Topological descriptions of complex linear differential equations

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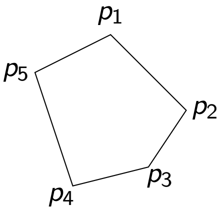
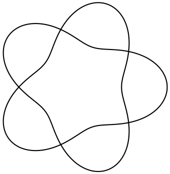
Teaser

How are these things related?

- The linear differential equation $y'' + x^3 y = 0$
- The polynomial equation $xyz + x + z + 1 = 0$
- A pentagon in the plane (with $p_i \neq p_{i+1}$)



Answer: via the picture



- Generalized Riemann-Hilbert correspondence: a complex linear ODE can be encoded by data of topological nature
- The form of these data depends on the *type of singularities* of the equation. For $y'' + x^3 y = 0$ this type is encoded by the picture above.
- There are several variants of topological data, e.g. starting from an equation of the same type as $y'' + x^3 y = 0$:
    - One variant gives a solution of $xyz + x + z + 1 = 0$,
    - The other variant gives a pentagon.

1. Complex linear ODEs and monodromy

2. Two frameworks for generalized monodromy data

3. Fourier transform

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Linear differential systems

We consider systems of linear ODEs on the complex plane $\mathbb{C}$

$$\frac{dY}{dz} = A(z)Y,$$

where

- $A(z)$ is a $n \times n$ matrix whose coefficients are meromorphic functions of z, with poles at points $a_1, \ldots, a_m \in \Sigma = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$;
- the unknown Y is a vector whose entries are holomorphic functions of z on $\Sigma^o := \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$.

Scalar linear ODEs

This framework also covers scalar ODEs, indeed

$$\left(\partial^n + a_{n-1}(z)\partial^{n-1} + \cdots + a_1(z)\partial + a_0(z)\right) y = 0$$

is equivalent to $Y' = A(z)Y$, with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & a_2 & \dots & -a_{n-1} \end{pmatrix}.$$

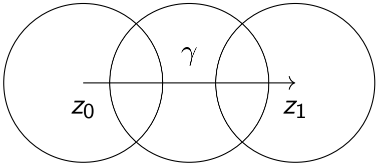
Example: $(\partial^2 + z^3)y = 0 \rightsquigarrow A(z) = \begin{pmatrix} 0 & 1 \\ -z^3 & 0 \end{pmatrix}$.

This has a unique singularity, at ∞ (to see this do the change of coordinate $\zeta := 1/z$).

Solutions

Cauchy theorem: given initial condition $Y_0 \in \mathbb{C}^n$ at z_0, unique solution Y in a neighbourhood of z_0 such that $Y(z_0) = Y_0$.

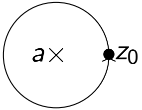
This allows us to integrate the equation along any path $\gamma : [0,1] \to \Sigma^0$



Local monodromy

Consider $Y = (Y_1|\ldots|Y_n)$ basis of solutions close to z_0.

Integrate along a path γ going around a singularity a.



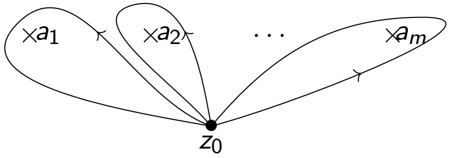
Get a new basis $\widetilde{Y}$ of solutions at z_0: $\exists\ M \in \mathrm{GL}_n(\mathbb{C})$ such that $\widetilde{Y} = YM$.

M is the monodromy of the equation around a

M is well-defined up to conjugation.

Global monodromy and character varieties

Choose some paths $\gamma_1, \ldots, \gamma_m$ around $a_1, \ldots, a_m$



The equation $Y' = A(z)Y$ has monodromy $M_i \in G = \mathrm{GL}_n(\mathbb{C})$ around γ_i.

The space of all possible monodromy data is the *character variety*

$$\mathcal{M}_B(\mathbb{P}^1, \mathbf{a}, G) = \{M_1, \ldots, M_m \,|\, M_m \ldots M_1 = 1\}/G.$$

In our example $(\partial^2 + z^3)y = 0$, since there is a unique singularity at infinity, we have $M_\infty = \mathrm{Id}_2$.

Regular vs irregular singularities

Regular singularities: simple poles $A(z) = \frac{A_1}{z} +$ holomorphic terms

Irregular singularities: higher order poles $A(z) = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z} + \ldots$

Simple rank one example:

- Regular $A(z) = \frac{\lambda}{z}$ Solution $z^\lambda = e^{\lambda \log z}$, monodromy $e^{2i\pi\lambda}$.
- Irregular $A(z) = \frac{dq}{dz} + \frac{\lambda}{z}$, with $q \in z^{-1}\mathbb{C}[z^{-1}]$.
  Solution $e^q z^\lambda$, monodromy also $e^{2i\pi\lambda}$, for any q.

Gauge equivalence

We can do a change of variable

$$Z := F(z)Y$$

with $F : \Sigma^\circ \to \mathrm{GL}_n(\mathbb{C})$ meromorphic

F is a "gauge transformation"

This changes the equation $Y' = A(z)Y$ into $Z' = B(z)Z$, with

$$B = FAF^{-1} - \frac{dF}{dz}F^{-1}.$$

We view the two equations as equivalent.

The Riemann-Hilbert correspondence

If the equation has only regular singularities, it is fully determined up to equivalence by its global monodromy.

(we can get a nice one-to-one correspondence [Deligne] if we introduce a slight generalization of linear ODEs: meromorphic connections)

For irregular singularities, the monodromy is not enough $\Rightarrow$ need generalized monodromy data (a.k.a Stokes data)!

1. Complex linear ODEs and monodromy

2. Two frameworks for generalized monodromy data

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Irregular type

Turritin-Levelt theorem: at a singularity a, one can "diagonalise" $A(z)$ with a formal gauge transformation $\widehat{F}(z_a)$ to a normal form

$$A^0 = \frac{dQ}{dz_a} + \frac{\Lambda}{z_a}, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z_a^{-1/r}\mathbb{C}[z_a^{-1/r}],$$

- q_i: exponential factors at a
- Q: irregular type at a, r ramification order, Q is twisted if $r > 1$.
- Regular singularity iff $Q = 0$.
- Λ: exponent of formal monodromy, $[\Lambda, Q] = 0$.

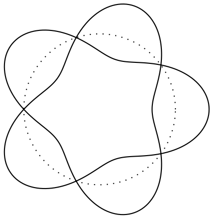
A formal fundamental solution of A at a is $Y^0 := \widehat{F}(z_a)e^Q z_a^\Lambda$.

In our example: $Q = \begin{pmatrix} \frac{2}{5}z^{5/2} & 0 \\ 0 & -\frac{2}{5}z^{5/2} \end{pmatrix}$, $\Lambda = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.

Stokes diagram

The growth rates of the terms e^{q_i} depend on the direction

Plot in polar coordinates the growth rate $e^{\mathrm{Re}(q_i(z))}$ for $|z| \to 0$ as a function of the direction $\theta = \arg(z)$.

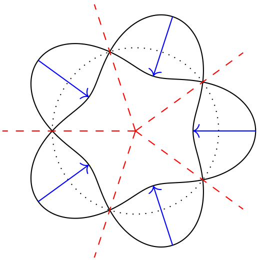


Picture here in our example: $q_1 = \frac{2}{5} z^{5/2}$, $q_2 = -\frac{2}{5} z^{5/2}$:
plot $\theta \mapsto e^{\mathrm{Re}(\pm \frac{2}{5} \cos\left(\frac{5}{2}\theta\right))}$

Stokes and anti-Stokes directions

Stokes directions: when two q_i exchange dominance (i.e. two strands cross on the diagram)

Anti-Stokes/singular directions: where the growth rates of two q_i are furthest apart



In our example: 5 Stokes directions, 5 singular directions.

Stokes arrow $i \leftarrow_d j$ at d if $(q_i - q_j)(z)$ has maximal decay when $z \to 0$ along d.

Two approaches towards Stokes data

The Stokes approach: try to "resum" the formal solutions into true solutions
$\Rightarrow$ the resummations jump at the anti-Stokes directions

The Poincaré approach: look at asymptotics of solutions
$\Rightarrow$ get jumps at the Stokes directions

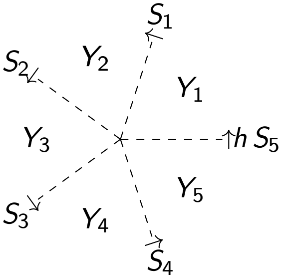
Assumption: from now on consider the situation where there is just one singularity, at infinity (in particular we have trivial monodromy $M_\infty = \mathrm{Id}$)

First approach: Stokes matrices

Resummation gives a preferred fundamental solution Y_i between consecutive singular directions d_{i-1} and d_i [Martinet, Ramis, Loday-Richaud]

The difference between Y_i and Y_{i+1} gives a **Stokes matrix** S_i

$$Y_i = Y_{i+1} S_i$$



h: monodromy of formal solution Y^0

Form of the Stokes matrices

The Stokes matrices have a particular form:

$$S_i = \begin{pmatrix} 1 & & & \\ & \ddots & * & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

- entries on the diagonal are 1.
- nonzero off-diagonal entry in position (k, l) iff there is a Stokes arrow $k \leftarrow_{d_i} l$ at d_i.

In our example:

- 5 Stokes matrices $S_{2i+1} = \begin{pmatrix} 1 & s_{2i+1} \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ s_{2i} & 1 \end{pmatrix}$,
- formal monodromy $h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$

Wild character variety

Irregular Riemann-Hilbert (first version): the Stokes matrices + formal monodromy characterize an equation of type Q up to equivalence

In our example, the space of all possible generalized monodromy data of type Q is the wild character variety

$$\mathcal{M}_B(\mathbb{P}^1, \infty, Q) = \{hS_5S_4S_3S_2S_1 = 1\}/H, \quad H = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\} \subset \mathrm{GL}_2(\mathbb{C})$$

Explicitly, we have

$$\mathcal{M}_B \simeq \{s_2, s_3, s_4 \in \mathbb{C} \mid s_2s_3s_4 + s_2 + s_4 + 1 = 0\}$$

$\Rightarrow$ Any differential equation of type Q determines a solution of the equation $xyz + x + z + 1 = 0$

Second approach: Stokes filtrations

Let V be the vector space of analytic solutions.

Consider a direction d which is not a Stokes direction.

For $j = 1, \ldots, n$, define $V_d(q_j)$ as the subspace of solutions Y such that $Ye^{-q_j(z)}$ has a most polynomial growth when $z \to_d 0$.

$\Rightarrow$ Get a **filtration** F_d of V, i.e. a collection of subspaces

$$0 \subset V_d(q_{i_1}) \subset \cdots \subset V_d(q_{i_k}) = V_d$$

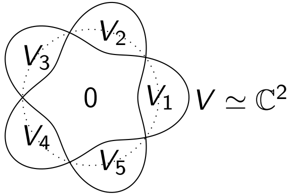
where we order the exponential factors $q_1, \ldots, q_n$ according to increasing growth rates $e^{\mathrm{Re}(q_i(z))}$ at d.

The filtration F_d jumps at Stokes directions

Irregular Riemann-Hilbert (second version) [Deligne-Malgrange]: the Stokes filtration characterizes an equation of type Q up to equivalence

Example

In our example, a Stokes filtration of type Q looks like



The filtration consists of 5 lines V_1, V_2, V_3, V_4, V_5 in $V \simeq \mathbb{C}^2$, such that V_i and V_{i+1} are not parallel.

Equations of type (s, r)

More generally consider

$$Q_{s,r} := \begin{pmatrix} z^{s/r} & & \\ & \ddots & \\ & & e^{2i\pi s(r-1)/r} z^{s/r} \end{pmatrix}$$

with s, r coprime, $s > r$

Our main example corresponds to $(s, r) = (5, 2)$.

Then one can actually reconstruct the Stokes filtration just from:

- Either the recessive solutions (pieces of dim. 1)
- Or the subdominant solutions (pieces of dim. $r - 1$)

Spaces of polygons

For $Q_{s,r}$, the recessive solutions give s lines in $V \simeq \mathbb{C}^r$, such that r consecutive lines are always linearly independent.

This gives s points in projective space $\mathbb{P}^{r-1} = \mathbb{C}^r/\mathbb{C}^*$, i.e. a nondegenerate s-gon in $\mathbb{P}^{r-1}$

The space of generalized monodromy data is isomorphic to the **moduli space of polygons**

$$\mathcal{M}_P(s,r) = \{p_1, \ldots, p_s \in \mathbb{P}^{r-1} \mid p_i, \ldots, p_{i+r-1} \text{ lin. indep. } \}/\mathrm{PGL}_r(\mathbb{C})$$

In our example $(s,r) = (5,2)$, get pentagons in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Fourier transform

The usual Fourier transform exchanges derivation and multiplication by x

$$\begin{cases} x & \mapsto -\partial_p \\ \partial_x & \mapsto p \end{cases}$$

We can imitate this to define a notion of Fourier transform for linear ODEs

$$P(z, \partial_z)y = 0 \rightsquigarrow P(-\partial_w, w)y = 0$$

Example: $(\partial_z^2 + z^3)y = 0 \rightsquigarrow (-\partial_w^3 + w^2)y = 0$

Question: can one express the Stokes data of the Fourier transform in terms of the initial Stokes data?

There are not many cases where one knows how to do this explicitly

Fourier transform of Stokes data

Assumptions:

- Only one singularity at infinity
- Exponential factors all of the form $q_i = \lambda_i z^{s/r}$, with $s > r$, s and r coprime, $|\lambda_i| = 1$

In this case, the Fourier transform also satisfies the assumption.

In particular this applies to the case $Q = Q_{s,r}$.

Then, using work of T. Mochizuki:

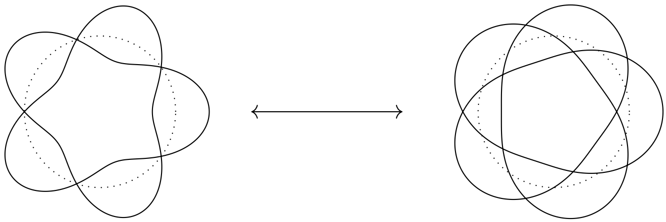
- [D.-Hohl 2025] Explicit recipe to compute the Stokes matrices of the Fourier transform
- [D. -Hohl, in progress] Simple relation between the polygons on both sides

Formal level: the stationary phase formula

The exponential factors of the Fourier transform are determined by those of the initial equations [Fang, Sabbah]

In particular (up to a constant coefficient): $Q_{s,r} \rightsquigarrow Q_{s,s-r}$

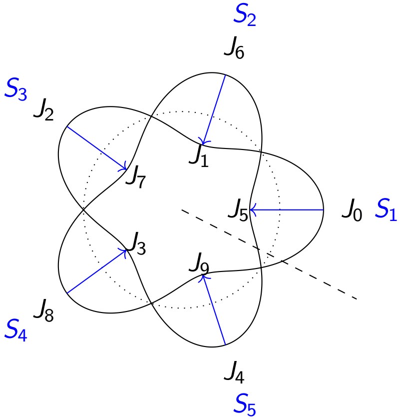
In our example: $(5,2) \rightsquigarrow (5,3)$



$(5,2)$ $(5,3)$

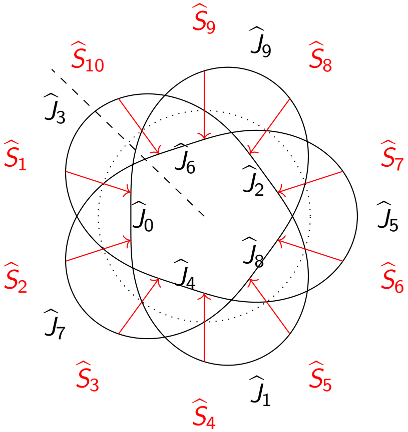
First approach: relating the Stokes matrices

$Q_{5,2}$



$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\}$ with $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \dots$

Fourier transform: $Q_{5,3}$



$$\mathcal{M}'_B = \left\{ \hat{h}\widehat{S}_{10}\ldots\widehat{S}_1 = 1 \right\} \text{ with } \hat{h} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \widehat{S}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & 0 & 1 \end{pmatrix}, \widehat{S}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}$$

$\mathcal{M}_B \simeq \mathcal{M}'_B$ via $\Phi : (s_1, s_2, s_3, s_4, s_5) \mapsto (-s_5, -s_2, s_4, s_1, -s_3, s_5, s_2, -s_4, -s_1, s_3)$

Computation of the new Stokes matrix entries

coefficient	Stokes arrow	Stokes matrix entry
t_1	$3 \to 0$	$-s_5$
t_2	$7 \to 0$	$-s_2$
t_3	$7 \to 4$	s_4
t_4	$1 \to 4$	s_1
t_5	$1 \to 8$	$-s_3$
t_6	$5 \to 8$	s_5
t_7	$5 \to 2$	s_2
t_8	$9 \to 2$	$-s_4$
t_9	$9 \to 6$	$-s_1$
t_{10}	$3 \to 6$	s_3

The Gale transform

Consider

- a collection of s vectors $v = (v_1, \dots, v_s)$ in $\mathbb{C}^r \rightsquigarrow r \times s$ matrix $X = (v_1 | \dots | v_s)$
- a collection of s vectors $w = (w_1, \dots, w_s)$ in $\mathbb{C}^{s-r} \rightsquigarrow (s-r) \times s$ matrix $Y = (w_1 | \dots | w_s)$

We say that they are Gale dual if there exists an invertible $n \times n$ diagonal matrix D such that

$$YDX^\top = 0$$

This induces a well-defined isomorphism of moduli spaces of s-gons

$$\mathcal{M}_P(s, r) \simeq \mathcal{M}_P(s, s-r)$$

Second approach: relating the recessive/subdominant solutions

Conjecture:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{eq. of type } Q_{s,r} \arrow[r, "\text{Fourier}"] \arrow[d, "\substack{\text{subdominant}\\ \text{solutions}}"'] & \text{eq. of type } Q_{s,s-r} \arrow[d, "\substack{\text{recessive}\\ \text{solutions}}"'] \\
s\text{-gon in } \mathbb{P}^{r-1} \arrow[r, "\text{Gale}"'] & s\text{-gon in } \mathbb{P}^{s-r-1}
\end{tikzcd}$$

In our example, the conjecture is true, we get

$$\text{pentagon in } \mathbb{P}^1 \rightsquigarrow \text{pentagon in } \mathbb{P}^2$$