiplicative quiver variety equal to $\mathbf{q} = (q_1, \dots, q_w)$ (the difference with the multiplicative quiver variety is that we do not perform the reduction with respect to the first node of the leg).

This discussion has an additive analogue for coadjoint orbits, with the multiplicative labels q_i replaced by additive ones $\lambda_i = \xi_i - \xi_{i-1}$, and the multiplicative moment map relations replaced by the additive ones.

Now we define the quiver $\Gamma(\boldsymbol{\mathcal{C}})$ associated to the formal data $\boldsymbol{\mathcal{C}}$ by gluing all the legs together: we fuse the first node of each leg associated to $\mathcal{C}_1, \dots, \mathcal{C}_m$ into a single node, with dimension n. We obtain a star shaped quiver $\Gamma(\boldsymbol{\mathcal{C}})$, with a dimension vector $\mathbf{d}$.

Theorem 3.4.3 (see [35])**.** *If the coadjoint orbits $\mathcal{O}_1, \dots, \mathcal{O}_m$ are closed, then the additive moduli space $\mathcal{M}^*$ is isomorphic to the Nakajima quiver variety $\mathcal{N}_{\Gamma(\boldsymbol{\mathcal{C}})}(\mathbf{d}, \lambda)$.*

We have the similar result for the multiplicative case.

Theorem 3.4.4 (see [36])**.** *If the conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_m$ are closed, then Betti moduli space $\mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}})$ is isomorphic to the multiplicative quiver variety $\mathcal{N}^m_{\Gamma(\boldsymbol{\mathcal{C}})}(\mathbf{d}, \mathbf{q})$.*

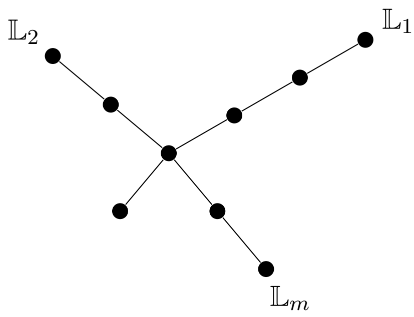


Figure 3.6: Star-shaped quiver associated to a connection with regular singularities.

3.4.3 The case of one irregular singularity

We now turn to the case where there is one irregular singularity, possibly together with simple poles. We assume that the irregular singularity is at the point at infinity $\infty \in \mathbb{P}^1$. We also assume that the active circles at infinity all have ramification order equal to 1. Let (E, ∇) an algebraic connection on the affine line (i.e. only having a singularity at infinity), $V^0 \to \mathcal{I}_\infty$ its formal local system at infinity, conditions, and $(\Theta_\infty, \boldsymbol{\mathcal{C}})$ its isomorphism class. Let us denote by $\langle q_1 \rangle, \dots, \langle q_k \rangle$ the active circles at infinity, and $n_1, \dots, n_k$ their multiplicities. We have $n_1 + \dots + n_k = n$, where n is the rank of the connection.

The structure of the diagram $\Gamma(\Theta, \boldsymbol{\mathcal{C}})$ associated to the formal data $(\Theta, \boldsymbol{\mathcal{C}})$ has the following structure: there is a *core diagram* $\Gamma_c(\Theta_\infty)$, only depending on the irregular class Θ_∞ at infinity, whose vertices correspond to the active circles at infinity. To this core diagram are glued legs corresponding to the conjugacy classes $\boldsymbol{\mathcal{C}}$ of the monodromies of the active circles at infinity, as well as "splayed" legs associated to the monodromies at the regular singularities at finite distance.

Definition 3.4.5 ([17])**.** The core diagram $\Gamma_c(\Theta_\infty)$ is the graph with set of vertices $I_\infty = \{\langle q_1 \rangle, \dots, \langle q_k \rangle\}$, such that the number of (unoriented) edges between $\langle q_i \rangle$ and $\langle q_j \rangle$ is given by

$$B_{i,j} = \deg(q_i - q_j) - 1. \tag{3.4.2}$$

Theorem 3.4.6. *If all the orbits are closed, then the moduli space $\mathcal{M}^*$ is isomorphic to the Nakajima quiver variety $\mathcal{N}_\Gamma(\mathbf{d}, \lambda)$*

Then, the leg $\mathbb{L}_i$ associated to the conjugacy class $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ of the monodromy of $V^0_{\langle q_i \rangle} \to \langle q_i \rangle$ is glued to the vertex $\langle q_i \rangle$, by fusing this vertex with the first vertex of the leg.

Let us denote by $a_1, \dots, a_m$ the regular singularities at finite distance. For $j = 1, \dots, m$, let $\mathcal{C}_j \subset Gl_n(\mathbb{C})$ the conjugacy class of the monodromy of $V^0_{a_j} \to \langle 0 \rangle_{a_j}$, and $\mathbb{L}_j$ the corresponding leg. We glue the leg $\mathbb{L}_j$ to the core diagram in the following way: we splay the first node of $\mathbb{L}_j$ (having dimension n) into k nodes with respective dimensions $n_1, \dots, n_k$, that we fuse with the core vertices $\langle q_1 \rangle, \dots, \langle q_k \rangle$ respectively. Finally, we obtain a full diagram $\Gamma(\Theta_\infty, \boldsymbol{\mathcal{C}})$ with set of vertices I together with a dimension vector $\mathbf{d} \in \mathbb{Z}^I$ and a vector of labels $\mathbf{q} \in (\mathbb{C}^*)^I$ in the multiplicative version, or $\lambda \in \mathbb{C}^I$ in the additive version.

Once again, the moduli spaces are related to quiver varieties associated to $\Gamma(\Theta_\infty, \boldsymbol{\mathcal{C}})$. On the additive side, we consider the moduli space $\mathcal{M}^*$ of meromorphic connections on the trivial rank n vector bundle on the affine line with formal local system V^0, of the form

$$\nabla = d - \left(\sum_{i=0}^{s} B_i z^i + \frac{A_1}{z - a_1} + \dots + \frac{A_m}{z - a_m} \right) dz,$$

with $\sum_{i=0}^s B_i z^i = \operatorname{diag}(q_1 \operatorname{Id}_{n_1}, \dots, q_k \operatorname{Id}_{n_k})$.

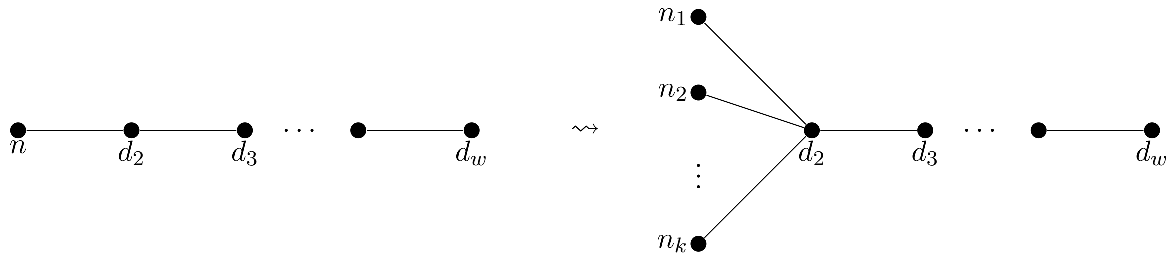


Figure 3.7: Splaying a leg corresponding to the conjugacy class of a regular singularity at finite distance, to glue it to the core diagram with vertices $\langle q_1 \rangle, \dots, \langle q_k \rangle$.

$\mathcal{M}^*$ can be expressed as a symplectic quotient

$$\mathcal{M}^* = (\mathcal{O}_B \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_m) /\!\!/ H,$$

with $\mathcal{O}_B$ an "extended orbit" (it plays the same role as $\mathcal{A}(V^0_\infty) /\!\!/ G$ on the Betti side).

Theorem 3.4.7. *The stable part $\mathcal{M}^*_s$ of $\mathcal{M}^*$ is isomorphic to the Nakajima quiver variety $\mathcal{N}_{\Gamma(V^0)}(\mathbf{d}, \lambda)$.*

This was proven in [17] in the case where the pole at infinity is of order at most 3, i.e. the q_i have slope less than 2, and in the general case in [57].

In the multiplicative case, in the setting of [17, 20] where the pole at infinity is of order at most 3, it is possible to define a notion of generalized multiplicative quiver varieties [22] such that the full Betti moduli space becomes a multiplicative quiver variety.

3.4.4 Different readings

One of the main properties of this relation between moduli spaces of connections and quivers is that several connections, with different formal data, can give rise to the same quiver. There are thus different ways to "read" a given quiver. When the moduli space is isomorphic to the quiver variety, these different readings yield isomorphisms between the moduli spaces.

The most interesting setting for which this happens is in the case studied in [17, 20, 22] where there is one irregular singularity at infinity of order less than 3 with unramified formal data, together with simple poles at finite distance. Let us look a the structure of the diagram in this situation.

Let $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ be the modified formal data of a connection satisfying these conditions. The exponential factors at infinity are of the following form

$$q = \lambda x^2 + \mu x, \tag{3.4.3}$$

with $\lambda, \mu \in \mathbb{C}$.

Let $I_\infty \subset \pi_0(\mathcal{I}_\infty)$ be the subset of active exponents at infinity, i.e. the support of V^0_∞. I_∞ is the set of vertices of the core diagram. We can partition I_∞ according to the coefficients of x^2 and x in the exponential factors. For $\lambda \in \mathbb{C}$, let $I_\lambda \subset I$ be the set of exponential factors of the form (3.4.3) for some $\mu \in \mathbb{C}$. There is a finite number of coefficients $\lambda \in \mathbb{C}$ such that I_λ, is non empty, which we denote $\lambda_1, \dots, \lambda_k$. We set $I_i := I_{\lambda_i}$ for $i = 1, \dots, r$. The sets I_i constitute a partition of I_∞.

Now let $\langle q \rangle$ and $\langle q' \rangle$ two distinct active circles at infinity. If $\langle q \rangle$ and $\langle q' \rangle$ belong to the same subset I_i, then from the definition of the core diagram, there is exactly one edge between $\langle q \rangle$ and $\langle q' \rangle$, since $\deg(q - q') = 2$. Otherwise, $\deg(q - q') = 1$ and there is no edge between $\langle q \rangle$ and $\langle q' \rangle$. The core diagram is thus a k-partite graph: each vertex in one of the k subsets I_i is related by exactly one edge to each vertex of the $k - 1$ other subsets I_j, for $j \neq i$.

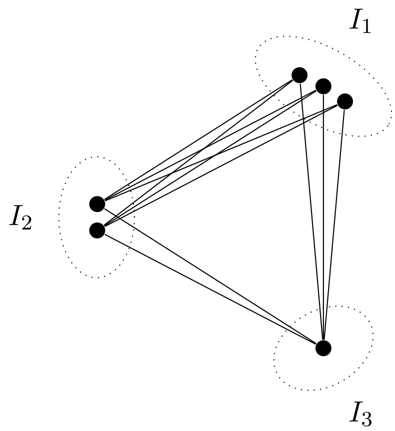
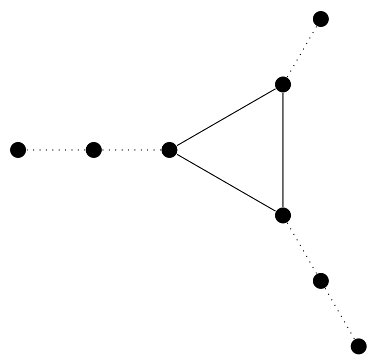
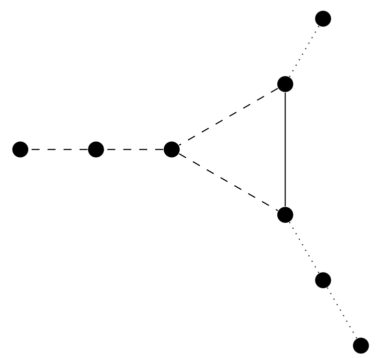


Figure 3.8: The k-partite core diagram (here with $k = 3$). Each blue oval corresponds to one of the k subsets I_i.



(a) One singularity at infinity with 3 active circles. The core diagram is in solid line, the legs are dotted.



(b) A singularity at infinity with 2 active circles, and a regular singularity at finite distance. In dashed lines, the splayed leg corresponding to the conjugacy class of the monodromy at the regular singularity at finite distance.

Figure 3.9: Two different readings of the same supernova quiver.

If there is no singularity at finite distance, the full diagram consists of this k-partite core diagram with legs attached, in the language of [17] this is a *supernova quiver*.

Now let us look at what happens if there are singularities $a_1, \dots, a_m$ at finite distance. In this situation, there are legs of length ≥ 2 $\mathbb{L}_1, \dots, \mathbb{L}_m$ which are splayed and glued to the core diagram. Let $I_{\neq\infty} \subset I$ the set whose elements are the second vertices of the $\mathbb{L}_j$ (i.e. the next vertices after those which are glued to the core). We notice that each vertex in $I_{\neq\infty}$ is linked by exactly one edge to each vertex of I_j for $j = 1, \dots, k$. This means that the subgraph of Γ with set of vertices $I \cup I_{\neq\infty}$ is a $k+1$-partite graph.

This implies that any supernova quiver, with a k-partite core corresponding to the partition $I_c = I_1 \sqcup \cdots \sqcup I_k$ of the set I_c of core vertices can be interpreted in $k+1$ different ways as coming from a formal local system $V^0 \to \mathcal{I}$. In the *generic reading*, all core vertices correspond to active circles at infinity, and the subsets I_i correspond to the different coefficients λ_i. For each $i = 1, \dots, k$, there is a nongeneric reading, where the subset $I_{\neq i} := I_c \setminus I_i$ is the set of active circles at infinity, whereas the subset I_i corresponds to the second vertices of legs associated to regular singularities at finite distance. In other words, each subset of the k-partite core may be seen as the set of legs associated to regular poles at finite distance.

Since in this case (if the connection is stable) the additive moduli space $\mathcal{M}^*$ is ismorphic to the quiver variety of the graph, these different readings give us isomorphism between moduli spaces of connections. This is also true in the multiplicative setting for the full Betti moduli spaces [22].

3.4.5 Weyl groups

One other property of the quivers associated to connections is that they carry Weyl group actions. Basically, the action of the Weyl group amounts to exchanging the order of the eigenvalues in the markings associated to the conjugacy classes.

Let Γ be an (unoriented) graph with set of vertices I, and $B \in M_{I\times I}(\mathbb{C})$ its adjacency matrix. The Cartan matrix of the quiver is defined as

$$C = 2\,\mathrm{Id} - B.$$

The Cartan matrix defines a bilinear form $(\cdot,\cdot)$ on the *root lattice* $\mathbb{Z}^I = \bigoplus_{i\in I}\mathbb{Z}\varepsilon_i$, given by

$$(\varepsilon_i, \varepsilon_j) = C_{i,j}.$$

To each vertex $i \in I$ is associated a *simple reflection* s_i defined by

$$s_i(\beta) = \beta - (\beta, \varepsilon_i)\varepsilon_i.$$

The simple reflections satisfy the relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } B_{i,j} = 0, \quad s_i s_j s_i = s_j s_i s_j \text{ if } B_{i,j} = 1.$$

The Weyl group of the graph is defined to be the group generated by those simple reflections. They are also dual simple Weyl reflections acting on the space $\mathbb{C}^I$ of additive labels, defined by

$$r_i(\lambda) = \lambda - \lambda_i \alpha_i,$$

for $\lambda = \sum_{i\in I}\lambda_i\varepsilon_i \in \mathbb{C}^I$, and $\alpha_i := \sum_j(\varepsilon_i,\varepsilon_j)\varepsilon_j \in \mathbb{C}^I$. The dual reflection is such that $s_i(\beta)\cdot r_i(\lambda) = \beta\cdot\lambda$, where $\cdot$ denotes the pairing such that $\varepsilon_i\cdot\varepsilon_j = \delta_{i,j}$.

For the multiplicative case, we have multiplicative dual reflections on $(\mathbb{C}^*)^I$, obtained from the additive ones by taking the exponential, that we still denote by r_i. If $\mathbf{q} = (q_i)_{i\in I} \in (\mathbb{C}^*)^I$, one has

$$(r_i(\mathbf{q}))_j = q_i^{-(\varepsilon_i,\varepsilon_j)} q_j.$$

The analogue of $s_i(\beta)\cdot r_i(\lambda) = \beta\cdot\lambda$ is then $\mathbf{q}^\beta = r_i(\mathbf{q})^{s_i(\beta)}$.

It is known that the Weyl reflections induce isomorphisms of Nakajima quiver varieties.

Theorem 3.4.8 ([76, 78])**.** *Let Γ be a quiver. If $\lambda_i \neq 0$, there is a natural isomorphism between the Nakajima quiver variety with dimension vector β and labels $\lambda = (\lambda_i)_i \in I$ and the one with dimension vector $s_i(\beta)$ and labels $r_i(\lambda)$:*

$$\mathcal{N}_\Gamma(\beta,\lambda) \cong \mathcal{N}_\Gamma(s_i(\beta), r_i(\lambda)).$$

These isomorphisms have a nice interpretation for the quivers associated to meromorphic connections: they correspond to acting on connections by twists, (i.e. tensoring by a rank one vector bundle with connection) that shift the eigenvalues of the formal monodromies.

In the multiplicative case, this can be used to show [22] that the Weyl reflections induce isomorphisms for multiplicative quiver varieties.

3.4.6 Deligne-Simpson problems

The (irregular) Deligne-Simpson problem asks whether, given some formal data, the corresponding moduli space of (stable) connections is nonempty. It has an additive version for connections on trivial bundles, as well as a multiplicative version involving the full moduli spaces. Remarkably, in several cases, a solution to this problem can be formulated in terms of the root system of the Kac-Moody algebra of the quiver associated to the connection.

Keeping the previous notations, the Kac-Moody root system associated to the graph Γ is a subset of the root lattice $\mathbb{Z}^I = \bigoplus_{i\in I} \mathbb{Z}\varepsilon_i$ defined as follows: the *simple roots* are the ε_i, for $i \in I$. The set of real roots is the Weyl group orbit of the simple roots. Next, define the fundamental region to be the set of non-zero elements $\beta \in \mathbb{Z}^I$, with support on a connected subgraph of Γ, such that $(\beta, \varepsilon_i) \geq 0$ for each $i \in I$. We define the set of imaginary roots to be the union of the Weyl group orbit of the fundamental region, and minus this orbit. The set of roots is the union of the sets of real and imaginary roots. A root is said to be positive if its coefficients (in the basis of the ε_i's) are all nonnegative.

The answer to the additive (irregular) Deligne-Simpson problem is formulated as follows:

Theorem 3.4.9. *Let $(\mathbb{P}^1, \{\infty\}, \Theta)$ a wild Riemann surface, consisting of at most one unramified irregular singularity at infinity, possibly together with regular singularities at finite distance. Let $\boldsymbol{\mathcal{C}}$ a collection of conjugacy classes for the irregular class Θ. Let $\Gamma = \Gamma(\Theta, \boldsymbol{\mathcal{C}})$ the corresponding quiver, with set of vertices I, $\mathbf{d} \in \mathbb{Z}^I$ its dimension vector, and $\boldsymbol{\lambda} \in \mathbb{C}^I$ its vector of (additive) labels. The moduli space $\mathcal{M}^*(\Theta, \boldsymbol{\mathcal{C}})$ of stable connections on trivial bundles with these formal data is non-empty if and only if*

1. *$\mathbf{d}$ is a positive root.*
2. *$\boldsymbol{\lambda} \cdot \mathbf{d} = 0$, and*
3. *If $\mathbf{d}$ decomposes in a non-trivial way as a sum $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_s = 0$, with $\boldsymbol{\lambda} \cdot \mathbf{d}_1 = \cdots = \boldsymbol{\lambda} \cdot \mathbf{d}_s = 0$, then $\Delta(\mathbf{d}) > \Delta(\mathbf{d}_1) + \cdots + \Delta(\mathbf{d}_s)$, where $\Delta(\mathbf{d}) := 2 - (\mathbf{d}, \mathbf{d})$.*

This result is due to Crawley-Boevey [35] for regular singular connections, Boalch [17, 20] in the simply laced case and Hiroe and Yamakawa [57] in the more general case. There is also a conjectural multiplicative analogue for the full moduli spaces [36, 22].

3.4.7 Allowing for ramification

These links between connections and quivers work under some hypotheses on the formal data of the connection: in all cases mentioned until now, there must be at most one irregular singularity, and the exponential factors are not allowed to have ramification. It is thus a natural question to ask whether it would be possible to extend part of this story for more general formal data. One direction is to allow for several irregular singularities, another one is to allow for ramification.

Whereas the first direction has been pursued by Hiroe, in the second direction the authors of [26] have recently proposed a definition of a diagram for connections with only one irregular singularity at infinity allowing for ramification.

Let $V^0 \to \mathcal{I}_\infty$ a formal local system at infinity and $(\Theta, \boldsymbol{\mathcal{C}})$ its formal data. We denote by $\langle q_1 \rangle, \dots, \langle q_r \rangle$ the active circles, and $\beta_i := \mathrm{ram}(q_i)$. In [26], the core diagram associated to the irregular class Θ is defined in the following way:

Definition 3.4.10. The *core diagram* $\Gamma_c(\Theta)$ has a set of nodes labelled by the active circles $\langle q_1 \rangle, \dots, \langle q_r \rangle$, and for $i, j = 1, \dots, r$, the number of arrows B_{ij} between $\langle q_i \rangle$ and $\langle q_j \rangle$ is given by

- If $i \neq j$ then
$$B_{ij} = A_{ij} - \beta_i \beta_j, \tag{3.4.4}$$
where $A_{ij} = \mathrm{Irr}(\mathrm{Hom}(\langle q_i \rangle, \langle q_j \rangle))$.

- If $i = j$, the number of oriented loops at $\langle q_i \rangle$ is given by

$$B_{ii} = A_{ii} - \beta_i^2 + 1. \tag{3.4.5}$$

The numbers of arrows may be negative. One has $B_{ij} = B_{ji}$ and B_{ii} is always an even number, so we can group the arrows two by two to get an unoriented diagram. The motivation for this definition comes from counting the number of appearances of blocks between graded parts of V^0 associated to the different active circles in the explicit presentation of the wild character variety. Recall that for a connection with only one singularity at infinity we have

$$\mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}}) := \mathrm{Hom}_{\mathbb{S}}(V) /\!\!/_{\mathcal{C}} H.$$

where $\mathrm{Hom}_{\mathbb{S}}(V^0) := \mathcal{A}(V^0) /\!\!/ G$ is the quasi-hamiltonian reduction with respect to G of the twisted quasi-hamiltonian $G \times H$-space

$$\mathcal{A}(V^0) := H(\partial) \times G \times \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d(V).$$

The positive terms in B_{ij} correspond to the matrix blocks in $\mathcal{A}(V^0)$, and the negative terms in the B_{ij} correspond to the relations given by the quasi-hamiltonian reduction.

Chapter 4

Diagrams for irregular connections on the Riemann sphere

In this chapter, we build up the ideas leading to our definition of a diagram associated to any algebraic connection on a Zariski open subset of the affine line. The main ingredient is the formal Fourier-Laplace transform, which allows to reduce to the setting of [26] where there is only one irregular singularity at infinity. The crucial point allowing this is that the diagram of [26] is invariant under the action of automorphisms of the Weyl algebra (including the Fourier-Laplace transform). Along the way, we establish explicit formulas for the number of edges and loops of the diagram, as well as for the images of exponential factors under formal Fourier transform, which are of independent interest.

4.1 Modified formal data

To arrive at our definition of the diagrams, we will need to use the Fourier-Laplace transform. However, the category which the Fourier-Laplace transform acts on is not the category of connections, but the category of modules on the Weyl algebra $A_1 = \mathbb{C}[z]\langle \partial_z \rangle$. This is the source of an important subtlety, pertaining to the fact that more formal data are needed to describe a $\mathcal{D}$-module than a connection. We review this in this paragraph, relying on [70, 89, 5, 87].

Let (E, ∇) an algebraic connection on a Zariski open subset $U = \mathbb{A}^1 \smallsetminus \mathbf{a}$ of the affine line (or, equivalently, a local system on U). One can associate [5, §5.5] to (E, ∇) its *minimal extension* M, which is a $\mathcal{D}_{\mathbb{A}^1}$-module. This is equivalent to the data of a module on the Weyl algebra. We want to describe the formal data of M.

The issue is local at the singularities. Let us thus recall a few results about the structure of holonomic $\mathcal{D}$-modules on the formal disk. Let $\widehat{\mathcal{D}} := \mathbb{C}[[z]]\langle \partial_z \rangle$ and M be a holonomic $\widehat{\mathcal{D}}$-module. Then M decomposes [70, IV.3 p.59] in a unique way as a direct sum $M = M_{irr} \oplus M_{reg}$ of an irregular part M_{irr} and a regular part M_{reg}. Furthermore, the irregular part M_{reg} is isomorphic to its localization, i.e. $M_{irr} \cong M_{irr}[z^{-1}]$, which implies that M_{irr} is a meromorphic connection on the formal punctured disk. There is thus no difference between $\widehat{\mathcal{D}}$-modules and meromorphic connections as far as the irregular part is concerned.

However, this is not the case for regular $\widehat{\mathcal{D}}$-modules. Indeed, we have the following descriptions of regular $\widehat{\mathcal{D}}$-modules and regular meromorphic connections on the formal punctured disk.

Let M be a regular holonomic $\widehat{\mathcal{D}}$-module. One associates [70, II.3] to M its local system of solutions $V \to \partial$ and its local system of microsolutions $W \to \partial$, where $\partial := \partial_0$ is the circle of directions at $z = 0$. Furthermore, we have two natural morphisms between them, the *canonical morphism* $can : V \to W$ and the *variation* $var : W \to V$. The monodromies of V and W are related to var and can by $T_V = 1 + var \circ can$, $T_W = 1 + can \circ var$. The data of V, W, can and var is sufficient to reconstruct M up to isomorphism. This can be represented by the diagram

$$V \underset{var}{\overset{can}{\rightleftarrows}} W$$

Choosing a base point $p \in \partial$, and setting $E := V_p$, $F := W_p$, $u := var_p : E \to F$ and $v := var_p$ leads to the following description: a regular holonomic $\widehat{\mathcal{D}}$-module M is determined by the data of a quadruplet (E, F, u, v), where E and F are finite dimensional vector spaces, and $u : E \to F, v : F \to E$ are linear applications, such that $1 + uv : F \to F$ (and $1 + vu : E \to E$) are invertible. This data is represented by the diagram:

$$E \underset{v}{\overset{u}{\rightleftarrows}} F$$

The category of regular holonomic $\mathcal{D}$-modules on the formal disk is in this way equivalent to the category whose objects are such quadruplets (E, F, u, v).

On the other hand, a regular meromorphic connection on the formal punctured disk is determined by its local system of solutions $V \to \partial$, and in turn, choosing a base point $p \in \partial$ and setting $E := V_p$, by a pair (E, T) where E is a finite dimensional vector space, and $T \in GL(E)$ is the monodromy operator.

In this picture, the minimal extension corresponds to the following data:

Lemma 4.1.1 ([6] (3.5))**.** *Let (E, ∇) a regular connection on the formal punctured disk, and (V, T) the corresponding vector space with automorphism. Its minimal extension M is a regular holonomic $\widehat{\mathcal{D}}$-module corresponding to the pair of vector spaces with maps*

$$V \underset{\iota}{\overset{T-1}{\rightleftarrows}} \operatorname{Im}(T-1)$$

where $T = 1 + vu \in GL(V)$, and $\iota : \operatorname{Im}(T-1) \to V$ is the inclusion.

Notice that we recover (E, ∇) from the data of (W, T_W), where $W := \operatorname{Im}(T-1)$ and $T_W = T_{|W}$, and of the rank of (E, ∇).

We can fit this description into the framework of $\mathcal{I}_0$-graded local systems. Let (E, ∇) be a connection on the punctured formal disk, and $V^0 \to \partial$ the associated $\mathcal{I}_0$-graded local system. The local system of solutions of its regular part corresponds to the graded piece $V^0_{\langle 0\rangle} \to \partial$ associated to the tame circle $\langle 0\rangle$. Let us denote by T its monodromy. Now let M be the minimal extension of (E, ∇), and $W := \operatorname{Im}(T-1) \subset V^0_{\langle 0\rangle}$ the local system of microsolutions of its regular part.

We will see later that W, and not $V^0_{\langle 0\rangle}$ belong to the data exchanged by the formal Fourier-Laplace transform. For this reason, we define a modified formal local system $\breve{V}^0 \to \mathcal{I}_0$ by setting

$$\breve{V}^0_{\langle 0\rangle} := W, \qquad \breve{V}_{\langle q\rangle} := V^0_{\langle q\rangle} \text{ if } \langle q\rangle \neq \langle 0\rangle.$$

$\breve{V}^0$ is an $\mathcal{I}_0$-graded local system, that we call the *modified formal local system* associated to the connection (E, ∇). What we have done is to replace the tame part $V_{\langle 0\rangle}$ of the local system by the local system of microsolutions $W \to \partial$, viewing it as a graded piece in the modified formal local system. As for V^0, we can also view $\breve{V}^0$ system as a local system on the topological space $\mathcal{I}_0$. Its irregular class is the *modified irregular class* associated to the connection (E, ∇).

Passing to the modified local system can be visualized in terms of the corresponding leg-shaped quivers encoding the conjugacy class of the monodromy. Let $\mathcal{C}_{\langle 0\rangle}$ the conjugacy class of the monodromy T of $V^0_{\langle 0\rangle}$, and $\breve{\mathcal{C}}_{\langle 0\rangle}$ the conjugacy class of the monodromy $T_{|V_{\langle 0\rangle}}$ of $V_{\langle 0\rangle}$. Let us choose a *special marking* $(\xi^0_1, \dots, \xi^0_w)$ of $\mathcal{C}_{\langle 0\rangle}$, i.e. a marking such that $\xi_1 = 1$. Then $(\xi^0_2, \dots, \xi^0_w) =: (\xi_1, \dots, \xi_{w-1})$ yields a marking of $\breve{\mathcal{C}}_{\langle 0\rangle}$. Let $\mathbb{L}_{\langle 0\rangle}$ and $\breve{\mathbb{L}}_{\langle 0\rangle}$ the legs respectively

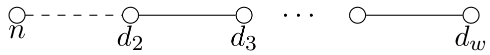


Figure 4.1: Passing to the modified formal local system amount to cut the first edge (dashed) of the leg $\mathbb{L}_{\langle 0\rangle}$ associated to the tame circle $\langle 0\rangle$, when using a special marking, to obtain the shorter leg $\breve{\mathbb{L}}_{\langle 0\rangle}$.

associated to $\mathcal{C}_{\langle 0\rangle}$ and $\breve{\mathcal{C}}_{\langle 0\rangle}$ with these choices of markings. The leg $\breve{\mathbb{L}}_{\langle 0\rangle}$ is simply obtained from $\mathbb{L}_{\langle 0\rangle}$ by deleting the first vertex and the first edge, as shown on figure 4.1.

We now return to the global situation. In the rest of the paper, we consider a genus zero Riemann surface $\Sigma \cong \mathbb{P}^1$. We fix a choice of isomorphism $\Sigma \cong \mathbb{P}^1$. This defines the point at infinity $\infty \in \Sigma$ and allows us to identify $\Sigma \setminus \{\infty\}$ to the affine line $\mathbb{A}^1$.

Let (E, ∇) an algebraic connection on a Zariski open subset $U = \mathbb{A}^1 \setminus \{a_1, \dots a_m\}$ of the affine line. It determines a modified formal local system in the following way. Let M be the minimal extension of (E, ∇) to $\mathbb{A}^1$ (but notice we do not take the extension at infinity). M is a module on the Weyl algebra. For each singularity a_i at finite distance, let $\breve{V}_{a_i} \to \mathcal{I}_{a_i}$ be the modified formal local system encoding the formal type of M at a as we just described. Then, let $V_\infty \to \mathcal{I}_\infty$ be the formal local system of (E, ∇) at ∞ (since we don't take the minimal extension at infinity). This gives us a global modified formal local system $\breve{V} := \bigoplus_{a \neq \infty} \breve{V}_a \oplus V_\infty$. Its isomorphism class is given by the couple $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ where $\breve{\Theta}$ is its irregular class and $\breve{\boldsymbol{\mathcal{C}}}$ is the collection of conjugacy classes of its monodromies.

Definition 4.1.2. $\breve{V} \to \mathcal{I}$ is the (global) modified formal system associated to (E, ∇) or to M. Its irregular class $\breve{\Theta} : \pi_0(\mathcal{I}) \to \mathbb{N}$ is the (global) modified irregular class of (E, ∇).

Notice that it satisfies the *compatibility condition*

$$\operatorname{rank}(\breve{V}_a) \leq \operatorname{rank} \breve{V}_\infty,$$

since for each $a \neq \infty$ replacing $V^0_{\langle 0\rangle}$ by $\breve{V}^0_{\langle 0\rangle}$ lowers the rank of the corresponding graded piece. The rank of the connection (E, ∇) is given by

$$\operatorname{rank}(E, \nabla) = \sum_{\langle q\rangle \in \pi_0(\mathcal{I}_\infty)} n_{\langle q\rangle} \operatorname{ram}(q).$$

If $\breve{V} \to \mathcal{I}$ is a formal local system, with isomorphism class $\breve{\boldsymbol{\mathcal{C}}}$, we say that V, or $\breve{\boldsymbol{\mathcal{C}}}$ is *compatible* if it satisfies this condition. If there exists a connection (E, ∇) on a Zariski open subset of the affine line such that $\breve{\boldsymbol{\mathcal{C}}}$ is its modified formal data, we say that $\breve{\boldsymbol{\mathcal{C}}}$ is *effective*. $\breve{\boldsymbol{\mathcal{C}}}$ cannot be effective if it is not compatible.

Notice also that specifying compatible formal data $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ associated to (E, ∇) is equivalent to specifying the corresponding non-modified formal data $(\Theta, \boldsymbol{\mathcal{C}})$. Indeed, from Θ_∞ one knows the rank of the connection, so the pieces $\breve{V}^0_{\langle 0\rangle_a} \to \langle 0\rangle_a$ determine the tame pieces $V^0_a \to \langle 0\rangle_a$.

Summary of data In the rest of the thesis, we fix an algebraic connection (E, ∇) on $\mathbb{A}^1 \setminus \mathbf{a}$, with modified irregular class $\breve{\Theta} : \pi_0(\mathcal{I}) \to \mathbb{N}$, and modified conjugacy classes $\breve{\boldsymbol{\mathcal{C}}}$. We fix the following notations.

- We denote by $a_1, \dots, a_m$ be the singularities at finite distance, so that $\mathbf{a} = \{a_1, \dots, a_m\}$
- For $k = 1, \dots, m$ let $\langle q_1^{(k)}\rangle_{a_k}, \dots \langle q_{r_k}^{(k)}\rangle_{a_k} \in \pi_0(\mathcal{I}_{a_k})$ denote the irregular active circles at a_k, $n_i^{(k)} \in \mathbb{N}$ their respective multiplicities, and $\beta_j^{(k)}$ their respective ramification orders, and $\alpha_j^{(k)}$ their irregularities, in an order such that the slopes satisfy

$$\frac{\alpha_1^{(k)}}{\beta_1^{(k)}} \geq \cdots \geq \frac{\alpha_{r_k}^{(k)}}{\beta_{r_k}^{(k)}}.$$

- For $k = 1, \ldots, m$ let m_k be the multiplicity of the tame circle $\langle 0 \rangle_{a_k}$ at a_k in the *modified* formal local system.
- At infinity, let $\langle q_1^{(\infty)} \rangle_\infty, \ldots \langle q_{r_\infty}^{(\infty)} \rangle_\infty \in \pi_0(\mathcal{I}_{a_k})$ denote the active circles (the tame circle is included), their $\beta_j^{(\infty)}$ respective ramification orders, and $\alpha_j^{(\infty)}$ their irregularities, in an order such that the slopes satisfy

$$\frac{\alpha_1^{(\infty)}}{\beta_1^{(\infty)}} \geq \cdots \geq \frac{\alpha_{r_\infty}^{(\infty)}}{\beta_{r_\infty}^{(\infty)}}.$$

The rank of any connection having these (modified) formal data is $\sum_{j=1}^{r_\infty} n_i^{(\infty)} \beta_j^{(\infty)}$.

4.2 Action of $SL_2(\mathbb{C})$

We now discuss the action of the Fourier-Laplace transform on modules on the Weyl algebra, and the stationary phase formula relating the formal data of an A_1-module and its Fourier transform.

4.2.1 The stationary phase formula

The Fourier transform is the automorphism of the Weyl algebra $A_1 = \mathbb{C}[z]\langle \partial_z \rangle$ defined by $z \mapsto -\partial_z$, $\partial_z \mapsto z$. It induces a transformation on modules on the Weyl algebra: the Fourier-Laplace transform $F \cdot M$ of a $\mathbb{C}[z]\langle \partial_z \rangle$-module M is the $\mathbb{C}[\xi]\langle \partial_\xi \rangle$-module obtained by setting: $z \mapsto -\partial_\xi$, $\partial_z \mapsto \xi$ (we use a dual variable ξ for the image in this paragraph). In particular, if $M = \mathcal{D}_z/\mathcal{D}_z p$ with $p \in \mathbb{C}[z]\langle \partial_z \rangle$, one has $F \cdot M = \mathcal{D}_\xi/\mathcal{D}_\xi . p'$, where p' is obtained from p by replacing z by $-\partial_\xi$ and ∂_z by ξ.

If (E, ∇) is an algebraic connection on a Zariski open subset $\Sigma^\circ \subset \mathbb{C}$, let M its minimal extension, and $M' = F \cdot M$. There exists [6] a meromorphic connection (E', ∇') on a Zariski open subset $\Sigma^{\circ\prime} \subset \mathbb{C}$ such that M' is the minimal extension of (E, ∇). It is the Fourier transform of (E, ∇) and we denote it by $F \cdot (E, \nabla)$.

The stationary phase formula [44, 91] states that the formal data of the Fourier transform (E', ∇') are determined by the formal data of (E, ∇).

Theorem 4.2.1. *There exists a bijection, that we also denote by F, from the set of effective formal data to itself, such that if (E, ∇) is a connection of a Zariski open subset of $\mathbb{C}$, $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ its modified formal data, (E', ∇') is the Fourier transform of (E, ∇) and $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ its modified formal data, the following diagram commutes:*

$$\begin{array}{ccc}
(E, \nabla) & \stackrel{F}{\longmapsto} & (E', \nabla') \\
\downarrow & & \downarrow \\
(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}) & \stackrel{F}{\longmapsto} & (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')
\end{array}$$

This map F is the formal Fourier transform. More precisely, the active exponents of $\breve{\Theta}'$ are related to the active exponents of $\breve{\Theta}'$ by a Legendre transform.

Idea The stationary phase formula is basically a consequence of the Laplace method for determining the asymptotic behaviour of oscillatory integrals. It basically says the following: if f is a function with a unique critical point at x_0, then one has

$$\int g(x)e^{\lambda f(x)} \underset{\lambda\to\infty}{\approx} \sqrt{\frac{2\pi}{\lambda|f''(x_0)|}}e^{\lambda f(x_0)}g(x_0),$$

i.e. the asymptotic behaviour of the integral $\lambda \to \infty$ is determined by what happens in the vicinity of the *stationary phase* x_0. In the situation we are interested in, if the connection ∇ has a pole at 0, any solution in a small sector around zero of the corresponding differential equation will be a linear combination of functions of the form

$$f(z) = e^q(z)g(z),$$

where q is one of the active exponents of ∇. Its Fourier transform is thus an integral of the form

$$\widehat{f}(\xi) = \int_\gamma e^{q(z)-\xi z}g(z),$$

over some contour γ. When $\xi \to \infty$, the behaviour of the integral is determined by the critical point z_0 of the exponential factor $q(z) - \xi z$, i.e. such that

$$\frac{\partial q}{\partial z}(z_0) = \xi.$$

Notice that z_0 is dependent on ξ. The Laplace method then gives for $\xi \to \infty$ the expression:

$$\widehat{f}(\xi) \approx \sqrt{\frac{2\pi}{|q''(z_0)|}}e^{q(z_0)-\xi z_0}.$$

A new exponential factor appears in this expression:

$$\widetilde{q}(\xi) = q(z_0(\xi)) - \xi z_0(\xi).$$

This exponent is nothing but the Legendre transform of q with respect to the conjugated variable ξ. This observation suggests that the exponential factors of the Fourier transform of M are obtained from the ones of M by a Legendre transform. The stationary phase formula of [91, 44] states that this is indeed the case.

The same reasoning can be carried out for any pole at finite distance $a \in \mathbb{C}$. Setting $z_a = z-a$, a component of a local solution at a has the form

$$f(z_a) = e^q(z_a)g(z_a).$$

Its Fourier-Laplace transform is thus

$$\widehat{f}(\xi) = \int e^{q(z_a)-\xi z}g(z_a) = e^{-a\xi}\int e^{q(z_a)-\xi z_a}g(z_a)$$

The exponential factor of $\widehat{f}$ will thus be $-a\xi + \widetilde{q}$, where $\widetilde{q}$ is given as previously by the Legendre transform.

Let us describe the formal Fourier transform and express it in our framework of local systems on $\mathcal{I}$. To this end, we formulate the Legendre transform as a homeomorphism sending collection of circles $\mathcal{I}$ to the dual collection $\mathcal{I}'$, obtained by replacing the variable z by the variable ξ. The Legendre transform takes different forms depending on the circles in $\pi_0(\mathcal{I})$. It is therefore necessary to distinguish several types of circles as follows:

Definition 4.2.2. Any circle in $\mathcal{I}$ belongs to one of the following five families:

1. The *pure circles* at infinity, of the form $\langle \alpha z \rangle_\infty$, with $\alpha \in \mathbb{C}$. There are of slope 1 if $\alpha \neq 0$, and 0 otherwise.
2. Other circles of slope ≤ 1 at infinity, of the form $\langle \alpha z + q \rangle_\infty$, with $\alpha \in \mathbb{C}$, and $q \neq 0$ of slope < 1,
3. Circles $\langle q \rangle_\infty$ of slope > 1 at infinity,
4. Irregular circles at finite distance $\langle q \rangle_a$, with $q \neq 0$, $a \in \mathbb{C} = \mathbb{P}^1 \smallsetminus \{\infty\}$.
5. The tame circles $\langle 0 \rangle_a$, $a \in \mathbb{C}$.

We will denote by $\mathcal{I}_1, \ldots, \mathcal{I}_5 \subset \mathcal{I}$ the corresponding collections of circles. In a similar way, we denote by $\mathcal{I}'_1, \ldots, \mathcal{I}'_5 \subset \mathcal{I}'$ the dual collections. The Legendre transform $\mathcal{L}$ yields homeomorphisms

$$\begin{aligned} \mathcal{L} &: \mathcal{I}_1 \to \mathcal{I}'_5, \\ \mathcal{L} &: \mathcal{I}_2 \to \mathcal{I}'_4 \\ \mathcal{L} &: \mathcal{I}_3 \to \mathcal{I}'_3, \\ \mathcal{L} &: \mathcal{I}_4 \to \mathcal{I}'_2, \\ \mathcal{L} &: \mathcal{I}_5 \to \mathcal{I}'_1, \end{aligned}$$

that we will all denote by $\mathcal{L}$ with a slight abuse of notation.

Let us first describe the Legendre transform in some detail for circles of type 4 at $z = 0$, that is irregular circles at 0. The other cases are easily deduced from this one by changes of variable. In this paragraph, we set $\partial := \partial_0$ the circle of directions at $0 \in \mathbb{P}^1_z$, $\mathcal{I} := \mathcal{I}_0$ the corresponding exponential local system, ∂' the circle of directions at $\xi = \infty$ in $\mathbb{P}^1_\xi$, and $\mathcal{I}' \to \partial'$ the corresponding exponential local system.

Definition 4.2.3. Let $U \subset \partial$ be a germ of sector, and $q_U \in \mathcal{I}(U)$ that is non-zero. For $z \in U$, we set

$$\phi(z) := \frac{dq}{dz}(z).$$

Choosing U small enough, ϕ induces a biholomorphism between U and a sector $U' \subset \partial'$. For $\xi \in U'$, we set

$$\widetilde{q}_{U'}(\xi) := q_U(\phi^{-1}(\xi)) - \xi\phi^{-1}(\xi).$$

This yields a section $\widetilde{q}_{U'} \in \mathcal{I}'(U')$, called the Legendre transform of q_U.

This definition of the Legendre transform is intrinsic. However, it will be useful to also have an algebraic way of computing the Legendre transform in local coordinates using explicit expressions for the exponential factors. Let us do this to find the slope and irregularity of the Legendre transform of an exponential factor.

Lemma 4.2.4. *Let $q \in z^{-1/r}\mathbb{C}[z^{-1/r}]$ be a nonzero exponential factor at 0, with ramification order r and irregularity s. Consider the algebraic system of equations*

$$\xi = \frac{\partial q}{\partial z}, \tag{4.2.1}$$

$$\widetilde{q} = q - z\xi. \tag{4.2.2}$$

corresponding to the Legendre transform. Using the first equation to express z as a function of ξ, as an element in the field of Puiseux series in the variable ξ, one gets $z \in \mathbb{C}((\xi^{1/(r+s)}))$. Using now the second equation to express $\widetilde{q}$ as a function of ξ, one gets $\widetilde{q} \in \mathbb{C}((\xi^{-1/(r+s)}))$. Keeping only the polar part of this expression, one on obtains $\widetilde{q} \in \xi^{1/(r+s)}\mathbb{C}[\xi^{1/(r+s)}]$, with ramification order $r + s$ and degree s.

Proof. Equation (4.2.1) implies

$$\xi = \sum_{i=1}^{s} \frac{-ia_i}{r} z^{-\frac{i}{r}-1} = -\frac{sa_s}{r} z^{-\frac{r+s}{r}} + \dots$$

Introducing a variable t such that $t^r = z$, and ζ such that $\zeta^{r+s} = \xi_\infty = 1/\xi$ this becomes

$$\zeta^{-(r+s)} = \sum_{i=1}^{s} \alpha_i t^{-(r+i)} = \alpha_s t^{-(r+s)} + \dots, \tag{4.2.3}$$

where we have set $\alpha_i := -\frac{a_i i}{r}$. This equation has a solution $t = P(\zeta)$ where $P = \sum_{i\geq 0} \lambda_i t^i \in \mathbb{C}[[t]]$, which is unique as soon as we fix a choice λ_0 of a $r+s$-Th root of α_s^{-1}. Equation (4.2.2) then yields $\widetilde{q}$ as an element of $\mathbb{C}((\zeta))$ of the form

$$\sum_{i\leq s} b_i \zeta^{-i}.$$

The polar part of this expression is $\widetilde{q} = \sum_{i=1}^{s} b_i \zeta^{-i}$. □

It follows from this computation that the image of an irregular circle $S = \langle q \rangle$ by the Legendre transform is the circle $S' = \langle \widetilde{q} \rangle$. The circle $\langle \widetilde{q} \rangle$ has ramification order $r+s$ and slope $s/r+s$. If we parametrize $\langle q \rangle$ by the argument $\arg t \in \mathbb{R}/2\pi\mathbb{Z}$, and $\langle \widetilde{q} \rangle$ by $\arg \zeta$, the relation $t = P(\zeta)$ yields for the homeomorphism between $\langle q \rangle$ and $\langle \widetilde{q} \rangle$ the expression

$$\arg t = \arg \lambda_0 + \arg \zeta.$$

Remark 1. Notice that, though the parametrization of $\langle \widetilde{q} \rangle$ that we obtain depends on the choice of the $r+s$-th root λ_0, so does the polynomial $\widetilde{q}$ inside the Galois orbit of q, in such a way that we have a bijection independent of the choices for the sections q_U on small sectors of ∂ and ∂', in agreement with the intrinsic definition.

The general case of circles of type 2,3,4 is similar to the case of a circle of type 4 at $z = 0$. Let us summarize the properties of the Legendre transform for the different types of circles.

- Type 4: if $a \in \mathbb{P}^1 \setminus \{\infty\}$ is a point at finite distance, the Legendre transform $\mathcal{L}$ yields an homeomorphism between a circle of type 4 of the form $\langle q \rangle_a$ with slope $\alpha/\beta \neq 0$ at a and the circle of type 2 $\langle -a\xi + \widetilde{q} \rangle_\infty$ with $\widetilde{q}$ of slope $\alpha/(\alpha+\beta) < 1$, once again determined by the same system of equations as before. The circle $\langle -a\xi + \widetilde{q} \rangle_\infty$ thus has slope 1 if $a \neq 0$, and slope < 1 if $a = 0$.

$$\begin{tikzcd}
\langle q \rangle_0 \arrow[r, "\mathcal{L}", "1:1"'] \arrow[d, "\pi", "\beta:1"'] & \langle \widetilde{q} \rangle_\infty \arrow[d, "\pi", "\alpha+\beta:1"'] \\
\partial_0 & \partial'_\infty
\end{tikzcd}$$

  This is obtained in a straightforward way from the special case $a = 0$ that we just detailed, by using the relation $z_a = z - a$ between the local coordinate z_a at a and the coordinate z on $\mathbb{P}^1$. The factor $\widetilde{q}$ is given by the same system of equations as previously, with z replaced by z_a.

- Type 2: conversely, if $\langle -ax + q \rangle_\infty$, with $a \in \mathbb{C}$ and q_1 of slope $\alpha/\beta < 1$ is a circle of type 2, the Legendre transform induces an homeomorphism between $\langle q \rangle_\infty$ and a circle of type 4 $\langle \widetilde{q} \rangle_a$ at the point $a \in \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$, with slope $\alpha/(\beta - \alpha)$. We thus have

$$\begin{tikzcd}
\langle q \rangle_\infty \arrow[r, "\mathcal{L}", "1:1"'] \arrow[d, "\pi", "\beta:1"'] & \langle \widetilde{q} \rangle_\alpha \arrow[d, "\pi", "\beta-\alpha:1"'] \\
\partial_0 & \partial'_\infty
\end{tikzcd}$$

- Type 3: If $\langle q\rangle$ is a circle at infinity of type type 3, of slope $\alpha/\beta > 1$, $\mathcal{L}$ induces a homeomorphism of $\langle q\rangle$ on the circle $\langle\widetilde{q}\rangle$ of slope $\alpha/\alpha-\beta > 1$, of type 3. The situation is the following:

$$\begin{tikzcd} \langle q\rangle_\infty \arrow[r, "\mathcal{L}", "1:1"'] \arrow[d, "\pi", "\beta:1"'] & \langle\widetilde{q}\rangle_\infty \arrow[d, "\pi", "\alpha-\beta:1"'] \\ \partial_0 & \partial'_\infty \end{tikzcd}$$

We now deal with circles of type 1 and 5. Let $a \in \mathbb{P}^1 \setminus \{\infty\}$, and consider the circles $\langle 0\rangle_a$ and $\langle -a\xi\rangle_\infty$. We define an homeomorphism ϕ between the circles of directions ∂_a and ∂_∞ as follows, for any argument $\theta \in \partial_a$, we draw the half-line l_θ starting from a with direction θ, and let $\phi(\theta) \in \partial_\infty$ be the direction at which l_θ approaches $z = \infty$. Notice that this inverses the sense of rotation, in agreement with Malgrange [70, e.g. p. 98]. Replacing z by the dual coordinate ξ to identify ∂_∞ to ∂'_∞, we get an homeomorphism $\phi' : \partial_a \to \partial'_\infty$ which lifts to an homeomorphism $\mathcal{L} : \langle 0\rangle_a \to \langle -a\xi\rangle_\infty$, as pictured on the diagram

$$\begin{tikzcd} \langle 0\rangle_a \arrow[r, "\mathcal{L}", "\cong"'] \arrow[d, "\pi"] & \langle -a\xi\rangle_\infty \arrow[d, "\pi"] \\ \partial_0 \arrow[r, "\phi'"'] & \partial'_\infty \end{tikzcd}$$

The Legendre transform induces a transformation of the local systems on $\mathcal{I}$: if $V \to \mathcal{I}$ is a local system, we define $\widetilde{V} := \mathcal{L}_*(V)$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}_*"] \arrow[d] & \widetilde{V} \arrow[d] \\ \mathcal{I} \arrow[r, "\mathcal{L}"] & \mathcal{I}' \end{tikzcd}$$

The previous construction accounts for what happens to the exponential factors in the Laplace method, but the square root term in the Gaussian integral remains to be taken into account. A way to do this is to define for any circle $\langle q\rangle$ of type 2,3,4 a local system $W_{\langle\widetilde{q}\rangle} \to \langle\widetilde{q}\rangle$ of one-dimensional vector spaces. In the case where $\langle q\rangle$ is a circle of type 2 at zero, the definition can formulated as follows.

Definition 4.2.5. Let $\langle q\rangle_0$ be an irregular circle at $z = 0$, $\langle\widetilde{q}\rangle_\infty$ its image by the Legendre transform. Let $r = \mathrm{ram}(q)$ and $s = \mathrm{Irr}(q)$. Let ζ a variable such that $\zeta^{r+s} = 1/\xi$ so that the Legendre transform $\widetilde{q}$ is a polynomial in ζ^{-1}, and the circle $\langle\widetilde{q}\rangle$ can be parametrized by the directions in the disk D_ζ. We define $W_{\langle\widetilde{q}\rangle} \to \langle\widetilde{q}\rangle$ to be the rank one local system with étalé space $W_{\langle\widetilde{q}\rangle}$ is $\langle\widetilde{q}\rangle \times \mathbb{C}$, and a flat section of which given by any determination of the germ of the function $\zeta \mapsto (\partial^2 q/\partial z^2)^{-1/2}(\zeta)$ on the disk D_ζ.

Lemma 4.2.6. *If q has slope α/β, with $\beta = \mathrm{ram}(q)$, then the monodromy of $W_{\langle\widetilde{q}\rangle}$ is $(-1)^\alpha$.*

Proof. One has $q(z) \sim z^{-\alpha/\beta}$, hence $\partial^2 q/\partial z^2 \sim z^{-(\alpha+2\beta)/\beta}$. In terms of the coordinate ζ, the Legendre transform implies $\zeta \sim z^\beta$, hence $\partial^2 q/\partial z^2 \sim \zeta^{\alpha+2\beta}$, hence $(\partial^2 q/\partial z^2)^{-1/2} \sim \zeta^{-(\alpha+2\beta)/2}$. Therefore, when going around $\zeta = 0$ in D_ζ, $(\partial^2 q/\partial z^2)^{-1/2}$ gets multiplied by $\exp\left(-2i\pi \times \frac{\alpha+2\beta}{2}\right) = (-1)^\alpha$. Notice this does not depend on the choice of determination of the square root $(\partial^2 q/\partial z^2)$, so that $W_{\langle\widetilde{q}\rangle}$ is well defined up to isomorphism. □

The definition of $W_{\langle q\rangle}$ in the general case is similar, and the lemma remains true.

Definition 4.2.7. Let $\breve{V}$ be a compatible local system on $\mathcal{I}$, with isomorphism class $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$. We set $F \cdot \breve{V} := \mathcal{L}_*(\breve{V}) \otimes W$, provided it is compatible. It is a local system on $\mathcal{I}'$. Let $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ be its isomorphism class. We define $F \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}) := (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$.

The operation F on modified formal local systems on $\mathcal{I}$ is the counterpart at the level of formal data of the Fourier-Laplace transform of A_1-modules, that is it is such that the theorem 4.2.1 is true.

Proof. It seems that this result was first understood by Malgrange. The case of irregular circles has been independently proven by Fang [44] and Sabbah [91]. An alternative proof was also given later by Graham-Squire [52]. To see that our formulation is indeed equivalent to the one in [91], it suffices to take the monodromy of the local systems. The case of regular circles can be extracted from Malgrange [70, see p129]: the Fourier transform exchanges the local system of regular microsolutions at finite distance and the local system of regular solutions at infinity. □

Remark 2. The definition of the formal Fourier transform $F \cdot (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ might still make sense if it is not compatible. We will rather consider that it is undefined in this case. If $F \cdot (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ is effective, the theorem guarantees that $F \cdot (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ is also effective. If $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ is compatible but not effective, $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ may be undefined (see [6]).

If $F \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}) = (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ the irregular class $\breve{\Theta}'$ only depends on $\breve{\Theta}$, so we set $F \cdot \breve{\Theta} = \breve{\Theta}'$. We will also set $F \cdot \langle q \rangle := \mathcal{L}(\langle q \rangle) \in \pi_0(\mathcal{I})$ for any circle $\langle q \rangle$. If $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ are effective formal data, $\langle q \rangle$ is an active circle of $\breve{\Theta}$ if and only if $F\langle q \rangle$ is an active circle of $\breve{\Theta}'$. Explicitly, if $\breve{\Theta} = n_1\langle q_1 \rangle + \dots n_r\langle q_r \rangle$, we have

$$\breve{\Theta}' = F \cdot \breve{\Theta} = n_1 F \cdot \langle q_1 \rangle + \cdots + n_r F \cdot \langle q_r \rangle.$$

Rank of the Fourier-Laplace transform From the stationary phase formula we obtain a formula for the rank of the Fourier-Laplace transform of a connection with formal data $(\Theta, \boldsymbol{\mathcal{C}})$.

Lemma 4.2.8. *Let* $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ *the modified formal data defined in section 4.1. and* $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}') := F \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$. *The rank of a connection with modified formal data* $(\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$, *if it exists, is*

$$\sum_k \left(m_k + \sum_{i=1}^{r_k} n_i^{(k)} (\alpha_i^{(k)} + \beta_i^{(k)}) \right) + \sum_{i, slope > 1} n_i^{(\infty)} (\alpha_i^{(\infty)} - \beta_i^{(\infty)}), \tag{4.2.4}$$

where the second sum is on all active circles $\langle q_i^{(\infty)} \rangle_\infty$ *at infinity with slope* $\alpha_i^{(\infty)} / \beta_i^{(\infty)} > 1$.

Proof. From the stationary phase formula the active circles of V' at infinity are the following:

- Images by Legendre transform of irregular circles $\langle q^{(k)_i} \rangle_{a_k}$, $k = 1, \dots, m$ i.e circles $\langle -a_k \xi + \tilde{q}^{(k)_j} \rangle_\infty$, with ramification $\alpha_i^{(k)} + \beta_i^{(k)}$ and multiplicity $n_i^{(k)}$.
- Images of the tame circles $\langle 0 \rangle_{a_k}$, $k = 1, \dots, m$: pure circles $\langle -a_k \xi \rangle_\infty$, with multiplicity m_k.
- Images of circles of slope > 1 at infinity among the $\langle q_i^{(\infty)} \rangle, i = 1, \dots, r_\infty$, of the form $\langle \tilde{q}_i^{(\infty)} \rangle$, with slope $\alpha_i^{(\infty)} - \beta_i^{(\infty)}$ and multiplicity $n_i^{(\infty)}$.

Adding the contribution of those circles to the rank gives the result. □

Remark 3. Notice that this is consistent with the formula given by Malgrange [70, p. 79] for the rank of the Laplace transform of modules over the Weyl algebra.

4.2.2 Symplectic transformations

The Fourier-Laplace transform is part of a larger group of transformations acting on modules over the Weyl algebra. Indeed, to any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbb{C})$, we can associate an automorphism of the Weyl algebra $A_1 = \mathbb{C}[z]\langle \partial_z \rangle$ given by $z \mapsto az + b\partial_z, \partial_z \mapsto cz + d\partial_z$. This induces an action of the group $SL_2(\mathbb{C})$ of symplectic transformations on modules over the Weyl algebra (see [70]).

The group of symplectic transformations is generated by three types of elementary transformations:

- The Fourier-Laplace transform F, corresponding to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- Twists at infinity T_λ, for $\lambda \in \mathbb{C}$, corresponding to the matrix $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.
- Scalings S_λ, for $\lambda \in \mathbb{C}^*$, corresponding to the matrix $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$.

The geometric interpretation of twists and scalings is the following. The twist T_λ corresponds to taking the tensor product with the rank one module $(\mathbb{C}[z], \partial_z + \lambda z)$, and the scaling S_λ corresponds to do the change of variable $z \mapsto z/\lambda$ on $\mathbb{P}^1$.

As for the Fourier transform, any element of $SL_2(\mathbb{C})$ induces a transformation on connections on Zariski open subsets on the affine line, and a corresponding formal transformation (that we will also denote by A) on compatible formal data, such that the diagram

$$\begin{tikzcd} (E, \nabla) \arrow[r, "A"] \arrow[d] & (E', \nabla') \arrow[d] \\ (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}) \arrow[r, mapsto, "A"] & (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}') \end{tikzcd}$$

commutes. To show this, it is enough to check this for elementary transformations. We have already dealt with the case of the Fourier transform, and for the twists and scalings we have:

Proposition 4.2.9. *Let (E, ∇) a connection on a Zariski open subset of the affine line, $\breve{V} \to \mathcal{I}$ its modified formal local system, and $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ its isomorphism class.*

- *Let $t_\lambda : \mathcal{I}_\infty \to \mathcal{I}_\infty$ be the homeomorphism of $\mathcal{I}_\infty$ defined by $q_U(1/z) \mapsto q_U(1/z) + \frac{\lambda}{2} z^2$, where q_U is a section of $\mathcal{I}_\infty$ over an open sector $U \subset \partial_\infty$. We extend t_λ to an homeomorphism of $\mathcal{I}$ by having it act trivially on $\bigsqcup_{a \in \mathbb{C}} \mathcal{I}_a$. Then the modified formal local system associated to $T_\lambda \cdot (E, \nabla)$ is isomorphic to $(t_\lambda)_* \breve{V}$.*
- *For $\lambda \in \mathbb{C}^*$, let $s_\lambda : \mathcal{I} \to \mathcal{I}$ be the homeomorphism of $\mathcal{I}$ defined in the following way: for $a \in \mathbb{C}$, if $q_{U_a}(z - a)$ is a section of $\mathcal{I}_a$ on the open sector $U \subset \partial_a$, its image by s_λ is the section $q_{U_a/\lambda}(\lambda(z - a/\lambda))$ over the sector $U_a/\lambda \subset \partial_{a/\lambda}$ which is the image of U by $z \mapsto z/\lambda$. If $q_U(1/z)$ is a section of $\mathcal{I}_\infty$, its image is $q_{U/\lambda}(\frac{1}{(\lambda z)})$. The modified formal local system associated to $S_\lambda \cdot (E, \nabla)$ is isomorphic to $(s_\lambda)_*(\breve{V})$.*

We thus define $T_\lambda \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ as the isomorphism class of $(t_\lambda)_*(\breve{V})$ for $\lambda \in \mathbb{C}$, and $S_\lambda \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ as the isomorphism class of $(s_\lambda)_*(\breve{V})$ for $\lambda \in \mathbb{C}^*$.

Proof. The twist T_λ consists of tensoring (E,∇) with the rank one connection $(\mathcal{O}, d+\lambda z dz)$, having a third order pole at infinity and no other singularity. Therefore, T_λ only modifies the formal data of ∇ at infinity. Since $\lambda z dz = -\lambda \frac{dz_\infty}{z_\infty^3}$, if A is the matrix of the formalization at infinity $\widehat{\nabla}_\infty$ in some basis, i.e. $\widehat{\nabla}_\infty$ corresponds $\partial_{z_\infty} + A$ in this basis, then the matrix of the formalization of $T_\lambda \cdot \nabla$ at infinity in the same basis is $A - \frac{\lambda}{z_\infty^3}$. Integrating this, we obtain that the exponential factors at infinity of $T_\lambda \cdot (E,\nabla)$ are obtained from the exponential factors of (E,∇) by adding $\frac{\lambda}{2z_\infty^2}$. The formal monodromies do not change.

The scaling S_λ corresponds to making the change of variable $z' = \lambda z$ The singularities of $S_\lambda \cdot (E,\nabla)$ are thus the images of the ones of (E,∇) by $z \mapsto z/\lambda$, and the exponential factors of $S_\lambda \cdot (E,\nabla)$ are obtained from the ones of (E,∇) by expressing them as a function of z', i.e. by substituting z with $\lambda z'$. □

As for the Fourier transform, if $A \cdot (\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}}) = (\breve{\Theta}', \breve{\boldsymbol{\mathcal{C}}}')$ the irregular class $\breve{\Theta}'$ only depends on $\breve{\Theta}$, so we set $A \cdot \breve{\Theta} = \breve{\Theta}'$. For any circle $\langle q \rangle$, there is also a well-defined circle $A \cdot \langle q \rangle \in \pi_0(\mathcal{I})$ such that, if $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$ are effective formal data, $\langle q \rangle$ is an active circle of $\breve{\Theta}$ if and only if $A \cdot \langle q \rangle$ is an active circle of $\breve{\Theta}'$. If $\breve{\Theta} = n_1 \langle q_1 \rangle + \dots n_r \langle q_r \rangle$, we have

$$\breve{\Theta}' = A \cdot \breve{\Theta} = n_1 A \cdot \langle q_1 \rangle + \dots + n_r A \cdot \langle q_r \rangle.$$

The circle $A \cdot \langle q \rangle$ can be determined in practice by factorizing A as a product of elementary operations: for $A = T_\lambda$ we have $A \cdot \langle q \rangle = t_\lambda(\langle q \rangle)$ and for $A = S_\lambda$ we have $A \cdot \langle q \rangle = s_\lambda(\langle q \rangle)$.

4.3 Invariance of the diagram for one irregular singularity

In this section, we discuss the properties of the diagram associated by [26] to an irregular class at infinity. We show that the diagram is invariant under Fourier-Laplace transform, and under $SL_2(\mathbb{C})$ transformations. To do this, we compute an explicit formula for the number of edges and loops of the diagram, as well as a formula for the terms appearing in the Legendre transform of an exponential factor.

4.3.1 Diagram for one irregular singularity

Let (E,∇) be an algebraic connection on the affine line. The only singularity is at infinity. Its formal local system $V^0 \to \mathcal{I}$ (which is also the modified formal local system) has support on $\mathcal{I}_\infty$. Let $(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty)$ denote its formal data. Let us fix a direction $d \in \partial$. We set $G := GL(V_d)$ and $H := \mathrm{GrAut}(V_d^0)$ the set of graded automorphisms of V_d. Recall that the conjugacy class $\mathcal{C}_\infty$ of the monodromy of $V^0 \to \partial_\infty$ is an element of a twist $H(\partial)$ of H. The wild character variety is in this case a multiplicative symplectic quotient

$$\mathcal{M}_B(E,\nabla) = \mathcal{M}_B(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty) = \mathrm{Hom}_{\mathbb{S}}(V^0) /\!\!/_{\mathcal{C}} H, \tag{4.3.1}$$

where

$$\mathrm{Hom}_{\mathbb{S}}(\Theta_\infty) = \mathcal{A}(V^0) /\!\!/ G, \tag{4.3.2}$$

is a twisted quasi-Hamiltonian H-space, itself obtained by symplectic reduction with respect to G from the twisted quasi-Hamiltonian $G \times H$-space

$$\mathcal{A}(V^0) = H(\partial) \times G \times \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d(V^0). \tag{4.3.3}$$

Let us denote by $\langle q_1 \rangle, \dots, \langle q_r \rangle$ the active circles, and $\beta_i := \mathrm{ram}(q_i)$. In this case, recall that we have from [26] the definition of a diagram, whose core is given by:

Definition 4.3.1. The *core diagram* $\Gamma_c^{BY}(\Theta_\infty)$ associated to the irregular class Θ_∞ has a set of nodes N given by the set of active circles $\langle q_1\rangle, \dots, \langle q_r\rangle$, and for $i, j = 1, \dots, r$, the number of arrows B_{ij} between $\langle q_i\rangle$ and $\langle q_j\rangle$ is given by

- If $i \neq j$ then
$$B_{ij} = A_{ij} - \beta_i\beta_j, \tag{4.3.4}$$
where $A_{ij} = \mathrm{Irr}(\mathrm{Hom}(\langle q_i\rangle, \langle q_j\rangle)$, and $\beta_i = \mathrm{ram}\langle q_i\rangle$.

- If $i = j$, the number of oriented loops at $\langle q_i\rangle$ is given by
$$B_{ii} = A_{ii} - \beta_i^2 + 1. \tag{4.3.5}$$

The numbers of arrows may be negative. One has $B_{ij} = B_{ji}$ and we will show that B_{ii} is always an even number, so we can group the arrows two by two to get an unoriented diagram. The motivation for this definition comes from counting the number of appearances of blocks between graded parts of V^0 associated to the different active circles in the explicit presentation of the wild character variety. The positive terms in B_{ij} correspond to the matrix blocks in $\mathcal{A}(V^0)$, and the negative terms in the B_{ij} correspond to the relations given by the quasi-Hamiltonian reduction.

The core diagram only depends on the irregular class Θ_∞, but it does not take into account the monodromies of the active circles. This can be done by adding legs to the core diagram encoding the conjugacy classes of the monodromies. For each circle $I \in \pi_0(\mathcal{I}_\infty)$ in the support of Θ_∞, with multiplicity $n(I)$, let $\mathcal{C}_I \subset GL_{n(I)}(\mathbb{C})$ be the corresponding conjugacy class. Recall that a choice of marking of $\mathcal{C}_I$ determines a leg $\mathbb{L}_I$, which comes together with its dimension vector $\mathbf{d}_L \in \mathbb{Z}^{N_\mathbb{L}}$, and vector of labels $\mathbf{q}_\mathbb{L} \in (\mathbb{C}^*)^{N_\mathbb{L}}$, where $N_\mathbb{L}$ denotes the set of vertices of $\mathbb{L}$.

Definition 4.3.2. Let us choose for each active circle I a special marking of the conjugacy class $\mathcal{C}_I$. The *full diagram* $\Gamma^{BY}(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty)$ associated to $(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty)$ is the diagram obtained by gluing for each active circle I the leg $\mathbb{L}(I)$ to the corresponding vertex v_I of $\Gamma_c^{BY}(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty)$.

The full diagram inherits in this way a dimension vector $\mathbf{d} \in \mathbb{Z}^N$, as well as a vector of multiplicative labels $\mathbf{q} \in (\mathbb{C}^*)^N$.

This recipe only enables us to define a diagram for connections with only one singularity at infinity. Our goal is to define a diagram for the general case, that is for connections with an arbitrary number of singularities. The task is not straightforward: although it is possible to draw a diagram corresponding to the formal data at each singularity, it is not clear that there is a natural way to somehow glue those diagrams together to define a global diagram having good properties. In other words, the way in which the different singularities should "interact" is not clear.

The idea, already present in [20], is thus to use the Fourier transform to reduce to the case where all active circles are at infinity, where we know how to draw the diagram. Indeed, the Fourier transform sends to infinity all active circles at finite distance. However, the Fourier transform also sends to finite distance some of the active circles at infinity, and we are back to the problem we started with. For this reason, we need before taking the Fourier transform to apply some operation to prevent the active circles at infinity to go to finite distance. This can be achieved by using more general $SL_2(\mathbb{C})$ transformations on modules over the Weyl algebra. As we shall see, any matrix in an open dense subset of $SL_2(\mathbb{C})$ sends all active circles to infinity.

We thus want to define the diagram associated to a connection as the diagram associated to its image under a generic $SL_2(\mathbb{C})$ transformation. For this to be well defined however, the diagram must not depend on the choice of $SL_2(\mathbb{C})$ transformation to bring all active circles to infinity, in other words the diagram has to be invariant under $SL_2(\mathbb{C})$ transformations. The goal of this section is to show that it is indeed the case. The main step will be to show that the diagram is invariant under Fourier transform.

4.3.2 An explicit formula for the number of edges

Let $\langle q\rangle$ and $\langle q'\rangle$ be two exponential factors at infinity, with ramification orders β, and β', with slopes α/β and α'/β'. We set

$$q=\sum_{j=0}^{p} b_j z_\infty^{-\alpha_j/\beta}, \qquad q'=\sum_{j=0}^{p'} b'_j z_\infty^{-\alpha'_j/\beta'}$$

the expression for q and q' where we write only the monomials with non-zero coefficients, i.e. $b_j\neq 0$ and $b'_j\neq 0$, $\alpha_0>\cdots>\alpha'_p$ and $\alpha'_0>\cdots>\alpha'_{p'}$. Because of ramification, a monomial appearing both in q and q' can give rise to Stokes arrows associated to the difference $q-q'$, since there are several leaves on the covers $I:=\langle q\rangle$ and $I':=\langle q'\rangle$. The leaves of $\langle q\rangle$ correspond to the images q under the action of the Galois group isomorphic to $\mathbb{Z}/\beta\mathbb{Z}$ arising from ramification, i.e. to the polynomials

$$q_i=\sum_{j=0}^{p} b_j\omega^{-\alpha_j} z_\infty^{-\alpha_j/\beta}, \quad i=0,\dots,\beta-1,$$

with $\omega=e^{2i\pi/\beta}$. In a similar way, the leaves of $\langle q'\rangle$ correspond to the polynomials

$$q'_i=\sum_{j=0}^{p'} b'_j\omega'^{-\alpha'_j} z_\infty^{-\alpha'_j/\beta'}, \quad i=0,\dots,\beta'-1,$$

with $\omega'=e^{2i\pi/\beta'}$.

Let r be the biggest integer such that $\sum_{j=0}^{r-1} b_j z_\infty^{-\alpha_j/\beta}$ and $\sum_{j=0}^{r-1} b'_j z_\infty^{-\alpha'_j/\beta'}$ have the same Galois orbit.

We set

$$q_c:=\sum_{j=0}^{r-1} b_j z_\infty^{-\alpha_j/\beta}, \qquad q'_c:=\sum_{j=0}^{r-1} b'_j z_\infty^{-\alpha'_j/\beta'},$$

and

$$q_d:=\sum_{j=r}^{p} b_j z_\infty^{-\alpha_j/\beta}, \qquad q'_d:=\sum_{j=r}^{p} b'_j z_\infty^{-\alpha'_j/\beta'},$$

We get a decomposition

$$q=q_c+q_d, \qquad q'=q'_c+q'_d,$$

of q and q' as the sum of a *common part* q_c and a *different part* q_d. Replacing q or q' by another element of their Galois orbit if necessary, we may assume that $q_c=q'_c$. In the case where q and q' do not have the same leading term, then $r=0$ and we have $q_c=q'_c=0$ and $q=q_d$, $q'=q'_d$. In particular, this happens when q and q' do not have the same slope. Otherwise, q and q' have a non-zero common part, in particular they have the same slope $\alpha_0/\beta=\alpha'_0/\beta'$. The slope of q_d is α_r/β, and the slope of q'_d is α'_r/β'. We use the usual notation $(\cdot,\cdot)$ to denote the greatest common divisor. With those notations now set, we are in position to state the formula for the number of edges between I and I'.

Lemma 4.3.3. • *Assume that and $\alpha_r/\beta\geq\alpha'_r/\beta'$. Then the number of edges between $I=\langle q\rangle$ and $I'=\langle q'\rangle$ is*

$$\begin{aligned}B_{I,I'}=&(\beta'-(\alpha'_0,\beta'))\alpha_0+((\alpha'_0,\beta')-(\alpha'_0,\alpha'_1,\beta'))\alpha_1+\cdots+((\alpha'_0,\dots,\alpha'_{r-2},\beta')-(\alpha'_0,\dots,\alpha'_{r-1},\beta'))\alpha_{r-1}\\&+(\alpha'_0,\dots,\alpha'_{r-1},\beta')\alpha_r-\beta\beta'.\end{aligned}$$

- *In particular, if q and q' have no common parts and $\alpha/\beta\geq\alpha'/\beta'$, then*

$$B_{I,I'}=\beta'(\alpha-\beta).$$

We have a similar result for the number of loops at a circle $\langle q \rangle$.

Lemma 4.3.4. *Let $q = \sum_{j=0}^{p} b_j z_\infty^{-\alpha_j/\beta}$ be an exponential factor of slope $\alpha_0/\beta > 1$ as before.*

- *One has*

$$B_{I,I} = (\beta-(\alpha_0,\beta))\alpha_0+((\alpha_0,\beta)-(\alpha_0,\alpha_1,\beta))\alpha_1+\cdots+((\alpha_0,\ldots,\alpha_{p-1},\beta)-(\alpha_0,\ldots,\alpha_p,\beta))\alpha_p-\beta^2+1. \tag{4.3.6}$$

- *In particular, if $(\alpha,\beta)=1$, then we have*

$$B_{I,I} = (\beta-1)(\alpha-\beta-1). \tag{4.3.7}$$

Proof. To compute the number of edges between $\langle q \rangle$ and $\langle q' \rangle$ we have to determine the local system $\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)$ as well as its irregularity. For this we have to look at the differences $q_i - q'_j$, with $i = 0, \ldots \beta-1, j = 0, \ldots, \beta'-1$ between all possible leaves $\langle q \rangle$ of $\langle q' \rangle$, find which connected components of $\mathcal{I}$ they fall into and what the irregularity of those circles is. The subtlety is that the degree of $q_i - q_j$ depends in general of i and j, so that the circles will not have the same irregularity.

Let μ be the smallest common multiple of β and $\widetilde{\beta}$. We set

$$\mu = k\beta, \qquad \mu = k'\beta'.$$

Let us also denote by δ the greatest common divisor of β and β'. We have

$$\beta = k'\delta, \qquad \beta' = k\delta.$$

For $0 \leq i \leq r$, let γ_i be the integer such that

$$\frac{\alpha_i}{\beta} = \frac{\alpha'_i}{\beta'} = \frac{\gamma_i}{\delta}.$$

Any difference $q_i - q'_j$ belongs to a circle $I = \langle q_i - q'_j \rangle$ which is a connected component of $\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)$. However, several differences $q_i - q'_j$ belong to the same connected component: each difference $q_i - q_j$ corresponds to exactly one leaf of such a circle. This implies the following formula:

$$\mathrm{Irr}(\mathrm{Hom}(\langle q \rangle, \langle q' \rangle)) = \sum_{i=0}^{\beta-1} \sum_{j=0}^{\beta'-1} \mathrm{slope}(q_i - q'_j). \tag{4.3.8}$$

Indeed, regrouping the terms of the sum according to the connected components, to each connected component I of ramification order r and irregularity s correspond r differences $q_i - q'_j$ in the sum, with slope equal to $\mathrm{slope}(q_i - q'_j) = s/r$. The total contribution of those terms is thus equal to $s = \mathrm{Irr}(I)$.

The computation can be simplified by noticing the following fact: for any $k \in \mathbb{Z}$,

$$\mathrm{slope}(q_i - q_j) = \mathrm{slope}(q_{i+k} - q'_{j+k}),$$

where the index $i+k$ is seen as an element of $\mathbb{Z}/\beta\mathbb{Z}$, and $j+k$ is seen as an element of $\mathbb{Z}/\beta'\mathbb{Z}$. Under this shifting action of $\mathbb{Z}$, the set $\mathbb{Z}/\beta\mathbb{Z} \times \mathbb{Z}/\beta'\mathbb{Z}$ is partitioned into δ orbits, each having cardinal μ. Furthermore, the differences

$$q_0 - q'_j, \; j = 0, \ldots, \delta - 1,$$

belong to distinct orbits and thus yield one representative of each orbit. The formula (4.3.8) therefore becomes

$$\begin{aligned}\operatorname{Irr}(\operatorname{Hom}(\langle q\rangle,\langle q'\rangle)) &= \mu\sum_{j=0}^{\delta-1}\operatorname{slope}(q_0-q_j)\\ &= \sum_{j=0}^{\delta-1}\deg_{z^{-1/\mu}}(q_0-q_j).\end{aligned}$$

The task is thus reduced to computing the degree of the differences $q_0-q'_j$, $j=0,\dots,\delta-1$ as polynomials in $z^{-1/\mu}$. It is the exponent of the largest monomial having different coefficients in q_0 and q'_j.

The common part $q_c=q'_c$ is given by

$$q_c=q'_c=\sum_{i=0}^{r} b_i z_\infty^{-\alpha_i/\beta}=\sum_{i=0}^{r} b_i z_\infty^{-\alpha'_i/\beta'}=\sum_{i=0}^{r} b_i z_\infty^{-\gamma_i/\delta}=\sum_{i=0}^{r} b_i z_\infty^{-kk'\gamma_i/\mu}.$$

We will start by determining the number of indices $j=0\in\{0,\dots,\delta-1\}$ such that this degree is the maximal possible degree $kk'\gamma_0$, then the number of indices for which it is $kk'\gamma_1$, etc.

Let us thus begin by computing the number of differences with degree $kk'\gamma_0$. We consider the coefficient of $z_\infty^{kk'\gamma_0/\mu}$ in the difference $q_0-q'_j$, $j=0,\dots,\delta-1$: the corresponding term is

$$a_0(z_\infty^{kk'\gamma_0/\mu}-e^{2i\pi j\gamma_0/\delta}z_\infty^{kk'\gamma_0/\mu})=a_0(1-e^{2i\pi j\gamma_0/\delta})z_\infty^{kk'\gamma_0/\mu}.$$

The factor $1-e^{2i\pi j\gamma_0/\delta}$ is zero if and only if j is an integer multiple of $\delta/(\gamma_0,\delta)$. There are thus (γ_0,δ) differences $q_0-q'_j$ having a degree strictly less than $kk'\gamma_0$. The $\delta-(\gamma_0,\delta)$ other differences $q_0-q'_j$ have degree $kk'\gamma_0$. Each one of these contributes to $kk'\gamma_0$ Stokes arrows from $\langle q\rangle$ to $\langle q'\rangle$.

Next, we compute the number of differences $q_0-q'_j, j=0,\dots\delta-1$, whose degree is $kk'\gamma_1$. In q_0-q_j the monomial of degree $kk'\gamma_1$ is

$$a_1(1-e^{2i\pi j\gamma_1/\delta})z_\infty^{kk'\gamma_1/\mu}.$$

As previously, this term is non-zero when j is an integer multiple of $\delta/(\gamma_1,\delta)$. It follows that $q_0-q'_j$ has degree strictly less than $kk'\gamma_1$ when j is both a multiple of $\delta/(\gamma_1,\delta)$ and of $\delta/(\gamma_0,\delta)$, i.e. is a multiple of their lowest common multiple $\operatorname{lcm}\left(\frac{\delta}{(\gamma_0,\delta)},\frac{\delta}{(\gamma_1,\delta)}\right)$. Since we have the equality

$$\operatorname{lcm}\left(\frac{\delta}{(\gamma_0,\delta)},\frac{\delta}{(\gamma_1,\delta)}\right)=\frac{\delta}{(\gamma_0,\gamma_1,\delta)},$$

we conclude that the number of differences $q_0-q'_j$, $j=0,\dots,\delta-1$ having degree strictly less than $kk'\gamma_1$ is equal to $(\gamma_0,\gamma_1,\delta)$. Therefore, the number of differences having degree equal to $kk'\gamma_1$ is $(\gamma_0,\delta)-(\gamma_0,\gamma_1,\delta)$. Each if these differences gives rise to $kk'\gamma_1$ Stokes arrows from $\langle q\rangle$ to $\langle q'\rangle$.

The same reasoning can be carried out for the next terms in $q_c=q'_c$. By induction, we find that for $1\leq i\leq r$, the number of differences $q_0-q'_j$, $j=0,\dots,\delta-1$ with degree $kk'\gamma_i$ is equal to the difference $(\gamma_0,\dots,\gamma_{i-1},\delta)-(\gamma_0,\dots,\gamma_i,\delta)$, and each of those circles gives rise to $kk'\gamma_i$ Stokes arrows between $\langle q\rangle$ and $\langle q'\rangle$.

There only remain $(\gamma_0,\dots,\gamma_{r-1},\delta)$ differences having degree strictly less than $kk'\gamma_{r-1}$. For those differences, we have $q_{c,0}-q'_{c,j}=0$, so only the different parts matter: $q_0-q'_j=q_{d,0}-q'_{d,j}$. Now, the leading term of q_d is $b_r z_\infty^{-\alpha_r/\beta}=b_r z_\infty^{-k\alpha_r/\mu}$, so q_d has degree $k\alpha_r$ as a polynomial in $z^{-1/\mu}$, and q'_d has degree $k'\alpha'_r$ in $z^{-1/\mu}$. It follows that the degree of $q_0-q'_j$ is $\max(k\alpha_r,k'\alpha'_r)$, since we have assumed that $\alpha_r/\beta\geq\alpha'_r/\beta'$, the maximum is $k\alpha_r$. Each of these thus accounts for $k\alpha_r$ Stokes arrows from $\langle q\rangle$ to $\langle q'\rangle$.

Adding all contributions, and subtracting the $\beta\beta'$ arrows appearing in the definition of $B_{\langle q\rangle,\langle q'\rangle}$, we get the following expression for the number of edges between $\langle q\rangle$ and $\langle q'\rangle$:

$$\begin{aligned} B_{\langle q\rangle,\langle q'\rangle} =&(\delta-(\gamma_0,\delta))kk'\gamma_0+((\gamma_0,\delta)-(\gamma_0,\gamma_1,\delta))kk'\gamma_1+\cdots+((\gamma_0,\ldots,\gamma_{r-2},\delta)-(\gamma,\ldots,\gamma_{r-1},\delta))kk'\gamma_{r-1}\\ &+(\gamma_0,\ldots,\gamma_{r-1},\delta)k\alpha_r-\beta\beta'. \end{aligned}$$

Since $k\delta=\beta'$, $k\gamma_i=\alpha_i'$, and $k'\gamma_i=\alpha_i$, this yields the desired formula.

The case where q and q' have no common part corresponds to having $r=0$. In the formula, all terms corresponding to the common part disappear, there only remains $B_{\langle q\rangle,\langle q'\rangle}=\beta'(\alpha-\beta)$. $\square$

The similar formula for the number of loops is

Lemma 4.3.5. *Let $q=\sum_{j=0}^{p} b_j z_\infty^{-\alpha_j/\beta}$ be an exponential factor of slope $\alpha_0/\beta>1$ as before.*

- *One has*

$$B_{q,q}=(\beta-(\alpha_0,\beta))\alpha_0+((\alpha_0,\beta)-(\alpha_0,\alpha_1,\beta))\alpha_1+\cdots+((\alpha_0,\ldots,\alpha_{p-1},\beta)-(\alpha_0,\ldots,\alpha_p,\beta))\alpha_p-\beta^2+1. \tag{4.3.9}$$

- *Otherwise, if $(\alpha,\beta)=1$, then we have*

$$B_{q,q}=(\beta-1)(\alpha-\beta-1). \tag{4.3.10}$$

Proof. The proof is similar to the case of two different circles. As previously we have

$$\mathrm{Irr}(\mathrm{End}(\langle q\rangle))=\sum_{i=0}^{\beta-1}\sum_{j=0}^{\beta-1}\mathrm{slope}(q_i-q_j)=\sum_{j=0}^{\beta-1}\mathrm{slope}(q_0-q_j).$$

Among those β differences, we determine how many have degree $\alpha_0,\ldots\alpha_p$, as polynomials in $z^{-1/\beta}$, and the remaining differences will have degree 0. We find that the number of differences with degree α_0 is $\beta-(\alpha_0,\beta)$, the number of differences with degree α_1 is $(\alpha_0,\beta)-(\alpha_0,\alpha_1,\beta)$, etc. One has $(\alpha_0,\ldots,\alpha_{p-1},\beta)-(\alpha_0,\ldots,\alpha_p,\beta)$ differences with degree α_p, and the $(\alpha_0,\ldots,\alpha_p,\beta)$ remaining differences belong to connected components that are copies of $\langle 0\rangle$. Each difference of degree α_i accounts for α_i Stokes arrows. The total number of (positive) Stokes arrows is thus

$$(\beta-(\alpha_0,\beta))\alpha_0+((\alpha_0,\beta)-(\alpha_0,\alpha_1,\beta))\alpha_1+\cdots+((\alpha_0,\ldots,\alpha_{p-1},\beta)-(\alpha_0,\ldots\alpha_p,\beta))\alpha_p.$$

To this we must subtract a number of arrows equal to $\beta(\beta-1)+(\beta-1)=\beta^2-1$, which gives the desired formula.

When α and β are relatively prime, only the first term remains: the number of positive Stokes arrows is $(\beta-1)\alpha$, and the conclusion follows. $\square$

Lemma 4.3.6. *The integer $B_{I,I}$ is even.*

Proof. Let us first assume that β is odd. Then all greatest common divisors appearing in the formula for $B_{I,I}$ are odd, so differences between two consecutive g.c.d.s are even, and all terms involving those differences are even. The sum of the remaining terms is $-\beta^2+1$ which is even, so the result follows.

Now, let us consider the case when β is even. Let us consider the sequence of greatest common divisors $\beta,(\alpha_0,\beta),\ldots,(\alpha_0,\ldots,\alpha_p,\beta)$. The first element β is even, the last one $(\alpha_0,\ldots,\alpha_p,\beta)$ is odd, and the sequence consists first of even integers until $(\alpha_0,\ldots,\alpha_{k_0},\beta)$ where k_0 is the smallest index k such that α_k is odd. Starting from this element, all g.c.d.s in the sequence are odd. As a consequence, all the terms involving differences of g.c.d.s in the formula for $B_{I,I}$ are even except $((\alpha_0,\ldots,\alpha_{k_0-1},\beta)-(\alpha_0,\ldots,\alpha_{k_0},\beta))\alpha_{k_0}$ which is odd. The sum of the remaining terms $-\beta^2+1$ is odd, and the conclusion follows. $\square$

This proves what was a statement in [26, p.3], and guarantees that we have a diagram with unoriented edges.

4.3.3 Form of the Legendre transform

In this paragraph, we compute the Legendre transform of an exponential factor q at infinity as explicitly as possible. We determine the monomials appearing (with non-zero coefficients) in the Legendre transform.

Proposition 4.3.7. *Let $q = \sum_{j=0}^{p} b_j z^{\alpha_j/\beta}$, with $b_j \neq 0$, be an exponential factor at infinity with ramification β. We set $\alpha := \alpha_0$, so the slope of q is $\alpha/\beta > 1$. Then the exponents of ξ possibly appearing with non-zero coefficients in its Legendre transform $\widetilde{q}$ are of the form $\frac{\alpha - k_1(\alpha-\alpha_1) - \cdots - k_p(\alpha-\alpha_p)}{\alpha-\beta}$, with $k_1, \dots, k_p \geq 0$. More precisely, if we set $E := \{\gamma \in \mathbb{N} \mid \exists k_1, \dots, k_p \in \mathbb{N}, \gamma = k_1(\alpha-\alpha_1) + \cdots + k_p(\alpha-\alpha_p)\}$ the Legendre transform has the form*

$$\widetilde{q}(\xi) = \sum_{\substack{\gamma \in E \\ \alpha-\gamma>0}} \widetilde{b}_\gamma \xi^{\frac{\alpha-\gamma}{\alpha-\beta}}, \tag{4.3.11}$$

where the sum is restricted to the terms such that the exponent $\frac{\alpha-\gamma}{\alpha-\beta}$ is positive. Furthermore, the coefficients $\widetilde{b}_{(\alpha-\alpha_i)}$ are non-zero for $i \geq 1$.

Lemma 4.3.8. *Let $q = \sum_{j=0}^{p} b_j z^{\alpha_j/\beta}$, with $b_i \neq 0$,be an exponential factor at infinity. We set $\alpha := \alpha_0$, so the slope of q is $\alpha/\beta > 1$. Then the exponents of ξ possibly appearing with non-zero coefficients in its Legendre transform $\widetilde{q}$ are of the form $\frac{\alpha - k_1(\alpha-\alpha_1) - \cdots - k_p(\alpha-\alpha_p)}{\alpha-\beta}$, with $k_1, \dots, k_p \geq 0$. More precisely, if we set $E := \{\gamma \in \mathbb{N} \mid \exists k_1, \dots, k_p \in \mathbb{N}, \gamma = k_1(\alpha-\alpha_1) + \cdots + k_p(\alpha-\alpha_p)\}$ the Legendre transform has the form*

$$\widetilde{q}(\xi) = \sum_{\substack{\gamma \in E \\ \alpha-\gamma>0}} \widetilde{b}_\gamma \xi^{\frac{\alpha-\gamma}{\alpha-\beta}}, \tag{4.3.12}$$

where the sum is restricted to the terms such that the exponent $\frac{\alpha-\gamma}{\alpha-\beta}$ is positive. Furthermore, the coefficients $\widetilde{b}_{(\alpha-\alpha_i)}$ are non-zero for $i \geq 1$.

Proof. The proof consists in computing the Legendre transform directly from the system of equations by which it is defined, in a way explicit enough to find the order of the terms which appear. The first equation of the system is

$$\frac{dq}{dz} = \xi, \tag{4.3.13}$$

it is interpreted as determining ξ as a function of z. We will show that this implies ξ is of the form

$$z = \sum_{\gamma \in E} c_\gamma \xi^{\frac{\beta-\gamma}{\alpha-\beta}}. \tag{4.3.14}$$

The second equation of the system then yields $\frac{d\widetilde{q}}{d\xi} = -z$, and the lemma follows by integrating.

Equation (4.3.13) has a solution $z(\xi)$ in the field of Puiseux series in the variable ξ which is unique once we fix a choice of $\alpha - \beta$-th root. To show (4.3.14), we thus can take this form of solution as an ansatz, and check that it gives a unique solution for the coefficients of this expression: this will automatically be the solution $z(\xi)$. So let us assume (4.3.14). We set

$$\frac{dq}{dz} = \sum_{j=0}^{p} b'_j z^{\frac{\alpha_j-\beta}{\beta}}.$$

This implies

$$\xi = \frac{dq}{dz} = \sum_{j=0}^{p} b'_j \left(\sum_{\gamma \in E} c_\gamma \xi^{\frac{\beta-\gamma}{\alpha-\beta}} \right)^{\frac{\alpha_j-\beta}{\beta}}.$$

We develop this expression to identify the coefficients. We have

$$
\begin{aligned}
\xi &= \sum_{j=0}^{p} b'_j \left(\sum_{\gamma} c_\gamma \xi^{\frac{\beta-\gamma}{\alpha-\beta}} \right)^{\frac{\alpha_j-\beta}{\beta}} \\
&= \sum_{j=0}^{p} b'_j \xi^{-\frac{\alpha_j-\beta}{\alpha-\beta}} \left(\sum_{\gamma} c_\gamma \xi^{\frac{-\gamma}{\alpha-\beta}} \right)^{\frac{\alpha_j-\beta}{\beta}} \\
&= \sum_{j=0}^{p} b''_j \xi^{1-\frac{\alpha-\alpha_j}{\alpha-\beta}} \left(1 + \sum_{\gamma\neq 0} c'_\gamma \xi^{\frac{-\gamma}{\alpha-\beta}} \right)^{\frac{\alpha_j-\beta}{\beta}} \\
&= \sum_{j=0}^{p} b''_j \xi^{1-\frac{\alpha-\alpha_j}{\alpha-\beta}} \left(\sum_{(l_\gamma)_{\gamma\neq 0}} \prod_{\{\gamma,\, l_\gamma\neq 0\}} A^{(j)}_{l_\gamma} {c'_\gamma}^{l_\gamma} \xi^{\frac{-l_\gamma\gamma}{\alpha-\beta}} \right) \\
&= \sum_{j,(l_\gamma)} b''_j \left(\prod_{\{\gamma,\, l_\gamma\neq 0\}} A^{(j)}_{l_\gamma} {c'_\gamma}^{l_\gamma} \right) \xi^{\left(1-\frac{(\alpha-\alpha_j)+\sum_\gamma l_\gamma\gamma}{\alpha-\beta}\right)} \\
&= \sum_{\delta\geq 0} d_\delta \xi^{1-\frac{\delta}{\alpha-\beta}},
\end{aligned}
$$

with

$$
d_\delta = \sum_{\substack{j,(l_\gamma) \\ \sum_\gamma l_\gamma\gamma+(\alpha-\alpha_j)=\delta}} b''_j \left(\prod_{\{\gamma,\, l_\gamma\neq 0\}} A^{(j)}_{l_\gamma} {c'_\gamma}^{l_\gamma} \right). \tag{4.3.15}
$$

In the calculation, we have set $b''_j = b'_j c_0^{(\alpha_j-\beta)/\beta}$, $c'_\gamma = c_\gamma/c_0$, and the $A^{(j)}_{l_\gamma}$ are the combinatorial coefficients appearing in the series expansion of the term $(1+\dots)^{(\alpha_j-\beta)/\beta}$.

The crucial point in this computation is that the integers $\delta = \sum_\gamma l_\gamma\gamma + (\alpha-\alpha_j)$ still belong to the set of exponents E. This is the reason why our ansatz is correct. This enables us to find the coefficients c_γ by induction:

- For $\delta = 0$, we have $d_0 = b''_0$. By comparing the terms in ξ, we find $1 = d_0 = b''_0$, then c_0. This is where the choice of $\alpha-\beta$-th root takes place.

- Let $\delta \in E$. Assume that we know all c_γ for $\gamma \in E$ such that $\gamma \leq \delta$. Let δ' be the smallest element of E strictly greater than δ. Equation (4.3.13) gives $d_{\delta'} = 0$. But δ' is a sum of terms featuring the c'_γ for $\gamma \leq \delta'$, and of a single term featuring d'_δ, equal to

$$
b''_0 A^{(0)}_1 c'_{\delta_1} = \frac{\alpha_0-\beta}{\beta} c'_{\delta'}
$$

  since we have $A^{(0)}_1 = \frac{\alpha_0-\beta}{\beta} \neq 0$. All the other terms being known by induction hypothesis, this determines $c'_{\delta'}$ and hence $c_{\delta'}$, in a unique way.

It remains to see that the first subleading term associated to each α_i, that is $\tilde{b}_{(\alpha-\alpha_i)}$, is non-zero. For this we look at the coefficient $\delta_{(\alpha-\alpha_j)}$. Equation (4.3.14) implies that $\delta_{(\alpha-\alpha_j)} = 0$. On the other hand $\delta_{(\alpha-\alpha_j)}$ is given by (4.3.15) with $\delta = \alpha-\alpha_j$. Let us write down this equation more explicitly. There are only two decompositions $\alpha-\alpha_j$ of the form

$$
\alpha - \alpha_j = \sum_{\gamma\in E} l_\gamma\gamma + (\alpha-\alpha_k),
$$

with $l_\gamma \geq 0, k \in \{0,\dots,p\}$:

$$\alpha - \alpha_j = 0 + (\alpha - \alpha_j), \text{ and } \alpha - \alpha_j = 1 \times (\alpha - \alpha_j) + \underbrace{(\alpha - \alpha_0)}_{=0}.$$

Equation (4.3.15) thus yields

$$0 = d_{\alpha-\alpha_j} = b''_j \times 1 + b''_0 A^{(j)}_{\alpha-\alpha_j} c'_{\alpha-\alpha_j}.$$

Since b''_0, b''_j and $A^{(j)}_{\alpha-\alpha_j}$ are non-zero, this implies that $c'_{\alpha-\alpha_j} \neq 0$, and the conclusion follows.

□

4.3.4 Invariance of the diagram under Fourier-Laplace transform

We now prove the invariance of the diagram under Fourier transform.

Theorem 4.3.9. *Let $\Theta_\infty : \pi_0(\mathcal{I}_\infty) \to \mathbb{N}$ an irregular class infinity, such that the slopes of all active circles are > 1. Then the core diagram $\Gamma_c^{BY}(\Theta_\infty)$ is invariant under Fourier-Laplace transform.*

Proof. The proof comes from combining lemma 4.3.3 giving the number of edges of the diagram, and lemma 4.3.7 giving the structure of the Legendre transform. This allows us to compute the number of edges in the diagrams $\Gamma_c(\Theta_\infty)$ and $\Gamma_c(\Theta'_\infty)$, where $\Theta'_\infty := F \cdot \Theta_\infty$, and check they are the same. Keeping the same notations as before, let $I := \langle q \rangle$ and $I' := \langle q' \rangle$ be two distinct active circles of Θ_∞. We want to compute the number of edges between their images $\widetilde{I} := \langle \widetilde{q} \rangle$ and $\widetilde{I}' := \langle \widetilde{q}' \rangle$. Lemma 4.3.7 implies that $\widetilde{q}$ is of the form

$$\widetilde{q} = \sum_{\gamma \in E} \widetilde{b}_\gamma \xi_\infty^{-\frac{\alpha-\gamma}{\alpha-\beta}},$$

with $E = \{\gamma \in \mathbb{N} \mid \gamma < \alpha,\ \exists k_1,\dots,k_p \geq 0,\ \gamma = k_1(\alpha_0 - \alpha_1) + \cdots + k_p(\alpha_0 - \alpha_p)\}$, and similarly

$$\widetilde{q}' = \sum_{\gamma' \in E'} \widetilde{b}'_{\gamma'} \xi_\infty^{-\frac{\alpha'-\gamma'}{\alpha'-\beta'}},$$

with $E' = \{\gamma' \in \mathbb{N} \mid \gamma' < \alpha',\ \exists k_1,\dots,k_p \geq 0,\ \gamma' = k_1(\alpha'_0 - \alpha'_1) + \cdots + k_p(\alpha'_0 - \alpha'_p)\}$. Let us denote by $0 = \gamma_0 < \cdots < \gamma_N$ the distinct elements of E and $0 = \gamma'_0 < \cdots < \gamma'_{N'}$ those of E'. We first determine the common parts and the different parts of $\widetilde{q}$ and $\widetilde{q}'$. The integers γ and γ' containing only terms common to q and q', i.e. of the form $\gamma = k_1(\alpha_0 - \alpha_1) + \cdots + k_{r-1}(\alpha_0 - \alpha_{r-1})$ and $\gamma' = k_1(\alpha'_0 - \alpha'_1) + \cdots + k_{r-1}(\alpha'_0 - \alpha'_{r-1})$, belong to the part common to $\widetilde{q}$ and $\widetilde{q}'$. Let us denote by $E_c \subset E$ and $E'_c \subset E'$ the corresponding subsets. Furthermore, the respective leading terms of the different parts of $\widetilde{q}_d$ and $\widetilde{q}'_d$ correspond to the first integers γ where the factors $(\alpha - \alpha_r)$ and $(\alpha' - \alpha'_r)$ appear, which are $\alpha - \alpha_r =: \gamma_R$, and $\alpha' - \alpha'_{r+1} =: \gamma'_R$ with $R \leq N$. It follows that $\widetilde{q}_d$ is of degree $\frac{\alpha_r}{\alpha-\beta}$ and $\widetilde{q}'_d$ is of degree $\frac{\alpha'_r}{\alpha'-\beta'}$. Since we have assumed $\alpha_r/\beta > \alpha'_r/\beta'$, we have $\frac{\alpha_r}{\alpha-\beta} > \frac{\alpha'_r}{\alpha'-\beta'}$. From lemma 4.3.3, the number of edges between $\langle \widetilde{q} \rangle$ and $\langle \widetilde{q}' \rangle$ is

$$B_{\widetilde{I},\widetilde{I}'} = ((\alpha' - \beta') - p'_0)\alpha_0 + \sum_{k=1}^{R-1} (p'_{k-1} - p'_k)(\alpha - \gamma_k) + p_R(\alpha - \gamma_R) - (\alpha - \beta)(\alpha' - \beta'), \quad (4.3.16)$$

where $p'_k = (\alpha'_0, \dots, \alpha'_0 - \gamma_k, \alpha' - \beta')$.

In this expression, let us examine the g.c.d.s p'_k and the differences $p'_{k-1} - p'_k$. Notice that one has $p'_{k-1} = p'_k$ as soon as any decomposition $\gamma_k = k_1(\alpha'_0 - \alpha'_1) + \cdots + k_p(\alpha'_0 - \alpha'_p)$ of γ_k only contains factors $(\alpha - \alpha_j)$ already present in in some p_l with $l \leq k-1$. In other words, we can have $p'_{k-1} > p'_k$ only when in γ_k a new factor $(\alpha - \alpha_j)$ appears, that is when $\gamma_k = \alpha - \alpha_j$ for some $j = 1, \dots, p$. The distinct values of p'_k are thus the g.c.d.s $(\alpha'_0, \dots, \alpha'_l, \alpha' - \beta') = (\alpha'_0, \dots, \alpha'_l, \beta')$. Therefore, the non-zero terms involving g.c.d.s in the formula for $B_{\langle\widetilde{q}\rangle,\langle\widetilde{q'}\rangle}$ are exactly the same as those for $B_{I,I'}$. The sum of the remaining terms is $\alpha'\beta - \beta\beta' = \beta(\alpha' - \beta')$ for $B_{I,I'}$, and $(\alpha' - \beta')\alpha_0 - (\alpha - \beta)(\alpha' - \beta') = \beta(\alpha' - \beta')$ for $B_{\widetilde{I},\widetilde{I'}}$: they are also the same. Finally, we have the equality $B_{I,I'} = B_{\widetilde{I},\widetilde{I'}}$ of the numbers of edges.

It remains to deal with the case of loops at $\langle q\rangle$ and $\langle\widetilde{q}\rangle$. In a similar way as the previous case, the non-zero differences of g.c.d.s appearing in the formula for $B_{\widetilde{I},\widetilde{I}}$ are exactly the same as those appearing in the formula for $B_{I,I}$. The sum of the remaining terms for $B_{I,I}$ is $\beta(\alpha - \beta) + 1$, and this expression is invariant under Fourier transform since $\alpha \mapsto \alpha$ and $\beta \mapsto \alpha - \beta$. We thus have $B_{I,I} = B_{\widetilde{I},\widetilde{I}}$, which completes the proof of the invariance of the graph. □

4.3.5 Invariance of the diagram under symplectic transformations

The goal of this section is to show:

Theorem 4.3.10. *Let Θ_∞ be an irregular class at infinity. There is an open dense subset $S_V \subset SL_2(\mathbb{C})$ such that for all $A \in S_V$, $A \cdot \Theta_\infty$ only has a singularity at infinity. Furthermore, we have $\Gamma_c^{BY}(\Theta_\infty) = \Gamma_c^{BY}(A \cdot \Theta_\infty)$.*

To prove this, the idea is to decompose the elements of $SL_2(\mathbb{C})$ as products of elementary operations.

Lemma 4.3.11. *Let Θ_∞ be an irregular class at infinity. Then*

- *For $\lambda \in \mathbb{C}$, $\Gamma_c(T_\lambda \cdot \Theta_\infty) = \Gamma_c^{BY}(\Theta_\infty)$.*
- *For $\lambda \in \mathbb{C}^*$, $\Gamma_c(S_\lambda \cdot \Theta_\infty) = \Gamma_c^{BY}(\Theta_\infty)$.*

Proof. The twist T_λ induces via t_λ a bijection between the active circles of Θ_∞ and $T_\lambda \cdot \Theta_\infty$. The differences of exponential factors $q_U(z_\infty) - q'_U(z_\infty)$ are invariant under a twist T_λ, so the number of Stokes arrows between two active circles and their images by t_λ are the same. This proves the first part of the lemma. In a similar way, S_λ induces via s_λ a bijection between the active circles of Θ_∞ and $S_\lambda \cdot \Theta_\infty$. The degree of a difference $q_U(z_a) - q'_U(z_a)$ of exponential factors at $a \in \mathbb{C}$ is the same as its image $q_U(\lambda(z_{a/\lambda})) - q'_U(\lambda(z_{a/\lambda}))$ by s_λ. This is also true for exponential factors at infinity. Therefore, the number of Stokes arrows between two active circles and their images by s_λ are the same, and the conclusion follows. □

Since we have already seen that the Fourier transform preserves the diagram when all circles are of slope > 1, i.e. when $F \cdot \Theta_\infty$ only has a singularity at infinity, all elementary transformations preserve the core diagram. However, deducing the theorem from this by factoring matrices of $SL_2(\mathbb{C})$ as products of elementary transformations is not entirely straightforward. Indeed, if $A \in SL_2(\mathbb{C})$ admits a factorization $A = A_k \dots A_1$ as a product of elementary transformations, it may occur that for some $1 \leq i \leq k$, $A_i \dots A_1 \cdot \Theta_\infty$ has singularities at finite distance. Therefore, we have to show that A always admits a factorization such that this doesn't happen. We will do this in several steps.

Lemma 4.3.12. *Let Θ_∞ be an irregular class at infinity. There exists $A \in SL_2(\mathbb{C})$ such that all active circles of $A \cdot \Theta_\infty$ have slope no greater than 2. More precisely, there exists such a matrix of the form FT_λ, for some $\lambda \in \mathbb{C}$.*

Proof. Let $\langle q\rangle_\infty$ be an irregular circle at infinity. Its image by T_λ is $\langle q + \frac{\lambda}{2}z^2\rangle$. The term $\frac{\lambda}{2}z^2$ is of slope 2. There are thus two possibilities: either q has slope strictly greater than 2 and $\mathrm{slope}(q) = \mathrm{slope}(q + \frac{\lambda}{2}z^2)$, or q has slope less than 2 and $\mathrm{slope}(q + \frac{\lambda}{2}z^2) = 2$, if we choose λ such that $-\lambda/2$ is not equal to the coefficient of slope 2 of q. We then apply the Fourier transform.

In the first case, if we denote $\mathrm{slope}(T_\lambda \cdot \langle q \rangle) = p/q > 2$, with $p, q \in \mathbb{N}$, one has $p - q > q$, so from the stationary phase formula $\mathrm{slope}(FT_\lambda \cdot \langle q \rangle) = \frac{p}{p-q} < 2$. In the second case we have $\mathrm{slope}(FT_\lambda \cdot \langle q \rangle) = \mathrm{slope}(T_\lambda \cdot \langle q \rangle) = 2$. As a consequence, if we choose λ such that $-\lambda/2$ is not equal to any of the coefficients of slope 2 of the active circles of Θ_∞, then $FT_\lambda \cdot \Theta_\infty$ only has active circles at infinity of slope no greater than 2. $\square$

Let $\langle q \rangle$ be a circle at infinity of slope ≤ 2. It is clear that this circle remains at infinity when a twist or a scaling is applied. It remains at infinity under Fourier transform if and only if its slope is strictly greater than 1. In particular, it remains at infinity if its coefficient of slope 2 is non-zero.

Proposition 4.3.13. *Let q be an exponential factor at infinity of slope ≤ 2. We write $q = -\frac{\lambda}{2} z^2 + q_{<2}$ with $q_{<2}$ of slope < 2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. Let $\langle q' \rangle$ be the image of $\langle q \rangle_\infty$ by A. Then $\langle q' \rangle$ is also at infinity unless $\mathrm{slope}(q_{<2}) \leq 1$ and $c\lambda + d = 0$. In this case, writing $q' = -\frac{\lambda'}{2} + q'_{<2}$ with $q_{<2}$ of slope < 2, the coefficients λ and λ' are related by*

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}. \tag{4.3.17}$$

To prove this, we begin with the case of elementary transformations.

Lemma 4.3.14. *The proposition is true if A is an elementary transformation*

Proof. The twist $T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ acts on q according to

$$T_\mu(q) = q - \frac{\mu}{2} z^2 = -\frac{\lambda + \mu}{2} z^2 + q_{<2}.$$

This amounts to $\lambda \mapsto \lambda + \mu$, so the property is verified. The scaling $S_\mu = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$ sends q to

$$S_\mu(q) = q(z/\mu) = -\frac{\lambda\mu^2}{2} z^2 + q_{<2}(z/\mu).$$

This amounts to $\lambda \mapsto \lambda\mu^2$, so the property is again verified. The Fourier transform $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts on q according to the Legendre transform: computing it explicitly we find

$$F(q) = \mathcal{L}q = \frac{1}{2\lambda} x^2 + \widetilde{q}_0,$$

where $\widetilde{q}_0$ has slope < 2. This amounts to $\lambda \mapsto -\frac{1}{\lambda}$ and the property is once again verified. $\square$

The idea is then to factorize A as a product of elementary operations, using the following lemma.

Lemma 4.3.15. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, and $h_A : \mathbb{P}^1 \to \mathbb{P}^1$ be the corresponding homography $z \mapsto \frac{az+b}{cz+d}$.*

- *If $h_A(\infty) = \infty$, then A admits a factorization as a product of elementary operations of the form $A = S_\nu T_\rho$, with $\nu \in \mathbb{C}^*, \rho \in \mathbb{C}$.*

- *Otherwise, if $h_A(\infty) \neq \infty$, A admits a factorization of the form*
$$A = T_\mu F S_\nu T_\rho, \tag{4.3.18}$$
with $\mu \in \mathbb{C}$, $\nu \in \mathbb{C}^*$, $\rho \in \mathbb{C}$.

Proof. We have $h_A(\infty) = \frac{a}{c} \in \mathbb{P}^1$, so h_A fixes infinity if and only if $c = 0$. This implies $a \neq 0$ and $d = a^{-1}$, and we have $A = S_\nu T_\rho$ with $\nu = a$, $\rho = b/a$. When $h_A(\infty) \neq \infty$,we can reduce to the case where h_A fixes infinity. Let $\mu := h_A(\infty) = \frac{a}{c}$. We introduce

$$B_\mu := \begin{pmatrix} 0 & -1 \\ 1 & -\mu \end{pmatrix} = F^{-1} T_{-\mu},$$

such that the corresponding homography h_{B_μ} sends μ to ∞. Then the homography associated to $B_\mu A$ sends ∞ to ∞. This implies that $B_\mu A$ has the form

$$B_\lambda A = \begin{pmatrix} \nu & \sigma \\ 0 & \nu^{-1} \end{pmatrix},$$

with $\nu = -c$ and $\sigma = -d$, so that $B_\mu A = S_\nu T_\rho$ with $\rho = \sigma/\mu$. Finally we get $A = B_\mu^{-1} S_\nu T_\rho = T_\mu F S_\nu T_\rho$. □

Proof of the proposition. If $c = 0$ then $d \neq 0$ and $c\lambda + d = d \neq 0$. In this case A factorizes as $A = S_\nu T_\rho$, so the circle $\langle q' \rangle$ is at infinity. Now if $c \neq 0$, from the lemma A factorizes as $A = T_\mu F S_\nu T_\rho$. Let $\langle q'' \rangle := S_\nu T_\rho \cdot \langle q \rangle_\infty$, and write $q'' = -\frac{\lambda''}{2} + q''_{<2}$ with the slope of $q''_{<2}$ being < 2. The circle $\langle q' \rangle$ is at infinity if and only if $\langle q'' \rangle$ is not sent to finite distance by the Fourier transform, i.e. using the stationary phase formula if we don't have $\lambda'' = 0$ and $\mathrm{slope}(q''_{<2}) \leq 1$. From the Legendre transform $\mathrm{slope}(q''_{<2}) = \mathrm{slope}(q_{<2})$. We have $S_\nu T_\rho = \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}$, so $\lambda'' = -\frac{c\lambda + d}{-c^{-1}}$. Therefore we have $\lambda'' \neq 0$ if and only if $c\lambda + d \neq 0$. It follows from this that, if we do not have $c\lambda + d = 0$ and $\mathrm{slope}(q_{<2}) \leq 1$, then at each step of the factorization the images of $\langle q \rangle$ remain at infinity. Since at each elementary operation the coefficient λ is transformed according to the corresponding homography, the conclusion follows.

□

We are now in position to prove the theorem.

Proof of the theorem. Let Θ_∞ be an irregular class at infinity. Up to applying a well-chosen symplectic transformation, we may assume that all active circles of Θ_∞ have slope less than 2. We denote by $-\lambda_i/2, i = 1, \ldots, r$ the coefficients of z^2 of these active circles. It follows from the previous proposition that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \in SL_2(\mathbb{C})$, $A \cdot \Theta_\infty$ only has a singularity at infinity if and only if $c\lambda_i + d \neq 0$ for $i = 1, \ldots, m$. This corresponds to an open dense subset of $SL_2(\mathbb{C})$. Furthermore A has a factorization $A = T_\lambda F S_\mu T_\nu$, such that when applying successively the elementary operations, Θ_∞ always remains with only one singularity at infinity. At each step, the diagram doesn't change, and the conclusion follows. □

Remark 4. It follows from this analysis that one may view the active circles at finite distance as circles at infinity of slope ≤ 2 for which the coefficient of slope 2 is infinite. We may call the copy of $\mathbb{P}^1$ where those coefficients live the *Fourier sphere*, as in [20]. Acting with symplectic transformations on circles at infinity of slope ≤ 2 amounts to acting on the Fourier sphere with the corresponding homographies. It is then clear that for a generic rotation none of the coefficients will be at infinity, i.e. there will only be one singularity at infinity.

4.4 Diagrams for general connections

4.4.1 Definition of the diagram

To define the diagram in the the general case where there are several irregular singularities, the idea is to use the Fourier-Laplace transform to reduce to the situation where there is only one singularity at infinity studied in the previous section. The invariance of the diagram under $SL_2(\mathbb{C})$ will ensure that this construction is well defined.

Theorem 4.4.1. *Let (E, ∇) an algebraic connection on a Zariski open subset of $\mathbb{A}^1$ with modified formal data $(\breve{\Theta}, \breve{\mathcal{C}})$. Then for any A in an open dense subset of $SL_2(\mathbb{C})$, the local system $A \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ only has support at infinity, and the diagram $\Gamma^{BY}(A \cdot (\breve{\Theta}, \breve{\mathcal{C}}))$ is independent of A.*

Proof. It is easy to find a matrix $A \in SL_2(\mathbb{C})$ such that $A \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ has support at infinity. For example we may take A of the form $A = FT_\lambda$, with $\lambda \in \mathbb{C}$: it suffices to choose the coefficient λ of the twist such that all the coefficients of the terms of slope 2 in the active circles of $T_\lambda \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ at infinity are non-zero. Applying then the Fourier transform, this guarantees that the active circles at infinity of $T_\lambda \cdot \breve{\Theta}$ remain at infinity while the active circles at finite distance are sent to infinity. The results of the previous section now imply that $BA \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ has support at infinity and that $\Gamma^{BY}(BA \cdot (\breve{\Theta}, \breve{\mathcal{C}})) = \Gamma^{BY}(A \cdot (\breve{\Theta}, \breve{\mathcal{C}}))$ for B in an open dense subset of $SL_2(\mathbb{C})$. $\square$

Definition 4.4.2. Let (E, ∇) a connection on a Zariski open subset of the affine line, $(\Theta, \mathcal{C})$ its formal data, and $(\breve{\Theta}, \breve{\mathcal{C}})$ the corresponding modified formal data. We define the diagram associated to (E, ∇) as

$$\Gamma(E, \nabla) = \Gamma(\Theta, \mathcal{C}) = \Gamma(\breve{\Theta}, \breve{\mathcal{C}}) := \Gamma^{BY}(A \cdot (\breve{\Theta}, \breve{\mathcal{C}}))$$

for any $A \in SL_2(\mathbb{C})$ such that $A \cdot (\breve{\Theta}, \breve{\mathcal{C}})$ has support at infinity.

4.4.2 Direct formula for the core diagram

The definition of the diagram involves a choice of $SL_2(\mathbb{C})$ transformation. The explicit formulas for the number of edges and our understanding of the Legendre transform allow us to give a direct expression of the core diagram, which we may then take as the definition of the core diagram as in the introduction.

Let $I = \langle q \rangle, I' = \langle q' \rangle$ be circles at points $a, a' \in \mathbb{P}^1$. Using similar notations as before, we write $\beta := \mathrm{ram}(q)$, $\beta' := \mathrm{ram}(q')$ and

$$q = \sum_{i=0}^{k} b_i z_a^{-\alpha_i/\beta}, \qquad q' = \sum_{i=0}^{k'} b'_i z_{a'}^{-\alpha'_i/\beta'},$$

where the coefficients $b_i, b'_i \in \mathbb{C}$ are non-zero, and set $\alpha := \alpha_0 = \mathrm{Irr}(q)$, $\alpha' := \alpha'_0 = \mathrm{Irr}(q')$. If q and q' are both at infinity, that is if $a = a' = \infty$, then we know from a previous lemma the number of edges $B_{\langle q \rangle, \langle q' \rangle}$, which depends only on the degrees of the terms of q and q'. Let us denote by $B^\infty_{\langle q \rangle, \langle q' \rangle}$ this formula.

Theorem 4.4.3. *Let $I = \langle q \rangle_a$ and $I' = \langle q' \rangle_{a'}$ be active circles. If $a = a'$, denoting as before by α_r/β and α'_r/β' the slopes of the different parts of q and q', we assume that $\frac{\alpha_r}{\beta} \geq \frac{\alpha'_r}{\beta'}$. The number of edges $B_{I,I'}$ between the vertices associated to I and I' following:*

1. *If $a = a' = \infty$ then $B_{I,I'} = B^\infty_{I,I'}$.*
2. *If $a = \infty$ and $a' \neq \infty$ then $B_{I,I'} = \beta(\alpha' + \beta')$.*
3. *If $a \neq \infty, a' \neq \infty$ and $a \neq a'$ then $B_{I,I'} = 0$.*

4. *If $a = a' \neq \infty$ then $B_{I,I'} = B^\infty_{I,I'} - \alpha\beta' - \alpha'\beta$.*

Proof. To prove this, we apply a generic $SL_2(\mathbb{C})$ twist at infinity acting on exponential factors at infinity as $q \mapsto q + \lambda z^2$, followed by Fourier transform. From our study of the Legendre transform we obtain the form of the images of q and q' under this process, then we apply the formula for the number of edges at infinity. Let us first deal with the cases where $q \neq q'$.

1. This case is just a consequence of the fact that the twist and the Fourier transform leave $B^\infty_{\langle q\rangle,\langle q'\rangle}$ invariant.

2. Let q_1 and q_1' be the images of q, q'. If $\frac{\alpha}{\beta} \leq 2$, then q_1 has ramification β and slope $2 = 2\beta/\beta$. If $\frac{\alpha}{\beta} > 2$, then q_1 has ramification $\alpha - \beta$ and slope $\frac{\alpha}{\alpha-\beta}$. On the other hand, q_1' is of the form
$$q_1' = -a'z + b'z^{\alpha'/(\alpha'+\beta')} + \dots$$
for some $b' \in \mathbb{C}$, $b' \neq 0$, so q_1' has ramification $\alpha' + \beta'$ and slope 1 (if $a \neq 0$), or $\frac{\alpha'}{\alpha'+\beta'}$ (if $a = 0$). When $\frac{\alpha}{\beta} \leq 2$, it follows from the formula for B^∞ that
$$B_{\langle q\rangle,\langle q'\rangle} = B^\infty_{\langle q_1\rangle,\langle q_1'\rangle} = (\alpha' + \beta')(2\beta - \beta) = \beta(\alpha' + \beta').$$
When $\frac{\alpha}{\beta} > 2$ the same formula gives
$$B_{\langle q\rangle,\langle q'\rangle} = B^\infty_{\langle q_1\rangle,\langle q_1'\rangle} = (\alpha' + \beta')(\alpha - (\alpha - \beta)) = \beta(\alpha' + \beta').$$
so both cases yield the expected result.

3. In this case q_1 and q_1' are of the form
$$q_1 = -az + bz^{\alpha/(\alpha+\beta)} + \dots, \qquad q_1' = -a'z + b'z^{\alpha'/(\alpha'+\beta')} + \dots,$$
that is q_1 and q_1 have respective ramification orders $\alpha + \beta$ and $\alpha' + \beta'$, have both slope 1 and non common part. The formula for B^∞ thus gives
$$B_{\langle q\rangle,\langle q'\rangle} = B^\infty_{\langle q_1\rangle,\langle q_1'\rangle} = (\alpha' + \beta')((\alpha + \beta) - (\alpha + \beta)) = 0.$$

4. If $a = a'$, q_1 and q_1' are of the form
$$q_1 = -az + \tilde{q}, \qquad q_1' = -az + \tilde{q}',$$
with $\tilde{q}$ having ramification $\alpha + \beta$ and slope $\frac{\alpha}{\alpha+\beta}$, and $\tilde{q}'$ having ramification $\alpha + \beta$ and slope $\frac{\alpha'}{\alpha'+\beta'}$. Let us first assume that q and q' have a common part. Then so do q_1 and q_1'. The slopes or their different parts are $\frac{\alpha_r}{\alpha+\beta} \geq \frac{\alpha_r'}{\alpha'+\beta'}$. The form of q_1 and q_1' is given by the lemma giving the form of the Legendre transform. The slopes or their different parts are $\frac{\alpha_r}{\alpha+\beta} \geq \frac{\alpha_r'}{\alpha'+\beta'}$ and the formula for $B^\infty_{\langle q_1\rangle,\langle q_1'\rangle}$ gives
$$\begin{aligned}
B_{\langle q\rangle,\langle q'\rangle} &= B^\infty_{\langle q_1\rangle,\langle q_1'\rangle} \\
&= ((\alpha' + \beta') - (\alpha_0', \alpha' + \beta'))\alpha_0 + \dots + (\alpha_0', \dots, \alpha_{r-1}', \alpha' + \beta')\alpha_r - (\alpha + \beta)(\alpha' + \beta') \\
&= ((\alpha' + \beta') - (\alpha_0', \beta'))\alpha_0 + \dots + (\alpha_0', \dots, \alpha_{r-1}', \beta')\alpha_r - (\alpha\alpha' + \beta\beta' + \alpha'\beta + \alpha\beta') \\
&= B^\infty_{\langle q\rangle,\langle q'\rangle} - \alpha\beta' - \alpha'\beta \\
&= B^\infty_{\langle q\rangle,\langle q'\rangle} - 2\alpha\beta',
\end{aligned}$$
where in the last step we have used that $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$ since q and q' have a common part. Now assume that q and q' have no common part. Then the slope of $\tilde{q}$ and $\tilde{q}'$ are respectively

$\frac{\alpha}{\alpha+\beta} \geq \frac{\alpha'}{\alpha'+\beta'}$ (the order is preserved by the Legendre transform), so from the formula for $B^{\infty}_{\langle\tilde{q}\rangle,\langle\tilde{q}'\rangle}$ we have

$$B_{\langle q\rangle,\langle q'\rangle} = -\beta(\alpha' + \beta').$$

On the other hand, we have $B^{\infty}_{\langle q\rangle,\langle q'\rangle} = \beta'(\alpha - \beta)$, so we again have $B^{\infty}_{\langle q\rangle,\langle q'\rangle} = B^{\infty}_{\langle q\rangle,\langle q'\rangle} - \alpha\beta' - \alpha'\beta$ as expected.

To complete the proof, it remains to deal with the case of loops. We check that one has $B_{\langle q\rangle,\langle q\rangle} = B^{\infty}_{\langle q\rangle,\langle q\rangle} - 2\alpha\beta$ when $a \neq \infty$, so the formula for $B_{\langle q\rangle,\langle q'\rangle}$ remains valid when $q = q'$. $\square$

Chapter 5

Dimension of the wild character variety

In this chapter, we prove that the dimension of the wild character variety is obtained from the diagram and its dimension vector. Let Γ be a diagram, possibly with loops, negative edges or negative loops. Let N be its set of vertices and k its cardinal. A dimension vector associated with Γ is an element $\mathbf{d} = (d_i)_{i\in N} \in \mathbb{N}^N$. Let $B \in M_k(\mathbb{Z})$ be the adjacency matrix of the diagram. The Cartan matrix of the diagram is defined by

$$C = 2\,\mathrm{Id} - B.$$

We denote by $(\cdot,\cdot)$ the bilinear form on $\mathbb{C}^k$ with matrix C.

We have the following result:

Theorem 5.0.1. *The dimension of the wild character variety $\mathcal{M}_B(\Theta,\boldsymbol{\mathcal{C}})$, when it is non-empty, is given by*

$$\dim \mathcal{M}_B(\Theta,\boldsymbol{\mathcal{C}}) = 2 - (\mathbf{d},\mathbf{d}), \tag{5.0.1}$$

where $(\cdot,\cdot)$ is the bilinear form associated with the diagram $\Gamma(\Theta,\boldsymbol{\mathcal{C}})$, and $\mathbf{d}$ is the dimension vector of $\Gamma(\Theta,\boldsymbol{\mathcal{C}})$.

Corollary 5.0.2. *If $(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}})$ are the modified formal data of a connection (E,∇) and $A \in SL_2(\mathbb{C})$, then the character varieties $\mathcal{M}_B(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}})$ and $\mathcal{M}_B(A\cdot(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}}))$, if nonempty, have the same dimension.*

This is consistent with the preservation of the index of rigidity by Fourier transform [12].

Proof. By construction of the diagram $(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}})$ and $A\cdot(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}})$ have the same diagram Γ. Therefore we have $\dim \mathcal{M}_B(\Theta,\boldsymbol{\mathcal{C}}) = 2 - (\mathbf{d},\mathbf{d}) = \dim \mathcal{M}_B(A\cdot(\breve{\Theta},\breve{\boldsymbol{\mathcal{C}}}))$. $\square$

To prove this, we will use the quasi-Hamiltonian description of the wild character variety. We briefly recall the main facts. Let us set $G := GL_n(\mathbb{C})$ where n is the rank of (E,∇), and keeping the previous notations let $a_1,\ldots,a_m \in \mathbb{P}^1$ be the singular points of any connection with set of formal data V. Let $V^0 \to \mathcal{I}$ be the (non-modified) local system on $\mathcal{I}$ giving rise to V. For $k = 1,\ldots,m$, we consider the restriction $V^0_{a_k} \to \partial_{a_k}$ of the local system system V^0, that we now see as a local system over the circles ∂_a, $a \in \mathbb{C}$. Let us choose a direction $d \in \partial_{a_k}$, and denote by $\rho_k \in GL(V^0_d)$ the monodromy of $V^0_{a_k}$. As before, we denote by $\langle q_1^{(k)}\rangle,\ldots\langle q_{r_k}^{(k)}\rangle \in \pi_0(\mathcal{I}_{a_k})$ the active circles at a_k, $n_i^{(k)} \in \mathbb{N}$ their respective multiplicities, and $\beta_j^{(k)}$ their respective ramification orders. Let $H_k := \mathrm{GrAut}(V^0_{a_k}) \cong \prod_{i=1}^{r_k} GL_{n_i^{(k)}}(\mathbb{C})^{b_i^{(k)}}$. The monodromy ρ_k is an element of the H_k-torsor $H_k(\partial_{a_k})$ (see [25]). Explicitly, an element of $H_k(\partial_{a_k})$ is of the form

$$h^{(k)} = \mathrm{diag}(h_1^{(k)},\ldots,h_{r_k}^{(k)}),$$

where $h_i^{(k)}$ is a square diagonal block matrix of size $\beta_i^{(k)}$ of the form

$$h_i^{(k)} = \begin{pmatrix} 0 & \dots & 0 & * \\ * & & & 0 \\ & \ddots & & \vdots \\ & & * & 0 \end{pmatrix},$$

with each block in $GL_{n_i^{(k)}}(\mathbb{C})$.

The wild character variety $\mathcal{M}_B(E, \nabla) = \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}})$ is obtained as a symplectic reduction of the twisted quasi-Hamiltonian manifold

$$\mathrm{Hom}_{\mathbb{S}}(V) \cong \mathcal{A}(V_{a_1}^0) \circledast \cdots \circledast \mathcal{A}(V_{a_m}^0) /\!\!/ G,$$

where $\circledast$ is the quasi-Hamiltonian fusion operation. Here, each piece $\mathcal{A}(V_{a_k}^0)$ is a twisted quasi-Hamiltonian $H_k \times G$-space:

$$\mathcal{A}(V_{a_k}^0) = H(\partial_{a_k}) \times G \times \prod_{d \in \mathbb{A}_{a_k}} \mathbb{S}\mathrm{to}_d,$$

where $\mathbb{A}_{a_k}$ is the set of singular directions of the connection at a_k, and $\mathbb{S}\mathrm{to}_d$ is the Stokes group associated to the singular direction $d \in \mathbb{A}_{a_k}$. Hence, $\mathrm{Hom}_{\mathbb{S}}(E, \nabla)$ is a twisted quasi-Hamiltonian $\mathbf{H} \times G$-space, where $\mathbf{H} = H_1 \times \cdots \times H_m$.

The formal monodromies $\rho(a_k) \in H(\partial_{a_k})$ determine twisted conjugacy classes $\mathcal{C}(\partial_{a_k}) \subset H(\partial_{a_k})$ which do not depend on the choice of direction $d \in \partial_{a_k}$ and the choice of isomorphism $V_d^0 \cong \mathbb{C}^{\sum_j n_i^{(k)} \beta_i^{(k)}}$. Finally, the wild character variety is the twisted quasi-Hamiltonian reduction of $\mathrm{Hom}_{\mathbb{S}}(E, \nabla)$ at the twisted conjugacy class $\boldsymbol{\mathcal{C}} := \prod_{k=1}^m \mathcal{C}(\partial_{a_k})$, i.e.

$$\mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}}) = \mathrm{Hom}_{\mathbb{S}}(E, \nabla) /\!\!/_{\boldsymbol{\mathcal{C}}} \mathbf{H}. \tag{5.0.2}$$

This description enables us to compute the dimension of $\mathcal{M}_B(E, \nabla)$. One has

$$\dim \mathcal{A}(V_{a_k}^0) = \dim G + \dim H(\partial_{a_k}) + \dim \prod_{d \in \mathbb{A}_{a_k}} \mathbb{S}\mathrm{to}_d\,. \tag{5.0.3}$$

The dimension of G is $\dim G = n^2$. One has $\dim H(\partial_{a_k}) = \sum_i \beta_i^{(k)} {n_i^{(k)}}^2$. The dimension of the product of the Stokes groups can be expressed as a function of the number of Stokes arrows between the active circles (see [21]):

$$\dim \prod_{d \in \mathbb{A}_{a_k}} \mathbb{S}\mathrm{to}_d = \sum_{1 \leq i,j \leq r_k} n_i^{(k)} n_j^{(k)} B_{i,j}^+, \tag{5.0.4}$$

where $B_{i,j}^+$ denotes the number of (positive) Stokes arrows between $\langle q_i^{(k)} \rangle$ and $\langle q_j^{(k)} \rangle$ in the Stokes diagram corresponding to the formal local system $V_{a_i}^0 \to \partial_{a_i}$.

The dimension of $\mathrm{Hom}_{\mathbb{S}}(V^0)$ is

$$\dim \mathrm{Hom}_{\mathbb{S}}(V^0) = \sum_{i=1}^m \dim \mathcal{A}(V_{a_k}^0) - 2 \dim G + 2 \dim Z(G).,$$

and the dimension of the wild character variety is then, taking into account the symplectic reduction at the conjugacy classes $\mathcal{C}(\partial_{a_k})$ is given by

$$\dim \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}}) = \sum_{k=1}^m \Big(\dim \mathcal{A}(V_{a_k}^0) + \dim \mathcal{C}(\partial_{a_k}) - 2 \dim H(\partial_{a_k}) \Big) - 2 \dim G + 2 \dim Z(G). \tag{5.0.5}$$

We now are in position to compute the quantities $D := 2 - (\mathbf{d}, \mathbf{d})$ and $D' := \dim \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}})$. We recall our notations:

- Let $a_1, \dots, a_m$ be the singularities at finite distance.
- For $k = 1, \dots, m$, let $\langle q_1^{(k)} \rangle_{a_k}, \dots \langle q_{r_k}^{(k)} \rangle_{a_k} \in \pi_0(\mathcal{I}_{a_k})$ denote the irregular active circles at a_k, $n_i^{(k)} \in \mathbb{N}$ their respective multiplicities, and $\beta_j^{(k)}$ their respective ramification orders, and $\alpha_j^{(k)}$ their irregularities, in an order such that the slopes satisfy

$$\frac{\alpha_1^{(k)}}{\beta_1^{(k)}} \geq \cdots \geq \frac{\alpha_{r_k}^{(k)}}{\beta_{r_k}^{(k)}}.$$

- For $k = 1, \dots, m$, let m_k be the multiplicity of the tame circle $\langle 0 \rangle_{a_k}$ in the modified formal local system at a_k.
- At infinity, let $\langle q_1^{(\infty)} \rangle_{\infty}, \dots \langle q_{r_\infty}^{(\infty)} \rangle_{\infty} \in \pi_0(\mathcal{I}_{a_k})$ denote the active circles (the tame circle is included), $\beta_j^{(\infty)}$ their respective ramification orders, and $\alpha_j^{(\infty)}$ their irregularities, in an order such that the slopes satisfy

$$\frac{\alpha_1^{(\infty)}}{\beta_1^{(\infty)}} \geq \cdots \geq \frac{\alpha_{r_\infty}^{(\infty)}}{\beta_{r_\infty}^{(\infty)}}.$$

Here we will assume that all Stokes arrows come from the leading terms in the exponential factors. In the language of the previous section, this means that the different exponential factors have no common part. This entails no loss of generality, indeed in the computation of the number of Stokes arrows we have seen that the terms involving subleading terms do not change under Fourier transform, so that those term will give the same contribution to $2 - (\mathbf{d}, \mathbf{d})$ and $\dim \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}})$.

Let us list the number of loops and edges between the different types of circles. From the formulas for the number of loops and edges, and the formulas the the transformation of slopes under Fourier transform, we get

- Loops at circles at infinity: $B_{\langle q_i^{(\infty)} \rangle, \langle q_i^{(\infty)} \rangle} = (\beta_i^{(\infty)} - 1)(\alpha_i^{(\infty)} - \beta_i^{(\infty)} - 1)$.
- Edges between different circles at infinity: $B_{\langle q_i^{(\infty)} \rangle, \langle q_j^{(\infty)} \rangle} = \beta_j^{(\infty)}(\alpha_i^{(\infty)} - \beta_i^{(\infty)})$ if $i < j$.
- Loops at irregular circles at finite distance: $B_{\langle q_i^{(k)} \rangle, \langle q_i^{(k)} \rangle} = (-\beta_i^{(k)} - 1)(\alpha_i^{(k)} + \beta_i^{(k)} - 1)$.
- Edges between different irregular circles at a same pole at finite distance: $B_{\langle q_i^{(k)} \rangle, \langle q_j^{(k)} \rangle} = -\beta_i^{(k)}(\alpha_j^{(k)} + \beta_j^{(k)})$ if $i < j$.
- Edges between the tame circle $\langle 0 \rangle_{a_k}$ and an irregular circle at a_k: $B_{\langle 0 \rangle_{a_k}, \langle q_i^{(k)} \rangle} = -\beta_i^{(k)}$.
- Edges between circles at two different poles at finite distance: they are none, i.e. $B_{\langle q_i^{(k)} \rangle, \langle q_j^{(l)} \rangle} = 0$
- Edges between an irregular circle at finite distance and a circle at infinity: $B_{\langle q_i^{(k)} \rangle, \langle q_j^{(\infty)} \rangle} = \beta_j^{(\infty)}(\alpha_i^{(k)} + \beta_i^{(k)})$.
- Edges between the tame circle $\langle 0 \rangle_{a_k}$ and a circle at infinity: $B_{\langle 0 \rangle_{a_k}, \langle q_i^{\infty} \rangle} = \beta_j^{(\infty)}$.

For each circle I, let $D_{\mathbb{L}_I}$ be the contribution to $2-(\mathbf{d},\mathbf{d})$ of the leg $\mathbb{L}_I$ associated to I. Putting everything together, this gives us the following lengthy expression for D:

$$\begin{aligned}
2-(\mathbf{d},\mathbf{d}) =& 2+\sum_{i^{(\infty)}}\left(-2+(\beta_i^{(\infty)}-1)(\alpha_i^{(\infty)}-\beta_i^{(\infty)}-1)\right){n_i^{(\infty)}}^2 \\
&+\sum_{i^{(\infty)}<j^{(\infty)}} 2\beta_j^{(\infty)}(\alpha_i^{(\infty)}-\beta_i^{(\infty)})n_i^{(\infty)}n_j^{(\infty)}+\sum_k\sum_{i^{(k)}}\left(-2+(-\beta_i^{(k)}-1)(\alpha_i^{(k)}+\beta_i^{(k)}-1)\right){n_i^{(k)}}^2 \\
&+\sum_k\sum_{i^{(k)}<j^{(k)}}-2\beta_i^{(k)}(\alpha_j^{(k)}+\beta_j^{(k)})n_i^{(k)}n_j^{(k)}+\sum_k\sum_{i^{(k)}}-2\beta_i^{(k)}m_kn_i^{(k)} \\
&+\sum_k\sum_{i^{(k)}<j^{(\infty)}}2\beta_j^{(\infty)}(\alpha_i^{(k)}+\beta_i^{(k)})n_i^{(k)}n_j^{(\infty)}+\sum_k\sum_{j^{(\infty)}}2\beta_j^{(\infty)}m_kn_j^{(\infty)}-2\sum_k m_k^2 \\
&+\sum_{i^{(\infty)}}D(\mathbb{L}_{\langle q_{i^{(\infty)}}\rangle})+\sum_k\sum_{i^{(k)}}D(\mathbb{L}_{\langle q_i^{(k)}\rangle})+\sum_k D(\mathbb{L}_{\langle 0\rangle_{a_k}}).
\end{aligned} \tag{5.0.6}$$

To compute the dimension of the wild character variety, we have to find the number of (positive) Stokes arrows at each singularity. We have

- Number of Stokes arrows between two irregular circles $\langle q_i^{(k)}\rangle$, $\langle q_i^{(k)}\rangle$ with $i<j$ at a_k: $\beta_j^{(k)}\alpha_i^{(k)}$
- Number of Stokes loops at the irregular circle $\langle q_i^{(k)}\rangle$: $\alpha_i^{(k)}(\beta_i^{(k)}-1)$.
- Number of Stokes arrows between the irregular circle $\langle q_i^{(k)}\rangle$ and the tame circle $\langle 0\rangle_{a_k}$ at a_k: $\alpha_i^{(k)}$.

We also have the similar numbers of arrows at infinity.

Let n_k be the multiplicity of the tame circle $\langle 0\rangle_{a_k}$ in the non-modified local system $V^0_{a_k}$, i.e. the rank of the regular part of the connection at a_k. The dimension of the torsor $H(\partial_{a_k})$ is

$$\dim H(\partial_{a_k})=\sum_{i^{(k)}}{n_i^{(k)}}^2\beta_i^{(k)}+{n_k}^2.$$

Therefore, we get from (5.0.3) the following expression for $\dim\mathcal{A}(V^0_{a_k})$:

$$\dim\mathcal{A}(V^0_{a_k})=n^2+\sum_{i^{(k)}}({n_i^{(k)}}^2\beta_i^{(k)}+{n_k}^2)+\sum_{i^{(k)}}\alpha_i^{(k)}(\beta_i^{(k)}-1){n_i^{(k)}}^2+\sum_{i^{(k)}<j^{(k)}}2\beta_j^{(k)}\alpha_i^{(k)}n_i^{(k)}n_j^{(k)}+\sum_{i^{(k)}}2\alpha_i^{(k)}n_i^{(k)}n_k. \tag{5.0.7}$$

Let $\mathcal{C}_i^{(k)}$ be the twisted conjugacy class of the monodromy of the irregular circle $\langle q_i^{(k)}\rangle$, and $\mathcal{C}_k^{\mathrm{reg}}$ the conjugacy class of the monodromy of the tame circle in $V^0_{a_k}$, so that

$$\mathcal{C}(\partial_{a_k})\cong\prod_{i^{(k)}}\mathcal{C}_i^{(k)}\times\mathcal{C}_k^{\mathrm{reg}}\subset H(\partial_{a_k}).$$

Using $\dim Z(G)=1$, the formula (5.0.5) for the dimension of the wild character variety then becomes

$$
\begin{aligned}
\dim \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}}) =& 2 - 2n^2 \\
&+ \sum_k \left(n^2 - \sum_{i^{(k)}} ({n_i^{(k)}}^2 \beta_i^{(k)} + {n_k}^2) + \sum_{i^{(k)}} \alpha_i^{(k)} (\beta_i^{(k)} - 1) {n_i^{(k)}}^2 \right. \\
&\left. + \sum_{i^{(k)} < j^{(k)}} 2\beta_j^{(k)} \alpha_i^{(k)} n_i^{(k)} n_j^{(k)} + \sum_{i^{(k)}} 2\alpha_i^{(k)} n_i^{(k)} n_k \right) \\
&+ n^2 - \sum_{i^{(\infty)}} {n_i^{(\infty)}}^2 \beta_i^{(\infty)} + \sum_{i^{(\infty)}} \alpha_i^{(\infty)} (\beta_i^{(\infty)} - 1) {n_i^{(\infty)}}^2 \\
&+ \sum_{i^{(\infty)} < j^{(\infty)}} 2\beta_j^{(\infty)} \alpha_i^{(\infty)} n_i^{(\infty)} n_j^{(\infty)} \\
&+ \sum_k \left(\sum_{i^{(k)}} \dim \mathcal{C}_i^{(k)} + \dim \mathcal{C}_k^{\mathrm{reg}} \right) + \sum_{i^{(\infty)}} \dim \mathcal{C}_i^{(\infty)} + \dim \mathcal{C}_\infty^{\mathrm{reg}}
\end{aligned}
\tag{5.0.8}
$$

We have to compare these two expressions. First, we relate the contributions of the legs to the dimension of the conjugacy classes.

Dimensions of conjugacy classes

We have to deal with a subtlety: the leg $\mathbb{L}_I$ which we glue to a circle I of ramification order β over the pole $a \in \mathbb{P}^1$ of the core of the diagram corresponds to the conjugacy class of the monodromy of the local system $V_I^0 \to I$, whereas the twisted conjugacy class $\mathcal{C}_I \subset H(I)$ is the one of the monodromy of $V_I^0 \to \partial_a$, i.e. the local system downstairs on the circle of directions ∂_a.

$$
\begin{tikzcd}
V_I^0 \arrow[r, "\phi"] \arrow[dr, "\psi"'] & I \arrow[d, "\pi"] \\
& \partial_a
\end{tikzcd}
$$

Let us fix a direction $d \in \partial_a$, let $d_0, \dots d_{\beta_1}$ be its preimages by the cover $\pi : I \to \partial_a$. The fibre of V_I^0 over d is the direct sum

$$V_{I,d}^0 = V_{I,d_0}^0 \oplus \cdots \oplus V_{I,d_{\beta-1}}^0,$$

where V_{I,d_i}^0 is the fibre of $\phi : V_I^0 \to I$ over d_i, and the indices i are in $\mathbb{Z}/\beta\mathbb{Z}$. The monodromy of $V_I^0 \to \partial_a$ is given by the collection of linear applications

$$\rho_i : V_{I,d_i}^0 \to V_{I,d_{i+1}}^0, \qquad i = 0, \dots, \beta - 1.$$

The monodromy of $V_I^0 \to I$ is then given by the product

$$\rho = \rho_{\beta-1} \dots \rho_0 : V_{I,d_0}^0 \to V_{I,d_0}^0$$

The group $H_I := GL(V_{I,d_0}^0) \times \cdots \times GL(V_{I,d_{\beta-1}}^0)$ acts on the monodromy by

$$(k_0, \dots, k_{\beta-1}).\rho_i = k_{i+1} \rho_i k_i^{-1},$$

where $\mathbf{k} = (k_0, \dots, k_{\beta-1}) \in H_I$. The twisted conjugacy class $\mathcal{C}_I$ is the orbit of $(\rho_0, \dots, \rho_{\beta-1})$ under this action. This induces on ρ the transformation

$$\mathbf{k}.\rho = k_0 \rho k_0^{-1}.$$

which is just the action of $GL(V^0_{I,d_0})$ by conjugation, so that its orbit, which we will denote $\widetilde{\mathcal{C}}_I$, is the conjugacy class of the monodromy of $V^0_I \to I$. Furthermore, each fibre of the map

$$\begin{aligned}\mathcal{C}_I &\to \widetilde{\mathcal{C}}_I\\ (\rho_i) &\mapsto \rho\end{aligned}$$

is isomorphic to $GL(V^0_{I,d_1}) \times \cdots \times GL(V^0_{I,d_{\beta-1}})$. It follows from this that the dimension of $\mathcal{C}_I$ is

$$\dim \mathcal{C}_I = n_I^2(\beta - 1) + \dim \widetilde{\mathcal{C}}_I, \tag{5.0.9}$$

where n_I is the dimension of each fibre V^0_{I,d_i}.

The legs of the diagram correspond to the reduced conjugacy classes $\widetilde{\mathcal{C}}_I$. The dimension of a conjugacy class is obtained from the associated leg in the following way (see again [20]):

Lemma 5.0.3. *Let $\mathcal{C} \subset GL_n(\mathbb{C})$ a conjugacy class, $\mathbb{L}$ the associated leg (after choosing a marking), with* $\mathbf{d}$ *its dimension vector. The dimension of $\mathcal{C}$ is given by*

$$\dim \mathcal{C} = 2n^2 - (\mathbf{d}, \mathbf{d}). \tag{5.0.10}$$

We can now compare the different terms appearing in the quantities $D = 2 - (\mathbf{d}, \mathbf{d})$ and $D' = \dim \mathcal{M}_B(\Theta, \boldsymbol{\mathcal{C}})$. Let us consider the terms in D involving the multiplicities m_k of the tame circles: their contribution to D is

$$\begin{aligned}&\sum_k \sum_{i^{(k)}} -2\beta_i^{(k)} m_k n_i^{(k)} + \sum_k \sum_{j^{(\infty)}} 2\beta_j^{(\infty)} m_k n_j^{(\infty)} - 2\sum_k m_k^2\\ &= -2\sum_k n_k(n - n_k) - 2m_k^2 + 2\sum_k m_k n\\ &= \sum_k 2m_k n_k - 2m_k^2,\end{aligned}$$

where we use that $n = n_k + \sum_{i^{(k)}} \beta_i^{(k)} n_i^{(k)}$ for each k, and the similar formula at infinity. We can recognize that $2m_k n_k - 2m_k^2$ is exactly the contribution of the first edge of the leg associated to a special marking of the conjugacy class of the monodromy of the tame circle $\langle 0 \rangle_k$; i.e. the edge corresponding to the eigenvalue 1. From this we deduce that

$$D(\mathbb{L}_{\langle 0 \rangle_{a_k}}) + 2m_k n_k - 2m_k^2 = \dim \mathcal{C}_k^{\mathrm{reg}}. \tag{5.0.11}$$

We also have for each irregular circle at finite distance

$$D(\mathbb{L}_{\langle q_i^{(k)} \rangle}) = \dim \widetilde{\mathcal{C}}_i^{(k)},$$

and similarly for the circles at infinity

$$D(\mathbb{L}_{\langle q_i^{(k)} \rangle}) = \dim \widetilde{\mathcal{C}}_i^{(k)}.$$

This already gives us equalities between some terms appearing in D and D'. Let us deal with the remaining terms. First, for each pole a_k at finite distance, we have an equality between the following term in D, which is the contribution of the irregular circles at a_k

$$\begin{aligned}&\sum_{i^{(k)}} \left(-2 + (-\beta_i^{(k)} - 1)(\alpha_i^{(k)} + \beta_i^{(k)} - 1)\right) {n_i^{(k)}}^2 + \sum_{i^{(k)} < j^{(k)}} -2\beta_j^{(k)}(\alpha_i^{(k)} + \beta_i^{(k)}) n_i^{(k)} n_j^{(k)}\\ &+ \sum_{i^{(k)}, j^{(\infty)}} 2\beta_j^{(\infty)}(\alpha_i^{(k)} + \beta_i^{(k)}) n_i^{(k)} n_j^{(\infty)},\end{aligned}$$

and the following term in D'

$$n^2 - \sum_{i^{(k)}} ({n_i^{(k)}}^2 \beta_i^{(k)} + {n_k}^2) + \sum_{i^{(k)}} \alpha_i^{(k)} (\beta_i^{(k)} - 1) {n_i^{(k)}}^2 \\ + \sum_{i^{(k)} < j^{(k)}} 2\beta_j^{(k)} \alpha_i^{(k)} n_i^{(k)} n_j^{(k)} + \sum_{i^{(k)}} 2\alpha_i^{(k)} n_i^{(k)} n_k + {n_i^{(k)}}^2 (\beta_i^{(k)} - 1),$$

where the term ${n_i^{(k)}}^2 (\beta_i^{(k)} - 1)$ comes from equation (5.0.9). To obtain this equality, we notice that since $n = \sum_{j^{(\infty)}} \beta_j^{(\infty)} n_j^{(\infty)}$,

$$\sum_{i^{(k)}, j^{(\infty)}} 2\beta_j^{(\infty)} (\alpha_i^{(k)} + \beta_i^{(k)}) n_i^{(k)} n_j^{(\infty)} = n \sum_{i^{(k)}, j^{(\infty)}} (\alpha_i^{(k)} + \beta_i^{(k)}) n_i^{(k)},$$

then we decompose on the both sides the total rank: $n = \sum_{i^{(k)}} \beta_i^{(k)} n_i^{(k)} + n_k$. It is then a somewhat tedious but straightforward check that we have the same terms on both sides.

It only remains to consider the contributions to D and D' of the circles at infinity: we have an equality between the following term in D

$$\sum_{i^{(\infty)}} \left(-2 + (\beta_i^{(\infty)} - 1)(\alpha_i^{(\infty)} - \beta_i^{(\infty)} - 1)\right) {n_i^{(\infty)}}^2 + \sum_{i^{(\infty)} < j^{(\infty)}} 2\beta_j^{(\infty)} (\alpha_i^{(\infty)} - \beta_i^{(\infty)}) n_i^{(\infty)} n_j^{(\infty)}$$

and the following term in D':

$$-2n^2 + n^2 - \sum_{i^{(\infty)}} {n_i^{(\infty)}}^2 \beta_i^{(\infty)} + \sum_{i^{(\infty)}} \alpha_i^{(\infty)} (\beta_i^{(\infty)} - 1) {n_i^{(\infty)}}^2 \\ + \sum_{i^{(\infty)} < j^{(\infty)}} 2\beta_j^{(\infty)} \alpha_i^{(\infty)} n_i^{(\infty)} n_j^{(\infty)} + \sum_{i^{(\infty)}} {n_i^{(\infty)}}^2 (\beta_i^{(\infty)} - 1),$$

where once again the term ${n_i^{(\infty)}}^2 (\beta_i^{(\infty)} - 1)$ comes from (5.0.9). These equalities exhaust all terms appearing in D and D', so this concludes the proof.

Chapter 6

Examples and fundamental representations

In this chapter, we list several examples to illustrate the construction of diagrams from singularity data $(\Theta, \mathcal{C})$ on the Riemann sphere. We will later discuss in more detail the properties of some of these examples. Since gluing on the legs from the conjugacy classes $\mathcal{C}$ is standard the main part is to describe core diagram. Thus we will list the modified irregular class $\breve{\Theta}$ and the core diagram it determines. In §6.2 we then discuss how each diagram determines several different representations, the "fundamental representations". We apply this to the study of different Lax pairs of several isomonodromy systems. This generalizes the different readings of the simply-laced cases [17, 20].

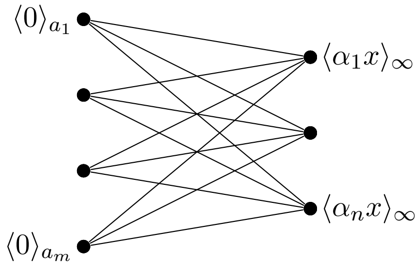
6.1 Examples of diagrams

We begin by reviewing known cases, before studying new ones provided by our more general construction of the diagrams.

Complete bipartite case Let us consider the case of a connection with a second order poles at infinity, together with simple poles $a_1, \dots, a_m$ at finite distance, with a global modified irregular class of the form

$$\breve{\Theta} = k_1 \langle \alpha_1 z \rangle_\infty + \cdots + k_n \langle \alpha_n z \rangle_\infty + l_1 \langle 0 \rangle_{a_1} + \cdots + l_m \langle 0 \rangle_{a_m},$$

with $k_i \in \mathbb{N}$, $l_j \in \mathbb{N}$, and $\alpha_i \in \mathbb{C}$ two by two distinct, $a_i \in \mathbb{C}$ two by two distinct. There are n active circles of slope 1 at infinity. The diagram is the following:



It is a complete bipartite graph, one part having n vertices and one part having m vertices. If $n = 1$ or $m = 1$ this is the star-shaped case.

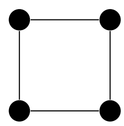
Simply laced case An important case to consider is the one where there is only one unramified irregular pole at infinity of order 3, together with simple poles at finite distance. This is the case extensively studied in [17, 20, 22], giving rise in those works to simply-laced supernova graphs, with core any complete multipartite graph. We can check that our definition gives the same diagrams.

In this case, the circles at infinity are unramified and have slope ≤ 2, i.e. they are of the form $\langle q_{i,j}\rangle_\infty$ with $q_{i,j} = \lambda_i z^2 + \mu_{i,j} z$, for $i = 1, \dots k$, $j = 1, \dots s_i$, where the coefficients $\lambda_1, \dots, \lambda_k$ are all different, as well as $\mu_{i,1}, \dots, \mu_{i,s_i}$ for all i. The simple poles correspond to tame circles $\langle 0\rangle_{a_1}, \dots, \langle 0\rangle_{a_t}$, where $a_1, \dots a_t \in \mathbb{C}$. Let $I_0 := \{\langle 0\rangle_{a_1}, \dots, \langle 0\rangle_{a_t}\}$ and for $i = 1, \dots k$, $I_i := \{\langle \lambda_i z^2 + \mu_{i,1} z\rangle, \dots, \langle \lambda_i z^2 + \mu_{i,s_i} z\rangle\}$. The sets $I_0, \dots, I_k$ constitute a partition of the set of active circles. The core diagram has the following structure: two active circles are either linked by no edge in the diagram when they belong to the same set I_k, otherwise they are linked by exactly one edge. The core diagram that we obtain is thus a $k+1$-partite graph. The diagram coincides with the one considered in [20, 22].

The standard rank two Lax representations of the Painlevé V and Painlevé IV equation fit into this setting. The representation for Painlevé V has one irregular singularity at infinity of order 2, together with two simple poles at finite distance. The modified irregular class is of the form

$$\breve{\Theta} = \langle \alpha z\rangle_\infty + \langle \beta z\rangle_\infty + \langle 0\rangle_a + \langle 0\rangle_b,$$

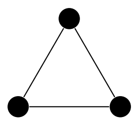
with $\alpha \neq \beta \in \mathbb{C}$, $a \neq b \in \mathbb{C}$. This gives the following diagram:



The standard representation for Painlevé IV has one irregular singularity at infinity of order 3, together with one simple pole at finite distance. The modified irregular class is of the form

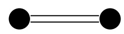
$$\breve{\Theta} = \langle \alpha z^2\rangle_\infty + \langle \beta z^2\rangle_\infty + \langle 0\rangle_a,$$

with $\alpha \neq \beta \in \mathbb{C}$, $a \in \mathbb{C}$. This gives the following diagram:

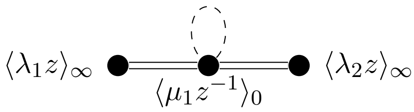


Other supernova graphs More general supernova graphs correspond to the case where there is one irregular singularity at infinity with all active circles being unramified, together with simple poles at infinity. This is the case considered in [17, appendix C] and [57]. If the active circles at infinity are $\langle q_1\rangle, \dots, \langle q_r\rangle$ we have $B_{ij} = \deg(q_i - q_j) - 1$.

The standard Painlevé II Lax pair fits into this framework. The (modified or not) irregular class is of the form $\Theta = \Theta_\infty = \langle \alpha z^3\rangle_\infty + \langle \beta z^3\rangle_\infty$ with $\alpha \neq \beta$. The diagram is



Painlevé III The standard Painlevé III Lax pair corresponds to a rank 2 connection with 2 irregular singularities. The active circles are two irregular circles of slope 1 at infinity and 2 irregular circles of slope 1 at $z = 0$, say $\langle \lambda_1 z\rangle_\infty$, $\langle \lambda_2 z\rangle_\infty$ $\langle \mu_1 z^{-1}\rangle_0$, $\langle \mu_2 z^{-1}\rangle_0$, with $\lambda_1 \neq \lambda_2$, $\mu_1 \neq \mu_2$. With no loss of generality, up to applying a twist by a rank one connection on the trivial bundle on $\mathbb{P}^1$, we may assume that $\mu_2 = 0$, so that we have the active circle $\langle 0\rangle_0$, and that the tame circle has trivial formal monodromy in the non-reduced formal local system, so that it has multiplicity zero in the associated modified formal local system. The diagram associated to the connection is given by

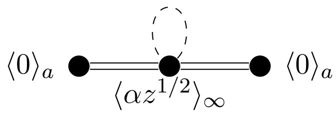


with dimension vector $\mathbf{d} = (1, 1, 1)$. This is exactly the same diagram as the one in [26].

Degenerate Painlevé V The representation of Painlevé III known as degenerate Painlevé V corresponds as for Painlevé V to a rank 2 connection with one order two pole at infinity and two simple poles at finite distance [63, p. 34]. The difference is that the leading term of the connection is nilpotent (this is the meaning of the word degenerate in this context) which implies there is just one ramified active circle at infinity of slope 1/2. The modified irregular class is of the form:

$$\breve{\Theta} = \langle \alpha z^{1/2}\rangle_\infty + \langle 0\rangle_a + \langle 0\rangle_b,$$

and this gives the diagram



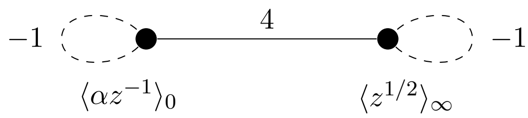
computed in [26]. This is the same diagram as for the standard Painlevé III Lax pair, as we shall explain later.

Degenerate Painlevé III The degenerate Painlevé III system admits, as for the standard Painlevé III representation, a representation corresponding to a rank 2 connection with two order two poles [80]. The difference is that the leading term of the connection matrix at one of the poles is nilpotent. Again, this corresponds to having one active circle with slope 1/2. The modified irregular class has the following form:

$$\breve{\Theta} = \langle z^{1/2}\rangle_\infty + \langle \alpha z^{-1}\rangle_0 + \langle \beta z^{-1}\rangle_0$$

with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$.

Up to performing a twist at zero, we may assume that $\beta = 0$, so that we have the tame circle at 0, and that this circle has trivial formal monodromy, so that the corresponding piece of the modified formal local system has multiplicity 0. This leads to the following diagram, with one negative loop at each vertex, and 4 edges between the two vertices.



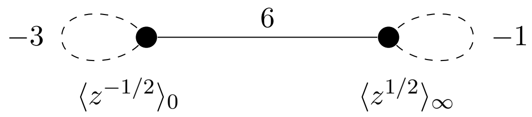
The Cartan matrix is

$$C = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}.$$

Doubly degenerate Painlevé III The doubly degenerate Painlevé III system admits a Lax representation corresponding to two irregular singularities of order two, each with nilpotent leading term leading to an active circle of slope 1/2 [80]. The modified irregular class is of the form

$$\breve{\Theta} = \langle \alpha z^{1/2}\rangle_\infty + \langle \beta z^{-1/2}\rangle_0$$

with α, β nonzero complex numbers. This gives the following diagram, where the number on the edges correspond to their multiplicities.



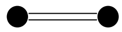
The Cartan matrix is

$$C = \begin{pmatrix} 8 & -6 \\ -6 & 4 \end{pmatrix},$$

Flaschka-Newell Lax pair for Painlevé II The Flaschka-Newell Lax pair for Painlevé II [45], also known as degenerate Painlevé IV, corresponds to a rank 2 connection with one irregular singularity at infinity with one active circle of slope $3/2$, together with one simple pole at finite distance. The modified irregular class is of the form

$$\breve{\Theta} = \langle \alpha z^{3/2}\rangle_\infty + \langle 0\rangle_0.$$

with $\alpha \neq 0$. This gives the diagram



that is the same diagram as for Painlevé II, as found in [26].

Painlevé I The standard Lax pair for the Painlevé I equation corresponds to a rank 2 connection with just one irregular singularity at infinity, with one active circle of slope $5/2$. The modified irregular class is of the form

$$\breve{\Theta} = \langle \alpha z^{5/2}\rangle_\infty,$$

with $\alpha \neq 0$. The corresponding diagram has one vertex and one loop.

H3 surfaces The Painlevé equations correspond to the simplest examples of non-trivial wild character varieties, since their moduli spaces are 2-dimensional. The corresponding nonabelian Hodge spaces $\mathfrak{M}$ are thus complete hyperkähler manifolds of real dimension four, so are examples of H3 surfaces in the terminology of [23] to designate the two-dimensional wild character varieties. Apart from the cases corresponding to Painlevé equations, there are 3 other known H3 surfaces whose standard representation correspond to fuchsian connection with 3 regular singularities. Since they have no deformation parameters, they do not give rise to an isomonodromy system. Thanks to our more general theory, we are now able to associate a diagram to all H3 surfaces. They are represented on fig. 6.1. The reader may check that in each case we have $2-(\mathbf{d},\mathbf{d}) = 2$ as expected. Notice that for Painlevé II, IV, V, VI, the diagram is exactly the affine Dynkin diagram corresponding to its Okamoto symmetries [82, 84, 83, 85, 80],

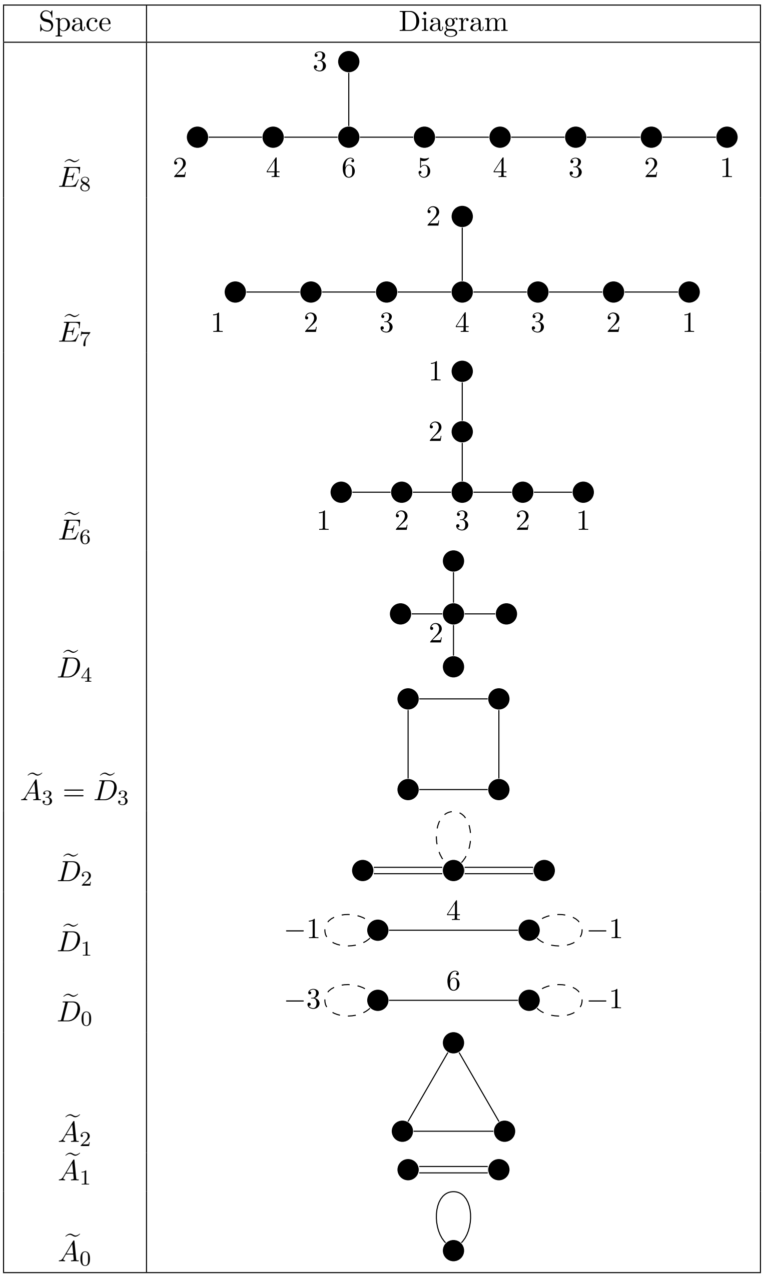


Figure 6.1: Diagrams associated to all known H3 surfaces. The names of the surfaces are as in [23]. All unspecified multiplicities are equal to 1.

h-Painlevé systems It is possible to define a notion of h-Painlevé systems $hP_k^{(n)}$, where h stands for higher or hyperbolic or Hilbert. Higher Painlevé systems $hP_k^{(n)}$ were introduced by Boalch [17, 20]: for each integer n there is a rank $2n$ Lax representation giving rise to a $2n$-dimensional higher Painlevé moduli space. All higher Painlevé systems of a given number have the same diagram, but the dimension vector is a function of n. More precisely, the diagram associated to a given higher Painlevé system is obtained by taking the original diagram with all multiplicities scaled by n, then adding a leg of length one, with its end vertex having multiplicity 1, as on fig. 6.2 for higher Painlevé IV,V,VI. The higher Painlevé diagrams are hyperbolic Dynkin diagrams since they are obtained by adding a vertex to an affine Dynkin diagram. In this sense they can be seen as the next simplest examples after the affine case. The higher Painlevé moduli spaces are (conjecturally) related to Hilbert schemes on n points on the corresponding H3 surface (see [20]).

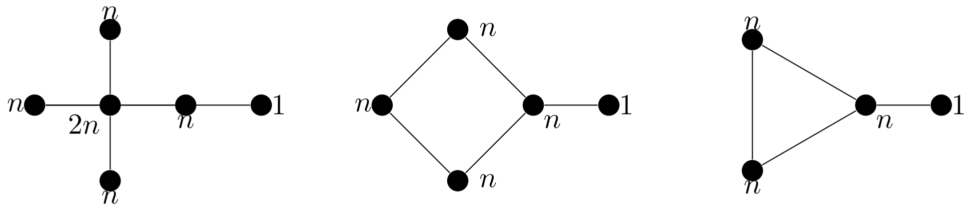
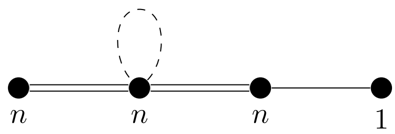


Figure 6.2: Diagrams for higher Painlevé systems $hP_{VI}^{(n)}$, $hP_{V}^{(n)}$, $hP_{IV}^{(n)}$.

The same recipe for Painlevé III yields the $hP_3^{(n)}$ diagram:



Some 4-dimensional cases Another point of view on higher dimensional isomonodromy systems consists in looking at degenerations of an equation coming via isomonodromy from a fuchsian connection. This approach is used in [65] to list some representations of some 4-dimensional isomonodromy systems: they are defined there as the systems coming via degeneration from the isomonodromic deformation equations associated to connections with regular singularities giving rise to 4-dimensional moduli spaces. Whereas for the usual 2-dimensional Painlevé equations there is only one possible choice of formal data for fuchsian connections with non-trivial admissible deformations and 2-dimensional moduli spaces, in the 4-dimensional case there are 4 possible choices. The full degeneration scheme is represented at [65, p. 40]. Each of the 4 fuchsian formal data gives rise via degeneration to a family of 4-dimensional Painlevé type equations.

The two points of view on higher-dimensional isomonodromy systems are actually not independent. The elements of the fourth degeneration family of 4-dimensional Painlevé-type equations in the degeneration scheme of [65] (called there matrix Painlevé systems) correspond to the higher Painlevé equations in the sense of [20] in the 4-dimensional case, and the master diagram $hP_6^{(2)}$ at the top of the coalescence cascade first appeared in the context of Painlevé systems in the list of 4-dimensional hyperbolic examples in [17, p. 12]. This notion of higher Painlevé systems is however more general than just this 4-dimensional case.

On figure 6.3 are drawn the diagrams associated by our approach to the 4-dimensional isomonodromy systems listed in [65] whose Lax representations feature several irregular singularities. These diagrams are to be contrasted with the shapes in [56, p.235].

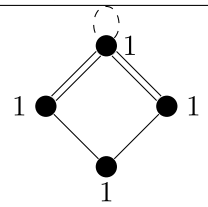
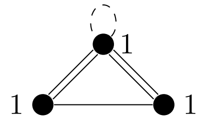
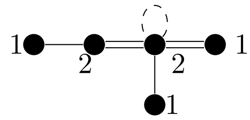
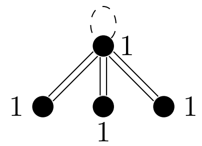
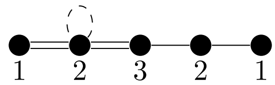
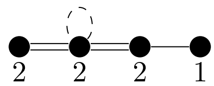
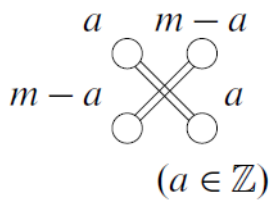
Name in [65]	Mod. irreg. class	Diagram
H_{Gar}^{2+2+1}	$\langle \alpha z\rangle_\infty+\langle \beta z\rangle_\infty+\langle \gamma z^{-1}\rangle_0+\langle 0\rangle_1$	
H_{Gar}^{3+2}	$\langle \alpha z^2\rangle_\infty+\langle \beta z^2\rangle_\infty+\langle \gamma z^{-1}\rangle_0$	
$H_{\text{FS}}^{A_3}$	$2\langle \alpha z\rangle_\infty+\langle \beta z\rangle_\infty+2\langle \gamma z^{-1}\rangle_0$	
$H_{\text{Gar}}^{3/2+1+1+1}$	$\langle \alpha z^{1/2}\rangle_\infty+\langle 0\rangle_a+\langle 0\rangle_b+\langle 0\rangle_c$	
$H_{\text{Ss}}^{D_4}$	$3\langle \alpha z\rangle_\infty+\langle \beta z\rangle_\infty+2\langle \gamma z^{-1}\rangle_0$	
$H_{III}^{\text{Mat}}(D_6)$	$2\langle \alpha z\rangle_\infty+2\langle \beta z\rangle_\infty+2\langle \gamma z^{-1}\rangle_0$	

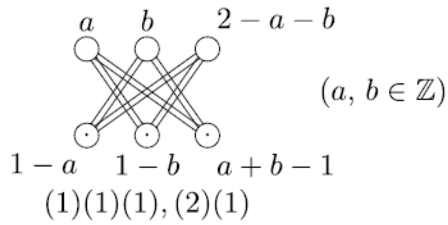
Figure 6.3: Diagrams for the Lax representations of 4-dimensional Painlevé-type equations of [65] featuring several irregular singularities.

Remark 5. Notice that the diagram associated to the Painlevé-type equation denoted $H_{III}^{\text{Mat}}(D_6)$ in [65] fits in the higher Painlevé picture in the sense of Boalch: it is $hP_3^{(2)}$, obtained from the Painlevé III diagram by taking all multiplicities equal to 2 and adding a vertex with multiplicity 1. Beware that the usual terminology “matrix Painlevé equations” [7] differs from that of [65].

Remark 6. It is interesting to compare our diagrams with those of Hiroe [57] and Hiroe and Oshima [56]. In [55], Hiroe defines diagrams associated to meromorphic connections with several unramified irregular singularities. When there is only one irregular singularity, these diagrams coincide with the ones of [20] and in turn with ours, however when they are several irregular singularities they differ from ours. Hiroe also introduces the notion of “shape”, and in their approach the moduli spaces are classified by these shapes. When there are several irregular singularities, the shape differs from the diagram. As an example, figure 6.4 shows the shapes associated to the third Painlevé equation and the 4-dimensional Painlevé-type equation whose Hamiltonian is denoted by $H_{\text{Gar}}^{\frac{3}{2}+1+1+1}$ in [65]. This approach however does not allow to define diagrams for the degenerate and doubly degenerate Painlevé equations.



(a) shape of Painlevé III



(b) shape associated to $H_{\mathrm{Gar}}^{\frac{3}{2}+1+1+1}$

Figure 6.4: Shapes associated to some Painlevé-type equations in [65, 55].

6.2 Fundamental representations of the diagrams

In this section, we discuss how a diagram can be “read” in different ways corresponding to connection on bundles of different ranks with different formal data, which we call the fundamental representations of the diagram.

This generalizes the k-partite cases considered in [20] reviewed in chapter 3. As an application, we discuss in the next section how many known different Lax pairs for Painlevé equations and higher Painlevé isomonodromy systems correspond to different fundamental representations of the same diagram. Many cases fit into the simply laced framework [17, 20], but our more general construction of the diagrams gives a few new examples.

The idea is the following. We have seen that the diagram associated to a meromorphic connection is invariant under the action of $SL_2(\mathbb{C})$. Each orbit under this action contains different formal data, with in general different numbers of singularities.

Let $\langle q\rangle_\infty$ be the circle at infinity associated with an exponential factor q. We have seen that $\langle q\rangle$ can be sent to finite distance by a symplectic transformation if and only if it is of the form

$$q = \lambda z^2 + q', \tag{6.2.1}$$

where q' is of slope ≤ 1. Furthermore, when this is the case, the position of the singularity at finite distance that we obtain is determined by the coefficient of z in q, i.e. the number $\mu \in \mathbb{C}$ such that

$$q = \lambda z^2 + \mu z + q'', \tag{6.2.2}$$

where q'' has slope < 1.

Let us consider formal data at infinity $(\Theta_\infty, \boldsymbol{\mathcal{C}}_\infty)$ at infinity. Let $N \subset \pi_0(\mathcal{I}_\infty)$ be the subset of active exponents, i.e. the support of Θ_∞. N is the set of vertices of the core diagram associated to Θ_∞. We can partition N according to the coefficients of z^2 and z in the exponential factors. Let $N_\infty \subset N$ be the set of exponential factors in N which are not of the form (6.2.1) for any $\lambda \in \mathbb{C}$. For $\lambda \in \mathbb{C}$, let $N_\lambda \subset N$ be the set of exponential factors of the form (6.2.1). For $\lambda, \mu \in \mathbb{C}$, let $N_{\lambda,\mu} \subset N_\lambda$ be the set of exponential factors of the form (6.2.2).

There is a finite number of coefficients $\lambda \in \mathbb{C}$ such that N_λ, is non empty, which we denote $\lambda_1, \dots, \lambda_k$. We set $N_i := N_{\lambda_i}$ for $i = 1, \dots, r$. Then, for each i, there is again a finite number of coefficients $\mu \in \mathbb{C}$ such that $N_{\lambda_i,\mu}$ is non-empty, which we denote $\mu_{i,1}, \dots, \mu_{k,s_i}$, and we set $N_{i,j} := N_{\lambda_i,\mu_{i,j}}$. For each pair i, j, let $t_{i,j} := \operatorname{Card} N_{i,j}$, and $q_{i,j,k}$, $k = 1, \dots, t_{i,j}$ be the exponential factors of slope < 1 such that the elements of $N_{i,j}$ are

$$\lambda_i z^2 + \mu_{i,j} z + q_{i,j,k}. \tag{6.2.3}$$

Let $\beta_{i,j,k} = \operatorname{ram} q_{i,j,k}$ be the ramification order of $q_{i,j,k}$, and $\alpha_{i,j,k}/\beta_{i,j,k}$ its slope. We thus have the following partition of the set of active circles:

$$N = N_\infty \cup N_1 \cup \dots N_r.$$

There are $r+1$ fundamental representations of diagram: the generic representation, corresponding to Θ, and r other representations, depending on which of the sets $N_1, \dots, N_r$ we choose to send to finite distance by a symplectic transformation. The generic representation corresponds to symplectic transformations $A \in SL_2(\mathbb{C})$ such that all circles remain at infinity under A. From the previous discussion of the action of $SL_2(\mathbb{C})$ on the coefficients of z^2 in the exponential factor, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the condition for this is $2c\lambda - d \neq 0$.

For each $i = 1, \dots, r$ the i-th representation of the diagram corresponds to acting on Θ_∞ with $A \in SL_2(\mathbb{C})$ such that $2c\lambda_i - d = 0$. Such a symplectic transformation sends all circles in N_i to finite distance. The formal data that we obtain have $s_i = \mathrm{Card}(N_i)$ poles at finite distance corresponding to the coefficients $\mu_{i,1}, \dots, \mu_{i,s_i}$ (and whose position depends on A). For each $j = 1, \dots, s_i$, the vertices in $N_{i,j}$ then correspond to the active circles of $A \cdot \Theta_\infty$ at the pole associated to $\mu_{i,j}$. It follows from the Legendre transformation that the image $A \cdot \langle q_{i,j,k} \rangle$ of the circle $\langle q_{i,j,k} \rangle$ has ramification order $\beta_{i,j,k} - \alpha_{i,j,k}$ and slope $\alpha_{i,j,k}/(\beta_{i,j,k} - \alpha_{i,j,k})$. This generalizes the different readings of the diagram considered in [17, 20, 22]. The situation considered in those works is the simply laced case where all circles at infinity are of the form

$$q = \lambda z^2 + \mu z,$$

with $\lambda, \mu \in \mathbb{C}$. In particular N_∞ is empty. In this situations, each circle can be sent by a symplectic transformation to a tame circle at finite distance.

An example is shown on fig. 6.5, to be contrasted with the simply laced situation of fig. 3.8.

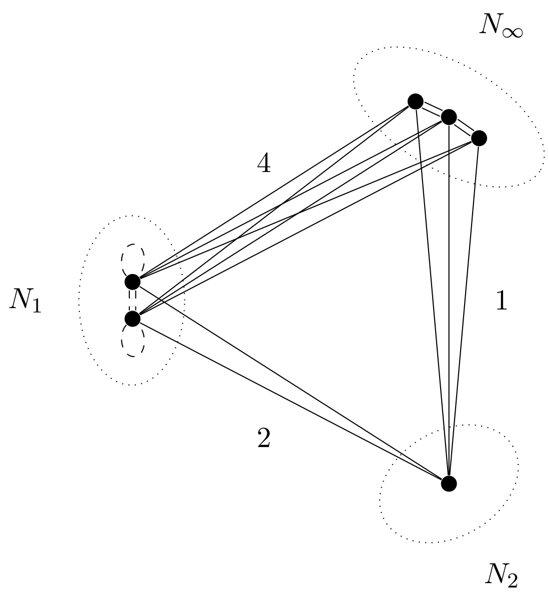
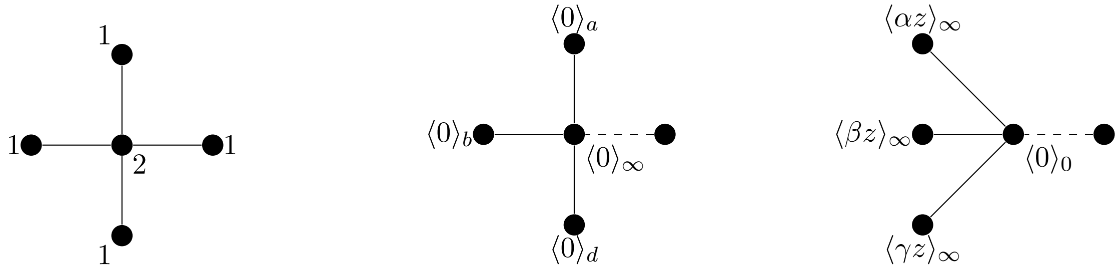


Figure 6.5: An example of core diagram with the partition of the set of nodes $N = N_\infty \cup N_1 \cup N_2$. The integers indicate the multiplicities of the edges.

6.3 Examples of different representations

We now discuss how the different fundamental representations of the diagrams also allow to recover many known different Lax representations for the Painlevé equations as well as 4-dimensional isomonodromy systems which do not fit into the simply laced case [17, 20, 22].

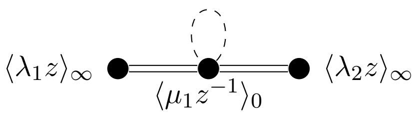
Painlevé VI Let us first mention a well-known example. Figure 6.6 shows two representations of the Painlevé VI moduli space, corresponding to two fundamental representations of its diagram. The second representation corresponds to a rank 3 connection obtained by Harnad duality [53] (see also [14]). The generic representation of the diagram corresponds to a connection of rank 5.



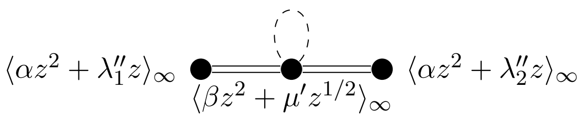
(a) Dimension vector (b) Standard representation (c) Harnad dual

Figure 6.6: Two Lax representations for Painlevé VI corresponding to several representations of the same diagram. The standard Lax representation corresponds to a rank 2 connection with 4 simple poles at a, b, c, ∞, the other Lax representation corresponds to a rank 3 connection with a second order pole at infinity and a simple pole at zero.

Painlevé III In a similar way, we can see that the standard Painlevé III Lax representation and the degenerate Painlevé V representation correspond to two fundamental representations of the Painlevé III diagram. Recall that the standard representation corresponds to a rank two connections, with two irregular circles of slope 1 at infinity and 2 irregular circles of slope 1 at $z = 0$, say $\langle \lambda_1 z\rangle_\infty$, $\langle \lambda_2 z\rangle_\infty$ $\langle \mu_1 z^{-1}\rangle_0$, $\langle \mu_2 z^{-1}\rangle_0$, with $\lambda_1 \neq \lambda_2$, $\mu_1 \neq \mu_2$. With no loss of generality, up to applying a twist by a rank one connection on the trivial bundle on $\mathbb{P}^1$, we may assume that $\mu_2 = 0$, so that we have the active circle $\langle 0\rangle_0$, and that the tame circle has trivial formal monodromy in the non-reduced formal local system, so that it has multiplicity zero in the associated modified formal local system. The diagram associated to the connection is given by

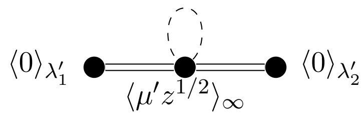


with dimension vector $\mathbf{d} = (1, 1, 1)$. The generic representation of the diagram corresponds to a rank 4 connection, with active circles of the following form



with $\alpha \neq \beta \in \mathbb{C}$ and $\lambda_1'' \neq \lambda_2'' \in \mathbb{C}$.

The other nongeneric representation of the diagram has circles of the form



and we see that it is precisely the alternative Lax representation for Painlevé III used there to obtain this diagram. The two Lax representations correspond to two fundamental representations of the diagram.

Painlevé II In the case of Painlevé II, we recover with our general theory the observation of [26] that the Painlevé II diagram comes from two representations: the standard one with one irregular singularity at infinity, corresponding to the generic representation of the diagram, and another representation corresponding to a rank 2 connection with one irregular pole at infinity with one active circle of slope 3/2, together with one simple pole at finite distance. More explicitly, the active circles of the usual Lax representation are of the form $\langle \alpha z^3\rangle_\infty, \langle \beta z^3\rangle_\infty$ with $\alpha \neq \beta$, each with multiplicity 1, whereas the active circles of the alternative Lax representation are of the form $\langle \lambda z^{3/2}\rangle_\infty$, and $\langle 0\rangle_0$, with $\langle \lambda z^{3/2}\rangle_\infty$ having multiplicity 1, and $\langle 0\rangle_0$ having multiplicity 1 (i.e. the formal monodromy at the simple pole has two distinct eigenvalues, up to a twist we may assume one of the eigenvalues is equal to 1 so that the multiplicity of $\langle 0\rangle_0$ is 1 in the modified formal local system associated to the connection). The generic representation corresponds to a rank 3 connection.

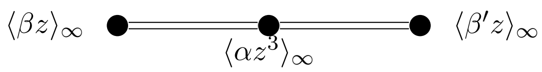
Remark 7. Notice that the rank 2 Lax representations discussed in this paragraph exhaust all cases listed in [81], looking at Painlevé equations from the perspective of degeneration of the fuchsian Painlevé VI Lax representation.

Higher isomonodromy systems As in the two-dimensional case, the diagrams give us a nice interpretation of many known different Lax representations of the same 4-dimensional isomonodromy system: they correspond again to different representations of the diagram. Many cases are already covered by the simply laced framework, but our more general construction allows us to extend this to a few new cases, so that:

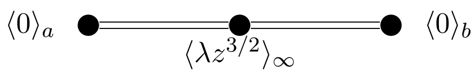
Theorem 6.3.1. *For each type of 4-dimensional isomonodromy system equation listed in [65, p. 40], the formal data associated to the different representations mentioned there correspond to different representations of the same diagram.*

Proof. Among all isomonodromy systems in figure of [65, p. 40], we have to consider the ones associated to several Lax representations. For all equations except two, the formal data involved satisfy the hypotheses of [17, 20]: they feature only one unramified irregular singularity together with regular singularities. In this case, it follows directly from the simply laced case that the different formal data correspond to different readings of the diagram.

Let us discuss in more detail the two cases involving several irregular singularities or ramification. The first case is the box corresponding to the Hamiltonian denoted by $H_{\mathrm{Gar}}^{\frac{5}{2}+1+1}$. It is associated to two different representations. The first one corresponds to a rank 3 connections with one irregular singularity at infinity, with 3 active circles of the form $\langle \alpha z^3\rangle_\infty$,$\langle \alpha' z^3 + \beta z\rangle$, $\langle \alpha' z^3 + \beta' z\rangle$, with $\alpha \neq \alpha'$ and $\beta \neq \beta'$. Up to performing a twist, we may assume that $\alpha' = 0$. The corresponding diagram is drawn below:

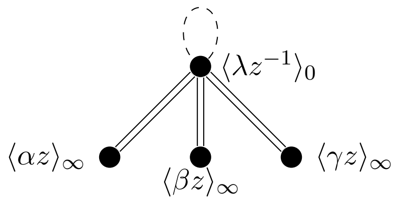


The second Lax representation corresponds to a rank two connection, with one irregular singularity with one active circle of slope 3/2, of the form $\langle \lambda z^{3/2}\rangle_\infty$ with multiplicity 1 together with two simple poles at finite distance, i.e. active circles of the form $\langle 0\rangle_a$, and $\langle 0\rangle_b$ with multiplicity 1 (as before, the formal monodromy has two distinct eigenvalues, up to a twist we may assume one of them is equal to 1 so the multiplicity of the tame circle is 1). This leads to the following diagram:

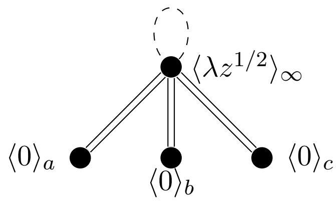


These two formal data give the same diagram. They correspond (up to twists) to several representations of the same diagram, and are related by the Fourier-Laplace transform. The rank 3 Lax representation corresponds to the generic representation of the diagram, while the rank 2 one corresponds to one of the non-generic representations.

The second case is the one with the Hamiltonian denoted by $H_{\mathrm{Gar}}^{\frac{3}{2}+1+1+1}$. It is associated to two different representations. The first one corresponds to a rank 3 connection with two irregular singularities. The first singularity, say at ∞, has three active circles of slope one, each with multiplicity 1, of the form $\langle \alpha z\rangle_\infty$, $\langle \beta z\rangle_\infty$, $\langle \gamma z\rangle_\infty$ with α, β, γ all distinct. The second singularity, say at 0, has two active circles of slope 1, of the form $\langle \lambda z^{-1}\rangle_0$ and $\langle \mu z^{1}\rangle_\infty$, with $\lambda \neq \mu$. The circle $\langle \lambda z^{-1}\rangle_0$ has multiplicity 1, whereas $\langle \mu z^{-1}\rangle_0$ has multiplicity two and trivial conjugacy class. Up to applying a twist at 0, we may assume that $\mu = 0$, so that the circle $\langle \mu z^{-1}\rangle_0$ is the tame circle $\langle 0\rangle_0$. Since its formal monodromy is trivial, its multiplicity in the associated modified irregular class is 0 and the corresponding vertex will not appear in the diagram. The diagram we obtain is the following (the dimension is 1 at all vertices):



On the other hand, the second representation corresponds to a rank two connection, with one irregular singularity with one active circle of slope 1/2, say $\langle \lambda z^{1/2}\rangle_\infty$, together with three simple poles at finite distance, which we denote by $a, b, c \in \mathbb{C}$. The formal monodromies at the simple pole have two distinct eigenvalues, so up to a twist we may assume one of them is equal to 1, so that in the modified irregular class the tame circles $\langle 0\rangle_a$, $\langle 0\rangle_b$, $\langle 0\rangle_c$ have multiplicity 1. This leads to the diagram:



This is the same diagram as for the first Lax representation. More precisely, the two modified irregular classes are related (up to twists and admissible deformations) by applying the Fourier transform. $\square$

Chapter 7

Weyl reflections and operations on connections

We have seen that the diagram associated to a connection (E, ∇) is invariant under $SL_2(\mathbb{C})$ transformations. We now turn to the study of the effect on the diagrams of other operations on connections: Möbius transformations, and more general twists than the ones being part of $SL_2(\mathbb{C})$. In the remainder of the thesis, we will refer to these operations as *basic operations*. We discuss how the diagrams are modified under these operations. We show that abstract simple Weyl reflections acting on the dimension vector and labels of the diagrams, with respect to some of the vertices, come from combinations of such operations, partly generalizing what happens in the simply laced case. As examples of these processes, we look at several diagrams corresponding to Lax pairs for the third Painlevé equation as well as for its degenerate versions. Finally, we study how the Okamoto symmetries of Painlevé III are related to the effect of operations on its Lax pairs.

7.1 Möbius transformations

In all the constructions that we have done, the point at infinity has played a special role. The basic reason for this is that we needed to remove a point to pass from $\mathbb{P}^1$ to the affine line and use the Weyl algebra. In the previous discussion, the point at infinity was fixed once and for all. However, we are free to choose how to identify the Riemann surface $\Sigma \cong \mathbb{P}^1$ to $\mathbb{P}^1$, in particular which point in $\mathbb{P}^1$ is at infinity. This is equivalent to acting on $\mathbb{P}^1$ with Möbius transformations to move the singular points of our connection according to the corresponding homography.

Let $\mu : z \to \frac{az+b}{cz+d}$, with $ad - bc = 1$ a Möbius transformation. If (E, ∇) is a connection on a Zariski open subset Σ° of $\mathbb{P}^1$, we define $\mu \cdot (E, \nabla)$, as the connection on $\mu(\Sigma^\circ)$ obtained by rotating $\mathbb{P}^1$ according to μ. If Θ is the irregular class of (E, ∇), the irregular class of $\mu \cdot \Theta$ of $\mu \cdot (E, \nabla)$ is such that

$$(\mu \cdot \Theta)_a = \Theta_{\mu(a)},$$

for any point $a \in \mathbb{P}^1$.

Beware that, since the point at infinity play a special role and because we take the minimal extension at the points at finite distance, the modified irregular classes are not transformed in as simple a way by a Möbius transformation. In particular, the modified irregular class $\breve{\Theta}'$ of $\mu \cdot (E, \nabla)$ does not just depend on the modified irregular class $\breve{\Theta}$ of (E, ∇): it also depends on the monodromies. Even more, their number of active circles are different in general: if $a = \mu^{-1}(\infty)$ is not a singular point of $E, \nabla)$, the tame circle $\langle 0 \rangle_a$ is not an active circle of Θ, but the tame circle at infinity $\langle 0 \rangle_\infty$ will be an active circle of $\breve{\Theta}'$ with trivial monodromy. Similarly, if $\langle 0 \rangle_\infty$ is an active circle of $\breve{\Theta}$ and has trivial monodromy, then if $a = \mu(\infty) \neq \infty$ $\langle 0 \rangle_a$ is not an active circle of $\breve{\Theta}'$.

However, if $\langle q \rangle_a$ is not a tame circle, it is an active circle of $\breve{\Theta}$ if and only if $\langle q \rangle_{\mu(a)}$ is an active circle of $\breve{\Theta}'$: we thus set $\mu \cdot \langle q \rangle_a = \langle q \rangle_{\mu(a)}$.

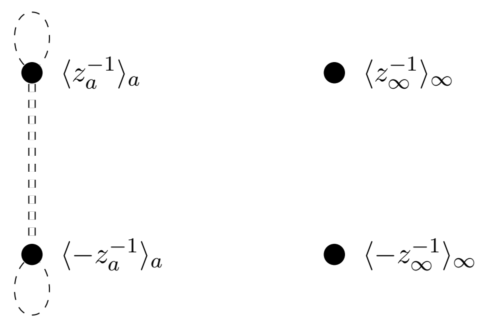
Let us see the effect of Möbius transformations on the diagrams. Let $a \in \mathbb{P}^1$ a point of the Riemann sphere, Θ_a an irregular class at a, and I_a the corresponding set of active circles. For any connection (E, ∇) with a singularity at a with irregular class Θ_a at a, the active circles of the corresponding modified irregular class $\breve{\Theta}_a$ at a define a subdiagram of the core diagram $\Gamma_c(E, \nabla)$ that we denote by $\Gamma_{c,a}(\Theta_a)$. From the direct formula for the diagram, it follows that $\Gamma_{c,a}(\Theta_a)$ only depends on whether a is at infinity or not. This motivates the following definition:

Definition 7.1.1. Let Θ a local irregular class, expressed in terms of some local coordinate z.

- The subdiagram at infinity $\Gamma_c^{\infty}(\Theta)$ associated to Θ is the diagram such that the number of edges between two active circles I and I' is $B^{\infty}_{I,I'}$.
- The subdiagram at finite distance $\Gamma_c^{\neq\infty}(\Theta)$ associated to Θ is the diagram such that the number of edges between two active circles I and I' with ramifications β, β' and slopes α/β, α'/β' is
$$B^{\neq\infty}_{I,I'} := B^{\infty}_{I,I'} - \alpha\beta' - \alpha'\beta.$$

If $a = \infty$ we have $\Gamma_{c,a}(\Theta_a) = \Gamma_c^{\infty}(\Theta_a)$, whereas is a is at finite distance, then $\Gamma_{c,a}(\Theta_a) = \Gamma_c^{\neq\infty}(\Theta_a)$.

Example 7.1.2. Let Θ_a an irregular class with two active circles $\langle z_a^{-1} \rangle_a$, $\langle -z_a^{-1} \rangle_a$ of slope 1. The infinity subdiagram and finite distance subdiagram associated to Θ_a are drawn below:



Now let $\mu \in PSL_2(\mathbb{C})$ a Möbius transformation, and Θ a global irregular class. Applying μ changes the position of the singularities. Assume that Θ has r distinct singularities $a_1, \dots, a_r \in \mathbb{P}^1$. Then, in a similar way as for the different representations of the diagram, we have $r+1$ different diagrams depending whether one of the singularities is at infinity.

The *generic position* corresponds to having ∞ different from all the poles $a_1, \dots, a_r$. In this case, in the diagram we have one extra vertex corresponding to the tame circle $\langle 0 \rangle_\infty$ at infinity, with multiplicity equal to the rank of (E, ∇) (and trivial monodromy). The subdiagrams corresponding to the irregular class at the poles a_i are the subdiagrams For $i = 1, \dots, r$, the i-th position corresponds to having $\infty = a_i$. In this case, in the diagram, the subdiagram corresponding to the circles at a_i coincides with the Stokes diagram of the irregular class at a_i, whereas for the other poles we have the subdiagrams at finite distance. The diagram has one vertex less than in the generic case.

7.2 Twists

Another operation to consider is to tensor a connection with a rank one meromorphic connection on $\mathbb{P}^1$. The symplectic twists that we have considered are special cases of this, where the rank one connection only has a singularity of slope 2 at infinity. However, we are free to choose more general connections with any number of singularities. At each singularity, the irregular class has to consist of only one circle, which must be unramified (otherwise the rank would be greater than one). More precisely, let (L, ∇_L) a meromorphic line bundle with connection on $\mathbb{P}^1$. The twist by L is the operation on connections given by

$$T_L : (E, \nabla) \mapsto (E \otimes L, \nabla \otimes \nabla_L). \tag{7.2.1}$$

For any point $a \in \mathbb{P}^1$ let $\langle p_a \rangle \in \pi^0(\mathcal{I}_a)$ the (unique) exponential factor of ∇_L at a, and $W_a^0 \to \langle p_a \rangle$ the local system corresponding to the formal data of (L, ∇_L) at a. On the level of the $\mathcal{I}$-graded local system $V_a^0 \to \partial_a$, the twist induces

$$\begin{tikzcd} V_a^0 \arrow[d, "\psi"] \arrow[r, phantom, "\overset{T_L}{\longmapsto}"] & V_a^0 \otimes W_a^0 \arrow[d] \\ \partial_a & \partial_a \end{tikzcd} \tag{7.2.2}$$

Let us describe how this operation behaves with respect to the $\mathcal{I}_a$-grading. The twist transforms any exponential factor q into $q + p_a$. As a consequence, it induces a transformation on the local system $V_a \to \mathcal{I}_a$ such that we have

$$\begin{tikzcd} V_a^0 \arrow[d] \arrow[r, phantom, "\overset{T_L}{\longmapsto}"] & V_a^0 \otimes W_a^0 \arrow[d] \\ \langle q \rangle & \langle q + p_a \rangle \end{tikzcd} \tag{7.2.3}$$

For each circle $\langle q \rangle \in \pi^0(\mathcal{I}_a)$.

Since W_a^0 is of rank one, its monodromy is given by a scalar $\lambda_a \in \mathbb{C}^*$. Let us fix a direction $d \in \partial_a$. For a circle $\langle q \rangle$ of ramification index β, the fibre of $V_{\langle q \rangle}^0$ at d decomposes as

$$V^0_{\langle q \rangle, d} = V^0_{\langle q \rangle, d_0} \oplus \cdots \oplus V^0_{I, d_{\langle q \rangle - 1}},.$$

The monodromy of $V^0_{\langle q \rangle} \to \partial_a$ around ∂_a is of the form

$$\rho_{\partial_a} = \begin{pmatrix} 0 & \dots & 0 & \rho_{\beta-1} \\ \rho_0 & & & 0 \\ & \ddots & & \vdots \\ & & \rho_{\beta-2} & 0 \end{pmatrix} \in GL(V^0_{\langle q \rangle, d}),$$

with $\rho_i : V^0_{\langle q \rangle, d_i} \to V^0_{\langle q \rangle, d_{i+1}}$ for $i = 0, \dots, \beta - 1$. The fibre over d of the tensor product $V_a^0 \otimes W_a^0$ at d is

$$(V_a^0 \otimes W_a^0)_d = V^0_{\langle q \rangle, d} \otimes W^0_{a,d} = V^0_{\langle q \rangle, d_0} \otimes W^0_{a,d} \oplus \cdots \oplus V^0_{\langle q \rangle, d_{\langle q \rangle - 1}}, \otimes W^0_{a,d},$$

and its monodromy over ∂_a is $\rho'_{\partial_a} = \rho_{\partial_a} \otimes \lambda \, \mathrm{Id}_{W^0_{a,d}}$. The twist thus sends ρ_i to $\rho'_i = \rho_i \otimes \lambda$. Since the monodromy of $V^0_{\langle q \rangle} \to \langle q \rangle$ is given by

$$\rho = \rho_{\beta-1} \dots \rho_0 \in GL(V^0_{\langle q \rangle, d_0}),$$

the monodromy of $V^0_{\langle q \rangle} \otimes W_a^0 \to \langle q + p_a \rangle$ is

$$\rho' = \rho \otimes \lambda^\beta \in GL(V^0_{\langle q \rangle, d_0} \otimes W^0_{a,d}). \tag{7.2.4}$$

How does a twist modify the diagram? We first consider what happens at infinity. Let p_∞ the exponential factor at infinity of a meromorphic rank one connection (L, ∇_L). The corresponding twists acts on the formal data of a connection (E, ∇) by adding p_∞ to each of its exponential factors. Therefore, the differences $q - q'$ between different exponential factors are infinity are unchanged by the twist. It follows that the number of Stokes arrows between two circles at infinity $\langle q\rangle$ and $\langle q'\rangle$ does not change when we apply a twist, hence the number of edges between the corresponding vertices remains the same (since we just subtract a number of arrows determined by $\operatorname{ram} q$ and $\operatorname{ram} q'$ to pass from the Stokes arrows to the edges of the diagram). The number of edges between a circle $\langle q\rangle$ at infinity and a circle at finite distance also does not change when $q \mapsto q + p_\infty$. This means that

Lemma 7.2.1. *A twist by a rank one connection with only one singularity at infinity does not change the diagram.*

The situation is different for singularities at finite distance: indeed, the number of edges between two circles $\langle q\rangle$ and $\langle q'\rangle$ at a same pole $a \in \mathbb{C}$ is determined by the Stokes arrows between the Legendre transforms $\langle \tilde{q}\rangle$ and $\langle \tilde{q}'\rangle$, and adding the same exponential factor p_a to q and q' does change the difference $\tilde{q}' - \tilde{q}$. Therefore, in general the graph does not remain the same under the twist by a connection having poles at finite distance.

An important particular case of twist at finite distance is the following : If $\langle q\rangle_a$ is an unramified circle at $a \in \mathbb{C}$, we can do a twist at finite distance by a connection (L, ∇_L) such that $p_a = -q$ to "cancel" q and transform $\langle q\rangle$ into the tame circle $\langle 0\rangle_a$ at a. This will modify the number of edges adjacent to the active circles at a.

7.3 Weyl reflections

We will now discuss how some operations on connections give rise to simple Weyl reflections with respect to some of the vertices of the diagram. This is a generalization of the reflections considered in [22, 20]. The general idea goes as follows: we get simple reflections when we exchange two consecutive eigenvalues in the marking corresponding to a leg of the full diagram. This gives reflections with respect to all vertices internal to some leg. But there is a subtlety involving the tame circles at finite distance, related to passing from the non-modified to the modified formal irregular class to obtain the diagram: a tame circle at finite distance can be viewed as the first internal vertex of a *splayed* leg with a *special marking*, i.e with the first eigenvalue equal to 1. This enables us to get reflections with respect to tame circles at finite distance. In turn, we also get reflections with respect to circles at infinity which can be brought to a tame circle at finite distance under a $SL_2(\mathbb{C})$ transformation, that is in an other representation of the diagram.

Definition 7.3.1. Let $I \in \pi^0(\mathcal{I})$ a circle. We say that I is simple if there exists a symplectic transformation $A \in SL_2(\mathbb{C})$ which sends I to a tame circle at finite distance $\langle 0\rangle_a$, for some $a \in \mathbb{C}$.

Since we know how $SL_2(\mathbb{C})$ transformations act on tame circles, we immediately get the following characterization:

Lemma 7.3.2. *I is simple if either $I = \langle 0\rangle_a$ for $a \in \mathbb{C}$, or $I = \langle \lambda x^2 + \mu x\rangle_\infty$, for $\lambda, \mu \in \mathbb{C}$.*

This means that the generic representative at infinity of a simple circle is of the form $\langle \lambda x^2 + \mu x\rangle_\infty$, that is an unramified circle of slope less that two.

Let (E, ∇) a connection of a Zariski open subset of the affine line with formal data$(\Theta, \boldsymbol{\mathcal{C}})$ and modified formal data $(\breve{\Theta}, \breve{\boldsymbol{\mathcal{C}}})$. Choosing a marking for each conjugacy class in $\breve{\boldsymbol{\mathcal{C}}}$, we have a triple $(\Gamma, \mathbf{d}, \mathbf{q})$ where $\Gamma = \Gamma(E, \nabla)$, $\mathbf{d}$ is the dimension vector, and $\mathbf{q}$ is the vector of multiplicative labels.

Theorem 7.3.3. *Let i be a vertex of the diagram $\Gamma(E,\nabla)$. If i is not in the core, or corresponds to a simple circle I, then there exists a change of marking of $\check{\mathcal{C}}_i$ or a combination of twists and Fourier transforms which acts on the diagram $\Gamma(E,\nabla)$ as the simple reflection with respect to i, i.e. which transforms the triple $(\Gamma, \mathbf{d}, \mathbf{q})$ into*

$$(\Gamma, s_i(\boldsymbol{d}), r_i(\mathbf{q})).$$

Proof. We first deal with the case where i is not in the core, i.e. is in the interior of a leg $\mathbb{L}$ with marking $(\xi_1, \dots, \xi_k)$. Then, the corresponding reflection simply arises by swapping the eigenvalues ξ_j and ξ_{j-1} such that the value of the component in $GL(\mathbb{C}^{d_i})$ of the moment map at which take the quasi-hamiltonian symplectic reduction is $\xi_{j-1}/\xi_j \operatorname{Id}_{\mathbb{C}^{d_i}}$. The details are exactly similar to the proof of theorem 10.2 in [22].

Now, if i corresponds to a simple circle I, the proof is again almost similar to the proof of [22] theorem 10.2. We apply a $SL_2(\mathbb{C})$ transformation to change the representation of the diagram and send I to a tame circle at finite distance $\langle 0\rangle_a$. Let ξ the first eigenvalue of the marking $(\xi, \dots)$ of $\check{\mathcal{C}}_{\langle 0\rangle_a}$ associated to the leg $\mathbb{L}_i$. Then $(1, \xi, \dots)$ is a (special) marking of the non-reduced conjugacy class $\mathcal{C}_{\langle 0_a\rangle}$. We swap the two eigenvalues 1 and ξ i.e. we do $(1, \xi, ...) \mapsto (\xi, 1, ...)$, then we do a twist by a regular connection at a to rescale $(1, \xi) \mapsto (\xi^{-1}, 1)$ so that we get a special marking of the new conjugacy class $\mathcal{C}'_{\langle 0\rangle_a}$.

The difference with [22] is in the way we check that this transformation acts as the desired reflection on the dimension vector and the multiplicative labels. Let $T_a \in GL(V^0_{\langle 0\rangle_a})$ the monodromy of the regular part of the connection at a. The dimension at i is

$$d_i = \operatorname{rank}(T_a - 1) = \dim V^0_{\langle 0\rangle_a} - \dim \operatorname{Ker}(T_a - 1).$$

Let $i+1$ be the vertex adjacent to i in the leg $\mathbb{L}_i$. We have

$$d_{i+1} = \operatorname{rank}(T_a - \xi)(T_a - 1) = \dim V^0_{\langle 0\rangle_a} - \dim \operatorname{Ker}(T_a - 1) - \dim \operatorname{Ker}(T_a - \xi)$$

The other vertices j adjacent to i in the diagram correspond to two types of circles :

- Circles at infinity $\langle q\rangle_\infty$, for which the number of edges between i and j is $B_{i,j} = \operatorname{ram} q$.
- Irregular circles $\langle q\rangle_a$ at a, for which for which the number of edges between i and j is $B_{i,j} = -\operatorname{ram} q$.

Therefore, the dimension at i after Weyl reflection is

$$\begin{aligned}
(r_i(\mathbf{d}))(i) &= d_i - \sum_{j\in I} C_{ij} d_j \\
&= d_i - 2d_i + d_{i+1} + \sum_{j\in I_\infty} B_{ij} d_j + \sum_{j \in I_a \smallsetminus i} B_{ij} d_j \\
&= -d_i + d_{i+1} + n - (n - n_a) \\
&= -(\dim V^0_{\langle 0\rangle_a} - \dim \operatorname{Ker}(T_a - 1)) + (\dim V^0_{\langle 0\rangle_a} - \dim \operatorname{Ker}(T_a - 1) - \dim \operatorname{Ker}(T_a - \xi)) + \dim V^0_{\langle 0\rangle_a} \\
&= \dim V^0_{\langle 0\rangle_a} - \dim \operatorname{Ker}(T_i - \xi) \\
&= \operatorname{rank}(T_i - \xi),
\end{aligned}$$

Indeed we have

$$\sum_{j\in I_\infty} B_{ij} d_j = \sum_{\langle q\rangle_\infty} \operatorname{ram}(\langle q\rangle) n_{\langle q\rangle} = \operatorname{rank} E = n,$$

and similarly

$$\sum_{j\in I_a \smallsetminus i} B_{ij} d_j = \sum_{\langle q\rangle_a, q\neq 0} \operatorname{ram}(\langle q\rangle) n_{\langle q\rangle} = n - n_k$$

is the rank of the irregular part of the connection at a.

Therefore, we get $(r_i(\mathbf{d}))(i) = \operatorname{rank}(T_i - \xi)$ which is exactly the new dimension at i after our operation.

Now for the multiplicative labels of $\mathbf{q} = (q_j)_{j\in I}$, our twist multiplies the monodromy around ∂_a by ξ^{-1} and the monodromy at infinity around ∂_∞ by ξ. This transforms the marking corresponding to the legs in the following way.

- The marking of the leg $\mathbb{L}_i$ at i is transformed according to $(\xi, \xi_{i2}, \dots) \mapsto (\xi^{-1}, \xi^{-1}\xi_{i2}, \dots)$.
- If $j \in I_a \smallsetminus i$ corresponds to a circle $\langle q\rangle_a$ at a of ramification r, the marking of the leg $\mathbb{L}_j$ is transformed according to $(\xi_{j1}, \xi_{j2}, \dots) \mapsto (\xi^{-r}\xi_{j1}, \xi^{-r}\xi_{j2}, \dots)$
- If $j \in I_\infty$ corresponds to a circle $\langle q\rangle_\infty$ at ∞ of ramification r, the marking of the leg $\mathbb{L}_j$ is transformed according to $(\xi_{j1}, \xi_{j2}, \dots) \mapsto (\xi^{r}\xi_{j1}, \xi^{r}\xi_{j2}, \dots)$

The markings of the other legs do not change. Recall that the labels q_j are defined by $q_j = \xi_{j1}$ if j is in the core, and $\mathbb{L}_j$ is the corresponding leg, and $q_{jk} = \xi_{jk}/\xi_{j(k-1)}$ where jk denotes the k-th vertex of the leg $\mathbb{L}_j$. This implies the following transformations of the labels:

- For the vertex i, the label q_i is transformed according to $\xi \mapsto \xi^{-1}$.
- For the vertex $i2$ next to i in the leg $\mathbb{L}_i$, q_{i2} is transformed according to $\xi_{i1}/\xi \mapsto \xi_{i1}$.
- If $j \in I_a \smallsetminus i$ corresponds to a circle $\langle q\rangle_a$ at a of ramification r, q_{j1} follows $\xi_{j1} \mapsto \xi^{-r}\xi_{j1}$, and for $k \geq 2$, q_{jk} follows $\xi_{jk}/\xi_{j(k-1)} \mapsto \xi_{jk}/\xi_{j(k-1)}$, i.e. does not change.
- If $j \in I_\infty$ corresponds to a circle $\langle q\rangle_\infty$ at ∞ of ramification r, q_{j1} follows $\xi_{j1} \mapsto \xi^{r}\xi_{j1}$, and for $k \geq 2$, q_{jk} follows $\xi_{jk}/\xi_{j(k-1)} \mapsto \xi_{jk}/\xi_{j(k-1)}$, i.e. does not change.

The other components do not change since they come from markings which don't. In all cases, we find that those transformations coincide with what is expected from the simple reflection, that is

$$q_j \mapsto \xi^{-C_{ij}} q_j.$$

□

Remark 8. Notice that the simply laced case considered in [20, 22] is exactly the case where all active circles are simple. We thus recover in our framework the result that in this case we get simple reflections with respect to *every vertex* of the diagram.

Combining twists with reflections Until now we have only used symplectic transformations and twists by regular rank one connections to get Weyl reflections of the diagram. We can get reflections with respect to some of the other vertices using more general twists.

Let $i \in I$ be a vertex corresponding to an unramified circle $\langle q\rangle_\infty$ at infinity. We can obtain a reflection with respect to i in the following way: we do a twist by a rank one connection with exponential factor $-q$ at infinity to cancel the exponential factor q, which transforms $\langle q\rangle_\infty$ into the tame circle $\langle 0\rangle_\infty$. As we have seen, this operation does not change the diagram. Now the vertex i corresponds to a simple circle, and we know how to get the reflection corresponding to i.

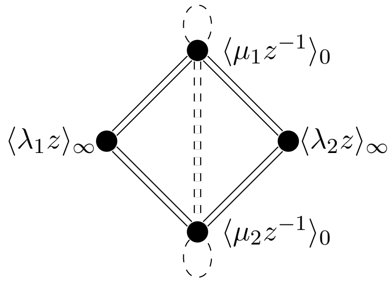
If $i \in I$ is in the core of the diagram and corresponds to an unramified irregular circle $\langle q\rangle_a$ at finite distance, we can cancel q using a twist, but we have seen that this changes the diagram. In the new diagram, i corresponds to the tame circle $\langle 0\rangle_a$, so we get a reflection with respect to i. We thus get a reflection also in this case, but not in the diagram we started with. Notice that, by an appropriate twist, one can cancel one unramified circle per pole, obtaining different new diagrams for each combination of choices. An important consequence is that in the case of (non simply laced) supernova diagrams, we obtain in this way simple reflections with respect to all vertices from operations on connections.

Remark 9. Notice that all the simple reflections that we have found are relative to vertices with no loops in the diagram. This seems to be consistent with having some kind of generalized Kac-Moody algebra associated to the diagram acting on connections. A natural question which arises is: can we obtain a simple reflection with respect to all loopless vertices from operations on connections?

7.4 Other diagrams for Painlevé equations

7.4.1 Painlevé III

We have already seen how to obtain the Painlevé III diagram from its standard Lax pair. As an example of the effects of operations on connections on the diagram, it is interesting to look at the diagram we obtain if we do not assume that $\mu_2 = 0$. In this case there are 4 active circles, and the diagram is represented below:

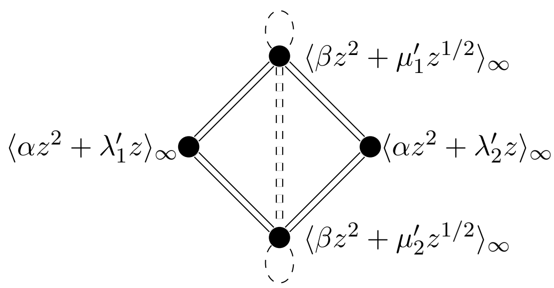


with $\mu_1 \neq \mu_2$, $\lambda_1 \neq \lambda_2$, and the dimension at every vertex being 1, i.e. $\mathbf{d} = (1, 1, 1, 1)$. The Cartan matrix is

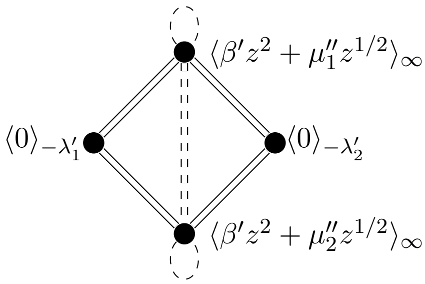
$$\begin{pmatrix} 4 & 2 & -2 & -2 \\ 2 & 4 & -2 & -2 \\ -2 & -2 & 2 & 0 \\ -2 & -2 & 0 & 2 \end{pmatrix},$$

and we have $2 - (\mathbf{d}, \mathbf{d}) = 2$ as expected since Painlevé equations correspond to two-dimensional moduli spaces.

Let us describe the different representations of the diagram. There are two other representations of the diagram, obtained by applying $SL_2(\mathbb{C})$ transformations. The generic representation corresponds to having circles of the form

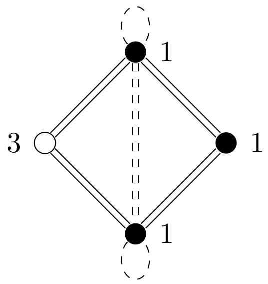


Indeed, the Legendre transform sends a circle of the form $\langle \mu z^{-1} \rangle_0$ of slope 1 at 0 to a circle $\langle \mu' z^{1/2} \rangle_\infty$ of slope 1/2 at infinity. The generic representation corresponds to a rank 6 connection with only one pole at infinity. The other nongeneric representation may then obtained by cancelling the term αz^2 by an appropriate twist and applying the Fourier-Laplace transform. The active circles are of the following form:



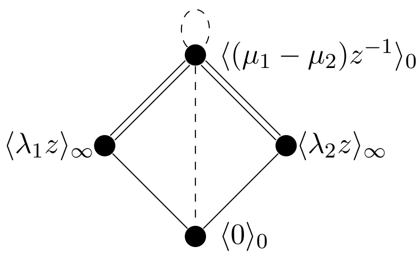
This is a rank 4 connection, with one irregular singularity at infinity, and two simple poles at finite distance where the monodromy has rank one.

We can do reflections with respect to the two loopless vertices of the diagram. The figure below shows the new dimensions after the reflection with respect to the vertex on the left on the figure below:



The operation on connections which gives rise to this reflection can be understood from the third representation of the diagram, the one featuring simple poles. In this representation, the (multiplicative) formal monodromy of the connection at the simple pole at $-\lambda_1$ is a 4 by 4 matrix with two eigenvalues 1 and $\lambda \neq 1$, with eigenspaces of respective dimensions 1 and 3. The reflection comes from swapping the two eigenvalues, hence the dimension of the eigenspaces. The same goes for the reflection with respect to the other vertex without loops. We can also check that we still have $2 - (\mathbf{d}, \mathbf{d}) = 2$.

The reduction from this diagram to the 3 vertex diagram of [26] can be decomposed in two steps. The first step consists in applying a (non $SL_2(\mathbb{C})$) twist at $z = 0$ to connections corresponding to the first representation of the graph to transform one of the circles at 0, say $\langle \mu_2 z^{-1} \rangle_0$ into the tame circle $\langle 0 \rangle_0$ without changing the formal monodromy. In the generic case where the monodromy around this circle is not trivial, this gives us the new diagram

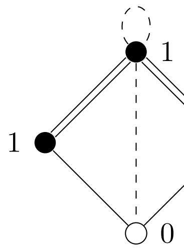


with multiplicity 1 at the tame circle $\langle 0 \rangle_0$. The Cartan matrix of this new diagram is

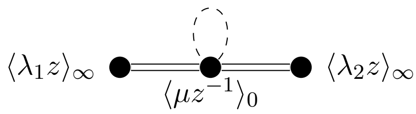
$$C' = \begin{pmatrix} 4 & 1 & -2 & -2 \\ 1 & 2 & -1 & -1 \\ -2 & -1 & 2 & 0 \\ -2 & -1 & 0 & 2 \end{pmatrix},$$

and we still have $2 - (\mathbf{d}, \mathbf{d}) = 2$.

In this diagram, it is now possible to do a reflection with respect to the vertex corresponding to the tame circle. This gives the new dimensions

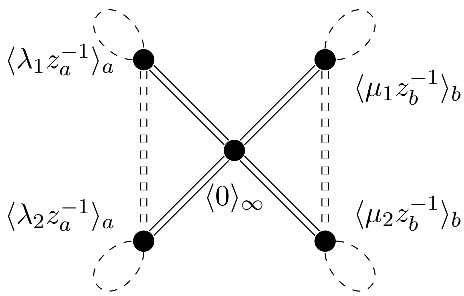


We notice that the mutliplicity at the tame becomes 0. Indeed, the operation corresponding to the reflection consists in setting to 1 the unique eigenvalue of multiplicity 1 of the monodromy of $\langle 0\rangle_0$, so the monodromy becomes trivial, and the reduced monodromy has rank 0. Therefore, we may erase this vertex to recover the 3 vertex diagram of [26] (with $\mu = \mu_1 - \mu_2$):



Remark 10. At this point, we could try to cancel the circle $\langle \mu z^{-1}\rangle_0$ by a twist at $z = 0$. However, if we do this, the tame circle $\langle 0\rangle_0$ with multiplicity 0, corresponding to $\langle 0\rangle_0$ with rank 1, becomes $\langle -\mu x^{-1}\rangle_0$ with multiplicity 1. In other words, the circle that we had just erased reappears. The simplified diagram still "knows" about the tame circle with trivial monodromy.

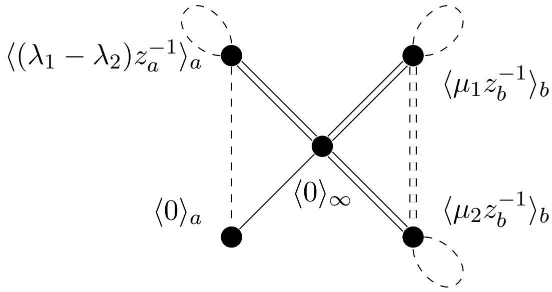
Möbius transformations We can also apply a Möbius transformation to the original Lax representation to have both poles at finite distance, say at $a, b \in \mathbb{C}$. We must then add to the diagram a vertex corresponding to the tame circle at infinity $\langle 0\rangle_\infty$, with multiplicity 2 and trivial monodromy. This gives the following diagram.



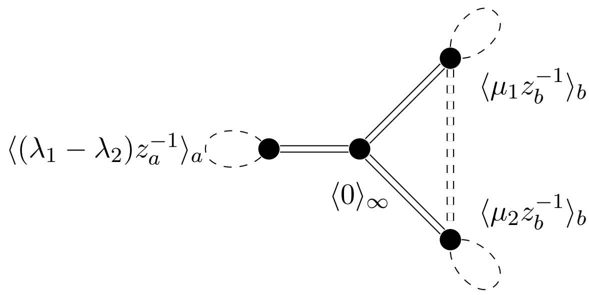
where the multiplicity of the central vertex is 2, and the other multiplicities are 1. The Cartan matrix is

$$\begin{pmatrix} 2 & -2 & -2 & -2 & -2 \\ -2 & 4 & 2 & 0 & 0 \\ -2 & 2 & 4 & 0 & 0 \\ -2 & 0 & 0 & 4 & 2 \\ -2 & 0 & 0 & 2 & 4 \end{pmatrix},$$

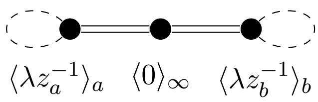
and again we check that $2 - (\mathbf{d}, \mathbf{d}) = 2$. As for the previous diagram, we can do a twist at finite distance to transform one of the circles at a into the tame circle $\langle 0\rangle_a$, say $\langle \lambda_2 z_a^{-1}\rangle_a$. In the generic case where the monodromy is not trivial, we get the following diagram



where $\langle 0\rangle_a$ has multiplicity 1. Once again, we can do a Weyl reflection with respect to the tame circle $\langle 0\rangle_a$. Its multiplicity then becomes zero, so we can erase the circle and get the simpler diagram.



We can do exactly the same thing for the circles at b. This leads to the diagram



The Cartan matrix is

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

and once again we check that $2-(\mathbf{d},\mathbf{d})=2$.

Remark 11. At this stage, we can make the following intriguing observation: the Cartan matrix of this last diagram and the Cartan matrix of the 3-vertex diagram of [26] are the symmetrized Cartan matrices coming from two non-symmetric Cartan matrix which are the transpose of each other, i.e. whose associated generalized Lie algebras are the Langlands dual of each other.

Indeed, these two diagrams have the following Cartan matrices:

$$C_0=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \qquad C_1=\begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

.

We have

$$C_0=\widehat{C}_0 D_0, \qquad C_1=\widehat{C}_1 D_1.$$

with

$$\widehat{C}_0=\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \widehat{C}_1=\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix},$$

and

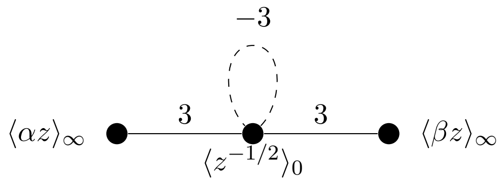
$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This means that C_0 and C_1 are symmetrized Cartan matrices coming from the non-symmetric Cartan matrices $\widehat{C}_0$, $\widehat{C}_1$. The observation is that $\widehat{C}_1$ is the transpose of $\widehat{C}_0$, i.e. they correspond to Langlands dual Lie algebras. In particular, this implies that the corresponding Weyl groups are isomorphic. It is unclear whether this fact is a mere coincidence or if it has a deeper meaning.

7.4.2 Degenerate Painlevé III equations

Degenerate Painlevé III The standard Painlevé III Lax pair has the following active circles: a circle of slope 1/2 at infinity, say $\langle z^{1/2}\rangle_\infty$ and two circles of slope 1 at 0, $\langle \alpha z^{-1}\rangle_0$, $\langle \beta z^{-1}\rangle_0$, with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$. We have already studied the corresponding diagram in the case where $\beta = 0$ and $\langle \beta z^{-1}\rangle_0$ has trivial monodromy.

We can perform a Möbius transformation to exchange 0 and ∞. If we do this we get the diagram drawn below, where the central vertex has 3 negative loops and is linked to the 2 other vertices by a triple edge.



The Cartan matrix is

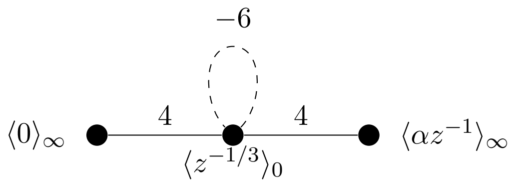
$$\begin{pmatrix} 8 & -3 & -3 \\ -3 & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix},$$

and we again have $2 - (\mathbf{d}, \mathbf{d}) = 2$ as expected. The generic representation corresponds to a rank 5 connection with one irregular singularity at infinity, with three active circles, one of them with ramification 3, the two others with ramification 1. There is a representation of this graph where the central vertex corresponds to a circle at infinity of slope 1/3 (the image of the slope 1/2 at 0 under formal Fourier transform) and two simple poles at infinity, at which the modified formal local system has rank 1.

Doubly degenerate Painlevé III The doubly degenerate Painlevé III admits a Lax representation with the following active circles, each having multiplicity 1: a circle of slope 1/2 at infinity, say $\langle z^{1/2}\rangle_\infty$, and circle of slope 1/2 at 0, say $\langle z^{-1/2}\rangle_0$. We have already studied the corresponding diagram.

We have seen that there is a representation of this diagram whose modified irregular class has an active circle of slope 1/3 at infinity, and a circle of slope 1 at 0, with multiplicity 1. In the non-modified irregular class, the tame circle $\langle 0\rangle_0$ has multiplicity 2 and trivial monodromy.

Now, performing a Möbius transformation to exchange 0 and ∞, we get formal data giving the following diagram, with $\langle 0\rangle_\infty$ having multiplicity 2, and the two other circles having multiplicity 1.



The Cartan matrix is

$$C = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 14 & -4 \\ 0 & -4 & 2 \end{pmatrix},$$

and once again we have $2 - (\mathbf{d}, \mathbf{d}) = 2$. There is a representation of this diagram corresponding to a rank 4 connection, with one circle of slope $1/4$ at infinity, and two simple poles at finite distance.

7.5 Okamoto symmetries of Painlevé III

In this paragraph, we consider the Okamoto symmetries of the parameter space of the Painlevé III equation, and investigate whether they can be obtained from basic operation on the Painlevé III Lax pair(s). This is in the same spirit as in [14], where the Okamoto affine Weyl group symmetries of Painlevé VI are realized from operations on different Lax pairs. More precisely, there are three sides to the story that we wish to compare.

1. Okamoto symmetries of the Painlevé III equations.
2. The abstract Weyl group defined from the Cartan matrix of the Painlevé III diagram.
3. Action of geometric operations on the Painlevé III Lax representation(s).

7.5.1 Okamoto symmetries

Let us first briefly review the third Painlevé equations and its Okamoto symmetries following [85, 98]. The third Painlevé equation admits a Hamiltonian formulation: the Painlevé III Hamiltonian $H(p, q)$ is given by

$$tH = q^2p^2 - (q^2 - (\alpha_1 + \beta_1)q - t)p - \alpha_1 q.$$

It depends on two parameters $\alpha_1, \beta_1 \in \mathbb{C}$. The space of parameters of the Painlevé III equations is thus a two dimensional vector space V. Setting $t = \tau^2$ and $y = q/\tau$, this gives for y the Painlevé III equation

$$\frac{d^2y}{d\tau^2} = \frac{1}{y}\left(\frac{dy}{d\tau}\right)^2 - \frac{1}{\tau}\frac{dy}{d\tau} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

where the four complex coefficients $\alpha, \beta, \gamma, \delta$ are related to α_1, β_1 by

$$\alpha = 4(\alpha_1 - \beta_1), \quad \beta = -4(\alpha_1 + \beta_1 - 1), \quad \gamma = 4, \quad \delta = -4.$$

Recall from e.g. [28] the root system of type B_2. If V is a two-dimensional vector space with scalar product $(\cdot,\cdot)$ and orthonormal basis $(\varepsilon_1, \varepsilon_2)$, the B_2 root system is the set $\mathcal{R} = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2$. The corresponding coroots are $\alpha_1^\vee = \varepsilon_1 - \varepsilon_2$, $\alpha_2^\vee = 2\varepsilon_2$. The B_2 Weyl group is generated by the simple Weyl reflections $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$. It isomorphic to the dihedral group with 8 elements D_8. The affine Weyl group of type B_2, on the other hand, is generated by the affine Weyl reflections

$$s_{\alpha,k} = t(k\alpha^\vee) \circ s_\alpha, \ \alpha \in \mathcal{R}, k \in \mathbb{Z}$$

where $t(k\alpha^\vee)$ denotes the translation by $k\alpha^\vee$. The affine group is generated by any subset of 3 reflections corresponding to the walls of an alcove.

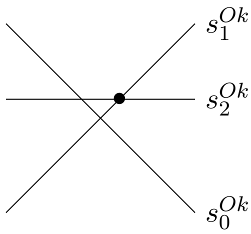
The Okamoto symmetries are birational canonical transformations preserving the Painlevé III equation, but changing the parameters. In his original article, Okamoto parametrizes the vector space V_{Ok} of Painlevé III parameters using the coordinates (v_1, v_2) with

$$v_1 = \beta_1 - \alpha_1, \quad v_2 = -\alpha_1 - \beta_1.$$

The group of transformations is generated by the three reflections $s_1^{Ok}, s_2^{Ok}, s_0^{Ok} \in GL(V_{Ok})$ defined by

$$s_1^{Ok} : (v_1, v_2) \mapsto (v_2, v_1), \tag{7.5.1}$$
$$s_2^{Ok} : (v_1; v_2) \mapsto (v_1, -v_2), \tag{7.5.2}$$
$$s_0^{Ok} : (v_1, v_2) \mapsto (-1 - v_2, -1 - v_1). \tag{7.5.3}$$

whose axes are represented on the figure below.



The two reflections s_1^{Ok}, s_2^{Ok} generate a group isomorphic to the Weyl group of type B_2, via the isomorphism $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$. Furthermore $s_1^{Ok}, s_2^{Ok}, s_0^{Ok}$ generate a group isomorphic to the *affine* Weyl group of type B_2. Indeed, their axes are the walls of an alcove.

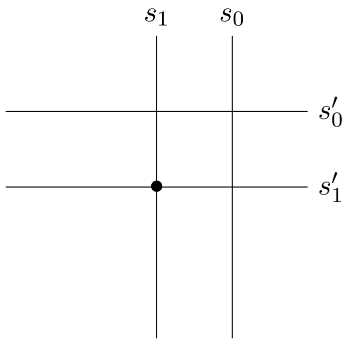
Remark 12. The article [98] on the other hand uses a different parametrization of the birational transformations, which involves the parameters α_1, β_1 along with $\alpha_0 := 1 - \alpha_1$ and $\beta_0 := 1 - \beta_1$. The group of transformations is described there as the extended Weyl group $\widetilde{W}((2A_1)^{(1)}) = \mathrm{Aut}((2A_1)^{(1)}) \ltimes W((2A_1)^{(1)})$ associated to two copies of the affine A_1 Dynkin diagram. The group $W((2A_1)^{(1)})$ is generated by four transformations s_0, s_0', s_1, s_1'. If we use (α_1, β_1) as coordinates on V, they are given by

$$\begin{aligned}
s_0 &= (\alpha_1, \beta_1) \mapsto (2 - \alpha_1, \beta_1),\\
s_1 &= (\alpha_1, \beta_1) \mapsto (-\alpha_1, \beta_1),\\
s_0' &= (\alpha_1, \beta_1) \mapsto (-\alpha_1, 2 - \beta_1),\\
s_1' &= (\alpha_1, \beta_1) \mapsto (\alpha_1, -\beta_1).
\end{aligned}$$

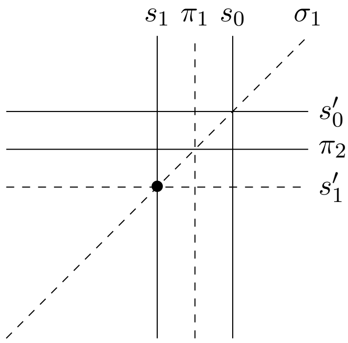
The group of automorphisms $\mathrm{Aut}((2A_1)^{(1)})$ is generated by π_1, π_2, σ_1, whose expressions are

$$\begin{aligned}
\pi_1 &: (\alpha_1, \beta_1) \mapsto (1 - \alpha_1, \beta_1),\\
\pi_2 &: (\alpha_1, \beta_1) \mapsto (\alpha_1, 1 - \beta_1),\\
\sigma_1 &: (\alpha_1, \beta_1) \mapsto (\beta_1, \alpha_1).
\end{aligned}$$

On the (α_1, β_1) plane, s_0, s_0', s_1, s_1' are reflections with respect to the lines on the figure below.



Adding the reflections generating $\mathrm{Aut}((2A_1)^{(1)})$ the figure becomes



It follows from this that the action of $\widetilde{W}((2A_1)^{(1)})$ on the parameters (α_1, β_1) is actually generated by three reflections only whose axes constitute the walls of an alcove, e.g. by σ_1, s'_1, π_1. Let us compare the group of transformations in those two descriptions. Using that

$$\begin{cases} \alpha_1 & = \frac{-v_1 - v_2}{2}, \\ \beta_1 & = \frac{v_1 - v_2}{2}, \end{cases}$$

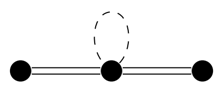
we can express s_0, s_1, s_2 in terms of the coordinates (α_1, α_2). Noticing that

$$\begin{aligned} s_1^{Ok} &: (\alpha_1, \beta_1) \mapsto (\alpha_1, -\beta_1) = s'_1 \\ s_2^{Ok} &: (\alpha_1, \beta_1) \mapsto (-\beta_1, -\alpha_1) = s'_1 \sigma_1 s'_1 \\ s_0^{Ok} &: (\alpha_1, \beta_1) \mapsto (1 - \alpha_1, \beta_1) = \pi_1. \end{aligned}$$

we conclude that the groups of transformations of the Painlevé III parameters in [85] and [98] are the same group $W_{Ok} = \langle \sigma, s'_1, \pi_1 \rangle$, isomorphic to the affine Weyl group of type B_2.

7.5.2 Abstract Weyl group of the Painlevé III diagram

We now discuss abstract the Weyl group corresponding to the Painlevé III diagram:



The Cartan matrix of the diagram is

$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{7.5.4}$$

As already mentioned, it is the symmetrization of the generalized Cartan matrix of type $\widetilde{C}_2$ (see e.g. [30, p. 576]). Let e_1, e_2, e_3 basis vectors corresponding to the three vertices of the diagram, and $\Gamma := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. Let $(\cdot,\cdot)$ the bilinear form on Γ defined by $(e_i, e_j) = C_{ij}$ for $i, j = 1, 2, 3$. It is degenerate, with its kernel is generated by the null root $\delta = (1, 1, 1)$. The Weyl group W of the diagram is the group generated by the simple Weyl reflections $r_i, i = 1, 2, 3$ with

$$r_i(d) = d - \frac{2(e_i, d)}{(e_i, e_i)} e_i. \tag{7.5.5}$$

In the diagrams coming from meromorphic connections, we have labels at each vertex corresponding to the eigenvalues of the formal monodromies. In our case, let $E = \Gamma \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ the vector space of labels at the three vertices. The dual Weyl reflections acting on the labels are defined by

$$s_i(\lambda) = \lambda - \lambda_i \sum_{j=1}^{3} \frac{2(e_i, e_j)}{(e_i, e_i)} e_j, \tag{7.5.6}$$

for $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in E$. Let $\cdot$ the inner product on E defined by $e_i \cdot e_j = \delta_{ij}$. The dual reflections satisfy

$$r_i(d) \cdot s_i(\lambda) = d \cdot \lambda, \quad d \in \Gamma, \lambda \in E.$$

In the Painlevé III case, the dimension vector of the diagram is the null root $(1, 1, 1)$. The fact that it lies in the kernel of the Cartan matrix has the following consequence.

Lemma 7.5.1. *Let $V_{Lie} \subset E$ the hyperplane defined by $V_{Lie} := \{(\lambda = \lambda_1, \lambda_2, \lambda_3) \in E, \lambda_1 + \lambda_2 + \lambda_3\}$. Then V_{Lie} is stable by s_i for all i.*

Proof. For any $\lambda \in E$, we have $\delta \cdot \lambda = r_i(\delta) \cdot s_i(\lambda) = \delta \cdot s_i(\lambda)$. Since $\lambda \in V_{Lie}$ means $\delta \cdot \lambda = 0$, this implies that λ is in V_{Lie} if and only if $s_i(\lambda)$ is in V_{Lie}. □

This is important for us since, as we have seen in the previous paragraph, the labels corresponding to the Lax representation of Painlevé III lie in V_{Lie}. The group to be compared to the Okamoto symmetries is thus the group generated by the restriction to V_{Lie} of the abstract Weyl reflections.

Lemma 7.5.2. *The group W_{Lie} generated by the restrictions $(s_i)_{|V_{Lie}}$ to V_{Lie} of the Weyl reflections is isomorphic to the Weyl group of type B_2.*

Proof. We use the following basis $(e_1 - e_2, e_3 - e_2)$ of V_{Lie}. The matrices of s_1, s_2, s_3 are respectively

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The group generated by these matrices is isomorphic to the Weyl group of type B_2. □

We have a similar discussion for the multiplicative Weyl reflections. Let $E^* = (\mathbb{C}^*)^3$ the multiplicative equivalent of the vector space E. We define Weyl reflections $\hat{s}_i, i = 1, 2, 3$ acting on multiplicative labels $q = (q_1, q_2, q_3) \in E^*$ by

$$(\hat{s}_i(q))_j = q_j \, q_i^{-\frac{2(e_i, e_j)}{(e_i, e_i)}}, \quad j = 1, 2, 3. \tag{7.5.7}$$

Let $T_{Lie} := \{q = (q_1, q_2, q_3) \in E^*, \; q_1 q_2 q_3 = 1\} \subset E^*$ the multiplicative equivalent of V_{Lie}. The fact that V_{Lie} is invariant under s_i implies by exponentiation that T_{Lie} is invariant under $\hat{s}_i$. The group $\widehat{W}_{Lie}$ generated by their restrictions on T_{Lie} is isomorphic to W_{Lie}.

Remark 13. We may ask whether considering the extended Weyl group, i.e. adding automorphisms of the diagram, results in a larger group of transformations. The matrix in the basis $(e_1 - e_2, e_3 - e_2)$ of V_{Lie} of the restriction to V_{Lie} of the automorphism exchanging the vertices 1 and 3 of the diagram is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We notice that this matrix belongs to W_{Lie}. This means that the extended Weyl group and nonextended Weyl group induce the same group of transformations of V_{Lie}.

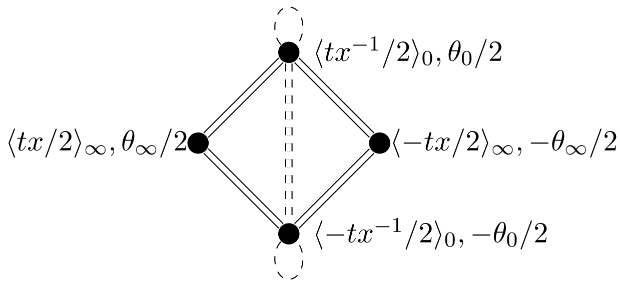
7.5.3 Derivation of the symmetries geometrically

The Jimbo-Miwa Lax representation for Painlevé III corresponds to a rank two connection on the Riemann sphere wtih two irregular singularities at 0 and ∞, of slope 1. In the parametrization of [62], the active circles are given by

- At ∞ : $\langle \frac{tx}{2} \rangle_\infty$, $\langle -\frac{tx}{2} \rangle_\infty$.
- At 0 : $\langle \frac{tx^{-1}}{2} \rangle_0$, $\langle -\frac{tx^{-1}}{2} \rangle_0$.

each where the parameter t becomes the time of the third Painlevé equation, and active circle has multiplicity 1. The data of the (multiplicative) formal monodromies at each active circle lives in a torus isomorphic to $(\mathbb{C}^*)^4$.

In the parametrization of [62], the (additive) exponents of formal monodromy associated to each active circles are the ones indicated on the figure below.



Since there are only two independent parameters only, the space of additive formal monodromies is a two-dimensional vector space V_{Lax_0} with coordinates $(\theta_0, \theta_\infty)$.

In the Betti picture, we rather consider the multiplicative formal monodromies, which are obtained from the additive ones by taking the exponential. This gives the point $(q_\infty, q_0, q_\infty^{-1}, q_0^{-1}) \in (\mathbb{C}^*)^4$, with $q_\infty := e^{i\pi\theta_\infty}$ and $q_0 := e^{i\pi\theta_0}$. The two independent formal monodromy parameters q_0, q_∞ live in a torus $T_{Lax_0} = (\mathbb{C}^*)^2$.

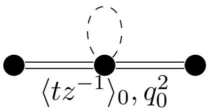
The irregular class as well as the formal monodromies change when applying basic operations on connections, i.e. combinations of twists, Fourier-Laplace and Möbius transformations. In particular, we have seen that Weyl reflections come from combinations of twists and $SL_2(\mathbb{C})$ transformations. We are interested in the action of these transformations on the Painlevé III Lax representation(s), i.e. in transformations that preserve the irregular class of the connection, but change the formal monodromies. Such transformations induce automorphisms of the formal monodromy space T_{Lax_0} which we want to determine.

We have already noticed that reflections on the 4-vertex Painlevé III diagram do not leave the dimension vector $d = (1, 1, 1, 1)$ invariant. For example the reflection with respect to the left vertex sends the dimension at this vertex to 3 (this comes from the fact that in the corresponding lecture of the graph, the connection has rank 4, and the left vertex corresponds to a regular pole whose monodromy has two eigenvalues with multiplicities 1 and 3). In order to find transformations on the standard Lax representation that preserve the dimension vector, it is necessary to pass to some auxiliary Lax representation(s) whose monodromy spaces admit automorphisms coming from geometry.

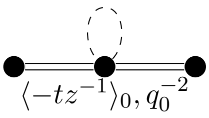
To this end, let us introduce some notations for twists: for $a \in \mathbb{P}^1$ and $q \in \mathbb{C}[x_a^{-1}]$, let T_q^a denote the operation of taking the tensor product with the trivial line bundle with connection $\nabla = d - dq$. It has the effect of adding q_a to each exponential factor at a. For $a \in \mathbb{C}$ and $\lambda \in \mathbb{C}$, let K_λ^a the operation of taking the tensor product with the trivial line bundle with regular connection $\nabla = d - \frac{\lambda}{x-a}dx$, having poles at a and ∞. It has the effect of multiplying the (multiplicative) formal monodromy of an active circle with ramification order β at a by q^β, and the monodromy of an active circle at ∞ with ramification order β by $q^{-\beta}$, where $q := e^{2i\pi\lambda}$. We may use such operations to pass to alternative Lax representations, as follows:

Lemma 7.5.3.

- *The operation* $\phi_1 := K^0_{\theta_0/2} T^0_{tz^{-1}/2}$ *transforms the irregular class and the multiplicative formal monodromies into those indicated on the figure below:*



- *The operation* $\phi_2 := K^0_{-\theta_0/2} T^0_{-tx^{-1}/2}$ *transforms the irregular class and the formal monodromies into those indicated on the figure below:*



Proof. For ϕ_1, the twist $T^0_{tz^{-1}/2}$ sends the irregular circle $\langle -tz^{-1}/2\rangle_0$ to the tame circle $\langle 0\rangle_0$, then the twist $K^0_{\theta_0/2}$ makes its formal monodromy trivial, so that the corresponding vertex disappears. This is similar for ϕ_2. □

Let T_{Lax_1} and T_{Lax_2} denote the spaces of (multiplicative) formal monodromies of these Lax representations. Both are readily identified via the diagram to the torus T_{Lie} of the previous paragraph, so that ϕ_1 and ϕ_2 induce two maps:

$$\hat{\phi}_1 : \begin{array}{ll} T_{Lax_0} & \to T_{Lax_1} \\ (q_0, q_\infty) & \mapsto (q_\infty q_0^{-1}, q_0^2, q_0^{-1} q_\infty^{-1}) \end{array}, \qquad \hat{\phi}_2 : \begin{array}{ll} T_{Lax_0} & \to T_{Lax_2} \\ (q_0, q_\infty) & \mapsto (q_\infty q_0, q_0^{-2}, q_0 q_\infty^{-1}) \end{array}.$$

In these Lax representations, the Weyl reflections with respect to the left and right vertices do not change the dimension vector, and the dual Weyl reflections transform the (multiplicative) labels. It follows from section 4.1 that the automorphisms $s_1^{Lax_i}, s_3^{Lax_i}$, $i = 1, 2$ of T_{Lax_1} and T_{Lax_2} corresponding to the simple abstract Weyl reflections s_1 and s_3 have a modular interpretation in terms of geometric operations on the Lax representation.

Lemma 7.5.4. *The automorphism* $\phi^{Lax_1} : (q_1, q_2, q_3) \mapsto (q_3, q_2, q_1)$ *of* T_{Lax_1} *comes from geometric transformations.*

Proof. We apply the Fourier transform to change the representation of the diagram: the circles $\langle tz/2\rangle_\infty$ and $\langle tz/2\rangle_\infty$ respectively become the tame circles $\langle 0\rangle_{-t/2}$ and $\langle 0\rangle_{t/2}$. Now, applying a Möbius transformation μ exchanging the points $t/2$ and $-t/2$ in $\mathbb{P}^1$ amounts to exchange the formal monodromies of these two circles. Taking the inverse Fourier transform to change the representation back to the original one gives the result. □

Let us denote by $\widehat{W}_{Lax_i} \subset \mathrm{Aut}(T_{Lax_i})$ the group generated by $s_1^{Lax_i}, s_3^{Lax_i}, \phi^{Lax_i}$. One has $\widehat{W}_{Lax_i} \cong D_8$.

We may now go back to the original Lax pair to obtain automorphisms of T_{Lax_0}.

Lemma 7.5.5. *The following automorphisms of* T_{Lax_0} *come from geometric operations:*

- $\tau_1^{Lax_0} : (q_0, q_\infty) \mapsto (q_\infty, q_0)$,
- $\tau_3^{Lax_0} : (q_0, q_\infty) \mapsto (q_\infty^{-1}, q_0^{-1})$,
- $\tau^{Lax_0} : (q_0, q_\infty) \mapsto (q_0, q_\infty^{-1})$,
- $\sigma^{Lax_0} : (q_0, q_\infty) \mapsto (-q_0, -q_\infty)$.

Proof. The automorphism $\tau_1^{Lax_0}$ my be obtained by applying ϕ_1 to pass to the corresponding Lax pair, applying the geometric transformation corresponding to $s_1^{Lax_1}$, then going back to the original Lax pair using ϕ_1^{-1}. For $\tau_3^{Lax_0}$ and τ^{Lax_0} the process is similar, only replacing $s_1^{Lax_1}$ by $s_3^{Lax_1}$ and ϕ^{Lax_1}. Finally, σ^{Lax_0} is obtained by taking the twist $K^0_{1/2}$, whose effect is to multiply by -1 the multiplicative formal monodromies at 0 and ∞. □

We have thus obtained some automorphisms of the two dimensional torus T_{Lax_0} of formal monodromies parameters of the usual Painlevé III Lax representation from geometric transformations. Let us denote by $\widehat{W}_{Lax_0} := \langle \tau_1^{Lax_0}, \tau_3^{Lax_0}, \tau^{Lax_0}, \sigma^{Lax_0} \rangle$ the group they generate.

7.5.4 Links between the different pictures

The three settings we have described each feature two-dimensional vector spaces (in the additive picture) and/or tori (in the multiplicative picture) and automorphisms of these spaces: for Okamoto transformations, the vector space V_{Ok} of Painlevé III parameters, for the Dynkin diagram the vector space V_{Lie} and the corresponding torus T_{Lie}, for the Lax representations the residue space V_{Lax_0} and the tori T_{Lax_i} of eigenvalues of (multiplicative) formal monodromy. We now discuss the dictionary relating these spaces and compare the automorphism groups on either side.

The dictionary between the Lax representation and the Painlevé parameters is given by the isomonodromic deformation equations. From [62], the isomonodromy equations for the Lax representation give rise to the Painlevé III equation with parameters

$$\alpha = 4\theta_0, \quad \beta = 4(1-\theta_\infty), \quad \gamma = 4, \quad \delta = -4. \tag{7.5.8}$$

This implies we have an isomorphism

$$\begin{aligned} \Psi : V_{Ok} &\to V_{Lax} \\ (\alpha_1, \beta_1) &\mapsto (\theta_0, \theta_\infty) = (\alpha_1 - \beta_1, \alpha_1 + \beta_1) \end{aligned}$$

between the space of Okamoto parameters V_{Ok} and the space of eigenvalues of the Painlevé III Lax representation V_{Lax_0} compatible with passing to the associated Painlevé equation. Passing to the multiplicative monodromies yields a map

$$\begin{aligned} \hat{\Psi} : V_{Ok} &\to T_{Lax_0} \\ (\alpha_1, \beta_1) &\mapsto (q_0, q_\infty) = (e^{i\pi(\alpha_1 - \beta_1)}, (e^{i\pi(\alpha_1+\beta_1)}). \end{aligned}$$

We have the induced maps on the automorphism groups $\hat{\Psi}_* : \mathrm{Aut}(V_{Ok}) \to \mathrm{Aut}(V_{Lie})$ and $\hat{\phi}_{1*}, \hat{\phi}_{2*} : \mathrm{Aut}(T_{Lax_0}) \to \mathrm{Aut}(T_{Lax_i})$. This enables us to compare the symmetry groups on either side.

Proposition 7.5.6. *The image $\hat{\Psi}_*(W_{Ok})$ of the group of Okamoto symmetries under $\hat{\Psi}_*$ is equal to $\widehat{W}_{Lax_0}$.*

Proof. Let us compute the images by $\hat{\Psi}_*$ of the automorphisms $s_1^{Ok}, s_2^{Ok}, \sigma_0^{Ok}$ which generate W_{Ok} . We have

$$\begin{aligned} \widehat{\Psi}_*(s_1^{Ok}) &= \tau_1^{Lax_0} \\ \widehat{\Psi}_*(s_2^{Ok}) &= \tau^{Lax_0} \\ \widehat{\Psi}_*(\sigma_1) &= \sigma^{Lax_0} \circ \tau_3^{Lax_0}, \end{aligned}$$

and these images generate $\widehat{W}_{Lax_0}$. □

Remark 14. One given element in $\widehat{W}_{Lax_0}$ has an infinite number of preimages by $\widehat{\Psi}_*$ differing from one another by translations. These translations correspond to Schlesinger transformations (see e.g. [60]), shifting the additive residues by integers.

Proposition 7.5.7. *$\widehat{W}_{Lax_i}$ is the image of $\widehat{W}_{Lax_0}$ by $\hat{\phi}_{i*}$, for $i = 1, 2$.*

Proof. We compute the images by $\hat{\phi}_{1*}$ of $\tau_1, \tau_3, \tau \in G$. We find

$$\hat{\phi}_{1*}(\tau_1^{Lax_0}) = \hat{s}_1^{Lax_1} \quad \hat{\phi}_{1*}(\tau_3^{Lax_0}) = \hat{s}_3^{Lax_1} \quad \hat{\phi}_{1*}(\tau^{Lax_0}) = \phi^{Lax_1},$$

This implies the result since $\widehat{W}_{Lie} = \langle \hat{s}_1^{Lax_1}, \hat{s}_3^{Lax_1}, \phi^{Lax_1} \rangle$. Things are similar for $\hat{\phi}_{2*}$. ◻

The situation is shown on the diagram below:

$$\begin{tikzcd}
 & W_{Ok} \arrow[d, "\widehat{\Psi}_*"] & \\
 & \widehat{W}_{Lax_0} \arrow[dl, "\hat{\phi}_{1*}"'] \arrow[dr, "\hat{\phi}_{2*}"] & \\
\widehat{W}_{Lax_1} \arrow[dr, "\cong"] & & \widehat{W}_{Lax_2} \arrow[dl, "\cong"] \\
 & \widehat{W}_{Lie} &
\end{tikzcd}$$

Remark 15. Each element of $\widehat{W}_{Lax_1}$ has exactly two preimages by $\hat{\phi}_{1*}$. This comes from the fact that $\hat{\phi}_{1*}(\sigma^{Lax_0})$ is the identity. Indeed, passing from the standard representation to the representation corresponding to the 3-vertex diagram involves taking the quotient of the formal monodromies at the two active circles at 0, and this quotient doesn't change when applying the twist giving rise to σ^{Lax_0}.

To summarize, we have shown that the Okamoto symmetries of the Painlevé III parameters, as well as the abstract Weyl group symmetries defined from the Painlevé III diagram (when restricted to the subspace T_{Lie}) are modular, i.e. correspond via the maps $\hat{\Psi}$, $\hat{\phi}_i$, $i = 1, 2$, from operations on some Painlevé III Lax representation. Notice that to get a modular interpretation of all Okamoto symmetries, it is necessary to pass between different Lax representations, as e.g. for the case of Painlevé V [15].

Remark 16. Notice that $\hat{\phi}_2^{-1} \circ \hat{\phi}_1 = (\hat{s}_2)_{|T_{Lie}}$, that is the reflection with respect to the central vertex of the affine D_2 diagram coïncides with the effect on labels of passing from one of the two possible 3-vertex diagrams to the other. This gives a modular interpretation for the abstract Weyl reflection (restricted to V_{Lie}) with respect to the vertex with a negative loop. It seems however that this observation does not extend to more general cases in a straightforward way. More work is needed to understand whether there is a suitable way to define from the diagram(s), in the spirit of [22], some kind of global Weyl group having a modular interpretation in terms of operations on connections.

Chapter 8

Towards classification for simple examples

We have seen that operations on connections such as $SL_2(\mathbb{C})$ transformations, twisting with a rank one connection or Möbius transformations play an essential role the construction of the diagrams, passing from one Lax representation to another, and interpreting the symmetries of Painlevé equations. In this chapter, we explore more systematically how exponential factors and formal data transform under the group generated by these transformations. In a spirit similar to the philosophy of Katz' middle convolution algorithm [64] for rigid regular connections, and its extension by Arinkin [6] to rigid irregular connections, we would like to be able to describe the formal data of the connections in an orbit of the group generated by these operations. One may hope there is in each orbit a minimal diagram in some sense, so that the orbits would be classified by these minimal diagrams, together with some extra data.

We take here a few modest first steps towards this goal: thanks to our understanding of the action of the Legendre transform on exponential factors, we give a characterization of the exponential factors which can be brought to a tame circle by iterated application of these operations. For the case of diagram with only one vertex (with loops), we arrive at a characterization of all exponential factors giving rise to such a diagram when the number of loops is 0, 1 or 2. In this case, we find that all exponential factors for a given number of loops are related by successive applications of twists and the Fourier transform.

8.1 Formal data and basic operations

8.1.1 Orbits of formal data

Let $\breve{\Theta}$ the modified irregular class of a connection (E, ∇) on a Zariski open subset of $\mathbb{P}^1$. We would like to answer the following questions: what are all the irregular classes possibly obtained by iterated application of elementary operations? Arinkin's generalized Katz' algorithm answers the case of rigid irregular connections:

Theorem 8.1.1 ([39, 6])**.** *If (E, ∇) is a rigid irregular connection on a Zariski open subset of $\mathbb{P}^1$, it can be brought to the trivial connection by successive application of basic operations.*

The proof consists in showing that for a rigid irregular connection of rank > 1 it is always possible to find a basic operation that lowers the rank.

An easy general observation to make is that the number of active circles in any orbit is unbounded:

Lemma 8.1.2. *In each orbit, there are modified formal irregular classes with an arbitrarily large number of active circles.*

Proof. Let (E, ∇) a connection with modified irregular class $\breve{\Theta}$. It is enough to show that there exists a basic operation increasing the number of active circles. The reason for this is that since we do not take the reduced monodromy for the tame circle at infinity, it is an active circle even when the connection has no singularity at infinity. Up to applying a twist at infinity, we may assume that the tame circle at infinity $\langle 0\rangle_\infty$ is not an active circle of $\breve{\Theta}$. Let then μ be a Möbius transformation such that $\mu^{-1}(\infty)$ is not a singularity of (E, ∇). Then $\mu \cdot \breve{\Theta}$ has one more active circle than $\breve{\Theta}$: indeed all active circles of $\breve{\Theta}$ are sent by μ to singularities at finite distance (no circle disappears since $\langle 0\rangle_\infty$ isn't active), and the tame circle at infinity gives an extra active circle for $\mu \cdot \breve{\Theta}$. The result follows by induction. □

It is thus straightforward to "complexify" a given irregular class. The more interesting question is the following: is there a systematic way, in the non-rigid case, to simplify an irregular class to obtain one with minimal number of active circles?

8.1.2 Levels of an exponential factor

We introduce the notion of *levels* of an exponential factor (see [24]).

Definition 8.1.3. The levels of $\langle q\rangle$ are the slopes of the active circles of $\mathrm{Hom}(\langle q\rangle, \langle q\rangle)$.

We denote by $\mathrm{Levels}(q) \subset \mathbb{Q}$ the set of levels of $\langle q\rangle$. Let $\langle q\rangle_a$ an exponential factor given by

$$q = \sum_{i=0}^{k} b_i z_a^{\alpha_i/\beta}$$

with $\beta = \mathrm{ram}(q)$ and $b_i \neq 0$ for $i = 1, \dots, k$. The formula for $B_{\langle q\rangle,\langle q\rangle}$ shows that not all degrees in the set of degrees of $\{\frac{\alpha_0}{\beta}, \dots, \frac{\alpha_k}{\beta}\} \subset \mathbb{Q}$ of the terms of $\langle q\rangle$ contribute to Stokes arrows: the term of slope $\frac{\alpha_i}{\beta}$ in q, with $i = 1, \dots, k$, contributes to $B_{\langle q\rangle,\langle q\rangle}$ only if

$$p_i < p_{i-1},$$

where $p_i = (\alpha_0, \dots, \alpha_i, \beta)$. This implies the following result:

Lemma 8.1.4. *the set of levels Levels(q) of q is the largest subset of $\{\frac{\alpha_0}{\beta}, \dots, \frac{\alpha_k}{\beta}\}$ such that the corresponding sequence of greatest common divisors is strictly decreasing.*

We will say that an exponential factor $\langle q\rangle$ is reduced if the sequence $(p_0, \dots, p_k)$ is strictly decreasing. In this case its levels are just the degrees of its terms.

A consequence of this is that the diagram associated to a meromorphic connection (E, ∇) with only one active circle $\langle q\rangle$, as well as the quasi-Hamiltonian structure of the wild character variety $\mathcal{M}_B(E, \nabla)$, only depend on the levels of the exponential factor.

Example 8.1.5. Let us look at a few simple examples.

- If $(\alpha_0, \beta) = 1$, q is reduced if and only if it has only one term. Its set of levels is $\{\frac{\alpha_0}{\beta}\}$.
- Consider $q = z^{5/3} + z^{4/3} + z^{1/2}$. We have $\mathrm{ram}(q) = 6$, and the sequence of g.c.d.s is $(2, 2, 1)$ so q is not reduced. Its set of levels is $\{\frac{5}{3}, \frac{1}{2}\}$.

Using this notion of levels, we can interpret the formula for the form of the Legendre transform in the following way:

Lemma 8.1.6. *Let $\langle q\rangle_\infty$ an exponential factor at infinity of slope > 1, and $\langle \tilde{q}\rangle_\infty$ its Legendre transform. There is a bijection between the sets of levels of q and $\tilde{q}$. If $\beta = \mathrm{ram}(q)$ and $\mathrm{Levels}(q) = \{\frac{\alpha_0}{\beta}, \dots, \frac{\alpha_l}{\beta}\}$, then*

$$\mathrm{Levels}(\tilde{q}) = \left\{\frac{\alpha_0}{\alpha+\beta}, \dots, \frac{\alpha_l}{\alpha+\beta}\right\}.$$

8.1.3 Action of basic operations on circles

To make an active circle disappear from the diagram, the only way is to transform it into a tame circle at finite distance, with trivial formal monodromy. We have already encountered several examples where some circles with ramification can be transformed into circles without ramification by applying some suitable operation. For example the circle $\langle z^{n/(n-1)}\rangle_\infty$ for any integer $n \geq 1$ is sent by Fourier transform to an unramified circle $\langle \lambda z^n\rangle$, with $\lambda \neq 0$. It is therefore natural to ask whether any circle can be sent by iterated application of basic operations to a tame circle (this is akin to some kind of "transitivity" property). We will show that the answer is no.

Theorem 8.1.7. *If α, β are two coprime integers such that $\beta \not\equiv \pm 1 \mod \alpha$, then the circle $\langle z^{\alpha/\beta}\rangle_\infty$ cannot be sent to an unramified circle by iterated application of basic operations.*

Lemma 8.1.8. *The circle $\langle z^{\alpha/\beta}\rangle_\infty$, with $\alpha, \beta \in \mathbb{N}$ coprime can be sent to an unramified circle by a combination of Fourier transform and Möbius transformations if and only if $\beta \equiv \pm 1 \mod \alpha$.*

Proof. The images of $\langle z^{\alpha/\beta}\rangle_\infty$ by combinations of Fourier-Laplace transform and Möbius transformations are of the form $\langle \lambda' z^{\alpha'/\beta'}\rangle_a$, with $\lambda \in \mathbb{C}^*$, $\alpha', \beta' \in \mathbb{N}$ and $a \in \mathbb{P}^1$. To determine whether we can obtain an unramified circle, it suffices to track the transformation of the couple (α, β). Möbius transformations only act on the location of the singularity and do not change (α, β). On the other hand, the transformation of (α, β) under Fourier transform is given by the stationary phase formula: there are three cases

$$\begin{aligned}
(\alpha, \beta) &\mapsto (\alpha, \alpha+\beta) \quad \text{when } a \neq \infty, \\
(\alpha, \beta) &\mapsto (\alpha, \alpha-\beta) \quad \text{when } a = \infty \text{ and } a/b < 1, \\
(\alpha, \beta) &\mapsto (\alpha, \beta-\alpha) \quad \text{when } a \neq \infty \text{ and } a/b > 1.
\end{aligned}$$

It follows that the possible couples (α', β') for images of $\langle z^{\alpha/\beta}\rangle_\infty$ by compositions of Fourier transform and Möbius transformations are the

$$(\alpha, k\alpha \pm \beta)$$

with $k \in \mathbb{Z}$ such that $k\alpha \pm \beta > 0$. This corresponds to an unramified circle if $k\alpha \pm \beta = 1$, and the conclusion follows. □

Proof. The previous lemma shows that the Fourier transform and Möbius transformations do not allow to "deramify" the circle $\langle z^{\alpha/\beta}\rangle_\infty$, but this doesn't include the case of twists. It might be a priori possible that, by applying some preliminary twist so that the term $z^{\alpha/\beta}$ becomes a subleading term, then applying Fourier transform and Möbius transformations, and repeating the process, we could send $\langle z^{\alpha/\beta}\rangle_\infty$ to the tame circle $\langle 0\rangle_\infty$. We have to show that this cannot actually happen.
Let us first look at the case where we apply one twist increasing the slope. Let q_0 an exponential factor with ramification order β_0 and ramified leading term $\lambda_0 z^{\alpha_0/\beta_0}$, that is with $\alpha_0 \in \mathbb{N}$ not multiple if β_0. Applying a twist by a (necessarily unramified) exponential factor

$$\lambda_k z^{n_k} + \cdots + \lambda_1 z^{n_1} + \text{ terms of order} < \frac{\alpha_0}{\beta_0}$$

with $n_k > \cdots > n_1 > \frac{\alpha_0}{\beta_0}$, of order $n := n_k$, we get

$$q_0' = \lambda_k z^{\frac{n_k\beta_0}{\beta_0}} + \cdots + \lambda_1 z^{\frac{n_1\beta_0}{\beta_0}} + \lambda_0 z^{\frac{\alpha_0}{\beta_0}} + \ldots$$

Now, we apply some combination of Fourier transform and Möbius transformations. It follows from the description of the orders of the terms appearing in the Legendre transform that the exponential factor q_1 that we obtain has the form

$$q_1 = \lambda_k' z^{\frac{n_k\beta_0}{pn\beta_0 \pm \beta_0}} + \cdots + \lambda_1' z^{\frac{n_1\beta_0}{pn\beta_0 \pm \beta_0}} + \lambda_0' z^{\frac{\alpha_0}{pn\beta_0 \pm \beta_0}} + \ldots,$$

where $p \in \mathbb{Z}$, $pn\beta_0 \pm \beta_0 > 0$, $n = n_k > \cdots > n_1 > \frac{\alpha_0}{\beta_0}$. At this stage we make two crucial observations:

- The term in $z^{\alpha_0/(pn\beta_0 \pm \beta_0)}$ has not disappeared, that is we have $\lambda_0' \neq 0$. Indeed, from the formula for the Legendre transforms the terms $z^{n_k}, \ldots, z^{n_1}$ give contributions to terms of order $(n\beta_0 - m_{k-1}(n - n_{k-1})\beta_0 - \cdots - m_1(n - n_1)\beta_0)/(pn\beta_0 \pm \beta_0)$, with $m_1, \ldots m_{k-1} \in N$. But $\alpha_0/(pn\beta_0 \pm \beta_0)$ isn't one of those terms since α_0 is not a multiple of β_0. As a consequence, the coefficient λ_0' comes entirely from the first subleading correction coming from the term $\lambda_0 z^{\alpha_0/\beta_0}$, and this contribution cannot be cancelled by higher order corrections coming from $\lambda_{k-1} z^{n_{k-1}} + \ldots \lambda_1 z^{n_1}$.

- Assuming we do not perform further twists which increase the slope of the exponential factor, the only way to go back from q_1 to a situation where the term $z^{\alpha_0/*}$ coming from z^{α_0/β_0} is the leading term is to apply a combination of Möbius transformations and Fourier transform which undoes the transformation $(n\beta_0, \beta_0) \mapsto (n\beta_0, pn\beta_0 \pm \beta_0)$. Indeed, when applying such an operation to q_2, the possible ramification orders are the $qn\beta_0 \pm \beta_0 > 0$, with $q \in \mathbb{Z}$, and among those numbers β_0 is the only one that divides $n\beta_0$ and brings us back to an unramified leading term. Then, performing a twist, we are back to an exponential factor with leading term z^{α_0/β_0}.

We have just shown that a sequence of basic operations featuring one slope increasing twist can neither allow to make the term z^{α_0/β_0} disappear, nor transform it into $z^{\alpha_0/\beta'}$ with $\beta' \neq \beta_0$. By induction on the number of twists, it follows from this that no sequence of basic operations (featuring an arbitrary number of twists) allows to make the term z^{α_0/β_0} disappear, or transform it into into $z^{\alpha_0/\beta'}$ with $\beta' \neq \beta_0$. The induction works by applying the same reasoning to q_1, etc. This concludes the proof. $\square$

Example 8.1.9. The circles $\langle z^{5/3} \rangle_\infty$, $\langle z^{7/2} \rangle_\infty$ cannot be transformed into unramified circles.

Thanks to Arinkin's extension [6] of Katz' algorithm to the irregular case, the theorem implies the following corollary.

Corollary 8.1.10. *Let (E, ∇) a rigid irregular connection on $\mathbb{P}^1$. Then its modified irregular class cannot admit any active circle of the form $\langle \lambda z^{\alpha/\beta} \rangle$ with α, β coprime and $\beta \not\equiv \pm 1 \mod \alpha$.*

Proof. If this was the case, since (E, ∇) is rigid by the irregular version of Katz' algorithm, there would exist a sequence of basic operations sending (E, ∇) to the trivial connection of rank one. In particular, it would send $\langle \lambda z^{\alpha/\beta} \rangle$ to a tame circle. The theorem implies that this is impossible. $\square$

Actually, we can characterize in a more explicit way the circles which can be brought to the tame circle in this way.

Definition 8.1.11. Let $\langle q \rangle \in \pi_0(\mathcal{I})$ an exponential factor. We say $\langle q \rangle$ is *tamable* is there exists a sequence of basic operations bringing $\langle q \rangle$ to a tame circle.

Definition 8.1.12. Let $\langle q \rangle \in \pi_0(\mathcal{I})$ an exponential factor. Let $\beta := \mathrm{ram}(q)$ and $\alpha := \mathrm{Irr}(q)$, so that $\mathrm{slope}(q) = \frac{\alpha}{\beta}$. The taming algorithm consists in repeatedly applying the following steps as long as possible:

- Let r the remainder in the euclidean division of β by α. If r or $\alpha - r$ is a divisor δ of α, apply a combination of Fourier-Laplace transforms and Möbius transformations acting on the irregularity and ramification as $(\alpha, \beta) \mapsto (\alpha, \delta)$.

- If α/β is an integer, apply a twist to cancel all terms of integer degree in q.

The algorithm terminates when neither action is possible.

Theorem 8.1.13. *An exponential factor $\langle q\rangle$ is tamable if and only if the taming algorithm applied to $\langle q\rangle$ terminates at a tame circle.*

Proof. The proof is almost exactly the same as for the case $\langle z^{\alpha/\beta}\rangle$ studied in the previous paragraph. If the taming algorithm terminates at a tame circle, then obviously $\langle q\rangle$ is tamable. Otherwise, the algorithm terminates at some circle $\langle q'\rangle$ with ramification order β' and irregularity α' such that, denoting by r' the remainder in the euclidean division of β' by α', neither r' nor $\alpha - r'$ divides α'. Then, an argument similar as in the proof of the previous theorem shows that no natural operation is able to make the term of level α'/β' disappear, and the conclusion follows. $\square$

Example 8.1.14. Let $\langle q\rangle$ an exponential factor with set of degrees $\{\frac{3}{5}, \frac{2}{21}, \frac{2}{105}\} = \{\frac{63}{105}, \frac{14}{105}, \frac{2}{105}\}$. Running the algorithm gives

$$\begin{aligned}\{63/105, 14/105, 2/105\} &\to \{63/21, 14/21, 2/21\}\\ &= \{3, 14/21, 2/21\}\\ &\to \{14/21, 2/21\}\\ &\to \{14/7, 2/7\}\\ &\to \{2/7\}\\ &\to \{2/1\}\\ &\to \{0\},\end{aligned}$$

which shows $\langle q\rangle$ is tamable.

8.2 One-vertex diagrams with small number of loops

We would like to explore to which extent the diagrams classify the formal data. A natural question is: given a diagram, what are all the formal data giving rise to this diagram, or that can be transformed into this diagram by iterated application of basic operations on connections. In this paragraph, we look at this in the simplest case, the case of (core) diagrams with only one vertex with loops. The question we ask is

Question Let $k \in \mathbb{Z}$. What are the exponential factors q at infinity such that the core diagram associated to a connections with only active circle $\langle q\rangle_\infty$ is a vertex with k loops? Can all such factors can be reduced to one or several "minimal" exponential factors q?

8.2.1 Simplification algorithm

Among all basic operations, we have to restrict to the only two types of operations such that $\langle q\rangle$ remains at infinity, so that at each step the connection we only have one active circle.

- Fourier transform, if q has slope $\alpha/\beta > 1$. It acts on slopes as $\frac{\alpha}{\beta} \mapsto \frac{\alpha}{\alpha-\beta}$.
- Twists.

Twists can be used to delete terms of q with integer degree. On the other hand, if q has slope $\frac{n+1}{n}$ for some $n \geq 1$, then applying the Fourier transform, we get an exponential factor with integer slope $n+1$, whose leading term can be deleted by a twist. This leads to the following algorithm for simplification of an exponential factor at infinity.

Definition 8.2.1. The simplification algorithm consists in applying the following operations as long as it is possible.

1. If q has slope $\frac{n+1}{n}$ for some integer $n \geq 1$, then apply the Fourier transform.
2. Otherwise apply a twist cancelling all terms of q having integer slope if there are any.

The algorithm terminates when we reach a *non-simplifiable* exponential factor.

Remark 17. When the algorithm terminates, up to applying the Fourier-Laplace transform we may assume that the exponential factor we arrive at has slope less than 2 (indeed if it has slope > 2, from the stationary phase formula, applying the Fourier-Laplace transform gives an exponential factor of slope < 2).

Example 8.2.2. Consider an exponential factor q with levels $\{4/3, 4/9, 1/8\}$. Its number of loops is $k = 0$. The algorithm applied to q gives

$$\begin{aligned}\{4/3, 4/9, 1/8\} &= \{96/72, 32/72, 9/72\}\\ &\to \{96/24, 32/24, 9/24\}\\ &= \{4, 4/3, 3/8\}\\ &\to \{4/3, 3/8\}\\ &\to \{32/8, 9/8\}\\ &\to \{9/8\}\\ &\to \{9\}\\ &\to \{0\}\end{aligned}$$

The algorithm terminates at $q = 0$. As a second example, consider an exponential factor q with levels $\{3/2, 4/7, 1/4\}$, with $k = 2$ loops. The algorithm gives

$$\begin{aligned}\{3/2, 4/7, 1/4\} &= \{42/28, 16/28, 7/28\}\\ &\to \{42/14, 16/14, 7/14\}\\ &= \{3, 8/7, 1/2\}\\ &\to \{8/7, 1/2\}\\ &\to \{8, 7/2\}\\ &\to \{7/2\}\\ &\to \{7/5\}\end{aligned}$$

8.2.2 Non-simplifiable exponential factors for small numbers of loops

From computing (with a computer) the number of slopes for exponential factors with all possible levels, for small number of levels and bounded ramification orders, it seems that we have the following conjecture.

Conjecture 8.2.3. The list of levels of non-simplifiable exponential factors of slope < 2 for small number of loops is the following.

number of loops	levels of exponential factors
0	0
1	$z^{5/3}$
2	$z^{7/5}$
3	$z^{9/7}, z^{7/4}, z^{5/3} + z^{1/6}$
4	$z^{11/9}, z^{8/5}, z^{5/3} + z^{1/2}$

Notice that for $k = 0, 1, 2$, there is only one non-simplifiable exponential factor, so that the conjecture means:

Theorem 8.2.4. *Let q an exponential factor at infinity and k the corresponding number of loops.*

- *$k = 0$ if and only if the algorithm applied to q terminates at the set of levels $\{0\}$.*
- *$k = 1$ if and only if the algorithm applied to q terminates at the set of levels $\{5/3\}$.*
- *$k = 2$ if and only if the algorithm applied to q terminates at the set of levels $\{7/5\}$.*

However, for $k \geq 3$ there are now several non-simplifiable exponential factors, some having several levels.

8.2.3 Proof of the conjecture for small numbers of loops

We now prove that the conjecture is true for $n = 0, 1, 2$.

Proof. For any exponential factor q, the algorithm applied to q terminates at a non-simplifiable exponential factor. It is therefore enough to show that there are no reduced levels of non-simplifiable exponential factors having a number of loops equal to $0, 1, 2$ other than the ones listed. Let q a non-simplifiable exponential factor with ramification order β, slope $\frac{\alpha}{\beta} < 1$, and reduced shape $\{\frac{\alpha}{\beta}, \dots, \frac{\alpha_k}{\beta}\}$.

We first show that if q has slope < 1 it has a negative number of loops.

$$\begin{aligned}
B_{\langle q \rangle, \langle q \rangle} &= (\beta - (\alpha, \beta))\alpha_0 + \cdots + ((\alpha_0, \dots, \alpha_{k-1}, \beta) - (\alpha_0, \dots, \alpha_k, \beta))\alpha_k - \beta^2 + 1 \\
&\leq (\beta - (\alpha, \beta))\alpha + \cdots + ((\alpha_0, \dots, \alpha_{k-1}, \beta) - (\alpha_0, \dots, \alpha_k, \beta))\alpha - \beta^2 + 1 \\
&\leq \alpha\beta - \beta^2 + 1 \\
&\leq -\beta + 1 \\
&\leq -1,
\end{aligned}$$

so the number of loops is negative.

Let us assume now that q has slope $1 < \frac{\alpha}{\beta} < 2$. If its reduced shape only has one level, then one has

$$B_{\langle q \rangle, \langle q \rangle} = (\beta - 1)(\alpha - \beta - 1).$$

Since q is non-simplifiable, we have $\alpha \geq \beta + 2$, so $B_{\langle q \rangle, \langle q \rangle} \geq \beta - 1$. Since $n = \frac{B_{\langle q \rangle, \langle q \rangle}}{2}$, $n = 0, 1, 2$ is possible only if $\beta \leq 5$, and the conclusion follows. Finally, if the number of levels of the reduced shape is $k \geq 2$, we have

$$\begin{aligned}
B_{\langle q \rangle, \langle q \rangle} &= (\beta - (\alpha, \beta))\alpha + \cdots + (\alpha_0, \dots, \alpha_k, \beta)\alpha_k - \beta^2 + 1 \\
&\geq (\beta - (\alpha, \beta))\alpha - \beta^2 + 1. \\
&= \beta(\alpha - \beta) - (\alpha, \beta)\alpha + 1
\end{aligned}$$

Let $d := (\alpha, \beta)$, and $\alpha = ad$, $\beta = bd$ so that $\frac{\alpha}{\beta} = \frac{a}{d}$ with $(a, b) = 1$. Since q is non-simplifiable, we have $a \geq b + 1$, so $\alpha \geq \beta + 2(\alpha, \beta)$. It follows that

$$\begin{aligned}
B_{\langle q \rangle, \langle q \rangle} &\geq 2\beta(\alpha, \beta) - \alpha(\alpha, \beta) + 1. \\
&= (2\beta - \alpha)(\alpha, \beta) + 1.
\end{aligned}$$

Now, $2\beta - \alpha > 0$ since $\frac{\alpha}{\beta} < 2$, and it is a multiple of (α, β), so $2\beta - \alpha \geq (\alpha, \beta)$. This implies

$$B_{\langle q \rangle, \langle q \rangle} \geq (\alpha, \beta)^2 + 1.$$

Since the reduced shape of q has two levels, α and β cannot be coprime, so $(\alpha, \beta) \geq 2$, and in turn $B_{\langle q \rangle, \langle q \rangle} \geq 5$ so that $n > 2$. This concludes the proof. $\square$

For small number of loops, the diagram thus classifies the exponential factors at infinity up to iterated application of basic operations.

Remark 18. For all nondegenerate Painlevé equations, the simplest diagram of a corresponding Lax pairs has one vertex more than the number of independent parameters of the equation [26]. The degenerate Painlevé equation has one parameter, and we have seen that it admits a Lax pair giving rise to a two vertex. However, the case $k = 1$ of the theorem implies that this does not hold for the doubly degenerate Painlevé equation. Indeed it has no parameter, so we might hope to get a diagram with only one vertex. The dimension of the moduli space being equal to two, this would be necessarily the diagram with one vertex and one loop. But because of the theorem, the corresponding Lax pair would be related by successive operations to the standard Lax pair of Painlevé I.

Remark 19. It is natural to expect that the Fourier-Laplace transform induces symplectic isomorphisms between wild character varieties. In some cases, there are some known isomorphisms compatible with the full hyperkähler structure [97]. One expects that such isomorphisms can be obtained by determining how the Stokes representations transform under Fourier-Laplace transform. In the complete bipartite case, the transformation of Stokes data under Fourier-Laplace transform is well understood (see [70, ch. 12]). The question has been adressed by several authors using different frameworks in the general case [70, 74, 75, 37], or in simple examples [92, 54, 37], but it seems very difficult to obtain an explicit enough answer apart from some simple cases. One idea would be to extend the approach of [22] which set up a new theory of multiplicative quiver varieties, such that the (simply laced) wild character varieties on either side of the Fourier-Laplace transform arise simply by reordering the same multiplicative symplectic quotient (analogously to [20] on the additive side).

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ÉCOLE DOCTORALE
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Hadamard (EDMH)

Titre: Vers la classification des variétés de caractères sauvages

Mots clés: Connexions méromorphes, variétés de caractères sauvages, singularités irrégulières, phénomène de Stokes, espaces de modules

Résumé: Cette thèse est motivée par la question de la classification des variétés de caractères sauvages, qui sont les espaces de modules de connexions irrégulières sur les courbes. Ces variétés dépendent du choix de données de singularité caractérisant la forme des singularités des connexions, et il arrive souvent que des données de singularité différentes, correspondant à des connexions de rangs différents, avec des nombres de singularités différents, donnent lieu à des espaces de modules isomorphes. Nous définissons un diagramme associé à une connexion algébrique quelconque sur un ouvert de Zariski de la droite affine, généralisant des constructions précédentes reliant les variétés de caractères sauvages aux carquois au cas où il y a plusieurs singularités irrégulières, possiblement ramifiées. L'idée de la construction est d'utiliser la transformation de Fourier-Laplace pour se ramener à la situation de Boalch-Yamakawa, où il y a seulement une singularité irrégulière. Le diagramme est invariant sous l'action des automorphismes symplectiques de l'algèbre de Weyl, de telle sorte qu'il y a plusieurs connexions, avec des données de singularité différentes, correspondant au même diagramme. D'autres propriétés des cas précédents restent vraies dans notre cadre plus général : ainsi la dimension de la variété de caractères sauvages est donnée par une formule faisant intervenir la matrice de Cartan du diagramme, et on obtient des réflexions de Weyl simples par rapport à certains sommets du diagramme en appliquant certaines opérations sur les connexions. Comme application de cette construction, nous pouvons voir beaucoup de représentations de Lax connues pour les équations de Painlevé, ainsi que pour des analogues en dimension supérieure, comme des représentations différentes du même diagramme. Nous classifions aussi les cas où le diagramme a un seul sommet, et moins de 2 boucles.

Title: Towards the classification of wild character varieties

Keywords: meromorphic connections, wild character varieties, irregular singularities, Stokes phenomenon, moduli spaces

Abstract: This thesis is motivated by the question of the classification of wild character varieties, which are moduli spaces of irregular connections on curves. These varieties depend on the choice of some singularity data characterizing the form of the singularities of the connections, and it often occurs that different singularity data, corresponding to connections with different ranks and different number of singularities, give rise to isomorphic moduli spaces. We define a diagram associated to any algebraic connection on a Zariski open subset of the affine line, generalizing previous constructions relating wild character varieties to quivers to the case where there are several irregular singularities, possibly ramified. The idea of the construction is to use the Fourier-Laplace transform to reduce to the setting of Boalch-Yamakawa, where there is only one irregular singularity. The diagram is invariant under symplectic automorphisms of the Weyl algebra, so that there are several connections, with different singularity data, giving rise to the same diagram. Some other properties of the previous cases still hold in our more general setting: the dimension of the wild character variety is given from the diagram by a formula involving its Cartan matrix, and simple Weyl reflections with respect to some vertices of the diagram are obtained by applying some operations on connections. As an application of this construction, we can view many known different Lax representations of Painlevé equations, as well as of some higher dimensional analogues, as different representations of the same diagram. We are also able to classify all cases for which the diagram has one vertex and less than two loops.

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