A CHARACTERIZATION OF RELATIVE KAZHDAN PROPERTY T FOR SEMIDIRECT PRODUCTS WITH ABELIAN GROUPS

YVES DE CORNULIER AND ROMAIN TESSERA

Abstract. Let A be a locally compact abelian group, and H a locally compact group acting on A. Let $G = H \ltimes A$ be the semidirect product, assumed σ-compact. We prove that the pair (G, A) has Kazhdan's Property T if and only if the only countably approximable H-invariant mean on the Borel subsets of the Pontryagin dual $\hat{A}$, supported at the neighbourhood of the trivial character, is the Dirac measure.

1. Introduction

Let G be a locally compact group and A a subgroup. Recall that the pair (G, A) has *Kazhdan's Property T* (or *relative Property T*, or *Property T*) if every unitary representation of G with almost invariant vectors admits a non-zero A-invariant vector. We refer to the book [BHV] for a detailed background.

In this paper, we focus on the special case where G is written as a semidirect product $H \ltimes A$, and A is abelian. Any unitary representation of such a group can be restricted to A and we can then use the spectral theorem to decompose it as an integral of characters. It was thus soon observed that relative Property T for the pair (G, A) is related to restrictions on invariant probabilities on the Pontryagin dual $\hat{A}$ of A. This was first used by D. Kazhdan [Kaz] in the case of $\mathrm{SL}_n(\mathbf{R}) \ltimes \mathbf{R}^n$ for $n \geq 2$. These ideas were then used in a more systematic way, notably by G. Margulis [Mar] and M. Burger [Bur]. It was in particular observed that if H is any locally compact group with a representation on a finite-dimensional vector space V over a local field, then $(H \ltimes V, V)$ has Property T if and only if H does not preserve any probability measure on the Borel subsets of the projective space $\mathbf{P}(V^*)$ over the dual of V (see [Cor2, Prop. 3.1.9] for the general statement; the "if" part follows from [Bur, Prop. 7]). The idea of using means (i.e. finitely additive probabilities) instead of probabilities is due to Y. Shalom [Sha, Theorem 5.5], who proved that if H preserves no invariant mean

Date: March 23, 2010.

2000 *Mathematics Subject Classification.* 43A05 (Primary); 22D10, 43A07, 43A25 (Secondary).

on $\hat{A} - \{1\}$, then $(H \ltimes A, A)$ has Property T and used related ideas in [Sha2] to prove Property T for such pairs as $(\mathrm{SL}_2(\mathbf{Z}[X]) \ltimes \mathbf{Z}[X]^2, \mathbf{Z}[X]^2)$. Our main result gives the first sufficient condition for relative Property T in terms of invariant means, which is also necessary.

We say that a Borel mean m on a locally compact space X is *countably approximable* if there exists a countable set $\{\nu_n : n \geq 0\}$ of Borel probability measures, whose weak-star closure in $\mathcal{L}^\infty(X)^*$ (each probability measure being viewed as a mean) contains m.

Theorem 1. *Let $G = H \ltimes A$ be a σ-compact locally compact group and assume that the normal subgroup A is abelian. We have equivalences*

($\neg$T) *The pair (G, A) does not have Kazhdan's Property T.*

(M) *There exists a countably approximable H-invariant mean m on $\mathcal{L}^\infty(\hat{A} - \{1\})$ such that $m(V) = 1$ for every neighbourhood V of $\{1\}$.*

(P) *There exists a net of Borel probability measures (μ_i) on $\hat{A}$ such that*
- (P1) $\mu_i \to \delta_1$ *(weak-star convergence in $\mathcal{C}_c(\hat{A})^*$);*
- (P2) $\mu_i(\{1\}) = 0$*;*
- (P3) *for every $h \in H$, $\|h \cdot \mu_i - \mu_i\| \to 0$, uniformly on compact subsets of H.*

Here, Condition (P1) means that $\mu_i(V) \to 1$ for every neighbourhood V of 1 in $\hat{A}$. Also note that since G is assumed σ-compact, the net in (P) can be replaced by a sequence. In the case of discrete groups, the implication ($\neg$T)$\Rightarrow$(P) has been independently obtained by A. Ioana [Ioa, Theorem 6.1], while its converse was obtained by M. Burger [Bur, Prop. 7].

Corollary 2. *If $H_1 \to H$ is a homomorphism with dense image between σ-compact locally compact groups, then $(H \ltimes A, A)$ has Property T if and only if $(H_1 \ltimes A, A)$ does.*

Moreover, if $(H \ltimes A, A)$ has Property T, then we can find a finitely generated group Γ and a homomorphism $\Gamma \to H$ such that $(\Gamma \ltimes A, A)$ has Property T.

The first statement of Corollary 2 shows that, in a strong sense, relative Property T for such a semidirect product only depends on the image of the action map $H \to \mathrm{Aut}(A)$, and does not detect if this action, for instance, is faithful. It typically applies when H is discrete and H_1 is a free group with a surjection onto H_1.

Corollary 3. *The equivalence between ($\neg$T) and (P) holds for G locally compact (without any σ-compactness assumption).*

The implication ($\neg$T)$\Rightarrow$(M), which uses standard arguments (similar to [Sha, Theorem 5.5]), is borrowed, in the discrete case, from [Cor, Section 7.6], and improves it in the case when A is not discrete.

We also give a relative version of Theorem 1, generalizing ideas from [CI]. The relative result actually follows as a corollary from the proof of Theorem 1. In what follows, all positive functions are assumed to take the value 1 at the unit element. We denote by $\widehat{\mu_i}$ the Fourier-Stieltjes transform of μ_i, which is the positive definite function on A defined as $\widehat{\mu_i}(a) = \int \chi(a) d\mu_i(\chi)$ (see [BHV, Appendix D]). If $X \subset G$ is a closed subset (G any locally compact group), we also say that (G, X) has relative Property T if for positive definite functions on G, convergence to 1 uniformly on compact subsets of G implies uniform convergence in restriction to X (this extends the previous definition when X is a subgroup, see [Cor2]). At the opposite, (G, X) has the relative Haagerup Property if there exists positive definite functions on G, arbitrary close to 1 for the topology of uniform convergence on compact subsets, but whose restriction to X are C_0, i.e. vanish at infinity. If G is σ-compact, then this is equivalent to the existence of an affine isometric action on a Hilbert space, whose restriction to X is proper: the proof uses the same argument as the original proof by Akemann and Walter of the equivalence between the unitary and affine definition of Haagerup's Property [AW].

Theorem 4. *Under the assumptions of Theorem 1, assume that H is discrete and suppose that $X \subset A$ is a closed subset. Then we have equivalences*

- *The pair (G, X) does not have Kazhdan's Property T.*
- *There exists a net of Borel probability measures (μ_i) on $\hat{A}$ satisfying (P) and such that the convergence of $\widehat{\mu_i}$ to 1 on X is not uniform.*

If moreover X is H-invariant, we have equivalences

- *The pair (G, X) has relative Haagerup's Property.*
- *There exists a net of Borel probability measures (μ_i) on $\hat{A}$ satisfying (P), with $\widehat{\mu_i}$ is C_0 on X.*

In particular, we deduce the following corollary, which generalizes [CI, Theorem 3.1].

Corollary 5. *Under the assumptions of Theorem 1, assume that H is discrete and that Λ is a normal subgroup of H whose action on A is trivial. If $X \subset A$, then $(H \ltimes A, X)$ has relative Property T if and only if $(H/\Lambda \ltimes A, X)$ has relative Property T. If moreover X is H-invariant, then $(H \ltimes A, X)$ satisfies relative Haagerup's Property if and only if $(H/\Lambda \ltimes A, X)$ does.* $\square$

Remark 6. It is of course better when the condition of Theorem 4 on the Fourier-Stieltjes transforms can be made explicit. When $X = A$, wee actually see that, for Borel probability measures on $\hat{A}$, the uniform convergence of $\widehat{\mu_i}$ to one is equivalent to the condition $\mu_i(\{1\}) \to 1$. This extends to the case of a subgroup B of A (not necessarily H-invariant), by the statement: the convergence of $\widehat{\mu_i}$ to 1 is uniform on B if and only if $\mu_i(B^\perp) \to 1$.

The condition that $\widehat{\mu_i}$ is C_0 on A is not easy to characterize but has been long studied (see for instance [Ry]). In view of the Riemann-Lebesgue Lemma, it can be viewed as a weakening of the condition that μ_i has density with respect to the Lebesgue measure.

To prove the different equivalences, we need to transit through various properties analogous to (P), essentially differing in the way the asymptotic H-invariance is stated. Theorem 7 below states all these equivalences and encompasses Theorem 1. Several of these implications borrow arguments from the proof of the equivalence between various formulations of amenability [BHV, Appendix G]. Section 2 begins introducing some more definitions, notably concerning means, measures, and convolution, and then formulates Theorem 7. Section 3 contains all proofs.

2. Equivalent formulations of relative property T for semidirect products

We need to introduce some notation. Let $X = (X, \mathcal{T})$ be a measurable space. Recall that a mean on X is a finitely additive probability measure on the measurable subsets of X. We denote by $\mathcal{L}^\infty(X)$ the space of bounded measurable Borel functions on X, endowed with the supremum norm $\|\cdot\|_\infty$. Recall that any mean on X can be interpreted as an element $\bar{m} \in \mathcal{L}^\infty(X)^*$ such that $\bar{m}(1) = 1$ and $\bar{m}(\phi) \geq 0$ for all non-negative $\phi \in \mathcal{L}^\infty(X)$, characterized by the condition $\bar{m}(1_B) = m(B)$ for every Borel subset B. By a common abuse of notation, we generally write m instead of $\bar{m}$, and similarly $\mu(f)$ instead of $\int f(x)d\mu(x)$ when μ is a measure on X, and f is an integrable function. Note that any mean m on X can be approximated, in the weak-star topology, by a net $(\nu_i)_{i\in I}$ of finitely supported probabilities (i.e. finite convex combinations of Dirac measures).

We fix a Haar measure λ for H. We use the notation $\int f(h)dh$ for the integral of $f \in L^1(H)$ against λ. Let X be a measurable space with a measurable action $H \times X \to X$ of H. For every mean ν on X, $h \in H$, and B Borel subset of X, we write $(\nu \cdot h)(B) = \nu(hB)$. Let $\mathcal{UC}_H(X)$ be the subspace of $\mathcal{L}^\infty(X)$ whose elements ϕ satisfy that $h \to h \cdot \phi$ is continuous from H to $\mathcal{L}^\infty(X)$. We also need to consider the convolution product

$$f * \phi(x) = \int f(h)\phi(h^{-1}x)dh$$

between functions in $L^1(H)$, or between $f \in L^1(H)$ and $\phi \in \mathcal{L}^\infty(X)$. Note that in the first case, $f * \phi \in L^1(H)$, whereas in the second case, $f * \phi \in \mathcal{L}^\infty(X)$ (see also Lemma 10). If μ is a measure on X, we can define the convolution product of μ and $f \in L^1(H)$ by $\mu * f(B) = \mu(f * 1_B)$. Using the Lebesgue monotone convergence theorem, we see it is σ-additive. It follows (using again the Lebesgue monotone convergence theorem for $\phi \geq 0$) that for all $\phi \in \mathcal{L}^\infty(X)$ we have

$$(\mu * f)(\phi) = \mu(f * \phi).$$

Let Y be a locally compact Hausdorff space, endowed with its σ-algebra of Borel subsets. Let $\mathcal{M}(Y)$ be the Banach space of signed Borel regular measures on Y ("regular" is redundant when Y is metrizable), equipped with the total variation norm (i.e. the norm in $\mathcal{C}_c(Y)^* = \mathcal{M}(Y)$). Note that for $f \in L^1(H)$ and $\mu \in \mathcal{M}(Y)$, we have $\|\mu * f\| \leq \|f\|_1\|\mu\|$.

Let $L^1(H)_{1,+}$ be the subset of $L^1(H)$ consisting of non-negative elements of norm 1. Let $\mathcal{C}_c(H)_{1,+}$ be the set of non-negative, continuous, compactly supported functions f on H such that $\int f(h)dh = 1$. Note that $L^1(H)_{1,+}$ and $\mathcal{C}_c(H)_{1,+}$ are stable under convolution.

Theorem 7. *Let $G = H \ltimes A$ be a σ-compact locally compact group, with A abelian. Equivalences:*

- ($\neg$T) *the pair (G, A) does not have Property T.*
- (M) *There exists a countably approximable H-invariant mean m on $\hat{A} - \{1\}$ such that $m(V) = 1$ for every neighbourhood V of $\{1\}$.*
- (MC) *There exists a Borel σ-finite measure γ on $\hat{A} - \{1\}$ and a mean m on $\mathcal{L}^\infty(\hat{A} - \{1\})$ belonging to $L^\infty(\hat{A}, \gamma)^*$, such that $m(V) = 1$ for every neighbourhood V of $\{1\}$, and such that for all $f \in \mathcal{C}_c(H)_{1,+}$ and $\phi \in \mathcal{L}^\infty(\hat{A})$,*

$$m(f * \phi) = m(\phi).$$

- (P) *There exists a net of Borel probability measures (μ_i) on $\hat{A}$ satisfying (P1), (P2), (P3).*
- (PC) *There exists a net of Borel probability measures (μ_i) on $\hat{A}$ satisfying (P1), (P2), (P3c), where (P3c) is defined as: $\|\mu_i * f - \mu_i\| \to 0$, for all $f \in \mathcal{C}_c(H)_{1,+}$.*

(PQ) *There exists a net of Borel probability measures* (μ_i) *on* $\hat{A}$ *satisfying (P1), (P2), (P3), with the additional property that* μ_i *is* H*-quasi-invariant for every* i*.*

In (P) and (PQ), the net can be chosen to be a sequence. Besides, when G *is a* σ*-compact locally compact group and* A *a closed abelian normal subgroup (not necessarily part of a semidirect decomposition), then (*$\neg T$*) implies all other properties (with* $H = G/A$*), which are equivalent.*

Remark 8. If $(H \ltimes A, A)$ does not have Property T, we do not necessarily have a net of probabilities (μ_i), as in any of the properties in Theorem 7, with *density with respect to the Haar measure*. A simple counterexample is given by $\mathrm{SL}_2(\mathbf{R}) \ltimes (\mathbf{R}^2 \times \mathbf{R})$ (with the trivial action on $\mathbf{R}$), or its discrete analogue $\mathrm{SL}_2(\mathbf{Z}) \ltimes (\mathbf{Z}^2 \times \mathbf{Z})$. Indeed, we could push this sequence forward to $\mathbf{R}^2$ (resp. $(\mathbf{R}/\mathbf{Z})^2$) and contradict relative Property T for $\mathrm{SL}_2(\mathbf{R}) \ltimes \mathbf{R}^2$ and $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$.

Remark 9. We could define (M') as the following weak form of (M): there exists an H-invariant mean m on $\hat{A} - \{1\}$ such that $m(V) = 1$ for every neighbourhood V of $\{1\}$. It can easily be shown to be equivalent to (P'), defined as the existence of a net of Borel probability measures (μ_i) satisfying (P1),(P2), and (P3'), where (P3') is defined as: $\mu_i - h\mu_i$ tends to zero in the weak-star topology of $\mathcal{L}^\infty(\hat{A})^*$. We are not able to determine if these properties imply ($\neg$T).

3. Proof of the results

In this section, we first develop a few preliminary lemmas, which hold in a more general context. Then we prove Theorem 7, and the corollaries.

Lemma 10. *Let* X *be measurable space with measurable action of* H*. For all* $f \in L^1(H)$ *and for all* $\phi \in \mathcal{L}^\infty(X)$*, we have* $f * \phi \in \mathcal{UC}_H(X)$*.*

Proof. If $h \in H$, we have $h \cdot (f * \phi) = (h \cdot f) * \phi$. Therefore, if $h' \in H$ we get

$$\begin{aligned}\|h \cdot (f * \phi) - h' \cdot (f * \phi)\|_\infty &= \|(h \cdot f - h' \cdot f) * \phi\|_\infty \\ &\leq \|h \cdot f - h' \cdot f\|_1 \|\phi\|_\infty.\end{aligned}$$

Since the left regular action of H on $L^1(H)$ is continuous, we deduce that $g \mapsto g \cdot (f * \phi)$ is continuous from G to $\mathcal{L}^\infty(X)$, that is, $f * \phi \in \mathcal{UC}_H(X)$. □

Lemma 11. *If* A *is* σ*-compact, Condition (P1) is equivalent to:*

(P1') *for* $a \in A$*, we have* $\int \chi(a) d\mu_i(\chi) \to 1$*, uniformly on compact subsets of* A*.*

Proof. This appears as [Par, Theorem 3.3] under the assumption that A is second countable (and actually the proof extends to any locally compact abelian group A); however we give here a much shorter proof.

Suppose that (P1) holds. Let K be a compact subset of A. There exists a neighbourhood V of 1 in $\hat{A}$ such that $|1-\chi(a)| \leq \varepsilon$ for all $\chi \in V$ and $a \in K$. For i large enough, $\mu_i(V) > 1-\varepsilon$, which implies, for all $a \in K$

$$\left|1 - \int \chi(a) d\mu_i(\chi)\right| \leq \int |1-\chi(a)| d\mu_i(\chi)$$

$$\leq \int_V |1-\chi(a)| d\mu_i(\chi) + \int_{V^c} |1-\chi(a)| d\mu_i(\chi) \leq 2\varepsilon.$$

The converse follows from the following claim: for every neighbourhood V of 1 in $\hat{A}$ and every $\varepsilon > 0$, there exists $\eta > 0$ and a compact set K in A such that for every Borel measure μ on $\hat{A}$ satisfying $\sup_{a\in K} |1 - \int \chi(a) d\mu(\chi)| \leq \eta$, we have $\mu(V) \geq 1-\varepsilon$.

Let us prove this claim. Let ϕ be a positive function in $L^1(A)$ with $\int \phi(a) da = 1$ (this exists because A is σ-compact). Set $F(\chi) = \int \phi(a)\chi(a) da$; this is the Fourier transform of ϕ. In particular, by the Riemann-Lebesgue Lemma, F is continuous and vanishes at infinity. Moreover, $F(1) = 1$ and since $\phi > 0$, $|F(\chi)| < 1$ for all $\chi \neq 1$. Therefore there exists $\rho > 0$ such that $\{|F| \geq 1-\rho\}$ is contained in V.

Define $\eta = \rho\varepsilon/3$. Let K be a compact neighbourhood of 1 in A such that $\int_K \phi(a) da \geq 1-\eta$. Let μ be a Borel probability on $\hat{A}$ such that

$$\left|1 - \int \chi(a) d\mu(\chi)\right| \leq \eta$$

for all $a \in K$. Set $\sigma(a) = \int (1-\chi(a)) d\mu(\chi)$. We have

$$\left|\int \phi(a)\sigma(a) da\right| \leq \left|\int_K \phi(a)\sigma(a) da\right| + \left|\int_{K^c} \phi(a)\sigma(a) da\right|$$

$$\leq \eta + 2\eta = 3\eta.$$

On the other hand,

$$\int \phi(a)\sigma(a) da = 1 - \int \left(\int \phi(a)\chi(a) d\mu(\chi)\right) da;$$

since the term in the double integral is summable, we can use Fubini's Theorem, giving

$$\int \phi(a)\sigma(a) da = 1 - \int F(\chi) d\mu(\chi),$$

where

$$\begin{aligned}1-\int \phi(a)\sigma(a)da &= \int F(\chi)d\mu(\chi)\\ &= \int_{\{|F|>1-\rho\}} F(\chi)d\mu(\chi) + \int_{\{|F|\leq 1-\rho\}} F(\chi)d\mu(\chi),\end{aligned}$$

thus

$$\begin{aligned}\left|1-\int \phi(a)\sigma(a)da\right| &\leq (1-\mu(\{|F|\leq 1-\rho\})) + (1-\rho)\mu(\{|F|\leq 1-\rho\})\\ &= 1-\rho\mu(\{|F|\leq 1-\rho\})\end{aligned}$$

so

$$\left|\int \phi(a)\sigma(a)da\right| \geq 1-|1-\int \phi(a)\sigma(a)da| \geq \rho\mu(\{|F|\leq 1-\rho\}).$$

Combining with the previous inequality, we obtain.

$$\mu(\{|F|\leq 1-\rho\}) \leq 3\eta/\rho = \varepsilon,$$

hence

$$\mu(V) \geq 1-\varepsilon. \quad \square$$

Lemma 12. *Let X be a measurable space with a measurable action of H, and m a mean on $\mathcal{UC}_H(X)$. For all $\phi \in \mathcal{UC}_H(X)$ and $f \in L^1(H)$, we have*

$$m(f*\phi) = \int f(h)m(h\cdot\phi)dh.$$

Proof. Fix some $\varepsilon > 0$. Let W be a neighbourhood of $1 \in H$ such that for every $h \in W$,

$$\|h\cdot\phi - \phi\|_\infty \leq \varepsilon. \tag{3.1}$$

We can write, in $L^1(H)$, f approximately as a finite sum of functions with small disjoint support, namely $f = \sum_{i=1}^k f_i + f_0$ with $\mathrm{Supp}(f_i) \subset h_i W$ for some $h_i \in H$ (when $i \neq 0$) and $\|f_0\|_1 \leq \varepsilon$ and $\|f\|_1 = \sum_j \|f_j\|_1$. Write for short ${}^h\phi$ for $h\cdot\phi$.

For given $i \neq 0$, we have

$$\begin{aligned} & \left|\int f_i(h)m({}^h\phi)dh - m\left(\int f_i(h)\,{}^h\phi\,dh\right)\right| \\ \leq \quad & \left|\int f_i(h)m({}^h\phi)dh - \int f_i(h)m({}^{h_i}\phi)dh\right| \\ + \quad & \left|\int f_i(h)m({}^{h_i}\phi)dh - m\left(\int f_i(h)\,{}^h\phi\,dh\right)\right| \\ = \quad & \left|\int f_i(h)(m({}^h\phi - {}^{h_i}\phi))dh\right| + \left|m\left(\int f_i(h)({}^h\phi - {}^{h_i}\phi)dh\right)\right| \\ \leq \quad & 2\|f_i\|_1\varepsilon \end{aligned}$$

and

$$\left|\int f_0(h)m({}^h\phi)dh - m\left(\int f_0(h)\,{}^h\phi\,dh\right)\right| \leq 2\varepsilon\|\phi\|_\infty$$

If we sum over i, we deduce

$$\left|\int f(h)m({}^h\phi)dh - m\left(\int f(h)\,{}^h\phi\,dh\right)\right| \leq 2(\|f\|_1 + \|\phi\|_\infty)\varepsilon.$$

Since this holds for any ε, we deduce

$$m(f * \phi) = m\left(\int f(h)\,{}^h\phi\,dh\right) = \int f(h)m({}^h\phi)dh. \quad \square$$

Lemma 13. *Let X be a measurable space with a measurable action of H by homeomorphisms, and m an H-invariant mean on $\mathcal{UC}_H(X)$. Fix $f_0 \in \mathcal{C}_c(H)_{1,+}$, define a mean by*

$$\tilde{m}(\phi) = m(f_0 * \phi),\ \phi \in \mathcal{L}^\infty(X).$$

Then for all $f \in \mathcal{C}_c(H)_{1,+}$ and $\phi \in \mathcal{L}^\infty(X)$,

$$\tilde{m}(f * \phi) = \tilde{m}(\phi).$$

Proof. First, $\tilde{m}$ is well-defined by Lemma 10. We have to show that $\tilde{m}(f * \phi) = \tilde{m}(\phi)$ for all $f \in \mathcal{C}_c(H)_{1,+}$ and $\phi \in \mathcal{L}^\infty(X)$.

Let (f_i) be a net in $\mathcal{C}_c(H)_{1,+}$ with $\mathrm{Supp}(f_i) \to \{1\}$. This implies that $\|f * f_i - f\|_1 \to 0$, and hence that $\|f * f_i * \phi - f * \phi\|_\infty \to 0$, for all $f \in \mathcal{C}_c(H)_{1,+}$, and $\phi \in \mathcal{L}^\infty(X)$. Accordingly $m(f*\phi) = \lim_i m(f*f_i*\phi)$, which by Lemma 12 equals $\lim_i m(f_i * \phi)$ (since $f_i * \phi \in \mathcal{UC}_H(X)$). This shows that $m(f * \phi) = m(f' * \phi)$ for all $f, f' \in \mathcal{C}_c(H)_{1,+}$, and all $\phi \in \mathcal{L}^\infty(X)$. Then for all $f \in \mathcal{C}_c(H)_{1,+}$ and all $\phi \in \mathcal{L}^\infty(X)$,

$$\tilde{m}(f * \phi) = m(f_0 * f * \phi) = m(f_0 * \phi) = \tilde{m}(\phi). \quad \square$$

Proof of Theorem 7. We are going to prove the implications

$$(\neg\mathrm{T}) \Rightarrow (\mathrm{P}) \Rightarrow (\mathrm{PQ}) \Rightarrow (\neg\mathrm{T}) \quad \text{and}$$

$$(\mathrm{P}) \Rightarrow (\mathrm{M}) \Rightarrow (\mathrm{MC}) \Rightarrow (\mathrm{PC}) \Rightarrow (\mathrm{PQ}) \Rightarrow (\mathrm{P}).$$

- $(\neg\mathrm{T})\Rightarrow(\mathrm{P})$. Let $(\pi, \mathcal{H})$ be a unitary representation of G such that $1 \prec \pi$ and such that A has no invariant vector. Let (K_n) be an increasing sequence of compact subsets of G whose interiors cover G. Let (ε_n) be a positive sequence converging to zero. For each n, let ξ_n be a (K_n, ε_n)-invariant vector. Let E be the projection-valued measure associated to

$\pi|_A$, so that $\pi(a) = \int_{\hat{A}} \chi(a) dE(\chi)$ for all $a \in A$. For every n, let μ_n be the probability on $\hat{A}$ defined by $\mu_n(B) = \langle E(B)\xi_n, \xi_n\rangle$. We have:

$$\|\pi(a)\xi_n - \xi_n\|^2 = \int_{\hat{A}} |1 - \chi(a)|^2 d\mu_n(\chi) \qquad \forall a \in A.$$

Therefore, (P1) results from the almost invariance of (ξ_n). Since π has no A-invariant vector, $\mu_n(\{1\}) = 0$ for all n. If f is a continuous function on $\hat{A}$, we define a bounded operator $\hat{f}$ on $\mathcal{H}$ by $\hat{f} = \int f(\chi) dE(\chi)$ (actually $\hat{f}$ is the element of the C^*-algebra of $\pi|_A$ associated to f); note that its operator norm is bounded above by $\|f\|_\infty$. For every $h \in H$ and any f, we have

$$h \cdot \mu_n(f) = \int f d(h \cdot \mu_n) = \langle \pi(h^{-1})\hat{f}\pi(h)\xi_n, \xi_n\rangle$$
$$= \langle \hat{f}\xi_n, \xi_n\rangle + \langle \hat{f}(\pi(h)\xi_n - \xi_n), \xi_n\rangle$$
$$+\langle \hat{f}\xi_n, \pi(h)\xi_n - \xi_n\rangle + \langle \hat{f}(\pi(h)\xi_n - \xi_n), \pi(h)\xi_n - \xi_n\rangle$$

Thus

$$|h \cdot \mu_n(f) - \mu_n(f)| \le 4\|f\|_\infty \|\pi(h)\xi_n - \xi_n\|,$$

so

$$\|h \cdot \mu_n - \mu_n\| \le 4\|\pi(h)\xi_n - \xi_n\|$$

which by assumption tends to zero, uniformly on compact subsets of H. So (P3) holds.

- (PQ)$\Rightarrow$($\neg$T). Consider the sequence of Hilbert spaces $\mathcal{H}_n = L^2(\hat{A}, \mu_n)$, and for every n, the unitary action of H on $\mathcal{H}_n$ defined by

$$(\pi_n(h)f)(\chi) = f(h \cdot \chi) \left(\frac{d(h \cdot \mu_n)}{d\mu_n}(\chi)\right)^{1/2}.$$

There is also a natural action of A on $L^2(\hat{A}, \mu_n)$ given by $\pi_n(a) \cdot f(\chi) = \chi(a) f(\chi)$, and since (by a straightforward computation) we have

$$\pi_n(h)\pi_n(a)\pi_n(h^{-1}) = \pi_n(h \cdot a) \quad \forall h \in H, a \in A,$$

so that π_n extends to a unitary action of the semidirect product $H \ltimes A$ on $L^2(\hat{A}, \mu_n)$. This action has no nonzero A-invariant vector. Indeed, let f be an invariant vector. So for every $a \in A$, there exists a Borel subset $\Omega_a \subset \hat{A}$ with $\mu_n(\Omega_a) = 1$ and for all $\chi \in \Omega_a$,

$$(\chi(a) - 1) f(\chi) = 0.$$

If $a \in A$, define its orthogonal $K_a = \{\chi : \chi(a) = 1\}$ for all $a \neq 0$. Recall that we assume that A is σ-compact. If we assume for a moment that A is also second countable, then A is separable; so there exists a sequence (a_n) in A such that $\bigcap_n K_{a_n} = \{1\}$. If we set $Z = \{f \neq 0\}$, we get $Z \subset K_{a_n} \cup W$, where W is the complement of $\bigcap_n \Omega_{a_n}$. We deduce that $Z \subset W$, which has μ_n-measure zero. So $f = 0$ in $L^2(\hat{A}, \mu_n)$. If A is only assumed σ-compact (i.e. $\hat{A}$ has an open second-countable subgroup), we proceed as follows: there exists a second-countable open subgroup B of $\hat{A}$ such that $\mu_n(B) > 0$ for n large enough (because μ_n concentrates on $\{1\}$). So we can work in B as we just did in $\hat{A}$ and thus $L^2(\hat{A}, \mu_n)$ has no A-invariant vector (at least for n large enough).

An immediate calculation gives, for $a \in A$

$$\|1_{\hat{A}} - \pi_n(a)1_{\hat{A}}\|_{L^2(\hat{A},\mu_n)} = 2\mathrm{Re}\left(1 - \int \chi(a) d\mu_n(\chi)\right),$$

which tends to zero, uniformly on compact subsets of A, when $n \to \infty$, by (P1). On the other hand, for every $h \in H$ we have

$$\|1_{\hat{A}} - \pi_n(h)1_{\hat{A}}\|_{L^2(\hat{A},\mu_n)} = \int_{\hat{A}} \left|1 - \left(\frac{d(h\cdot\mu_n)}{d\mu_n}(\chi)\right)^{1/2}\right|^2 d\mu_n(\chi),$$

so using the inequality $|1 - \sqrt{u}| \leq \sqrt{1 - |u|}$ for all $u \geq 0$ we get

$$\begin{aligned}\|1_{\hat{A}} - \pi_n(h)1_{\hat{A}}\|_{L^2(\hat{A},\mu_n)} &\leq \int_{\hat{A}} \left|1 - \frac{d(h\cdot\mu_n)}{d\mu_n}(\chi)\right| d\mu_n(\chi) \\ &= \|\mu_n - h\cdot\mu_n\|,\end{aligned}$$

which tends to zero, uniformly on compact subsets of H, when $n \to \infty$, by (P3). Accordingly, if we consider the representation $\bigoplus \pi_n$, which has no A-invariant vector, then the sequence of vectors (ξ_n) obtained by taking $1_{\hat{A}}$ in the nth component, is a sequence of almost invariant vectors.

- (P)$\Rightarrow$(M). View μ_n as a mean on Borel subsets of $\hat{A} - \{1\}$. Let $m = \lim_\omega \mu_n$ be an accumulation point (ω some ultrafilter) in the weak-star topology of $\mathcal{L}^\infty(\hat{A} - \{1\})$. (P3) immediately implies that m is H-invariant. (P1) implies that $\int \chi(a) dm(\chi) = 1$ for all $a \in A$. So for every $\varepsilon > 0$, we deduce that $m(\{|\chi - 1| < \varepsilon\}) = 1$. In case A is discrete, since those subsets form a prebasis of the topology of $\hat{A}$, we deduce that $m(V) = 1$ for every neighbourhood V of 1 in $\hat{A}$. Hence (M) follows.

When A is not discrete, we need to appeal to Lemma 11, which implies that $\mu_n(V) \to 1$ (hence $m(V) = 1$) for every Borel neighbourhood V of 1 in $\hat{A}$.

- (M)⇒(MC). Let m be an invariant mean as in (M). Define $\tilde{m}$ as in Lemma 13, which provides the convolution invariance. Clearly, $\tilde{m}(\{1\}) = 0$. Besides, if V is a closed subset of $\hat{A}$ not containing 1, we see that $f_0 * 1_V$ is supported by the closed subset $\mathrm{Supp}(f_0)V$, which does not contain 1 either. So $\tilde{m}$ is supported at the neighbourhood of 1. The argument in the proof of Corollary 2 shows that $\tilde{m}$ also lies in the closure of a countable set $\{\nu_n : n \geq 0\}$ of probability measures on $\hat{A} - \{1\}$. If we set $\gamma = \sum 2^{-n}\nu_n$, then ν_n, viewed as a mean, belongs to $L^\infty(\hat{A} - \{0\}, \gamma)^*$ (i.e. vanishes on γ-null sets), so m also lies in $L^\infty(\hat{A} - \{0\}, \gamma)^*$.
- (MC)⇒(PC). Let m be a mean as in (MC) and let (ν_i) be a net of Borel probabilities on $\hat{A} - \{1\}$, converging to m in $\mathcal{L}^\infty(\hat{A})^*$ for the weak-star topology, with ν_i having density with respect to γ. We can suppose that γ is a probability measure. Let us show that for any $\varepsilon > 0$, any compact subset K of A, and any finite subset Ω of $\mathcal{C}_c(H)_{1,+}$, one can find an element μ in

$$W = \left\{\nu \in \mathcal{M}(\hat{A} \setminus \{1\}) : \mathrm{Re}\left(\int \chi(a) d\nu(\chi)\right) \geq 1 - \varepsilon, \forall a \in K\right\},$$

such that $\|\mu * f - \mu\| \leq \varepsilon$ for all $f \in \Omega$. This is exactly, in view of Lemma 11, what is required to produce a net (μ_i) satisfying (PC). First define $\gamma' = \gamma + \sum_{f\in\Omega} \gamma * f$, so that $\gamma'(\{0\}) = 0$ and each $\mu_i * f$ belongs to $L^1(\hat{A} - \{0\}, \gamma')$. For every $f \in \mathcal{C}_c(H)_{1,+}$, the net $(\nu_i * f - \nu_i)$ converges to 0 for the weak-star topology in $\mathcal{L}^\infty(\hat{A})^*$. Since γ' is σ-finite, the dual of $L^1(\hat{A}, \gamma)$ is equal to $L^\infty(\hat{A}, \gamma')$. So the convergence of $(\nu_i * f - \nu_i)$ to 0 holds in $L^1(\hat{A} - \{0\}, \gamma')$.

  Besides, (ν_i) satisfies (P1) and therefore, by the easy part of Lemma 11, we have the convergence $\int \chi(a) d\nu_i(\chi) \to 1$, uniformly on compact subsets of A. Note that W is a closed and convex subset of $\mathcal{M}(\hat{A} \setminus \{1\})$. Fix i_0 such that for all $i \geq i_0$, we have $\nu_i \in W$. Consider the (finite) product

$$E = L^1(\hat{A} \setminus \{1\}, \gamma)^\Omega,$$

  equipped with the product of norm topologies. Let Σ be the convex hull of

$$\{(\nu_i * f - \nu_i)_{f\in\Omega},\ i \geq i_0\} \subset E.$$

  Since $(\nu_i * f - \nu_i)$ converges to 0 in the weak topology of E, the convex set Σ contains 0 in its weak closure. As E is locally convex, by Hahn-Banach's theorem[1], the weak closure of Σ coincides with its closure in the original topology of E. Hence there exists μ in the convex hull of $\{\nu_i : i \geq i_0\}$ such that $\|\mu * f - \mu\| \leq \varepsilon$ for all $f \in \Omega$; since W is convex, we have $\mu \in W$.

- (PC)$\Rightarrow$(PQ). Let (μ_i) be as in (PC). By density of compactly supported continuous functions, for all $f \in L^1(H)_{1,+}$, we have $\|f * \mu_i - \mu_i\| \to 0$. This convergence is uniform on each compact subset K of $L^1(H)_{1,+}$: this is a trivial consequence of the fact that $(f, \mu) \mapsto f * \mu$ is 1-Lipschitz for every μ.

  Now fix $f_0 \in L^1(H)_{1,+}$ and set $\mu_i' = f_0 * \mu_i$. It is easy to check that it satisfies (P1) and (P2).

  By a direct computation, we have, for any $h \in H$ and $\nu \in \mathcal{M}(\hat{A})$, the equality $h \cdot (\nu * f) = \Delta(h)\nu * f_0^h$, where f_0^h is the *right* translate of f_0, given by $f_0^h(g) = f_0(gh)$. Note that $\Delta(h)f_0^h \in L^1(H)_{1,+}$. Then for $h \in H$ we have

$$\begin{aligned}
\|h \cdot \mu_i' - \mu_i'\| &= \|h \cdot (\mu_i * f_0) - \mu_i * f_0\| \\
&= \|\mu_i * (\Delta(h)f_0^h) - \mu_i * f_0\| \\
&\leq \|\mu_i * (\Delta(h)f_0^h) - \mu_i\| + \|\mu_i * f_0 - \mu_i\|.
\end{aligned}$$

  Since the right regular representation of H on $L^1(H)$ is continuous, the function $h \mapsto \Delta(h)f_0^h$ is continuous as well so maps compact subsets of H to compact subsets of $L^1(H)_{1,+}$; therefore the above term converges to zero, uniformly on compact subsets of H. So (μ_i') satisfies (P3).

  Now suppose that we have chosen $f_0 > 0$ everywhere; this is possible since H is σ-compact. Let us show that (μ_i') satisfies (PQ): it only remains to prove that each μ_i' is quasi-invariant. Since $h \cdot \mu_i' = \mu_i * (\Delta(h)f_0^h)$, we have to show that the measures $\mu_i * f$, for positive $f \in L^1(H)$, all have the same null sets. If B is a Borel subset of $\hat{A}$ and $x \in \hat{A}$, we have

$$\begin{aligned}
f * 1_B(x) = 0 \quad &\Leftrightarrow \quad \int f(h) 1_B(h^{-1}x) dh = 0 \\
&\Leftrightarrow \quad \lambda(\{h : f(h)1_B(h^{-1}x) \neq 0\}) = 0 \\
&\Leftrightarrow \quad \lambda(\{h : 1_B(h^{-1}x) \neq 0\}) = 0
\end{aligned}$$

[1] The Hahn-Banach Theorem works because we are working with the weak topology (and not the weak-star). This is the reason why we need all the measures ν_i to have density with respect to a given measure γ. We are not able to bypass this argument.

(since f does not vanish) and this condition does not depend on f, provided $f > 0$. Thus we have

$$\begin{aligned}\mu_i * f(B) = 0 &\Leftrightarrow \mu_i(f * 1_B) = 0\\ &\Leftrightarrow \mu_i(\{x : f * 1_B(x) \neq 0\}) = 0\end{aligned}$$

and this condition does not depend on f. So $\mu_i * f$ and $\mu_i * f'$ are equivalent measures.

- (PQ)⇒(P) is trivial.

Let us justify the statement about nets and sequences for (P) (the proof for (PQ) being the same). Since G is assumed σ-compact, there is an increasing sequence (K_n) of compact subsets whose interiors cover G. In view of Lemma 11, Condition (P) can be written as: for every $\varepsilon > 0$ and every n, there exists a Borel probability $\mu_{n,\varepsilon}$ on $\hat{A} - \{1\}$ such that $\int \chi(a) d\mu_i(\chi) \geq 1 - \varepsilon$ for all $a \in K_n$. So the sequence $(\mu_{n,1/n})$ satisfies the required properties.

For the last statement, first observe that the proof of (¬T)⇒ (P) works without assuming that A is part of a semidirect decomposition. Now all properties except (¬T) only refer to the action on A, so their equivalence follows from the theorem applied to the semidirect product $(G \ltimes A, A)$. □

Proof of Corollary 2. We use Characterization (M). The "if" part is trivial. Conversely, suppose that $(H_1 \ltimes A, A)$ does not have Property T. So there exists an H_1-invariant mean on $\mathcal{L}^\infty(\hat{A})$, with $m(1_{\{0\}}) = 0$ and $m = \lim_\omega \nu_n$ with $\nu_n(V_n) = 0$ for some neighbourhood V_n of 1. Consider the restriction m' of m to $\mathcal{UC}_H(\hat{A})$. Since the action of H on $\mathcal{UC}_H(\hat{A})$ is separately continuous (that is, the orbital maps $H \to \mathcal{UC}_H(\hat{A})$ are continuous), the action on $(\mathcal{UC}_H(\hat{A}), \text{weak*})$ is continuous as well. So the stabilizer of m' is closed in H; since it contains the image of H_1 in H, this shows that m' is H-invariant. Fix $f \in C_c(H)_{1,+}$. Thanks to Lemma 10, we can define, for $\phi \in \mathcal{L}^\infty(\hat{A})$,

$$m''(\phi) = m'(f * \phi).$$

Clearly, m'' is an H-invariant mean on $\hat{A}$. Moreover, $m''(1_{\{1\}}) = m'(f * 1_{\{1\}}) = m'(1_{\{1\}}) = 0$, so m'' is not the Dirac measure at 1. Finally we have $m'' = \lim_\omega \nu'_n$ in the weak-star topology, where $\nu'_n(\phi) = \nu_n(f * \phi)$, and ν'_n is a probability on $\hat{A} - \{1\}$.

For the second statement, assume that $(H \ltimes A, A)$ has Property T. There exists a compact normal subgroup K in $G = H \ltimes A$ such that G/K is separable [Com, Theorem 3.7]. Consider a countable subgroup S of G whose image into G/K is dense, and let T be the closure of S in G. Set $K' = K/(A \cap K)$. Then G/K' is generated by T/K' and K/K'. Since K/K' centralizes A/K', the means preserved by G and by T on the Pontryagin dual of A/K' are the same. This Pontryagin dual is an open subgroup of $\hat{A}$, so the means preserved by G and by T at the neighbourhood of 1 in $\hat{A}$ are the same. Therefore $(T \ltimes A, A)$ has Property T. Now by the first statement of the Corollary, if S is endowed with the discrete topology, then $(S \ltimes A, A)$ has Property T. Finally by [Cor, Theorem 2.5.2], there exists a finitely generated subgroup Γ of S such that $(\Gamma \ltimes A, A)$ has Property T. □

Proof of Corollary 3. We first deal with the case when A is not σ-compact. First, this condition easily implies $(\neg\mathrm{T})$ (see for instance [Cor, Lemma 2.5.1]). It also implies (P). Indeed, let (G_i) be an increasing net of open, σ-compact subgroups of G and $A_i = G_i \cap A$, $H_i = A_i \cap G_i$. Let μ_i be the Haar measure on the orthogonal of A_i in $\hat{A}$; note that μ_i is H_i-invariant and $\mu_i(\{1\}) = 0$ since A_i has infinite index in A. So (μ_i) satisfies (P).

Now suppose that A is σ-compact. If either $(\neg\mathrm{T})$ or (P) is true for $H \ltimes A$, then it also holds for $L \ltimes A$ for any open subgroup L of H. Let us check that conversely, if it fails for $H \ltimes A$, then it fails for some σ-compact open subgroup $L \ltimes A$, so that the corollary reduces to the σ-compact case from the theorem. This is immediate for (P). For $(\neg\mathrm{T})$, if $(H \ltimes A, A)$ has Property T, by [Cor, Theorem 2.5.2], there exists an open compactly generated subgroup L of H, containing A, such that (L, A) has Property T. □

Proof of Theorem 4. In either case, suppose that the first condition is satisfied. We have a net (φ_i) of positive definite functions on G, converging to 1 uniformly on compact subsets of G, satisfying some additional condition on X. The proof of $(\neg\mathrm{T})\Rightarrow$(P) of Theorem 7 constructs a net of Borel measures (μ_i) on $\hat{A}$, with $\widehat{\mu_i} = \varphi_{|A}$ and $\|\mu_i - h\mu_i\| \to 0$. So we exactly get the second condition.

Conversely, suppose that the second condition is satisfied. Let Γ be the subgroup generated by an arbitrary finite subset S of H. Denote by T the average operator by S. Then $\widehat{T\mu} = T\widehat{\mu}$. If $\widehat{\mu_i}$ is C_0 on X and X is H-invariant, then $\widehat{T\mu_i}$ is also C_0 and $T\mu_i$ is also Γ-quasi-invariant, we can then follow the proof of (PQ)$\Rightarrow(\neg\mathrm{T})$ of Theorem 7 to obtain a net (φ_i) of positive definite functions on $\Gamma \ltimes A$ whose restriction to A is $\widehat{\mu_i}$.

On the other hand, suppose that the convergence of $\widehat{\mu_i}$ to 1 is not uniform on X. Then the convergence of $\widehat{T\mu_i}$ to one is also non-uniform on X (by an obvious positivity argument using that positive definite functions are bounded by one). Again, apply the proof of (PQ)$\Rightarrow(\neg\mathrm{T})$ to obtain the desired net.

In both cases, we obtain a net on a subgroup of the form $\Gamma \ltimes A$. These functions can be extended to positive definite functions [BHV, Exercise C.6.7] on $H \ltimes A$

by taking the value zero elsewhere. If we define the resulting functions as a net indexed by both the indices i and Γ, the resulting net exactly gives the relative Haagerup Property for (G, X), resp. the negation of relative Property T. $\square$

References

[AW] C. A. Akemann, M. E. Walter. *Unbounded negative definite functions*. Canad. J. Math. **33**, no 4, 862-871, 1981.

[BHV] B. Bekka, P. de la Harpe, and A. Valette. "Kazhdan's Property (T)". New mathematical monographs: 11, Cambridge University Press, 2008.

[Bur] M. Burger. *Kazhdan constants for* $\mathrm{SL}_3(\mathbf{Z})$, J. Reine Angew. Math. **431**, 36-67, 1991.

[CI] I. Chifan, A. Ioana. *On Relative Property (T) and Haagerup's Property.* ArXiv:0906.5363, to appear in Trans. Amer. Math. Soc.

[Com] W. Comfort. *Topological groups.* 1143–1263 in: "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, Edt.), North Holland, Amsterdam, 1984.

[Cor] Y. Cornulier. *On Haagerup and Kazhdan properties.* Thèse EPFL, no 3438 (2006). Dir.: Peter Buser, A. Valette.

[Cor2] Y. Cornulier. *Relative Kazhdan Property.* Annales Sci. École Normale Sup. 39(2), 301-333, 2006.

[Ioa] A. Ioana. *Relative Property (T) for the subequivalence relations induced by the action of* $\mathrm{SL}_2(\mathbf{Z})$ *on* $\mathbf{T}^2$. Preprint 2009, to appear in Adv. Math., arXiv 0901.1874 (v1).

[Kaz] D. Kazhdan. *On the connection of the dual space of a group with the structure of its closed subgroups.* Funct. Anal. Appl. **1**, 63-65, 1967.

[Mar] G. Margulis. "Discrete subgroups of semisimple Lie groups". Springer, 1991.

[Par] K. Parthasarathy. "Probability measures on metric spaces". Probability and Mathematical Statistics, No. 3 Academic Press, Inc., New York-London, 1967.

[Ry] R. Ryan. *Fourier transforms of certain classes of integrable functions.* Trans. Amer. Math. Soc. **105**, 1962 102–111.

[Sha] Y. Shalom. *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's property (T).* Trans. Amer. Math. Soc. **351**, 3387-3412, 1999.

[Sha2] Y. Shalom. *Bounded generation and Kazhdan's Property (T).* Publ. Math. Inst. Hautes Études Sci. **90**, 145-168, 1999.

IRMAR, Campus de Beaulieu, 35042 Rennes CEDEX, France
E-mail address: yves.decornulier@univ-rennes1.fr

ÉNS Lyon, 46, allée d'Italie, 69364 Lyon CEDEX 07, France
E-mail address: Romain.Tessera@umpa.ens-lyon.fr