COMMENSURATING ACTIONS FOR GROUPS OF PIECEWISE CONTINUOUS TRANSFORMATIONS

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ABSTRACT. We use partial actions, as formalized by Exel, to construct various commensurating actions. We use this in the context of groups piecewise preserving a geometric structure, and we interpret the transfixing property of these commensurating actions as the existence of a model for which the group acts preserving the geometric structure. We apply this to many groups with piecewise properties in dimension 1, notably piecewise of class $\mathcal{C}^k$, piecewise affine, piecewise projective (possibly discontinuous).

We derive various conjugacy results for subgroups with Property FW, or distorted cyclic subgroups, or more generally in the presence of rigidity properties for commensurating actions. For instance we obtain, under suitable assumptions, the conjugacy of a given piecewise affine action to an affine action on possibly another model. By the same method, we obtain a similar result in the projective case. An illustrating corollary is the fact that the group of piecewise projective self-transformations of the circle has no infinite subgroup with Kazhdan's Property T; this corollary is new even in the piecewise affine case.

In addition, we use this to provide of the classification of circle subgroups of piecewise projective homeomorphisms of the projective line. The piecewise affine case is a classical result of Minakawa.

1. INTRODUCTION

The goal of the "survey" [Cor1] was to emphasize the notion of commensurating action, which previously appeared in various contexts, but not systematically translated in the natural framework of group actions. The study of commensurating actions is closely related to group actions on CAT(0) cube complexes, which have been developed in the last two decades and now plays a prominent role in geometric group theory.

The purpose of this work is to show how the notion of partial group action, formalized by Exel [Ex], naturally fits into this context, and obtain various applications, notably to group actions on the circle "piecewise" preserving some geometric structure, such as piecewise affine or piecewise projective actions. The use of the formalism of partial actions punctually appeared in geometric group

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theory already, namely in B. Steinberg's work [Ste], but not in the context of commensurating actions.

1.A. **Pseudogroups.** The notion of space modeled on a "model" space A is conveniently encoded using the notion of pseudogroup. In examples, let us stick to the case of the circle $A = \mathbf{R}/\mathbf{Z}$: a pseudogroup over $\mathbf{R}/\mathbf{Z}$ is a certain set of partial homeomorphisms between open subsets, with suitable stability conditions. Examples are, when $A = \mathbf{R}/\mathbf{Z}$, or more generally when meaningful:

- the largest one: the pseudogroup of local homeomorphisms of A;
- the pseudogroup $\mathbf{Or}_A$ of orientation-preserving local homeomorphisms;
- the pseudogroup $\mathcal{C}^k$ of diffeomorphisms of class $\mathcal{C}^k$;
- the pseudogroup $\mathbf{Iso} = \mathbf{Iso}_A$ of isometric local homeomorphisms;
- the pseudogroup $\mathbf{Aff} = \mathbf{Aff}_A$ affine local homeomorphisms;
- the pseudogroup $\mathbf{Proj} = \mathbf{Proj}_A$ of projective local homeomorphisms (i.e., given in local coordinates as homographies);
- the pseudogroup of affine local homeomorphisms with dyadic slopes and translations lengths;
- for any S, the intersection $S+$ of S with $\mathbf{Or}$;

From each pseudogroup S, we have a notion of S-modeled space X due to Ehresmann. This is a topological space X endowed with charts, with open domain, valued in open subsets of the given model space with the axiom that changes of charts are in S, and that X is covered by the domain of charts.

For instance A itself is S-modeled, and any open subset of an S-modeled space is canonically S-modeled. A space modeled over the pseudogroup of orientation-preserving local homeomorphisms of $\mathbf{R}/\mathbf{Z}$ is an oriented purely 1-dimensional manifold. A space modeled over the pseudogroup of affine local homeomorphisms is a 1-dimensional affine manifold.

An S-modeled space X is called finitely-charted if has a finite covering by domains of charts. This is automatic if X is compact. Note that this is highly sensitive to the choice of model. For instance, as an $\mathbf{Aff}_{\mathbf{R}/\mathbf{Z}}$-modeled curve, $\mathbf{R}$ is not finitely-charted (while tautologically it is finitely-charted as an $\mathbf{Aff}_{\mathbf{R}}$-modeled curve). Similarly, the universal covering of $\mathbf{P}^1_{\mathbf{R}}$ is not finitely-charted as a $\mathbf{Proj}_{\mathbf{P}^1_{\mathbf{R}}}$-modeled curve.

1.B. **Parcelwise, piecewise groups.** Given an S-modeled Hausdorff topological space X with no isolated point, we define the group $\mathrm{PC}_S(X)$ of parcelwise-S self-transformations of X as the set of S-preserving homeomorphisms between two cofinite subsets of X, modulo coincidence on a smaller cofinite subset. This forms a group; it admits as a subgroup the subset $\mathrm{PC}^0_S(X)$ of elements having a representative that is a self-homeomorphism of X.

The subgroup $\mathrm{PC}^{\wp}_S(X)$ of piecewise-S self-transformations is defined with a little additional hypothesis, which, in the case of the pseudogroup of class $\mathcal{C}^k$, requires that the functions are left and right k-differentiable at each point (for suitable choices of the value at the given point). See §2.G.4 for the general definition. In the affine or projective context, parcelwise and piecewise being equivalent, we will use the more common word "piecewise". When talking informally, by "piecewise groups" we refer in general to these various groups $\mathrm{PC}_S^{\wp}(X)$ and $\mathrm{PC}_S(X)$, which are the guest stars of this article.

Define $\mathrm{PC}_S^0(X)$ as the set of homeomorphisms in $\mathrm{PC}_S(X)$. More generally, when S is contained in the class-$\mathcal{C}^k$ pseudogroup, it endows X with a structure of a $\mathcal{C}^k$-manifold. Thus, for $i \le k$ it makes sense to consider the subgroup $\mathrm{PC}_S^i(X)$ of $\mathcal{C}^i$-diffeomorphisms in $\mathrm{PC}_S(X)$.

Three famous nested families of piecewise groups play an important role in 1-dimensional dynamics. They were often originally motivated by two-dimensional dynamics. In what follows, we choose the circle $A = \mathbf{R}/\mathbf{Z}$ as a model space, endowed with various pseudogroups.

- Considering the pseudogroup of local orientation-preserving isometries. The corresponding group $\mathrm{PC}_{\mathbf{Isom}+}(X)$ is known as the group IET^+ of interval exchange transformations, introduced by Keane [Ke]. For the pseudogroup of all local isometries, the group $\mathrm{IET}^{\pm} = \mathrm{PC}_{\mathbf{Isom}}(X)$ is sometimes referred to as group of interval exchanges with flips, or group of linear involutions. Note that $\mathrm{PC}^0_{\mathbf{Isom}}(X)$ is reduced to isometries of X. their continuous analogues are tiny: these are the group of isometries. The study of these groups is largely motivated by two-dimensional dynamics (translation surfaces).
- The piecewise affine group $\mathrm{PC}_{\mathbf{Aff}+}(X)$ is sometimes referred as group of affine interval exchanges of X. Its study is often focussed on the continuous version: the group $\mathrm{PC}^0_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ of piecewise affine self-homeomorphisms of $\mathbf{R}/\mathbf{Z}$. This group is particularly known because of the popularity of R. Thompson's groups. As a whole, it has notably been studied by Brin and Squier [BrS].
- The continuous piecewise projective versions appeared in work of Strambach and Betten [Str, Be], identifying $\mathrm{PC}_{\mathbf{Proj}}(\mathbf{P}^1_{\mathbf{R}})$ as automorphism groups of a Moulton plane (an affine plane in the sense of incidence geometry, for which Desargues's theorem does not hold). A survey of these results can be found in [Low]. Its derived series is computed in [BeW].

  Another classical occurrence of the group $\mathrm{PC}^0_{\mathrm{Proj}}(\mathbf{P}^1_{\mathbf{R}})$ come from the fact that it includes a subgroup isomorphic to Thompson's group T, namely its subgroup for which breakpoints are in $\mathbf{P}^1_{\mathbf{Q}}$ and modeled on the pseudogroup of rational projective transformations. The continuous group $\mathrm{PC}^1_{\mathbf{Proj}+}(\mathbf{P}^1_{\mathbf{R}})$ was further investigated by Greenberg, in relation to foliations. He also introduced its class-$\mathcal{C}^1$ subgroup $\mathrm{PC}^1_{\mathbf{Proj}+}(\mathbf{P}^1_{\mathbf{R}})$.

  A recent renewal of the interest on the piecewise projective self-homeomorphism groups comes from Monod's remarkable observation [Mon] that the stabilizer of ∞ in the group $\mathrm{PC}^0_{S(R)}(\mathbf{P}^1_{\mathbf{R}})$ is non-amenable and has no free subgroup. Lodha and Moore [LM] used this to produce explicit finitely presented subgroups with the same property.

Let X be the circle, or more generally an oriented curve. We can cheat in the definition of $\mathrm{PC}^+(X)$ by choosing left-continuous elements, which allows to define $\mathrm{PC}^+(X)$ as a subgroup of the permutation group X, rather than using a quotient construction. For $\mathrm{PC}(X)$ and its various non-orientation-preserving subgroups such as $\mathrm{IET}^{\pm}$, one cannot define it in this way. This nuance, which purely lies at the level of definitions, is possibly a reason why authors frequently restrict to the piecewise orientation-preserving groups.

1.C. **Commensurating actions.** Given a group G acting on a set Y, a subset X is commensurated if, $\triangle$ denoting the symmetric difference, $X \triangle gX$ is finite for every $g \in G$. We call (Y, X) a commensurating action. A stronger condition is being transfixed: this means that there exists a G-invariant subset X_0 such that $X \triangle X_0$ is finite.

Being transfixed is the "obvious" reason for being commensurated and the richness of the theory comes from the failure of the converse. The simplest example of a non-transfixing commensurating action is the action of $\mathbf{Z}$ on itself by translation, commensurating $\mathbf{N}$. The notion, more generally, underlies the whole theory of ends of groups and ends of coset spaces initiated by Freudenthal, pursued by Hopf and Specker and many others since then, see notably the Dicks-Dunwoody book [DD].

1.D. **Partial actions.** Exel [Ex] defined a notion of partial action α of a group G on a set: each group element g acts as a bijection $\alpha(g) : D_g \to D_{g^{-1}}$ between subsets of X, with the axioms that $D_1 = \mathrm{id}_X$ and that whenever $\alpha(g)(\alpha(h)x)$ and $\alpha(gh)x$ are both defined, they are equal. The subset D_g is called the domain of g.

An example consists in starting from an action α of G on a set X, considering a subset Y yields by restriction a partial action α_Y on Y: $\alpha_Y(g)y$ is defined if and only if $\alpha(g)y \in Y$ and $\alpha_Y(g)y = \alpha(g)y$. Conversely every partial action occurs this way, and that moreover there is one universal way to do so, in a categorical sense. This is called the universal globalization $X \subset \hat{X}$; it is not obvious to whom this construction can be attributed but it seems natural to quote Megrelishvili [Me1, Me2], Abadie [Aba1, Aba2] and Kellendonk-Lawson [KL], see Theorem 2.10 and Remark 2.12. An important feature we will use, also established in [Aba1, Aba2, KL], is that the universal globalization works at a topological level (see Theorem 2.13). When X carries more structure, preserved by the partial action, the universal globalization thus inherits the structure; however it often fails to be Hausdorff even starting from a "gentle" space X such as the circle: Abadie [Aba1, Aba2] provided a general criterion for this.

A partial action of a group G on a set X for which all group elements have a cofinite domain is called a cofinite-partial action. (Cofinite means with finite complement.) In this context, X is a commensurated subset of $\hat{X}$. Therefore, it makes sense to say that X is transfixed. This can be restated with no reference to $\hat{X}$: this means that $X \smallsetminus D_g$ has a cardinal bounded independently of g, where D_g is the domain of definition of g.

1.E. **Partial actions of piecewise groups.** It is not clear from the definition of $\mathrm{PC}_S(X)$ that it comes with a natural partial action on X. Indeed, $\mathrm{PC}_S(X)$ was defined as equivalence classes of maps while a partial action comes with an assignment of domain of definitions. This is an essential difference. However, in some specific cases, relying only on a local topological assumption on the underlying topological space, there is a canonical way to do so.

Lemma 1.1 (Main Lemma)**.** *Let A be a Hausdorff topological space with no isolated point.*

Let S be a pseudogroup over A. Let X be a Hausdorff, S-modeled topological space. Then there is a naturally defined cofinite-partial action α_S of $\mathrm{PC}_S(X)$ on X.

Let us not formalize "naturally defined", just mentioning that the construction (§2.F and §2.G.6) is not based on any choice. It satisfies the condition that $\alpha_S(\sigma)$ is a representative of σ, for every $\sigma \in \mathrm{PC}_S(\sigma)$.

Let us illustrate the lemma for a few pseudogroups on the circle. For a function f defined modulo indeterminacy on finite subsets, we denote by $f(x^{\pm})$ the right/left limit of f at x (keep in mind that f is not defined at x, but its germ on a neighborhood of x minus x is well-defined);

- Case of the pseudogroup of piecewise isometries: the domain of definition of σ is the set of outer continuity points, that is, the set of points x for which $\sigma(x^+) = \sigma(x^-)$;
- Case of the pseudogroup of local affine homeomorphisms: For a piecewise affine self-transformation on the circle $\mathbf{R}/\mathbf{Z}$, the domain of definition is the set of points x for which both $\sigma(x^+) = \sigma(x^-)$ and $\sigma'(x^+) = \sigma'(x^-)$, where σ' is the derivative;
- For the pseudogroup of local projective homeomorphisms: idem, but with also the same condition on the second derivative.

For various pseudogroups, this lemma therefore yields a cofinite-partial action, which, thanks to the existence of a universal globalization, naturally yields a commensurating action (which in practice can be complicated to "compute"). Therefore, we can exploit our knowledge about commensurating actions to obtain results.

For a commensurating action (Y, X) of a group G, we have counting results: for instance, writing $\ell^-(g) = |X \smallsetminus g^{-1}X|$, we have the existence of an integer $m_g \in \mathbf{N} = \{0, 1, 2, \dots\}$ such that $n \mapsto \ell^-(g^n) - m_g n$ is bounded independently of n (see Proposition 2.1). In particular, $n \mapsto \ell^-(g^n)$ has either linear growth, or is bounded.

Let us apply this to the $\mathcal{C}^k$-pseudogroup. Let $k \in \mathbf{N} = \{0, 1, \dots\}$ be an integer. For σ be a parcelwise-$\mathcal{C}^k$ self-transformation of $\mathbf{R}/\mathbf{Z}$ (or of an open interval), define $K_0(\sigma)$ as the subset of outer discontinuity points of σ (that is, the set of x such that $\sigma(x^+) \neq \sigma(x^-)$. For $x \notin K_0(\sigma)$, we can choose the value of $\sigma(x)$ as the one making σ continuous at x. For $i \in \{1, \dots, k\}$ let $K_i(\sigma)$ be the subset of those $x \notin K_{i-1}(\sigma)$ at the neighborhood of which σ is not of class $\mathcal{C}^i$. For $i \in \{0, \dots, k\}$, let $k_i(\sigma)$ be the cardinal of $K_i(\sigma)$ and $k_{\le i}(\sigma) = \bigoplus_{j=0}^{i} k_j(\sigma)$. Note that $k_i(\sigma) = k_i(\sigma^{-1})$.

Corollary 1.2 (See Corollary 4.6)**.** *For every $k \in \mathbf{N}$ and parcelwise-$\mathcal{C}^k$ self-transformation σ as above, there exist integers $0 \le m_0 \le \cdots \le m_k$ and bounded non-negative even functions $b_i : \mathbf{Z} \to \mathbf{N}$ such that for all $i \in \{0, \dots, k\}$, we have $k_{\le i}(\sigma^n) = m_i|n| + b_i(n)$.*

In particular, $k_{\le i}$ and k_i have the property of growing either linearly or being bounded.

This lets us retrieve or improve some known results in a unified way. Applied when $k = 0$ and in the case of interval exchanges, this concerns the discontinuity growth for interval exchanges and this counting result was obtained in [Nov, DFG]. When $k = 1$, in the case of piecewise affine self-transformations, it was obtained by Guelman and Liousse [GuL, §4] that the sequences $(k_1(\sigma^n))_{n\ge0}$ (and $(k_0(\sigma^n))_{n\ge0}$) are either bounded of have linear growth.

1.F. **Interpretation of the transfixing condition.** In a next step, we provide conjugacy results for actions $G \to \mathrm{PC}_S(X)$, under the assumption that G transfixes X for the partial action α_S. Then we can exploit result about commensurating actions, to obtain cases when it is automatically transfixing.

Namely, a group G has Property FW if every commensurating action of G is transfixing. More generally, given a subgroup H of G, the pair (G, H) has relative Property FW if every commensurating action of G is transfixing in restriction to H. This is notably the case

- when H has Property FW;
- when (G, H) has relative Property FH (in the sense that any isometric action of G on a Hilbert space has an H-fixed point;
- when H is cyclic and distorted (see Corollary 2.2);
- when $H = \langle c \rangle$ is cyclic and unboundedly divisible (in the sense that for every m there exists $n \ge m$ and $\gamma \in G$ such that $\gamma^n = c$), see Corollary 2.2;
- when H is cyclic and normal in G with G finitely generated, and H not a direct factor in any finite index subgroup in which H is a direct summand [Cor1, Proposition 6A8];
- when G is finitely generated, polycyclic and H is the largest normal subgroup such that G/H is virtually abelian and has no nontrivial finite

normal subgroup (this is essentially due to Houghton [Ho], see [Cor1, Proposition 6C5]).

Property FW is established for various lattices in semisimple groups in [Cor2], including cases without Property T.

The main conjugacy theorem is Theorem 4.7. Here we state it not providing all definitions, and focus on its corollaries obtained by applying it to various pseudogroups.

Theorem 1.3. *Let X be a Hausdorff S-modeled space. Let G be a group with a topological S-preserving partial action on X. Suppose that G transfixes X for the partial action α_S of §2.G.6. Then there exists a Hausdorff S-modeled space X' endowed with an S-preserving continuous G-action, and a cofinite S-preserving G-biequivariant isomorphism from X to X'.*

By curve we mean a topological space homeomorphic to a finite disjoint union of copies of $\mathbf{R}$ and the circle. We start with the application to the $\mathcal{C}^k$-pseudogroup.

Corollary 1.4. *Fix $k \in \mathbf{N}$. Let G act by piecewise-$\mathcal{C}^k$ (resp. parcelwise-$\mathcal{C}^k$) self-transformations on $\mathbf{R}/\mathbf{Z}$. Suppose that G transfixes $\mathbf{R}/\mathbf{Z}$ for the corresponding partial action. Then there exists a cofinite subset $Y \subset \mathbf{R}/\mathbf{Z}$, a curve X' with a structure of a $\mathcal{C}^k$-manifold and a G-action by $\mathcal{C}^k$-diffeomorphisms, a cofinite subset $Y' \subset X'$ and a G-biequivariant piecewise-$\mathcal{C}^k$ (resp. parcelwise-$\mathcal{C}^k$) homeomorphism $h : Y \to Y'$.*

Note that G-biequivariant is meant in the sense of partial actions: it means the homeomorphism conjugates one partial G-action to another.

For $k = 0$, the conclusion essentially says that we can find a "continuous model". The result is flexible, since we can apply it to the pseudogroup we wish. Notably, if the action is piecewise orientation-preserving (resp. piecewise isometric, resp. piecewise orientation-preserving and isometric), the conjugating map h can be chosen with the same property. In the latter case, this is a result of conjugacy in the group of interval exchanges, which is known [Nov, DFG] in one particular case: when G is cyclic. In this case the transfixing condition is reflected in the condition that its powers have a uniformly bounded number of discontinuity points.

Now applying the theorem to the affine or projective pseudogroup, we obtain:

Corollary 1.5. *Let G act by piecewise affine (resp. piecewise projective) self-transformations on $\mathbf{R}/\mathbf{Z}$. Suppose that G transfixes $\mathbf{R}/\mathbf{Z}$ for the corresponding partial action $\alpha_{\mathbf{Aff}}$ (resp. $\alpha_{\mathbf{Proj}}$). Then there exists a cofinite subset $Y \subset \mathbf{R}/\mathbf{Z}$, a Hausdorff finitely-charted affine (resp. projective) curve X' with a G-action by affine (resp. projective) automorphisms, a cofinite subset $Y' \subset X'$ and a G-biequivariant affine (resp. projective) (in the sense of partial actions) homeomorphism $h : Y \to Y'$.*

(See §1.A for the meaning of finitely-charted.) We explicitly mention the Hausdorff hypothesis, because there are many non-Hausdorff curves, and they turn out to play an essential intermediate role in the proof (see Remark 2.27).

In the continuous case, we can improve the conclusion to assume that Y and Y' are G-invariant subsets, which allows to assume $Y' = X'$ (at the cost of making the homeomorphism piecewise affine/projective). Explicitly:

Corollary 1.6. *For $k \in \mathbf{N}$. Let G act by piecewise affine (resp. piecewise projective) self-homeomorphisms on $\mathbf{R}/\mathbf{Z}$. Suppose that G transfixes $\mathbf{R}/\mathbf{Z}$ for the corresponding partial action $\alpha_{\mathbf{Aff}}$ (resp. $\alpha_{\mathbf{Proj}}$). Then there exists a cofinite subset $Y \subset \mathbf{R}/\mathbf{Z}$, a Hausdorff finitely-charted affine (resp. projective) curve X' with a G-action by affine (resp. projective) automorphisms, and a G-biequivariant piecewise affine (resp. piecewise projective) homeomorphism $h : Y \to X'$.*

Remark 1.7. Although in a different context, let us mention another result concluding the existence of an invariant projective structure. Navas [Nav2, Proposition 2.1] proves, for a group of $\mathcal{C}^3$-diffeomorphisms of the circle, assuming that it has no invariant probability measure and a certain technical condition, concludes that there is an invariant projective structure. The technical condition roughly says that some cocycle, called Liouville cocycle, related to the distortion of cross-ratios, is a coboundary, As in the current paper, he uses this proposition in combination with the knowledge of the automorphism group of projectively modeled curves, namely the fact that the orientation-preserving automorphism group is metabelian unless the projectively modeled curve is a finite covering of the projective line.

1.G. **More specific corollaries.**

Corollary 1.8. *(a) (See Corollary 4.12) The group $\mathrm{PC}_{\mathbf{Aff}}(X)$ of piecewise affine self-transformations of $X = \mathbf{R}/\mathbf{Z}$ has no infinite subgroup with Property FW (and hence none with Kazhdan's Property T).*

(b) (See Corollary 4.19) The group $\mathrm{PC}_{\mathbf{Proj}}(X)$ of piecewise projective self-transformations of $X = \mathbf{R}/\mathbf{Z}$ has no infinite subgroup with Property T.

See also Corollary 4.18 for strong restrictions on possible subgroups with Property FW in the group of piecewise projective self-transformations of $\mathbf{R}/\mathbf{Z}$.

In [LMT, Corollary 1.3], written independently of this paper, the authors prove Corollary 1.8(b) in the continuous case, relying on a commensurating action allowing to reduce to a theorem of Navas saying that the group of C^2-diffeomorphisms of the circle has no infinite subgroup with Kazhdan's Property T. They also provide an alternative more direct argument applying to the continuous piecewise affine case.

While our results in the piecewise affine case are partly new but accessible by approaches by hand, the projective case is certainly much more difficult to reach in such a way. Indeed, the conjugacy results in the piecewise isometric or piecewise affine case essentially implicitly use the classification of curves modeled on the pseudogroup of local isometries or local affine maps. The classification in the isometric case is trivial, and the change of model works by plain "cut-and-paste". In the affine context, we have to deal with non-standard affine structure on the circle, which can be obtained from modding out $\mathbf{R}_{>0}$ by the cyclic subgroup $\langle t\rangle$ generated by a nontrivial homothety. Along with the standard circle $\mathbf{R}/\mathbf{Z}$, this is a complete classification. This arouses an issue in gluings when it comes to "closing a circle". The artifact to rule out this issue by hand is to allow continuous piecewise affine self-transformations having at most two singularities by circle component. The use of affinely modeled structures avoids the use of this artifact.

The classification of 1-dimensional projective structures (see Appendix A) becomes too complicated to be attainable in such a disguise. It is thanks to this classification that we can obtain Corollary 1.8(b), using that the automorphism group of a projectively modeled curve cannot contain an infinite subgroup with Property T.

Example 1.9. Using Corollary 1.5, one obtains that the group $\mathrm{PSL}_2(\mathbf{Z}[i,\sqrt{2}])$ does not embed into $\mathrm{PC}_{\mathbf{Proj}}(\mathbf{R}/\mathbf{Z})$. Indeed, to start with, it has Property FW: this uses bounded generation by elementary unipotent elements due to Carter-Keller [WiM, Theorem 25.11]. Given this, the easy argument to deduce Property FW is the same as the one [Cor1] for $\mathrm{PSL}_2(\mathbf{Z}[i,\sqrt{2}])$. Then using Corollary 1.5, if there were such an embedding, we would obtain a finite index subgroup Γ with a homomorphism with infinite image into the automorphism group of a connected projectively modeled curve. Such automorphism groups are either metabelian, or isomorphic to a finite covering of $\mathrm{PSL}_2(\mathbf{R})$. That Γ (which is an irreducible arithmetic lattice in $\mathrm{PSL}_2(\mathbf{C})^2$) has no homomorphism with infinite image into $\mathrm{PSL}_2(\mathbf{R})$ follows from Margulis' superrigidity.

Recall that an element g of a group G is distorted if there exists a finite subset S of G such that $g\in\langle S\rangle$ and $\lim_n |g^n|_S/n=0$, where $|\cdot|_S$ denotes word length with respect to S. For instance elements of finite order are distorted. A cyclic subgroup $\langle g\rangle$ is by definition distorted in G if g has infinite order and is distorted (so finite order elements are distorted but finite cyclic subgroups are undistorted!).

Corollary 1.10. *For any distorted element σ in $\mathrm{PC}_{\mathrm{Aff}}(\mathbf{R}/\mathbf{Z})$ (resp. $\mathrm{PC}_{\mathrm{Proj}}(\mathbf{R}/\mathbf{Z})$) there exists a cofinite subset $Y\subset\mathbf{R}/\mathbf{Z}$, an affine (resp. projective) curve X' with a cofinite subset Y', and an affine (resp. projective) homeomorphism $\mathbf{R}/\mathbf{Z}\to Y'$, such that σ induces an affine (resp. projective) automorphism of X'.*

Moreover, in the affine case, we can suppose that X' is a standard curve (i.e., a finite disjoint union of isomorphic copies of $\mathbf{R}/\mathbf{Z}$ and $]0,1[$).

This statement is a corollary of the suggested method, except the last assertion, which requires an additional argument, essentially to rule out irrational rotations of non-standard affine circles. The affine case of Corollary 1.10 is close to a result recently obtained by Guelman and Liousse [GuL] while this paper was in preparation: they obtain it up to some minor differences: they work with piecewise orientation-preserving piecewise affine self-transformations, and they obtain the conjugation for some power of σ.

Question 1.11. Let X be a disjoint union of two standard circles. Let r be an irrational rotation of the first circle (acting trivially on the complement). Is r distorted in $\mathrm{PC}_{\mathbf{Aff}}(X)$?

Actually, it is unknown whether the group $\mathrm{PC}_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ admits a distorted cyclic subgroup; this question is originally due to Navas [HL, Cor. 1.10]. Let us observe that this is equivalent (see Corollary 4.15, which includes more restatements) to a negative answer to Question 1.11 for some r (a closely related statement is implicit in [GuL]). Actually, the answer to Question 1.11 is unknown for any r.

1.H. **Exotic circles in the group of piecewise projective self-homeomorphisms of the circle.** For the circle $X = \mathbf{R}/\mathbf{Z}$, Minakawa [Mi] solved the question of classifying up to conjugacy the subgroups of $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(X) = \mathrm{PC}_{\mathbf{Aff}}(X) \cap \mathrm{Homeo}^+(X)$, that are conjugate in $\mathrm{Homeo}^+(X)$ to the group $\mathbf{R}/\mathbf{Z}$ of translations. His classification depends on a positive parameter $t > 0$. It turns out that our approach allows to retrieve this result, but also to observe that this parameter $t > 0$ naturally corresponds to the much older classification by Kuiper of oriented affine structures on the circle, mentioned above. This also allows to settle the corresponding problem in the piecewise projective setting.

For the notation: write $\Theta_t = \mathbf{R}_{>0}/\langle t\rangle$ (non-standard affine circle) and $\Theta_1 = \mathbf{R}/\mathbf{Z}$. Also, define Σ_∞ as the universal covering of the pointed space $(\mathbf{P}^1_{\mathbf{R}}, \infty)$. Lift the 1-parameter group of 2-dimensional rotations $(\bar{\xi}_r)_{r\in\mathbf{R}/2\mathbf{Z}}$ to a one-parameter subgroup (ξ_r) in the universal covering of $\mathrm{SL}_2(\mathbf{R})$, and for $r \neq 0$ define Ξ_r as the "metaelliptic" projectively modeled curve, namely the quotient $\Sigma_\infty/\langle\xi_r\rangle$. Note that for $n \in \mathbf{N}_{>0}$, the metaelliptic curve Ξ_n is an n-fold covering of $\mathbf{P}^1_{\mathbf{R}}$. The projectively modeled curves $(\Theta_t)_{t\geq 1}$ and $(\Xi_r)_{r>0}$ are pairwise non-isomorphic. These turn out to be the only connected projectively modeled curves whose automorphism group contains a copy of the circle group $\mathbf{R}/\mathbf{Z}$, see the appendix.

Theorem 1.12 (Theorem 5.8)**.** *Fix $X = \mathbf{R}/\mathbf{Z}$. Let G be a closed subgroup of* $\mathrm{Homeo}(X)$ *isomorphic to $\mathbf{R}/\mathbf{Z}$. Suppose that $G \subset \mathrm{PC}_{\mathbf{Proj}}(X)$. Then*

(1) G preserves a unique projective structure π on the topological curve X such that the identity map $X \to (X,\pi)$ is piecewise projective;
(2) the projectively modeled curve (X,π) is isomorphic to either Θ_t for some $t \geq 1$ or Ξ_r for some $r > 0$: we thus say that (X,π) is of type Θ_t or Ξ_r;
(3) the type of (X,π) determines G up to conjugation in $\mathrm{PC}^0_{\mathbf{Proj}}(X)$, or indifferently in $\mathrm{PC}^{0,+}_{\mathbf{Proj}}(X)$;

(4) *if* $G \subset \mathrm{PC}^1_{\mathbf{Proj}}(X)$ *(the group of* $\mathcal{C}^1$*-diffeomorphisms in* $\mathrm{PC}^1_{\mathbf{Proj}}(X)$*) the type of* (X, π) *determines* G *up to conjugation in* $\mathrm{PC}^1_{\mathbf{Proj}}(X)$*, or indifferently in* $\mathrm{PC}^{1,+}_{\mathbf{Proj}}(X)$*).*

The group $\mathrm{PC}^1_{\mathbf{Proj}}(X)$ of piecewise projective $\mathcal{C}^1$-diffeomorphisms was introduced by Greenberg [Gre]. The question of classifying exotic circles within this group is explicit in Sergiescu's notes [Ser].

At a topological level, the only other connected Lie group actions on the circle are given by the action of $\mathrm{PSL}_2(\mathbf{R})$ on the projective line and its finite coverings. These actions preserve a projective structure, by definition. Answering a question in [LMT], we show here there are no "exotic" versions of such actions:

Theorem 1.13 (see Corollary 5.10)**.** *Let* G *be the image of a continuous and injective homomorphism from the* n*-fold covering* $\mathrm{PSL}_2^{(n)}(\mathbf{R})$ *to* $\mathrm{Homeo}(\mathbf{R})$*. Suppose that* $G \subset \mathrm{PC}_{\mathbf{Proj}}(X)$*. Then* G *is uniquely determined up to conjugation. If moreover* $G \subset \mathrm{PC}^1_{\mathbf{Proj}}(X)$ *then* G *is uniquely determined up to conjugation in* $\mathrm{PC}^1_{\mathbf{Proj}}(X)$*.*

We finish this introduction by repeating [Cor2, Question 1.19(2)]:

Question 1.14. Does there exist an infinite, finitely generated amenable group with Property FW?

I owe to Nicolás Matte Bon the remark that the absence of infinite Property FW groups in IET (and in the piecewise affine group), established here, discards the most natural candidates for such groups. Note that such a group would even have no homomorphism with infinite image in a piecewise projective group: indeed, in the piecewise projective group, any subgroup with Property FW and not virtually metabelian virtually admits a homomorphism with Zariski-dense image into $\mathrm{PSL}_2(\mathbf{R})$ and hence has a nonabelian free subgroup (see Corollary 4.18).

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2. Preliminaries

2.A. **Commensurating actions.** Denote by $\triangle$ the symmetric difference between subsets of a set, and $|\cdot|$ the cardinal function.

Given a group G acting on a set Y, a subset $X \subset Y$ is commensurated if $X \triangle gX$ is finite for every $g \in G$. We then call (Y, X) a commensurating action of G. We say that the commensurating action (Y, X) is transfixing, or that the subset $X \subset Y$ is transfixed, if there exists a G-invariant subset X_0 such that $X \triangle X_0$ is finite.

Transfixed implies commensurated, and is actually equivalent [BPP] to the boundedness of the function $\ell_X : g \mapsto |X \triangle gX|$. The latter is called the cardinal definite function of (Y, X)).

Continue with an arbitrary subset X of Y. Let us denote $\ell_X^+ : g \mapsto |X \smallsetminus gX|$ and $\ell_X^-(g) = \ell_X^+(g^{-1}) = |X \smallsetminus g^{-1}X|$. Call ℓ_X^+ and ℓ_X^- the loanshark and the prodigal semi-index functions of (Y, X). Note that Y being commensurated means that either of these functions takes finite values, and transfixed means that either of these function is bounded. Each of these functions, say ℓ, satisfies $\ell(1) = 0$ and $\ell(gh) \le \ell(g) + \ell(h)$ for all $g, h \in G$. Note that $\ell_X = \ell_X^+ + \ell_X^-$. (When X is G-commensurated, the difference $g \mapsto \ell^+(g) - \ell^+(g^{-1})$ is a well-defined map $G \to \mathbf{Z}$, and actually a group homomorphism, called index character of (Y, X), see [Cor1, §4.H].)

Proposition 2.1. *1)* [Cor1, Cor. 6.A.2] *Every cardinal definite function on* $\mathbf{Z}$ *has the form* $n \mapsto m|n| + b(n)$ *for some unique* $m \in \mathbf{N}$ *and bounded non-negative function* $b : \mathbf{Z} \to \mathbf{N}$ *(where* $\mathbf{N} = \{0, 1, \dots\}$*).*

2) For every commensurating action of $\mathbf{Z}$*, the corresponding prodigal semi-index function has, in restriction to* $\mathbf{N}$*, the form* $n \mapsto m'|n| + b(n)$ *for some unique* $m' \in \mathbf{N}$ *and bounded non-negative function* $b' : \mathbf{Z} \to \mathbf{N}$*.*

Proof. 2) Let (Y, X) be a commensurating action of $\mathbf{Z} = \langle u \rangle$. Let (Y, X) be a commensurating action of $\mathbf{Z} = \langle u \rangle$ and X a commensurating subset; let $\ell = \ell_{Y,X}$ be the corresponding cardinal definite function and $\ell^+ = \ell_{Y,X}^+$.

We start with the case when Y is $\langle u \rangle$-transitive, that is, consists of a single cycle. If Y is finite, then ℓ is bounded. Otherwise, $\mathbf{Z}$ acts simply transitively on Y and hence we can suppose that $Y = \mathbf{Z}$ with $u(n) = n + 1$. Then X is a subset with finite boundary, which therefore has a finite symmetric difference with some $X' \in \{\emptyset, -\mathbf{N}, \mathbf{N}, \mathbf{Z}\}$. So $\ell_X^+ - \ell_{X'}^+$ is bounded. We have $\ell_{X'}^+|_{\mathbf{N}} = 0$ for $X' \in \{\emptyset, -\mathbf{N}, \mathbf{Z}\}$, and $\ell_{\mathbf{N}}^+(n) = n$ for all $n \in \mathbf{N}$. We can thus write $\ell^+(n) = m^+n + b(n)$ with $m^+ \in \{0, 1\}$, for all for $n \in \mathbf{N}$. (Note that we have a similar formula for ℓ^- with some $m^- \in \{0, 1\}$ and that $(m^+, m^-) \in \{(0, 0), (1, 0), (0, 1)\}$.) A simple argument using sub-additivity of ℓ^+ shows that $b \ge 0$.

Adding over finitely many orbits, we obtain the result when Y consists of finitely many orbits.

In general, let W be the union of orbits of elements of the finite subset $X \triangle uX$; it consists of finitely many orbits W_i. Then $X \cap W^c$ is invariant, and hence we have $\ell = \ell_{W, X\cap W}$ and $\ell^+ = \ell_{W, X\cap W}$. This reduces to the case when there are finitely many orbits, which has been settled.

1) From (2) and since $\ell(n) = \ell^+(n) + \ell^+(-n)$, we can write $\ell(n) = m_\pm n + b_\pm(n)$ for all $n \in \pm\mathbf{N}$. Since $\ell(n) = \ell(-n)$ for all n, taking the limit of $(\ell(n) - \ell(-n))/n$ when $n \to \infty$ yields $m_+ = m_-$, and in turn we deduce $b_+ = b_-$. □

The first consequence below was originally observed as a consequence of a more difficult result of Haglund [Hag] on isometries of CAT(0) cube complexes.

Corollary 2.2. *Let G be a group and $\langle c\rangle$ a cyclic subgroup. Suppose that c is distorted, or c is unboundedly divisible. Then $(G, \langle c\rangle)$ has relative Property FW.*

Proof. Let ℓ be a cardinal definite function on G.

Suppose that c is distorted in some finitely generated subgroup Γ of G, and let $|\cdot|$ be the word length on Γ with respect to some finite generating subset. Since ℓ is subadditive, there exists C such that $\ell(g) \le C|g|$ for all $g \in \Gamma$. In particular, $\ell(c^n) \le C|c^n|$. If c is distorted, then $\lim |c^n|/n = 0$, and we deduce $\lim \ell(c^n)/n = 0$. By Proposition 2.1, it follows that $\sup_n \ell(c^n) < \infty$.

Also by Proposition 2.1, the limit $m(g) = \ell(g^n)/n$ belongs to $\mathbf{N}$ for all $g \in G$. Clearly, $m(g^k) = km(g)$. It immediately follows that if c has roots of unbounded order, then $m(c) = 0$, and hence, again by Proposition 2.1, we have $\sup_n \ell(c^n) < \infty$. □

Proposition 2.3. *Let A be a finitely generated abelian group and ℓ a cardinal-definite. Then there exist subgroups B, A' of A such that $B \cap A' = \{0\}$, $B + A'$ has finite index in A, while the length ℓ is bounded on B and has growth equivalent to the word length on C.*

Proof. Passing to a finite index subgroup, we can suppose that $A = \mathbf{Z}^d$. Let B be the maximal subgroup on which A is bounded. Let A' be any maximal subgroup among those with $B \cap A' = \{0\}$. Then $B + A'$ has finite index. Replacing A with A' boils down to proving that if $B = \{0\}$ then ℓ is equivalent to the word growth.

Let f be a cardinal-definite function on A. First suppose that it is associated to a transitive commensurating action A/E, with commensurated subset M, and that f is unbounded. Then A/E has more than one end, and hence is 2-ended. Let $\chi : A/E \to \mathbf{Z}$ a surjective homomorphism (this is unique up to sign). Then, up to replace χ with $-\chi$, the subset M has finite symmetric difference with $\chi^{-1}(M)$, and then $\ell(v) = m|\chi(v)| + O(1)$, where m is the cardinal of the kernel of χ.

In general, as in the proof of Proposition 2.1, f is equal to the cardinal-definite function associated to an action with finitely many orbits (this actually holds, by the same arguments for arbitrary finitely generated groups). Hence there exist k and homomorphisms $A \to \mathbf{Z}$ such that we have $f(v) = \sum_{i=1}^k |\chi_i(v)| + O(1)$.

Note that χ extends to a linear form on $\mathbf{R}^d$ with integral coefficients. Then $\nu = \sum_{i=1}^k |\chi_i|$ defines a seminorm on $\mathbf{R}^d$, whose vanishing subspace is $\bigcap_i \mathrm{Ker}(\chi)_i$. The latter is a rational subspace. Since ℓ is unbounded on any nonzero subgroup of $\mathbf{Z}^d$, we deduce that this rational subspace is zero. Hence ν is a norm, hence $\nu \geq c\|\cdot\|_1$ for some $c > 0$. Since the ℓ^1-norm $\|\cdot\|_1$ coincides with the word length on $\mathbf{Z}^d$, we thus have the required inequality. □

The following lemma is far from optimal but we find convenient to write it for reference.

Lemma 2.4. *Let G be a Lie group of dimension 1, of the form $G^\circ \rtimes F$ with F either trivial or cyclic of order 2, acting by inversion on the abelian unit component G°. Then G has a finitely generated dense subgroup Γ such that (G, Γ) has relative Property FW as abstract group.*

Proof. Let f be a cardinal definite function on G. Choose $u, v \in G^\circ$ generating a dense subgroup Λ of G°, and set $\Gamma = \Lambda \rtimes F$. Since each of u, v is divisible in G, we have f bounded on both $\langle u \rangle$ and $\langle v \rangle$ by Corollary 2.2. Since f is subadditive and since every element of Γ can be written as $u^n v^m k$ with $(n, m, k) \in \mathbf{Z}^2 \times F$, we deduce that f is bounded. □

2.B. **Using lemmas of B. Neumann and P. Neumann.** We use the following lemma holding for arbitary group actions.

Lemma 2.5. *Let G be a group and Z a G-set. Let U be a cofinite subset of Z including all finite G-orbits. Then for any finite subset F of Z there exists $g \in G$ such that $gF \subset U$.*

Proof. A result of P. Neumann [Neu2, Lemma 2.3] states $(*)$ that for every G-set W with no finite orbit and every finite subset F' of W, there exists $g \in G$ such that $gF' \cap F'$ is empty. (This is an easy consequence of B. Neumann's result [Neu1] that a group is never covered by finitely many left cosets of infinite index subgroups.)

Let Z_∞ be the union of all infinite G-orbits; by assumption $U \cup Z_\infty = Z$. Define $F' = (Z \smallsetminus U) \cup (F \cap Z_\infty)$. Then $F' \subset Z_\infty$, and we can apply $(*)$ to $W = Z_\infty$ and F': there exists $g \in G$ such that $gF' \cap F' = \emptyset$. In particular, $gF' \subset U$. Since $g(F \smallsetminus Z_\infty) \subset Z \smallsetminus Z_\infty \subset U$ by assumption, we deduce that $gF \subset U$. □

Corollary 2.6. *Let G be a group and Y a topological space on which G acts by self-homeomorphisms. Let U be a cofinite Hausdorff subset of Y including all finite G-orbits. Then Y is Hausdorff.*

Proof. Apply Lemma 2.5 to 2-element subsets of Z. □

Lemma 2.7. *Let G be a (discrete) group acting continuously on a topological space Y. Let X, X' be subsets of Y, such that X is G-essential (i.e., X meets all G-orbits), X is open, X' is G-invariant, and the symmetric difference $X \triangle X'$ is* *finite. Suppose that X is Hausdorff. Then X' has a cofinite G-invariant subset X'' that is Hausdorff and open in Y.*

Proof. Let F be the union of finite G-orbits in X' meeting $X' \smallsetminus X$. Define $X'' = X' \smallsetminus F$. Hence X'' is a cofinite G-invariant subset of X' and every element of $X'' \smallsetminus X$ belongs to an infinite G-orbit. By Corollary 2.6 applied to $Y = X''$ and $U = X \cap X''$, we infer that X'' is Hausdorff. □

Proposition 2.8. *Consider a topological partial action of a (discrete) group G on a Hausdorff topological space X. Suppose that X is G-transfixed. Then there exists a G-invariant subset X' of $\hat{X}$ with $X' \triangle X$ finite, that is G-invariant, Hausdorff and open in $\hat{X}$.*

Proof. Since X is transfixed, it is transfixed as a subset of its universal globalization $\hat{X}$. Let X'' be a G-invariant subset of $\hat{X}$, such that $X \triangle X''$ is finite. By Lemma 2.7, X'' has a G-invariant, cofinite subset that is both Hausdorff and open in $\hat{X}$. □

We will also use Lemma 2.5 in the proof of Proposition 2.14.

2.C. **Inverse symmetric monoids.** A semigroup is a set endowed with an associative binary law. A monoid is a semigroup endowed with a unit element (which is then unique). In a semigroup, a preinverse of an element x is an element y such that $xyx = x$ and $yxy = y$. A semigroup (resp. monoid) is called an inverse semigroup (resp. inverse monoid) if every element has a unique preinverse. Homomorphisms of monoids (resp. inverse semigroups, resp. inverse monoids) are required to map unit to unit (resp. preinverse to preinverse, resp. both). In inverse semigroup theory, preinverses are often called "inverses" but we rather use the more usual terminology of inverses in monoids (an inverse for x is y such that $yx = xy = 1$; such y is unique and is then a preinverse of x).

Given sets X, Y, the set $\mathcal{P}(X \times Y)$ of subsets of $X \times Y$ (the set of binary relations on X, Y) is endowed with its usual composition: given $A \subset X \times Y$ and $B \subset Y \times Z$,

$$A \circ B = \{(x, z) \in X \times Z : \exists y \in Y : (x, y) \in A, (y, z) \in B\}.$$

In particular, this makes $\mathcal{P}(X^2)$ a monoid (the diagonal id_X being the unit element). For $u \in \mathcal{P}(X \times Y)$, define D_u and D'_u as its projections on X and Y, called its domain and range. Denote by $s : A \mapsto A^{-1}$ the involution $\mathcal{P}(X \times Y) \to \mathcal{P}(Y \times X)$ induced by $(x, y) \mapsto (y, x)$. Beware that it is not an inverse map on $\mathcal{P}(X^2)$ as soon as X is nonempty, and not a preinverse map as soon as X contains two distinct elements.

Let $\mathcal{I}(X, Y)$ be the set of subsets of X^2 both of whose projections are injective. These are called partial bijections of X, namely each $u \in \mathcal{I}(X)$ is a bijection between its domain D_u and its range D'_u. These are stable under composition. In particular, $\mathcal{I}(X)$ is a well-known submonoid of X^2, called inverse symmetric monoid of X. Indeed, this is an inverse monoid: the preinverse of u being u^{-1}.

Let $\mathcal{I}^{\mathrm{cof}}(X)$ be the set of partial bijections of $\sigma \in \mathcal{I}(X)$ with cofinite domain and range. This is an inverse submonoid of $\mathcal{I}(X)$.

Given a topological space A, let A_δ be the underlying set (that is, A endowed with the discrete topology). Let $\mathcal{I}(A,B)$ be the subset of $\mathcal{I}(A_\delta, B_\delta)$ consisting of those u such that both D_u and D'_u are open subsets of A, and such that u induces a homeomorphism $D_u \to D'_u$. In particular, $\mathcal{I}(A)$ is an inverse submonoid of $\mathcal{I}(A_\delta)$.

2.D. **Partial actions.**

2.D.1. *Definition.*

Definition 2.9 (Exel [Ex]). A partial action of a group G on a set X is a map $\alpha : G \to \mathcal{I}(X)$ satisfying:

(1) $\alpha(1) = \mathrm{id}_X$;
(2) $\alpha(g^{-1}) = \alpha(g^{-1})$ for all $g \in G$;
(3) for all $g, h \in G$, $\alpha(g) \circ \alpha(h) \subset \alpha(gh)$.

We say that (X, α) is a partial G-set (and if $\alpha(G) \subset \mathfrak{S}(X)$, the group of permutations of X, we say that it is a global G-set, or just G-set).

When α is valued in $\mathcal{I}^{\mathrm{cof}}(X)$, it is called a cofinite-partial action.

Given a partial action of G on X, a topology $\mathcal{T}$ on X is said to be preserved by the partial action if for every $g \in G$, we have $\alpha_g \in \mathcal{I}(X_\mathcal{T})$. Here $X_\mathcal{T}$ means X endowed with the topology $\mathcal{T}$ (mostly the topology is implicit and is omitted from the notation). When X is endowed with $\mathcal{T}$, we call it a topological partial action of G on X; we call X a partial topological G-space, and (global) topological G-space when the underlying partial action is an action.

A homomorphism between partial G-sets (X, α), (Y, β) is a map $f : X \to Y$ that is G-equivariant, in the sense that $\alpha(g) \subset (f \times f)^{-1}(\beta(g))$ for all $g \in G$. That f is G-equivariant just means that for every $g \in G$ every $x \in X$ such that $\alpha(g)x$ is defined, then $\beta(g)f(x)$ is also defined and $f(\alpha(g)x) = \beta(g)f(x)$.

We say that such a homomorphism f is full if the above inclusion is an equality: $\alpha(g) = (f \times f)^{-1}(\beta(g))$ for all $g \in G$. We then say that f is fully G-equivariant.

The homomorphism f is essential if for every $y \in Y$ there exists $g \in G$ such that $\beta(g)y$ is defined and belongs to $f(X)$.

A bijection $(X, \alpha) \to (Y, \beta)$ is called G-biequivariant if both f and f^{-1} are G-equivariant. For a bijective homomorphism, this means that f is fully G-equivariant. (Beware that a bijective G-equivariant map can fail to be G-biequivariant, unlike in the setting of global actions.)

2.D.2. *Universal globalization.* Given a partial G-set (X, α), partial G-sets (Y, β) endowed with a homomorphism $(X, \alpha) \to (Y, \beta)$ form a category, whose maps are the G-equivariant maps so that the obvious triangle commutes. It has a full subcategory, consisting of those (global) G-sets (Y, β) endowed with a homomorphism $(X, \alpha) \to (Y, \beta)$. An initial element in this category is called a universal globalization for (X, α).

Theorem 2.10 (Megrelishvili [Me1, Me2], Abadie [Aba1, Aba2], Kellendonk-Lawson [KL]). *Every partial G-set (X, α) admits a universal globalization $\iota : (X, \alpha) \to (\hat{X}, \hat{\alpha})$. Moreover*

(1) the map ι is injective;

(2) every $\hat{\alpha}(G)$-orbit in $\hat{X}$ meets $\iota(X)$ (that is, ι is essential).

In detail, the first assertion means that there is a set $\hat{X}$, a map $\iota : X \to \hat{X}$, a G-action β on X such that ι is G-equivariant and such that for every other G-set Y and G-equivariant map $f = X \to Y$, there exists a unique G-equivariant map $u_f : \hat{X} \to Y$ such that $f = u_f \circ \iota$.

Let us recall the simple construction. Denote by $D_g \subset X$ the domain of $\alpha(g)$. Start from $\tilde{X} = G \times X$ and the map $X \to \tilde{X}$ given by $x \mapsto (1, x)$. Endow $\tilde{X}$ with the G-action given by $g \cdot (h, x) = (gh, x)$. Define $\hat{X}$ by modding out by the equivalence relation $\sim$ given by $(h, x) \sim (k, y)$ if $x \in D_{k^{-1}h}$ and $\alpha(k^{-1}h)(x) = y$. The G-action passes to the quotient $\hat{X}$ and the resulting map $X \to \hat{X}$ is a universal globalization. This can be checked as an exercise: indeed, the virtue of the theorem is, first and foremost, to have been formulated.

The following easy lemma is a convenient way to recognize a universal globalization.

Lemma 2.11 ([KL, Prop. 3.5]). *Any full and injective homomorphism from a partial G-action to a global G-action is a universal globalization* [KL, Prop. 3.5]. □

Remark 2.12. Megrilishvili [Me1, Me2] used a less general, related formalism of "preactions" and then constructed a "universal action" with the same construction as the one used for the universal globalization in [Aba1, Aba2, KL] in the framework of partial actions. Abadie's result was written in 1999 in his PhD [Aba1] and published only in 2003 [Aba2]. The construction of the universal globalization was independently later obtained, using the same construction, by Kellendonk and Lawson [KL] (published in 2004, but quoted as a preprint in [Me2] published in 2000).

Theorem 2.13 (Abadie [Aba1, Aba2, KL]). *Let (X, α) be a partial G-set and $\iota : (X, \alpha) \to (\hat{X}, \hat{\alpha})$ a universal globalization. For every topology $\mathcal{T}$ on X preserved by $\alpha(G)$, there is a unique topology $\hat{\mathcal{T}}$ on $\hat{X}$ preserved by $\hat{\alpha}(G)$ such that ι is an open continuous map. Moreover, $(\hat{X}, \hat{\alpha})$ is an initial element in the category of topological G-sets (topological spaces endowed with a G-action by self-homeomorphisms) endowed with a continuous homomorphism of partial actions from (X, α).*

Actually, Abadie's result is directly stated while defining the globalization in the context of actions on topological spaces. But the underlying action only depends on the underlying partial action, so we have found convenient to restate it as above, which is closer to the subsequent formulation from [KL, §3.2].

Note that in partial actions, we always consider the acting group as a discrete group, although Abadie [Aba2] treats partial actions in the more general setting of continuous actions topological groups.

2.E. **Encoding commensurating actions as partial actions.** Given a partial action α of G on a set X, let us define $\ell_X^-(g)$ as the cardinal of the complement $X \smallsetminus D_{\alpha(g)}$; define $\ell_X^+(g) = \ell_X^-(g^{-1})$. Call ℓ_X^+ and ℓ_X^- the loanshark and prodigal semi-index functions of the partial G-set X. This can actually be interpreted in previous setting: indeed, consider a universal globalization $X \to Y$. Then $\ell_X^-(g)$ coincides with its definition as in §2.A, in the setting of commensurating actions.

We say that the partial action commensurates X if ℓ_X^+ (or equivalently ℓ_X^-) takes finite values. We say that it transfixes X if it takes bounded (finite) values. For a partial action, the condition of commensurating X is equivalent to be a cofinite-partial action. This shows that all the theory of commensurating actions translates in the setting of cofinite-partial actions. In particular, given a subgroup H of G, relative Property FW of (G, H) can be restated as: every cofinite-partial action of G is transfixing in restriction to H.

Proposition 2.14. *Let G be a group and α an action of G on a set Y. Consider a cofinite-partial action β of G on Y such that the identity map $(Y, \beta) \to (Y, \alpha)$ is G-equivariant. Suppose that Y is transfixed for the partial action β. Then there exists an $\alpha(G)$-invariant cofinite subset X of Y such that $(X, \beta) \to (X, \alpha)$ is a universal globalization.*

In particular, for every cofinite subset W of X, denoting by W' the smallest $\alpha(G)$-invariant subset of X including W, the embedding $(W, \beta) \to (W', \alpha)$ is a universal globalization.

Proof. Let $i : Y \to \hat{Y}$ be a universal globalization of β. By the universal property, there exists a G-equivariant map $\pi : (\hat{Y}, \hat{\beta}) \to (Y, \alpha)$ such that $\pi \circ i = \mathrm{id}_Y$.

Since Y is transfixed for β, there exists a $\hat{\beta}(G)$-invariant subset Y' of $\hat{Y}$ such that $Y' \triangle i(Y)$ is finite. Define F as the union of finite $\hat{\beta}(G)$-orbits in Y' meeting $Y' \smallsetminus i(Y)$; it is finite and $\hat{\beta}(G)$-invariant.

Then $\pi(F)$ is a finite union of $\alpha(G)$-orbits in Y. Define $X = Y \smallsetminus \pi(F)$ and let Y' be the set of elements of $\hat{Y}$ whose G-orbit meets $i(X)$. By Lemma 2.11, the map $i : (X, \beta) \to (Y', \hat{\beta})$ is a universal globalization. Since $Y' \subset Y'$, we have $Y' \smallsetminus i(X)$ finite.

Let F' be a finite $\hat{\beta}(G)$-orbit in Y'. By definition of Y', we have $F' \cap i(X)$ nonempty. So, for some $y_0 \in X$, we have $i(y_0) \in F'$; hence $y_0 = \pi(i(y_0)) \in \pi(F')$. If $F' \subset F$, we get $y_0 \in \pi(F)$, a contradiction. So F' is not contained in F; this means that F' does not meet $Y' \smallsetminus i(Y)$; in other words, $F' \subset i(Y)$. Using that $i(Y) = \{z \in \hat{Y} : z = i(\pi(z))\}$, we deduce that $F' = i(\pi(F'))$; note that $\pi(F')$ is an $\alpha(G)$-orbit, in addition it contains y_0. Since $\pi(F)$ is $\alpha(G)$-invariant and does not contain y_0, the orbit of y_0 is entirely contained in its complement: $\pi(F') \subset Y \smallsetminus \pi(F) = X$. Since $F' = i(\pi(F'))$, we deduce $F' \subset i(X)$.

To conclude, we have to show that i is a bijection $X \to Y'$, which amounts to checking that π is injective on Y'. Indeed, consider $u_1, u_2 \in Y'$ with $\pi(u_1) = \pi(u_2)$. Hypotheses of Lemma 2.5 apply: there exists $g \in G$ such that $\hat{\beta}(g)$ maps $\{u_1, u_2\}$ into $i(X)$. Write $\hat{\beta}(g)u_i = i(y_i)$. Then for $j = 1, 2$,

$$y_j = \pi(i(y_j)) = \pi(\hat{\beta}(g)u_j) = \alpha(g)\pi(u_j),$$

whence $y_1 = y_2$, and hence $u_1 = u_2$. Thus $(X, \beta) \to (X, \alpha)$ is a universal globalization.

The last statement follows by applying Lemma 2.11. □

2.F. **Canonical partial action of self-homeomorphism groups.** Let X be a topological space.

Identifying two elements in $\mathcal{I}^{\mathrm{cof}}(X)$ whenever they coincide outside a finite subset, we obtain a group, which we denote by $\mathrm{PC}(X)$, and call it the group of parcelwise continuous self-transformations of X.

Denote by π the projection $\mathcal{I}^{\mathrm{cof}}(X) \to \mathrm{PC}(X)$. For $g \in \mathrm{PC}(X)$, define $\alpha(g) = \bigcup_{\sigma \in \pi^{-1}(\{g\})} \sigma$. Here the union is understood among subsets of X^2.

Lemma 2.15. *Suppose that X is Hausdorff with no isolated point. Then $\alpha(g)$ is a partial bijection.*

Proof. It is clear that $\alpha(g)$ has cofinite (hence open) projections. So we have to show that projections are injective, and by symmetry it is enough to check for the first projection. To show injectivity, consider $(x, y), (x, y') \in \alpha(g)$. Hence $(x, y) \in \sigma$ and $(x, y') \in \sigma'$ for some $\sigma, \sigma' \in \pi^{-1}(\{g\})$. Since every cofinite subset of X is dense, there exists a net (x_i) in the domain of $\sigma \cap \sigma'$, tending to x. Then, by continuity of both σ and σ' and using that Y is Hausdorff, we have $y = \lim_i \sigma(x_i) = \lim_i \sigma'(x_i) = y'$. □

It is easy to check failure of the conclusion of Lemma 2.15 when X is discrete. Thanks to Lemma 2.15, we have:

Proposition 2.16. *Let X be a Hausdorff topological space with no isolated point. Then $g \mapsto \alpha(g)$ defines a topological partial action of $\mathrm{PC}(X)$ on X, which is a splitting for the canonical projection $\mathcal{I}^{\mathrm{cof}}(X) \to \mathrm{PC}(X)$.* □

Remark 2.17. The above splitting is not a monoid homomorphism in general: for instance $\mathrm{id}_X = \alpha(1) = \alpha(g^{-1}g) \neq \alpha(g^{-1})\alpha(g) = \mathrm{id}_{D_g}$ when $D_g \neq X$.

Remark 2.18. From a categorist's point of view, it would be more natural and general to formulate this in a groupoid context, where a partial groupoid action is defined assigning to each object a set and to each arrow a partial bijection between the corresponding sets, with the analogous axioms. Then Proposition 2.16 adapts to this more general setting: every parcelwise continuous map $g : X \to Y$ between Hausdorff topological spaces without isolated point has a canonical representative $\alpha(g)$ in $\mathcal{I}^{\mathrm{cof}}(X)$, satisfying the axioms of groupoid partial action. We sticked to the group case only for the sake of conciseness, and because the generalization requires no further ingredient.

Definition 2.19. Consider a function f on a Hausdorff topological space defined modulo coincidence on finite subsets, valued in another Hausdorff topological space (typically, itself, or the space of real numbers). Say that f is outer continuous at x if x is not isolated and the restriction of some/any representative of f to some neighborhood of x (minus $\{x\}$) has a limit y at x. Call y the outer limit of f at x: it does not depend on the choice of representative. Let D'_f be the set of points at which f is outer continuous.

Suppose that X has no (or finitely many) isolated points. Then, for $\sigma \in \mathrm{PC}(X)$, the set D°_σ of points on which σ is outer continuous is cofinite, since it contains the domain of definition of any representative $\hat{\sigma}$ (minus isolated points). In the setting of Proposition 2.16, $D_{\alpha(\sigma)}$ is thus contained in D°_σ. Under a strong assumption, we have a converse. Define a topological space X as locally saturated if for every open subset U of X, every continuous injective map $U \to X$ is open.

For instance, topological manifolds of pure dimension n are locally saturated, by Brouwer's invariance of domain theorem (we will use it for $n = 1$, in which case this is obvious).

Proposition 2.20. *Let X be a Hausdorff topological space with no isolated point. Suppose that X is locally saturated. Then for every $\sigma \in \mathrm{PC}(X)$, the domain of definition of $\alpha(\sigma)$ coincides with the subset D°_σ of outer continuity of σ.*

Proof. We only have one inclusion to check. Suppose that σ is outer continuous at x. Let $\tilde{\sigma}$ be a lift of σ in $\mathcal{I}^{\mathrm{cof}}(X)$. If $\tilde{\sigma}$ is defined at x, then $x \in D_{\alpha(\sigma)}$ (by construction of the latter). Otherwise, let y be the outer limit of f at x. Define $f = \hat{\sigma} \cup \{(x, y)\}$ (viewed as subsets of X^2). Then f is a partial map and we have to check that f is injective and f^{-1} is continuous on its domain D.

Define a neighborhood $U \subset D$ of x as follows: if f is injective, choose $U = D$. Otherwise, there exists a unique $x' \in D \smallsetminus \{x\}$ such that $f(x) = f(x')$. Since X is Hausdorff, let U be any open neighborhood of x in D whose closure does not contain x'. Hence f is injective on U in all cases. Since X is locally saturated, $f(U)$ is open. Hence $V = \sigma^{-1}(f(U))$ is open; we have

$$V = \sigma^{-1}(f(U \smallsetminus \{x\}) \cup \{y\}) = (U \smallsetminus \{x\}) \cup \sigma^{-1}(\{y\}).$$

If by contradiction x' exist, $\sigma^{-1}(\{y\}) = \{x'\}$, and hence we deduce that $(U \smallsetminus \{x\}) \cup \{x'\}$ is open. Intersecting with the complement of the closure of U, we deduce that $\{x'\}$ is open, a contradiction.

Now let us show that f^{-1} is continuous at y. Indeed, $f(U)$ is a neighborhood of y for every neighborhood $U \subset D$ of x and this precisely establishes continuity of f^{-1}. ☐

2.G. **Pseudogroups, modeled structures, piecewise and parcelwise monoids.**

2.G.1. *Definition.* We define a pseudogroup over a topological space A is an inverse submonoid S of $\mathcal{I}(A)$ such that $\{U : \mathrm{id}_U \in S\}$ is a basis of the topology. Note that the latter condition implies that the topology is determined by S.

The completed pseudogroup $\hat{S}$ consists of the set of elements of $u \in \mathcal{I}(A)$ that can be written, for any index set I, as a union $\bigcup_{i\in I} u_i$ with $u_i \in S$ for all i. When $S = \hat{S}$, the pseudogroup is called complete.

For instance, if G is a subgroup of the group of self-homeomorphisms of X, we can define the pseudogroup induced by G, starting from those restrictions of elements of G to open subsets.

2.G.2. *S-structures, S-modeled topological spaces.* Let S be a pseudogroup on a topological space A.

Let X be a topological space. An S-atlas on X is a subset $\mathcal{H}$ of $\mathcal{I}(X, A)$, whose elements are called charts, such that $\emptyset \in \mathcal{H}$ and such that for any $f, g \in \mathcal{H}$, we have $g \circ f^{-1} \in \hat{S}$.

An S-atlas is called complete if it satisfies the following two additional conditions:

- for every $f \in \mathcal{H}$ and $u \in S$, we have $u \circ f \in \mathcal{H}$;
- for any index set I and family $(g_i)_{i\in I}$ in $\mathcal{H}$ such that $g := \bigcup_{i\in I} g_i$ belongs to $\mathcal{I}(X, A)$, we have $g \in \mathcal{H}$.

The data of a complete S-atlas $\mathcal{H}$ on X is called an S-structure on X. A topological space endowed with an S-structure is called an S-modeled topological space. We call it finitely-charted if X has a finite covering by domains of charts.

2.G.3. *Parcelwise-S inverse monoid and group.* Now consider an S-modeled topological space X, with complete atlas $\mathcal{H} = \mathcal{I}_S(X, A)$.

We then denote by $\mathcal{I}_S(X)$ the set of elements $h \in \mathcal{I}(X)$ that can be written, for some index set I and families $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ in $\mathcal{H}$, as $h = \bigcup_{i\in I} g_i^{-1} \circ f_i$.

Then $\mathcal{I}_S(X)$ is an inverse submonoid of $\mathcal{I}(X)$ (and thus a pseudogroup), and $\mathcal{I}_S(X, A)$ is stable under precomposition with $\mathcal{I}_S(X)$ and postcomposition with $\hat{S} = \mathcal{I}_S(A)$. By composition, any topological space endowed with a $\mathcal{I}_S(X)$-structure inherits a canonical S-structure.

Consider the inverse submonoid $\mathcal{I}_S^{\mathrm{cof}}(X) = \mathcal{I}_S(X) \cap \mathcal{I}^{\mathrm{cof}}(X_\delta)$, that is, the set of elements in $\mathcal{I}_S(X)$ with cofinite domain and range. We call it the parcelwise-S inverse monoid of X. Identifying two elements of $\mathcal{I}_S^{\mathrm{cof}}(X)$ whenever they coincide on a cofinite subset, we obtain a subgroup $\mathrm{PC}_S(X)$ of $\mathrm{PC}(X)$, called group of parcelwise-S self-transformations of X.

When X has no isolated point, the canonical homomorphism $\mathrm{Homeo}(X) \to \mathrm{PC}(X)$ is injective. The inverse image of $\mathrm{PC}_S(X)$ in $\mathrm{Homeo}(X)$ is denoted by $\mathrm{PC}^0_S(X)$.

In particular, we define $S^\wp$ as the pseudogroup over the model space A of restrictions of elements of $\mathrm{PC}^0_S(A)$. It is easy to check that $\mathrm{PC}_S(X) = \mathrm{PC}_{S^\wp}(X)$.

2.G.4. *Piecewise-S inverse monoid and group.* Define $\mathcal{I}_{S\sharp}^{\mathrm{cof}}$ as the set of $f \in \mathcal{I}_S^{\mathrm{cof}}(X)$ such that there exists an index set I and a family $(f_i)_{i\in I}$ in $\mathcal{I}_S(X)$ such that $\bigcup_{i\in I} D_{f_i} = \bigcup_{i\in I} D'_{f_i} = X$ and $f \subset \bigcup_{i\in I} f_i$. Note that we can assume that I finite (indeed, one can assume that f belongs to the family, and add finitely many f_i to get the missing points in the projections). It is easy to check that this is an inverse submonoid of $\mathcal{I}_S^{\mathrm{cof}}(X)$. We call it the piecewise-S inverse monoid of X. Its image in $\mathrm{PC}(X)$ is a subgroup $\mathrm{PC}_{S\sharp}(X)$ of $\mathrm{PC}(X)$, called group of piecewise-S self-transformations of X.

When S consists of all local homeomorphisms, we denote these by $\mathcal{I}_\sharp^{\mathrm{cof}}(X)$ and $\mathrm{PC}_\sharp(X)$.

2.G.5. *General comments comparing these groups.* Given a Hausdorff topological space X, we have inclusions $\mathrm{PC}(X) \subset \mathrm{PC}_\sharp(X) \subset \mathrm{Homeo}(X)$. When X is S-modeled, it induces inclusions $\mathrm{PC}_S(X) \subset \mathrm{PC}_{S\sharp}(X) \subset \mathrm{PC}_S^0(X)$.

We say that X has no local cut point if every $x \in X$, the set of neighborhoods V of x such that $x \in \overline{V \smallsetminus \{x\}}$ and $V \smallsetminus \{x\}$ is connected, is a basis of neighborhoods of x. For instance, this holds if X is a topological manifold with no component of dimension ≤ 1, or more generally if X is locally homeomorphic to a locally finite simplicial complex in which every vertex or edge belongs to a triangle, and in which the link at every vertex is connected.

Remark 2.21. Suppose that X is Hausdorff compact with no local cut point, in the sense that for every $x \in X$ and every neighborhood W of x. Then the inclusion $\mathrm{PC}(X) \subset \mathrm{Homeo}(X)$ is an equality.

Indeed, given that X is Hausdorff compact, the assumption that X has no local cut point is equivalent to the condition that for every the embedding $X \smallsetminus F \to X$ is the end compactification (in the sense of Specker [Spe]) of $X \smallsetminus F$.

Remark 2.22. Suppose that X is Hausdorff with no local cut point. Then the inclusion $\mathrm{PC}_\sharp(X) \subset \mathrm{Homeo}(X)$ is an equality.

Indeed, consider $f \in \mathcal{I}_\sharp^{\mathrm{cof}}(X)$ and $x \in X$. As in the definition, write $f \subset \bigcup f_i$ where $\bigcup_i D_{f_i} = X$. So $x \in D_{f_i}$ for some i. Since X is Hausdorff without local cut point and f, f_i have a cofinite domain, there exists a neighborhood V of x contained in $(D_f \cap D_{f_i}) \cup \{x\}$ such that $V \smallsetminus \{x\}$ is connected and contains x in its closure. Then $f_i(x)$ is the limit of $f_i(x') = f(x')$ when $x' \in V \smallsetminus \{x\}$ tends to x. In particular $f_i(x)$ does not depend on i; denote by $\bar{f}(x)$ its value; then $\bar{f} : X \to X$ is continuous. Then for any $g, f \in \mathcal{I}_\sharp^{\mathrm{cof}}(X)$, we have $\bar{g} \circ \bar{f} = \overline{g \circ f}$: indeed, this equality holds on a cofinite subset and hence holds everywhere since X is Hausdorff with no isolated point. In particular, for g inverse of f we deduce that $\bar{g}$ is inverse of $\bar{f}$, and hence have the desired equality.

Moreover, if $f \in \mathcal{I}_{S\sharp}^{\mathrm{cof}}(X)$ for some S, then by construction $\bar{f}$ is S-preserving, so the above then yields the equality $\mathrm{PC}_{S\sharp}(X) = \mathrm{PC}_S^0(X)$.

Remark 2.23. If $A = \mathbf{R}/\mathbf{Z}$ and S is one of the pseudogroups **Isom**, **Aff**, **Proj**, the whole pseudogroup of local homeomorphisms, or one of their oriented counterparts, then for every S-modeled topological space X, we have $\mathrm{PC}_S(X) = \mathrm{PC}_{S^\sharp}(X)$.

Continuing with $\mathbf{R}/\mathbf{Z}$, examples of S for which $\mathrm{PC}_S(X) \neq \mathrm{PC}_{S^\sharp}(X)$ are the group of local C^k-diffeomorphisms for $k > 0$, the pseudogroups $\mathbf{Aff}^\wp$ and $\mathbf{Proj}^\wp$ of piecewise affine/projective local homeomorphisms.

Remark 2.24. The equality $\mathrm{PC}(X) = \mathrm{Homeo}(X)$ is a rigidity property, which means, in a sense, that the parcelwise continuous group $\mathrm{PC}(X)$ does not deserve a specific study, and illustrates by contrast, the richness of the context of purely 1-dimensional topological manifolds X. Note however that even when $\mathrm{PC}(X) = \mathrm{Homeo}(X)$, we can have $\mathrm{PC}_S(X) \neq \mathrm{PC}_S^0(X)$. This is for instance the case when S is the pseudogroup of C^k-diffeomorphisms on $\mathbf{R}^d$ for $d \geq 2$ and $k > 0$.

In higher dimension, of course there are other natural interesting ways of considering "piecewise" properties, which allow infinite subsets (typically "codimension 1") subsets of "singular points". Such a study goes beyond the scope of this paper.

2.G.6. *Partial action α_S.* Let S be a pseudogroup on a topological space A, and let X be a S-modeled topological space. Denote by π_S the projection $\mathcal{I}_S^{\mathrm{cof}}(X) \to \mathrm{PC}_S(X)$. By definition, a topological partial action on X is S-preserving if it maps G into $\mathcal{I}_S(X)$.

To any topological S-preserving partial action of G on X, we can consider the induced homomorphism $G \to \mathrm{PC}_S(X)$.

Now assume that X is Hausdorff and has no isolated point. For $g \in \mathrm{PC}_S(X)$, define $\alpha_S(\sigma) = \bigcup_{\sigma \in \pi_S^{-1}(\{g\})} \sigma$. It is contained in $\alpha(g) = \bigcup_{\sigma \in \pi^{-1}(\{g\})} \sigma$ (i.e., where we take the union over the whole preimage in $\mathcal{I}^{\mathrm{cof}}(X)$). It follows from Lemma 2.15 that if X is Hausdorff and has no isolated points, then $\alpha(g)$ is a partial bijection, and hence $\alpha_S(g)$ is a partial bijection.

This shows that for X Hausdorff and without isolated point, the above mapping from the set of topological S-preserving partial actions of G on X to $\mathrm{Hom}(G, \mathrm{PC}_S(X))$ has a canonical section, and in particular is surjective.

It can happen, and it is one interest of the construction, that $\alpha_S(g)$ is properly contained in $\alpha(g)$. For instance, if g is a piecewise affine homeomorphism and S is the pseudogroup of local affine homeomorphisms, then $\alpha(g)$ is defined everywhere, while $\alpha_S(g)$ is defined outside singular points.

2.G.7. *Transfer of S-structure to the globalization.* The following simple proposition plays an essential role; it is also paramount to *not* assume Y to be Hausdorff. For this reason, we provide a detailed proof.

Proposition 2.25. *Let S be a pseudogroup on a topological space A. Let G be a group with a topological S-preserving partial action on an S-modeled space X. Consider a partial topological G-space Y with an injective open full essential* *homomorphism of partial actions $X \to Y$. Then there is a unique S-structure on Y extending the S-structure on X such that the partial action of G is S-preserving.*

Proof. View $X \to Y$ as an open inclusion, and denote by α the partial action on Y, and β the partial action on X. For every $y \in Y$, the inclusion being essential, there exists $g \in G$ such that $\alpha(g)y \in X$. Since X and the $D_{\alpha(g)}$ are open and $\alpha(g)$ is continuous on its domain, there exists $(*)$ an open neighborhood U of y included in $D_{\alpha(g)}$ such that $\alpha(g)U \subset X$. Given $(*)$, the uniqueness follows.

For the existence, it is enough to endow Y with a structure on the pseudogroup $\mathcal{I}_S(X)$. Namely, define an atlas where the charts are indexed by the pairs (g,U), where g ranges over G and U among open subsets of Y such that $\alpha(g)U \subset X$. Such a chart $\phi_{g,U}$ has domain U and is simply given by $\alpha(g)|_U$. That the domain of charts cover Y follows from $(*)$.

Now let us check the compatibility condition in the definition of an atlas. Consider two charts $\phi_{g,U}$ and $\phi_{h,V}$: we have to show that $\phi_{h,V} \circ \phi_{g,U}^{-1}$ belongs to $\mathcal{I}_S(X)$. Define $W = \alpha(g)U \cap \alpha(h)V$. Define $U' = \alpha(g)^{-1}(W) \subset U$ and $V' = \alpha(h)^{-1}(W) \subset V$. Then, adding subscripts to restrict the domain and range of the given partial bijections, we have

$$\phi_{h,V} \circ \phi_{g,U}^{-1} = \phi_{h,V'} \circ \phi_{g,U'}^{-1} = \big(\alpha(h)_{V' \to W}\big) \circ \big(\alpha(g)_{U' \to W}^{-1}\big) = \alpha(h \circ g^{-1})_{W \to W}.$$

The first equality just uses the definition of composition of partial maps; the second is just the definition, and the third follows from the definition of partial maps and the definition of partial action. Now it follows from the inclusion $X \to Y$ being full that $\alpha(h \circ g^{-1})_{W \to W}$ is equal to $\beta(h \circ g^{-1})_{W \to W}$. Therefore, the change of charts $\alpha(h \circ g^{-1})_{W \to W}$ belongs to $\mathcal{I}_S(X)$. This proves that we have an atlas.

Let us finally check that the partial action α is S-preserving. Indeed, since this is a local condition, we can check for a given g at given points $y, y' = \alpha(g)y$; we can choose an open neighborhood U of y which is the domain of a chart $\phi_{U,h}$, and such that $V = \alpha(g)U$ is domain of a chart $\phi_{V,k}$. Then, on $h(U)$, the composition $\phi_{V,k} \circ (\alpha(g)_{U \to V}) \circ \phi_{U,h}^{-1}$ (which describes $\alpha(g)_{U \to V}$ in charts) equals $\alpha(kgh^{-1})_{h(U) \to k(V)}$ which preserves S. Hence $\alpha(g)$ is S-preserving at the neighborhood of y. □

Corollary 2.26. *Given a group G with a topological S-preserving partial action on an S-modeled topological space X, there exists a unique G-invariant S-structure on the universal globalization $\hat{X}$ extending the original S-structure on X.*

Remark 2.27. Beware that even if X is Hausdorff, $\hat{X}$ is often far from Hausdorff: indeed the construction of $\hat{X}$ typically glues copies of open subsets of X along open intersections.

For this reason, and because of our extensive use of Corollary 2.26, discussions about Hausdorffness of spaces are important unavoidable issues: even if the ultimate goal is to deal with Hausdorff spaces and produce Hausdorff spaces, we have to accept the presence of non-Hausdorff spaces among our tools.

2.H. **Curves and doubling tricks.** By curve we mean a purely 1-dimensional Hausdorff paracompact topological manifold with finitely many connected components. A connected curve is homeomorphic to the circle or an open interval.

Start from a curve A with a pseudogroup S. A finitely-charted S-modeled curve is a curve endowed with an S-structure definable by finitely many charts (the condition is automatic for compact S-modeled curves).

By standard curve we mean finite disjoint union of open bounded intervals and circles (where a circle is a copy of $\mathbf{R}/a\mathbf{Z}$ for some $a > 0$). A more intrinsic (but somewhat using more formalism than necessary) is to define a standard curve as a Riemannian purely 1-dimensional oriented manifold, with finitely many components.

Given a standard curve X, local orientation-preserving measure-preserving homeomorphisms yield a canonical IET^+-structure on X. Therefore, for every pseudogroup S over the circle including the local motions, it endows X with a canonical S-structure.

The idea of doubling points in one-dimensional dynamics is very classical, and often attributed to Denjoy. In a group-theoretic context it was used by Keane [Ke] to find a continuous action IET of interval exchanges naturally reflecting the (non-continuous) action on the circle.

The idea underlying the following lemma appears in the context of interval exchanges in Danthony-Nogueira's article [DN]. I thank Thierry Bousch for bringing this important observation to my attention.

Lemma 2.28. *For any curve X endowed with an orientation, endow $X^{\pm} = X \times \{1, -1\}$ with the product topology and an orientation: that of X on $X \times \{1\}$ and the reverse one on $X \times \{-1\}$.*

Then there is a natural injective group homomorphism Φ of $\mathrm{PC}(X)$ into $\mathrm{PC}^+(X^{\pm})$, whose image is the centralizer of the involution $s : (x, 1) \leftrightarrow (x, -1)$ in $\mathrm{PC}^+(X^{\pm})$, and which makes the projection $X^{\pm} \to X$ equivariant.

If moreover X is modeled on a pseudogroup S on $\mathbf{R}/\mathbf{Z}$, then Φ induces an isomorphism from $\mathrm{PC}_S(X)$ onto the centralizer of s in $\mathrm{PC}^+_S(X^{\pm})$.

Proof. For $f \in \mathcal{I}^{\mathrm{cof}}(X)$ and $x \in D_f$, define the reduced derivative $f^{\imath}(x)$ as equal to 1 or -1 according to whether f is locally orientation-preserving or orientation-reversing at x (this is well-defined because X is endowed with an orientation). It can be thought of as the sign of the derivative, but is defined for arbitrary piecewise strictly monotonic functions.

It satisfies the same property as the derivative for composition: when defined, we have $(f_2 \circ f_1)^{\imath}(x) = f_1^{\imath}(x) f_2^{\imath}(f_1(x))$. With a suitable derivability assumption we would obtain an action on the tangent bundle, and then, modding out by the action of positive scalars, an action on the orientation bundle. This idea works directly thanks to the above formula. Namely, for any $f \in \mathcal{I}^{\mathrm{cof}}(X)$, and $(x, \varepsilon) \in X \times \{\pm 1\}$, define $\tau(f)(x, \varepsilon) = (f(x), f^{\imath}(x)\varepsilon)$. Note that it obviously commutes with s, and preserves the given orientation. That it defines a partial action with domain of definition $D_f \times \{\pm 1\}$ is immediate from the composition formula. It then induces a group homomorphism of $\mathrm{PC}(X)$ into $\mathrm{PC}^+(X \times \{1, -1\})$, which is clearly injective. That the image consists of the centralizer of s is straightforward, as well as the additional statement. □

Remark 2.29. In the context of piecewise isometric maps, the elements in the image of Φ are called "linear involutions" in [DN]; thus "linear involutions" denote one way to represent the group of "interval exchanges with flips", this encoded as interval exchanges in this model.

Remark 2.30. It is interesting to note that the underlying action of $\mathrm{PC}(X)$ on $X^{\pm}$ preserves two very different locally compact topologies on $X^{\pm}$ (both compact if X is compact): the product topology as above, but also the topology of the (local) ordering (in the case of the interval, it consists of the lexicographic order consisting in splitting each x into two consecutive elements). This is the topology used by Keane in [Ke]. Unlike the product topology, it is not metrizable.

3. Non-distortion phenomena

This shorter section is independent of others, and only referred to in the non-distortion corollaries. Indeed, while commensurated actions typically allow to prove that distorted elements preserve some given geometric structure, an additional step is understand when such automorphisms are distorted within the whole piecewise group.

We indicate a way to systematically tackle such problems, with a limited technical cost, thanks to the language of pseudogroups.

Let S be a pseudogroup on a topological space X. Whenever we refer to sx for $(s, x) \in S \times X$, it is understood that we mean "(such that) x belongs to the domain D_s of s". We freely consider s as a subset of X^2.

If Y, Z are open subsets of X, we denote by $S_{Y,Z}$ the set of those $s \in S$ that are included in $Y \times Z$. Write $S_Y = S_{Y,Y}$ (so $S = S_X$); note that S_Y is a pseudogroup over Y, and is an inverse subsemigroup of $\mathcal{I}(X)$.

Given a subset $T \subset \mathcal{I}(X)$, define a graph structure on X, with one edge (x, gx) for all $g \in T$ and $x \in X$. Let d_T be the corresponding graph "distance" (allowing the value ∞) on X. Note that $d_T = d_{T \cup T^{-1}}$, so it is generally no restriction to assume T symmetric.

Lemma 3.1. *Let X be a set with a subset Y. Let S be a pseudogroup over X.*

Let T be a subset of S and K a subset of $S_{X,Y}$, such that $\bigcup_{f\in K} D_f = X$. Define $T' = KTK^{-1}$ (which is contained in S_Y). Then for all $y, y' \in Y$ we have $d_{T'}(y, y') \le 3d_T(y, y')$.

Note that assuming $\mathrm{id}_Y \in K$ yields $d_T(y, y') \le d_{T'}(y, y')$; the interest of Lemma 3.1 is to provide an inequality in the reverse direction. This can be thought as a non-distortion property: we can replace a path of size n (in X) with a path of size $3n$ (within Y).

Proof. Since $(T')^{-1} = K(T \cup T^{-1})K^{-1}$, we can suppose that T is symmetric. Consider $y, y' \in Y$ with $d_T(y, y') = n$, so we can write $y' = t_n \dots t_1 y$ with $t_i \in T$. Write $x_j = s_j \dots s_1 y \in X$. For each j, there exists $k = k_j \in K$ such that $x_j \in D_{k_j}$. Then

$$y' = (s_n k_{n-1}^{-1})(k_{n-1}s_{n-1}k_{n-2}^{-1}) \dots (k_2^{-1}s_2k_1^{-1})(k_1s_1)y = \tau_n \dots \tau_1 y,$$

with $\tau_i = k_i s_i k_{i-1}^{-1} \in T'$ (and $k_0, k_n = \mathrm{id}_X$). Then $\tau_j \dots \tau_1 y \in Y$ for all j. So $d_{T'}(y, y') \le 3n$. □

Let X be a standard curve. Define a small interval in X as a subset I either empty or homeomorphic to an open interval, with the additional requirement that if I belongs to a circle component of length a, then the length of I is $\le a/2$. This ensures that the intersection of any two small intervals is a small interval. Define the pseudogroup of isometries S as consisting of those isometries between two small intervals of X.

Lemma 3.2. *Fix $z \in Y^{\pm}$ and let Y be the component of X containing y_0; assume that Y is a topological circle. Let Q_z be the set of $\sigma \in \mathrm{PC}_S(X)$ such that $\sigma(z) \in Y^{\pm}$. For $\sigma \in Q_z$, define $h = g_z(\sigma)$ as the unique isometry h of Y such that $h(z) = \sigma(z)$.*

Then for every finite subset W of $\mathrm{PC}_S(X)$, there exists a finite subset W' of $\mathrm{Isom}(Y)$ such that for every $z \in Y^{\pm}$ and every $\sigma \in Q_z$, we have $|\sigma|_W \ge \frac{1}{3}|g_z(\sigma)|_{W'}$.

Proof. Let W be a finite subset of $\mathrm{PC}_S(X)$. Let T be a finite subset of S such that every element of W is the union of finitely many elements of T. Let K a finite subset of S such that the range of every $k \in K$ is included in Y, and the domains of $k \in K$ cover X. Set $T' = KTK^{-1}$. For $t \in S_Y \smallsetminus \{\emptyset\}$, define $\Psi(t)$ as the unique self-isometry of Y extending t; define $W' = \Psi(T')$. Then, for every $z \in Y^{\pm}$ and $\sigma \in Q_z$, we have

$$|\sigma|_W \ge d_T(z, \sigma z) \ge \frac{1}{3} d_{T'}(z, \sigma z)) = \frac{1}{3}|g_z(\sigma)|_{W'}.$$

Let us justify each of the (in)equalities above. The middle inequality is provided by Lemma 3.1. The left-hand inequality follows from the case when $\sigma \in W$, in which case it holds by definition of T. For the right-hand equality, the inequality $\le$ is easy and not needed, so let us only justify $\ge$. Indeed, suppose that $d_{T'}(z,\sigma z)=n$. Then we can write $\sigma(z)=t_n\dots t_1 z$ with $t_i\in T'$. Write $\tau_i=t_i\dots t_1$ and $z_i=\tau_i z$. So

$$g_z(\tau_i)z=z_i=t_i\tau_{i-1}z=\Psi(t_i)g_z(\tau_{i-1})z.$$

Thus $g_z(\sigma)z=\Psi(t_n)\dots\Psi(t_1)z$, and hence, again using that the isometry group of Y acts freely on Y, we deduce $g_z(\sigma)=\Psi(t_n)\dots\Psi(t_1)$. Hence $|g_z(\sigma)|_{W'}\le n$. ☐

Proposition 3.3. *Let X be a standard curve and Z a clopen subset of X. Then* $\mathrm{Isom}(Z)$ *is undistorted in* $\mathrm{PC}_S(X)=\mathrm{IET}^{\pm}(X)$. *More generally, let Γ be any subgroup of* $\mathrm{Isom}(Z)$ *and homomorphism $q=\Gamma\to\mathrm{IET}^{\pm}(X\smallsetminus Z)$, and denote by Γ_q the image of Γ in $\mathrm{IET}^{\pm}(X)$ by the homomorphism* $\mathrm{id}\times q$. *Then Γ_q is undistorted in* $\mathrm{IET}^{\pm}(X)$.

Proof. We can suppose, passing to a subgroup of finite index, that Γ belongs to the unit component $\mathrm{Isom}(Z)^{\circ}$. In particular, Γ preserves each component of Z, and acts trivially on any component of Z that is not a topological circle. Let $Y_1,\dots,Y_k$ be the circle components of Z; we can view $\mathrm{Isom}(Z)^{\circ}$ as the product $\prod_j \mathrm{Isom}(Y_j)^{\circ}$. Fix $z_j\in Y_j$.

Let W be a finite subset of $\mathrm{PC}_S(X)$. For each j, apply Lemma 3.2, outputting a finite subset W_j of $\mathrm{Isom}(Y_j)$; we can suppose that W_j is symmetric and contains 1. For $\gamma\in\Gamma$, let γ_q be its image in $\mathrm{IET}^{\pm}(X)$. Then $\gamma_q\in Q_z$ and hence Lemma 3.2 says that $|\gamma_q|_W\ge\frac{1}{3}|g_{z_j}(\gamma_q)|_{W_j}$. Since $g_{z_j}(\gamma_q)=\gamma_j$, this yields $|\gamma_q|_W\ge\frac{1}{3}|\gamma_j|_{W_j}$. Write $W'=\prod_j(\{1\}\cup W_j)$; then for every γ we have $|\gamma|_{W'}=\sup_j|\gamma_j|_{W_j}$. Hence we deduce $|\gamma_q|_W\ge\frac{1}{3}|\gamma|_{W'}$. If $\Gamma=\mathrm{Isom}(Y)$, this gives the non-distortion result. In general, this follows by using that in a virtually abelian group, all subgroups are undistorted: precisely, the non-distortion ensures that there exists a finite subset W'' of Γ and $C>0$ such that for all $g\in\Gamma$, we have $|g|_{W'}\ge C|g|_{W''}$. ☐

Corollary 3.4 (Novak [Nov]). *Let X be a standard curve and f an self-isometry of X of infinite order. Then f is undistorted in* $\mathrm{IET}^{\pm}(X)$. ☐

This is, modulo the formulation, due to Novak in the context of (piecewise orientation-preserving) interval exchanges. Precisely, Novak's proof that IET has no distorted cyclic subgroups consists, in a first step, in showing that every distorted element has an isometric model and then the next step is to prove (by hand: [Nov, §4]) a result akin to the previous corollary. Using Proposition 3.3, Novak's result is improved in Corollary 4.9.

We now consider affine curves.

Proposition 3.5. *Let X be an affine curve and Z a clopen subset of X. Then the embedding of* $\mathrm{PC}^0_{\mathbf{Aff}}(Z)$ *into* $\mathrm{PC}_{\mathbf{Aff}}(Z)$ *is, in restriction to* $\mathrm{Aut}^+_{\mathbf{Aff}}(Z)$, *undistorted.*

Proof. Here given groups $G_2\subset G_1\subset G_0$, we say that $G_1\subset G_0$ is undistorted in restriction to G_2 if for every finite subset W of G_0 there exists $C>0$ and a finite subset W' of G_1 such that $|g|_W\ge C|g|_{W'}$ for all $g\in G_2$.

We define small intervals with the artifact of choosing a Riemannian metric on each circle component (small intervals having by definition at most half the length of the circle), and on an interval component, small intervals are required to have a compact closure. Let S be the pseudogroup consisting of affine isomorphisms between small intervals (this is not a natural definition at all, since the notion of small interval depends on choices, but this does not matter).

We first need a substitute for Lemma 3.2. The main difference is that there is no good analogue of Ψ: an affine isomorphism between small intervals of a component usually does not extend to an affine automorphism of the component. Another issue, more superficial, is that non-standard circles have no orientation-reversing automorphisms, so we cannot define g_z in the orientation-reversing context. As a substitute, every orientation-preserving affine isomorphism t between small intervals of a component extend to a piecewise affine self-homeomorphism $\hat{t}$ of the component (far from unique).

First, we fix an orientation on X, and choose a component Y_j. For $z \in Y_j^+$, we define $Q'_z = \{\sigma : \sigma(z) \in Y^+\}$, and for $\sigma \in Q'_z$ we define $g_z(\sigma)$ as the unique element of $h \in \mathrm{Aut}_{\mathbf{Aff}}(Y)$ such that $\sigma(z) = h(z)$.

Then we have to rework at the level of Lemma 3.1: we redefine K: we require that $K = K^+ \cup K^-$, where the domains of both K^+ and K^- cover X, that elements of K^+ are orientation-preserving and elements of K^- are orientation-reversing. This ensures that elements of $T' = T'_j$ are orientation-preserving. Then following the proof of Lemma 3.2, redefining W_j as $\{\hat{t} : t \in T'_j\}$, yields:

For every finite subset W of $\mathrm{PC}_{\mathbf{Aff}}(X)$ and every j, there exists a finite subset W_j of $\mathrm{PC}^0_{\mathbf{Aff}+}(Y_j)$ such that for every $z \in Y_j^{\pm}$ and every $\sigma \in \mathrm{PC}_{\mathbf{Aff}}(X)$ with $\sigma(z) \in Y_j^+$, we have $|\sigma|_W \geq \frac{1}{3}|g_z(\sigma)|_{W_j}$. We can suppose that W_j is symmetric and contains 1.

Denoting by Y_j the connected components of Z and $W = \prod W'_j$, if we apply this to $\sigma \in \mathrm{Aut}_{\mathbf{Aff}+}(Y)$, we get $|\sigma|_W \geq \frac{1}{3}|\sigma|_{W'}$. □

Corollary 3.6. *Under the assumptions of Proposition 3.5, assuming that all components of Z are non-standard circles,* $\mathrm{Aut}_{\mathbf{Aff}}(Z)$ *is undistorted in* $\mathrm{PC}_{\mathbf{Aff}}(X)$.

Proof. Proposition 3.5 boils down to proving that $\mathrm{Aut}_{\mathbf{Aff}}(Z)$ is undistorted in $\mathrm{PC}^0_{\mathbf{Aff}}(Z)$; it is enough to check this at the level of the finite index subgroups preserving all components along with their orientation, which immediately reduces to the case when Z is connected.

Write $Z = \mathbf{R}_{>0}/\langle s\rangle$. Let W be a finite subset of $\mathrm{PC}^0_{\mathbf{Aff}}(Z)$. Let $\bar{\tau}_i$ be a finite list of those partial affine isomorphisms of Z involved in W. Let $\tau_i : t \mapsto a_i t + b_i$ with $a_i > 0$ and $b_i \in \mathbf{R}$ be the affine extension of one lift of τ_i to $\mathbf{R}_{>0}$. Let W' be the set of induced automorphisms $q_i : t \mapsto a_i t$ of $\mathbf{R}_{>0}/\langle s\rangle$.

Consider an element $\sigma \in \mathrm{Aut}_{\mathbf{Aff}}(Z)^\circ$. Suppose that $|\sigma|_W \leq n$. Write $\sigma = \sigma_n \dots \sigma_1$ with $\sigma_i \in W$. There exists a nonempty open interval and $i_n, \dots, i_1$ such that σ coincides with $\bar{\tau}_{i_n} \dots \bar{\tau}_{i_1}$ on this interval. Lift σ to $\tilde{\sigma}$ on $\mathbf{R}_{>0}$: note that $\tilde{\sigma}$ is a homothety. Hence, there exists m such that $\tilde{\sigma}$ coincides with $s^m\tau_{i_n}\dots\tau_{i_1}$ on some nonempty open interval in $\mathbf{R}_{>0}$. Taking the slopes, this yields $\tilde{\sigma} = s^m a_{i_n}\dots a_{i_1}$. Then $\sigma = q_{i_n}\dots q_{i_1}$. Hence $|\sigma|_{W'} \le n \le |\sigma|_W$. ☐

This yields the following corollary which, modulo the formulation, is due to Guelman-Liousse [GuL, §7].

Corollary 3.7. *[Guelman-Liousse] Let X be a finitely-charted affinely modeled curve. Let Y a component of X that is a nonstandard circle, i.e., $Y \simeq \mathbf{R}_{>0}/\langle s\rangle$ with $s > 1$. Let g be an affine automorphism X acting on Y as an irrational rotation, i.e., is the map induced by the homothety $x \mapsto \eta x$ with $\eta > 0$ and $\log(\eta)/\log(t)$ irrational. Then g is undistorted in $\mathrm{PC}_{\mathbf{Aff}}(X)$.*

The robustness of the method allows to apply it in some other cases. For instance, a related argument can show that irrational rotations of non-standard circles are undistorted in the group of piecewise projective self-transformations. The proof is a little more difficult: the basic idea is to use non-distortion of homotheties in $\mathrm{PSL}_2(\mathbf{R})$ rather than in an abelian group. Actually, this should be performed in a systematic study of distortion in groups of piecewise projective self-transformations.

4. The main theorem and applications

4.A. **Powers and the first corollaries.** We first use the interpretation as partial action to provide counting results for the number of "singularities" of various maps, in various senses.

4.A.1. *Using the partial action α.* Let X be a Hausdorff topological space with no isolated point. Proposition 2.16 applies: $\mathrm{PC}(X)$ has a canonical partial action α on X. Call the finite complement of the domain of definition of $\sigma \in \mathrm{PC}(X)$ its domain of indeterminacy. It contains the subset of outer discontinuity points of σ, defined as the complement of the set of outer continuity points x of σ. If in addition X is locally saturated, Proposition 2.20 applies and these two finite subsets coincide for every σ.

When X is an oriented 1-dimensional manifold, not necessarily connected, outer discontinuity points of σ are the same as discontinuity points of the unique left-continuous representative of σ. (Beware that mapping σ to its unique left-continuous representative of σ is not a monoid homomorphism; yet it is a monoid homomorphism in restriction to piecewise orientation-preserving elements).

Corollary 4.1. *Let X be a Hausdorff topological space with no isolated point. Consider the cofinite-partial action of $\mathrm{PC}(X)$ on X.*

(1) *the prodigal semi-index function $\ell^-(\sigma)$ of this partial action (and hence of its restriction to any subgroup) coincides with the function mapping $\sigma \in \mathrm{PC}(X)$ to the number of indeterminacy points of σ;*

(2) for every $\sigma \in \mathrm{PC}(X)$ there exists $m_\sigma \in \mathbf{N}$ and a bounded function $b : \mathbf{N} \to \mathbf{N}$ such that $\ell^-(\sigma^n) = m_\sigma n + b(n)$ for all $n \in \mathbf{N}$.

(3) Let G be a group with a homomorphism into $\mathrm{PC}(X)$. Then X is transfixed by G if and only if the number of indeterminacy points of g is bounded independently of $g \in G$;

Proof. The first fact is immediate and the third immediately follows. The second fact follows from Proposition 2.1. □

This has the following addendum (for which we did not attempt to find optimal hypotheses):

Corollary 4.2. *In the setting of Corollary 4.1(2), assume that X is a topological manifold with no boundary and finitely many ends. Then $\ell^-(\sigma) = \ell^-(\sigma^{-1})$ for all σ. In particular, we have $m_\sigma = m_{\sigma^{-1}}$ for all σ. In other words, for every $\sigma \in \mathrm{PC}(X)$ there exists $m_\sigma \in \mathbf{N}$ and an even bounded function $b : \mathbf{Z} \to \mathbf{N}$ such that $\ell^-(\sigma^n) = m_\sigma|n| + b(n)$ for all $n \in \mathbf{Z}$.*

Proof. This reflects the fact that domains of definition of σ and σ^{-1} have complements of the same cardinal. In turn, this follows from the fact that the complement of m and m' points in X are homeomorphic only if $m = m'$. Let us check the latter assertion.

First suppose that X has constant dimension. Write $\theta = 2$ if $\dim(X) = 1$ and $\theta = 1$ if $\dim(X) \geq 2$. Let k be the number of ends of X. Then the number of ends of X minus m points is $k + \theta n$. This number retains m, when X is given. (If X has dimension 0, the condition of having finitely many ends means that X is finite and the result holds too.) The case when X has variable dimension immediately follows. □

Remark 4.3. In the case when $X = \mathbf{R}/\mathbf{Z}$ and in the context of interval exchanges, Corollary 4.2, was essentially established independently in [Nov, Prop. 2.3] and [DFG, Corollary 2.5]. Although not stated, the behavior of the form $n \mapsto kn + O(1)$ for $n \to +\infty$ with $k \in \mathbf{N}$ is established explicitly in [DFG] and follows from the proof in [Nov].

The symmetry established in Corollary 4.2, as well as the non-negativity of b seem to be (minor) new observations. The generalization to $\mathrm{PC}(\mathbf{R}/\mathbf{Z})$ is significant; however it seems that the methods used in both references can be applied with minor changes, at least in the piecewise orientation-preserving case.

4.A.2. *Using the partial action α_S.* We now use the partial action α_S introduced in §2.G.6 to obtain a result of the same flavor as Corollary 4.1. For an element g of $\mathrm{PC}(X)$, we call points of S-indeterminacy of $\alpha_S(g)$ the elements outside its domain of definition.

Corollary 4.4. *Let X be a Hausdorff topological space with no isolated point. Consider the cofinite-partial action α_S of $\mathrm{PC}_S(X)$ on X.*

(1) *the prodigal semi-index function* $\ell_S^-(\sigma)$ *of this partial action (and hence of its restriction to any subgroup) coincides with the function mapping* $\sigma \in \mathrm{PC}(X)$ *to the number of* S*-indeterminacy points of* σ*;*
(2) *for every* $\sigma \in \mathrm{PC}_S(X)$ *there exists* $m_{S,\sigma} \in \mathbf{N}$ *and a bounded function* $b : \mathbf{N} \to \mathbf{N}$ *such that* $\ell^-(\sigma^n) = m_{S,\sigma}n + b(n)$ *for all* $n \in \mathbf{N}$*.*
(3) *Let* G *be a group with a homomorphism into* $\mathrm{PC}_S(X)$*. Then* X *is transfixed by* G *if and only if the number of* S*-indeterminacy points of* g *is bounded independently of* $g \in G$*;*

Moreover, if X *is a topological manifold with no boundary and finitely many ends, then* $\ell_S^-(\sigma) = \ell_S^-(\sigma^{-1})$ *and* $m_{S,\sigma} = m_{S,\sigma^{-1}}$ *for all* $\sigma \in \mathrm{PC}_S(X)$*.*

Proof. The proof follows the same (two!) lines as that of Corollary 4.1 (applied to α_S instead of α). The last statement rather follows from the easy fact, checked in the proof of Corollary 4.2, that the complement of n and m points in such a topological manifold X are never homeomorphic for $n \neq m$. □

In cases such as the pseudogroup of local isometries of $\mathbf{R}/\mathbf{Z}$, we have $\alpha = \alpha_S$ so in this case Corollary 4.4 does not provide anything new. On the other hand, it yields something when α_S is finer than α. Let us provide some illustrations:

Example 4.5. Fix $k \in \mathbf{N}$. Let $\mathcal{C}^k$ be the pseudogroup of local diffeomorphisms of class $\mathcal{C}^k$ on the circle $\mathbf{R}/\mathbf{Z}$. Then the $\mathcal{C}^k$-indeterminacies of σ are the set of points at which either σ or one of its derivatives $\sigma^{(i)}$ for some $i \in \{1, \dots, k\}$ has no outer limit at x; call this k-singular points.

Another example is the pseudogroup **Aff** of local affine homeomorphisms; we have $\mathbf{Aff} \subset \mathcal{C}^1$ and these two pseudogroups have the same indeterminacies (since, a neighborhood of x (minus $\{x\}$), a piecewise affine σ coincides with a local affine homeomorphism at a neighborhood of x if and only it coincides with a local $\mathcal{C}^1$-diffeomorphism). In the piecewise affine context, 1-singular points are often called breakpoints. This proves that the number of 1-singular points of f^n, for f piecewise affine and $n \in \mathbf{Z}$, can be written as $k|n| + b(n)$ with b bounded and $k \in \mathbf{N}$. In particular, this retrieves Guelman and Liousse's result [GuL, Proposition 4.1] that this number, when $n \to \infty$, grows linearly as soon as it is unbounded.

One more example, for which computations by hand are even more complicated, is when $k = 2$: here the model $\mathbf{P}^1_{\mathbf{R}}$ with the pseudogroup **Proj** consisting of restrictions of projective transformations (that is, homographies). On the circle (or any standard curve), the isometric charts in $\mathbf{R}$ being isometric, they endow it with a **Proj**-modeled structure, which itself defines a $\mathcal{C}^2$-structure, with the same S-indeterminacy points for both S: this reflects the fact that the germ at a neighborhood x (minus $\{x\}$) of a piecewise projective transformation σ at a point x coincides with a germ of projective transformation if and only if x is not 2-singular at x.

Note that the pseudogroup **Proj** transfers as a pseudogroup on $\mathbf{R}/\mathbf{Z}$ (local homeomorphisms that are written locally as homographies), so it is less natural but harmless to stick to pseudogroups on $\mathbf{R}/\mathbf{Z}$.

We now apply this to counting singularities of piecewise continuous or differentiable self-transformations. We use the notation introduced before Corollary 1.2.

Corollary 4.6. *For every $k \in \mathbf{N}$ and every parcelwise-$\mathcal{C}^k$ self-transformation σ of $\mathbf{R}/\mathbf{Z}$, there exist integers $0 \le m_0 \le \cdots \le m_k$ and bounded non-negative even functions $b_i : \mathbf{Z} \to \mathbf{N}$ such that for all $i \in \{0, \dots, k\}$, we have $k_{\le i}(\sigma^n) = m_i|n| + b_i(n)$.*

In particular, $k_i(\sigma^n) = (m_i - m_{i-1})|n| + O(1)$ and $k_{\le i}$ and k_i have the property of growing either linearly or being bounded.

As mentioned in the introduction, in the piecewise affine case Guelman and Liousse [GuL, §4] proved that $k_{\le 1}(\sigma^n)$ (and $k_0(\sigma^n)$) are either bounded of have linear growth. Their proof, more precisely, consists in

- proving that either $k_0(\sigma^n)$ is either bounded, or it belongs to $[n - c, Cn]$ for some constants c, C and all $n \in \mathbf{N}$;
- if $k_0(\sigma^n)$ is bounded, proving that $k_1(\sigma^n)$ is either bounded, or it belongs to $[n - c', C'n]$ for some constants c', C' and all $n \in \mathbf{N}$.

4.B. **The main theorem.** Let S be a pseudogroup on a topological space A.

Consider two S-modeled spaces X, X' with topological S-preserving partial action. A cofinite S-preserving G-biequivariant isomorphism is the data of cofinite subsets $Y \subset X$, $Y' \subset X'$ (thus endowed with the corresponding partial actions of G by partial automorphisms of S-modeled space) and a G-biequivariant isomorphism $\psi : Y \to Y'$ of S-modeled spaces.

Thanks to all the preparatory work, we can formulate and obtain:

Theorem 4.7. *Let X be a Hausdorff S-modeled space. Let G be a group with a topological S-preserving partial action on X. Suppose that G transfixes X. Then there exists a Hausdorff S-modeled space X' endowed with an S-preserving continuous G-action, and a cofinite S-preserving G-biequivariant isomorphism from X to X'.*

Moreover, we can require that every finite G-orbit in X' is included in Y'.

Proof. Let $\hat{X}$ be the universal globalization of X (§2.D.2). By Corollary 2.26, $\hat{X}$ canonically inherits an S-structure. Since X is transfixed, by Proposition 2.8 there exists an open Hausdorff G-invariant subset X' of $\hat{X}$ such that $X \bigtriangleup X'$ is finite. Removing all finite G-orbits meeting $X' \smallsetminus X$ if necessary, we can ensure that $X' \smallsetminus X$ only meets infinite G-orbits. Then we have reached the conclusion, with $Y = Y' = X \cap X'$ and ψ being the identity map from $X \cap X'$. (It is on purpose that the statement of the theorem does not refer to the universal globalization $\hat{X}$, so we do not view X and X' as subsets of the same space.) $\square$

As a first corollary, we have the following. Recall that $S^{\wp}$ denotes the pseudogroup of parcelwise-S local homeomorphisms §2.G.3.

Corollary 4.8. *Let X be a curve. Let G be a group with a homomorphism $G \to \mathrm{PC}_S(X)$. Suppose that G transfixes X for the partial action α (restricted from $\mathrm{PC}(X)$). Then there exists an S-modeled compact curve X' endowed with an $S^{\wp}$-preserving continuous G-action, and a cofinite $S^{\wp}$-preserving G-biequivariant isomorphism from X to X'.*

Proof. We use Proposition 2.16 to have a cofinite-partial action, and thus apply Theorem 4.7. This yields the result, except compactness of X'. First obtain X'' possibly not compact; 1-point compactify each component to obtain a space X', which is naturally a curve. Extend the continuous action; since $S^{\wp}$ is stable under concatenation, the resulting action is $S^{\wp}$-preserving. □

4.C. **Applications of Corollary 4.8.** Let us provide applications of Corollary 4.8. It is mostly interesting when $S = S^{\wp}$. First, recall from Corollary 4.1(3) that the transfixing property is equivalent to the boundedness of the number of discontinuities.

4.C.1. *Pseudogroup of all local homeomorphisms.* Then $S = S^{\wp}$; here an S-modeled curve is just a curve. So Corollary 4.8 concerns homomorphisms into $\mathrm{PC}(X)$. It says that when the partial action of G is transfixing, we can find a continuous action on another compact curve that coincides with the original one on a cofinite subset.

4.C.2. *Pseudogroup of all local orientation-preserving homeomorphisms.* Again, $S = S^{\wp}$. This concerns homomorphisms into $\mathrm{PC}^+(\mathbf{R}/\mathbf{Z})$. Here an S-modeled curve is just an oriented curve. Corollary 4.8 then says that the conjugation can be chosen to be piecewise orientation-preserving.

4.C.3. *Local isometries.* Then $S = S^{\wp}$. This is the study of homomorphisms into $\mathrm{IET}^{\pm}$. Here an S-modeled curve is the same as a 1-dimensional Riemannian manifold with finite volume (thanks to the finiteness assumption in the definition of S-model); in particular every S-modeled curve is S-preserving homeomorphic to a standard curve. Hence Corollary 4.8 says that for a a homomorphism $G \to \mathrm{IET}^{\pm}$, the transfixing condition implies a piecewise isometric conjugation to an isometric action on a compact curve.

This has various consequences: for instance if the original homomorphism $G \to \mathrm{IET}^{\pm}$ is transfixing and injective, then this forces G be be virtually abelian. For instance, this implies that $\mathrm{IET}^{\pm}$ has no infinite subgroup with Property FW. In particular, it has no infinite subgroup with Kazhdan's Property T. The latter fact was established in [DFG, Theorem 6.1] for IET^+ by a distinct method (rather related to amenability); it implies the result for $\mathrm{IET}^{\pm}$ because $\mathrm{IET}^{\pm}$ embeds into IET^+ as the centralizer of $x \mapsto -x$ (see Lemma 2.28).

This notably applies when Γ is cyclic and distorted in G. In the case of IET or $\mathrm{IET}^{\pm}$, this narrows possibilities for distorted cyclic subgroups, but is not enough to discard them; additional work [Nov, §4], conceptualized in §3 allows to conclude that cyclic subgroups of $\mathrm{IET}^{\pm}$ are undistorted.

Corollary 4.9. *Every finitely generated abelian subgroup of* $\mathrm{IET}^{\pm}$ *is undistorted.*

Proof. Let A be a finitely generated abelian subgroup, with a finite generating subset W'. If A is distorted, then there exists a finite subset W in $\mathrm{IET}^{\pm}$ and a sequence (n_i) tending to infinity, a sequence (a_i) in A such that $|a_i|_{W'} \simeq n_i$ and $\lim |a_i|_W/n_i = 0$. Let ℓ be the cardinal-definite function associated to this partial action. Consider $B \oplus A'$ as in Proposition 2.3; we can suppose that $a_i = (b_i, a_i') \in B \oplus A'$ for all i. Then $\ell(b, a') \geq c|c|_{W'} - C$ for some some $c > 0$, $C \in \mathbf{R}$ and all $(b, a') \in B \times A'$. In particular, $\ell(a_i) \geq c|a_i'|_{W'} - C$ for all n. It follows that $\lim |a_i'|_{W'}/n_i = 0$, and in turn that $\liminf |b_i|_{W'}/n_i > 0$. Given that $\lim |a_i'|_W/n_i = 0$ and $\lim |a_i|_W/n_i = 0$, we deduce that $\lim |b_i|_W/n_i = 0$. Thus B is a distorted subgroup.

Since B transfixes X, after changing the model and using Corollary 4.8, we can suppose that B acts by isometries. Hence it is undistorted by Proposition 3.3. This reaches a contradiction. □

An immediate consequence is that the only virtually polycyclic groups embedding into $\mathrm{IET}^{\pm}$ are the virtually abelian ones; the latter fact being proved in [DFG2] with another method, handling more general virtually torsion-free solvable groups. Indeed, it is an easy exercise to show that a virtually polycyclic group is virtually abelian if and only if all its abelian subgroups are undistorted (beware that there exist non-virtually-abelian polycyclic groups of the form $\mathbf{Z}^4 \rtimes \mathbf{Z}$ in which all cyclic subgroups are undistorted).

4.C.4. *Local motions.* Again, $S = S^{\wp}$, and we obtain the same with the bonus of a piecewise orientation-preserving conjugation.

4.D. **Refinement of the partial action.** Applying Theorem 4.7 with α_S instead of α, we often obtain stronger conclusions:

Corollary 4.10. *Let X be a finitely-charted S-modeled curve. Let G be a group with a homomorphism $G \to \mathrm{PC}_S(X)$. Suppose that G transfixes X for the partial action α_S. Then there exists a finitely-charted S-modeled curve X' endowed with an S-preserving continuous G-action, and a cofinite S-preserving G-biequivariant isomorphism from X to X'.*

Proof. This is a direct application of Theorem 4.7, applied to the partial action α_S. □

For continuous actions, we can get a better control on the change of model. Denote by $\mathrm{PC}_S^0(X)$ the set of self-homeomorphisms of X that induce an element of $\mathrm{PC}_S(X)$. If X has no isolated point, the canonical map $\mathrm{PC}^0_S(X) \to \mathrm{PC}_S(X)$ is injective.

Corollary 4.11. *Fix two pseudogroups $S \subset T$ on $\mathbf{R}/\mathbf{Z}$. Let X be a finitely-charted S-modeled curve. Let G be a group with a homomorphism $\alpha : G \to \mathrm{PC}^0_S(X) \cap \mathrm{Aut}_T(X)$. Suppose that G transfixes X for the partial action α_S. Then there exists a cofinite G-invariant subset Y of X, finitely-charted S-modeled curve X' endowed with an S-preserving continuous G-action, and a T-preserving and parcelwise S-preserving G-biequivariant homeomorphism from Y to X'.*

In particular, if G has no finite orbit on X, then $Y = X$.

Proof. Corollary 4.10 yields a cofinite subset V of X, an S-modeled curve W with an S-preserving continuous G-action, a cofinite subset V' of W, and a G-biequivariant S-preserving homeomorphism $f : V \to V'$. Here X is endowed with the partial action α_S, while W is endowed with its given action, and the cofinite subsets V and V' are endowed with the induced partial actions.

We can apply Proposition 2.14 to X. It ensures that there is a cofinite G-invariant subset Z_0 of X such that for every cofinite subset Z_1 of Z_0, denoting $Z_1^\vee$ the smallest G-invariant subset including Z_1, the inclusion map $(Z_1, \alpha_S) \to (Z_1^\vee, \alpha)$ is a universal globalization. Similarly, we apply Proposition 2.14 to W (for W, we view the global action as a partial action as well): it yields a G-invariant subset Z'_0 of W with the analogous property.

We can suppose, replacing simultaneously V and V' with smaller subsets, that $V \subset Z_0$ and $V' \subset Z'_0$. Then the inclusions $V \to V^\vee$ and $V' \to (V')^\vee$ are universal globalizations. Then the isomorphism $f : V \to V'$ of topological partial actions extends, by the universal property, to a G-biequivariant homeomorphism $V^\vee \to (V')^\vee$. So we obtain the conclusion, with $Y = V^\vee$ and $X' = (V')^\vee$.

Let us observe that f is T-preserving. Indeed, this follows from the construction and the proof of Theorem 4.7: we have inclusions $X \to \hat{X} \leftarrow W$. The G-invariant T-structure is inherited by $\hat{X}$ and then by W, still G-invariant, and hence by the G-invariant subsets Y on the one hand and X' on the other hand. The map $f : Y \to X'$, constructed inside $\hat{X}$, is just the identity map. Hence it is T-invariant.

This argument for T does not apply to S, but applies to the pseudogroup $S^\wp$ of local parcelwise-S homeomorphisms. Hence f is T-preserving and parcelwise S-preserving. □

We now apply it to various pseudogroups S over $\mathbf{R}/\mathbf{Z}$. In each case the study has several steps: identifying domains of definition for α_S, identifying S-modeled curves, and drawing consequences.

4.E. **Pseudogroup of local affine homeomorphisms.** Here $\mathrm{PC}_S(X)$ is the group of piecewise affine self-transformations. The domain of definition is the set of non-singular points, which is the same as points at which both the function and its derivative have an outer limit. Hence, transfixing is equivalent to have a uniformly bounded number of singularities.

Classifying affinely modeled curves is not hard, see Appendix A. In particular the affine automorphism group of any affinely modeled finitely-charted curve is virtually abelian. As a consequence, we have:

Corollary 4.12. *Let* Γ *be a group with a faithful piecewise affine action on a standard curve, and a subgroup* Λ*. Suppose that* (Γ, Λ) *has relative Property FW. Then* Λ *is virtually torsion-free abelian. In particular, if* Γ *has Property FW (e.g., Kazhdan's Property T) then it is finite.*

Andrés Navas informed me that even the failure of Property T is a new result: for instance the above applies to the subgroup $V_{\{2,3\}}$ of piecewise affine maps with slopes in the multiplicative group $\langle 2, 3\rangle$ and singularities in $\mathbf{Z}[1/6]$, as well as its subgroup $T_{\{2,3\}}$ of elements acting continuously on the circle; the failure of Property T was unknown for both groups.

Lodha, Matte Bon and Triestino have independently obtained Corollary 4.12 in the case of continuous piecewise affine maps (thereby also proving the failure of Property T for $T_{\{2,3\}}$).

In the case of distortion, we deduce the following, which is essentially due to Guelman and Liousse [GuL].

Corollary 4.13. *For any distorted cyclic subgroup* $\langle c\rangle$ *of the group of piecewise affine self-transformations, there is a cofinite piecewise affine conjugation to an affine action on a standard curve. In the piecewise orientation-preserving case, we can choose the cofinite conjugation to be piecewise orientation-preserving.*

Proof. Since $\langle c\rangle$ is distorted, its cofinite-partial action is transfixing. Therefore, by Corollary 4.10, there a cofinite piecewise affine conjugation to an affine action on an affinely modeled curve C. Passing to a finite index subgroup, we can suppose that each component is preserved. Then Corollary 3.7 ensures that it acts with finite order on any non-standard circle occuring in C. Hence, removing a finite $\langle c\rangle$-invariant subset, we can suppose that there is no non-standard circle in C. □

The minor nuance is that Guelman and Liousse work in the piecewise orientation-preserving case, and obtain the conjugation after passing to a subgroup of finite index.

Also in the continuous case, we can refine the conjugacy and deduce the following:

Corollary 4.14. *For any distorted cyclic subgroup* $\langle c\rangle$ *of the group of piecewise affine self-homeomorphisms of the circle* $\mathrm{PC}^0_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$*, there exists a piecewise self-homeomorphism of* $\mathbf{R}/\mathbf{Z}$ *conjugating* $\langle c\rangle$ *to a cyclic group of irrational rotations.*

Proof. By Corollary 4.11, there is a cofinite $\langle c\rangle$-invariant subset Y of X, another affinely modeled curve X' and and a piecewise affine homeomorphism $Y \to X'$ conjugating the $\langle c\rangle$-action to an affine action on Y.

If $Y \neq X$, then X' is homeomorphic to a finite disjoint union of intervals, each being finitely-charted; hence each is isomorphic as affinely modeled curve to $]0,1[$ and hence has a finite automorphism group; since $\langle c\rangle$ is distorted, it is infinite and we get a contradiction.

So $Y = X$. Thus X' is an affinely modeled curve homeomorphic to the circle, and $\langle c\rangle$ acts as an irrational rotation. By Corollary 3.7, irrational rotations of non-standard affine circles are undistorted. Hence X' is a standard circle, and hence we can suppose (conjugating with an affine isomorphism $X' \to X$) that $X' = X$. □

Let us provide different equivalent restatements of Question 1.11.

Corollary 4.15. *Denote by θ_r the rotation $x \mapsto x + r$ on $\mathbf{R}/\mathbf{Z}$. Let $X_d \simeq \mathbf{R}/Z$ be a disjoint union of d copies of standard circles $Y_1, \dots, Y_d$. The following are equivalent.*

(i) The group $\mathrm{PC}_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ admits a distorted cyclic subgroup;
(ii) The group $\mathrm{PC}^+_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ admits a distorted cyclic subgroup;
(iii) There exists $r \in \mathbf{R}/\mathbf{Z} \smallsetminus \mathbf{Q}/\mathbf{Z}$ such $\theta_r \sqcup \mathrm{id}_Y$, is a distorted element in $\mathrm{PC}_{\mathbf{Aff}}(X_2)$.
(iv) There exists $r \in \mathbf{R}/\mathbf{Z} \smallsetminus \mathbf{Q}/\mathbf{Z}$ such $\theta_r \sqcup \theta_r \sqcup \mathrm{id}_{Y_3}$, is a distorted element in $\mathrm{PC}^+_{\mathbf{Aff}}(X_3)$.

Also, the following are equivalent:

(i') The group $\mathrm{PC}^0_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ admits a distorted cyclic subgroup;
(ii') There exists $r \in \mathbf{R}/\mathbf{Z} \smallsetminus \mathbf{Q}/\mathbf{Z}$ such that θ_r is a distorted element in $\mathrm{PC}^0_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$.

What comes out of [GuL] is the equivalence between (ii) and a slightly weaker analogue of (iv): the existence of n and a non-identity self-homeomorphism f of X_n acting on each Y_i as either the identity or an irrational rotation, such that f is distorted in $\mathrm{PC}^+_{\mathbf{Aff}}(X_n)$.

In the continuous case we did not make the orientation-preserving case explicit, because this is a trivial reduction, since the orientation-preserving subgroup has index 2. This is in contrast to the first case, where the piecewise orientation-preserving subgroup has infinite index.

Proof of Corollary 4.15. For the second equivalence, one implication is trivial, and the other follows from Corollary 4.14.

Let us prove the first equivalence. (i)⇐(iii)⇐(iv)⇒(ii)⇒(i) is trivial. That (i) implies (ii) follows from the group embedding

$$\mathrm{PC}_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z}) \to \mathrm{PC}^+_{\mathbf{Aff}}((\mathbf{R}/\mathbf{Z})^{\pm}) \simeq \mathrm{PC}^+_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$$

from Lemma 2.28, and where the second, non-canonical isomorphism is induced by a piecewise affine transformation between $(\mathbf{R}/\mathbf{Z})^{\pm}$ and $\mathbf{R}/\mathbf{Z}$. To obtain (iii)⇒(iv), we again use this embedding but more carefully: for $X = X_2$, it maps $\theta_r \sqcup \mathrm{id}_{Y_2}$ to (after suitable identifications) the element $\theta_r \sqcup \mathrm{id}_{Y_2} \sqcup \theta_r \sqcup \mathrm{id}_{Y_4}$ of $\mathrm{PC}^+(X_4)$. After conjugating $Y_2 \sqcup Y_4$ to Y_3 by a piecewise affine transformation and Y_3 to Y_2 by an affine transformation, we get the result.

It remains to prove (i)⇒(iii). First, one uses Corollary 4.13 to show that there exists a standard curve X with with $c \in \mathrm{PC}^0_{\mathbf{Aff}}(X)$ such that the cyclic subgroup $\langle c \rangle$ is distorted $\mathrm{PC}_{\mathbf{Aff}}(X)$. Replacing c with a power, we can suppose that c preserves each component Y of X, and that as an automorphism of Y, c is either the identity or has infinite order. Hence c is the identity on each noncompact component of X, and is an irrational rotation on every compact component of X. Then we fix one component Y of X on which c acts as an irrational rotation, and we let w be an isometry of X, acting as a reflection on Y and as the identity outside. Let q be equal to c on Y and the identity elsewhere. Then $c^n w c^{-n} w^{-1} = q^{2n}$. Hence c being distorted, so is q. Then conjugating the complement of Y by an affine transformation to a single standard circle, we obtain (iii). □

In the case of the pseudogroup $\mathcal{C}^k$ of local diffeomorphisms of class $\mathcal{C}^k$, the classification of $\mathcal{C}^k$-modeled curves is "trivial" in the sense that there are only two such connected curves up to isomorphism: the open interval and the circle. In particular, Corollary 4.10 yields Corollary 1.4.

In the continuous case, this yields the following, with a little refinement to pass from parcelwise to piecewise conjugacy:

Corollary 4.16. *Let G be group acting continuously on $\mathbf{R}/\mathbf{Z}$, with no finite orbit. Suppose the action is parcelwise-$\mathcal{C}^k$ (i.e., G maps into $\mathrm{PC}^0_{\mathcal{C}^k}(\mathbf{R}/\mathbf{Z})$. If G transfixes $\mathbf{R}/\mathbf{Z}$ for the corresponding partial action $\alpha_{\mathcal{C}^k}$, then the action is conjugate (in $\mathrm{PC}^0_{\mathcal{C}^k}(\mathbf{R}/\mathbf{Z})$) to a $\mathcal{C}^k$-action.*

If moreover G acts by piecewise-$\mathcal{C}^\ell$ self-transformations for some $\ell \leq k$, then the conjugating map can be assumed to have the same property.

Proof. Both are direct applications of Corollary 4.11 with S being the $\mathcal{C}^k$ pseudogroup. In the first case, we consider no T (technically, this means we take T as the pseudogroup of all local homeomorphisms). In the second case, we take T as the pseudogroup of all self-homomorphisms that piecewise-$\mathcal{C}^\ell$. □

Lodha, Matte Bon and Triestino [LMT] obtain a result which is very close to Corollary 4.16. Actually the result of Corollary 4.16 (at least both the first statement and the second for $\ell = k$) is a consequence of their work, but they mainly formulate results for groups with Property FW or T, which, using Thurston's stability theorem on the one hand and Navas' theorem about groups with Property T of class $\mathcal{C}^{3/2}$ on the other hand, yield stronger conclusions [LMT, Corollaries 1.3, 1.4].

As regards the piecewise projective case, let us write the corollaries:

Corollary 4.17. *Let G be a group with a piecewise projective action on a finitely-charted projectively modeled curve X. Suppose that the corresponding partial action transfixes X. Then there is another finitely-charted projectively modeled curve X', cofinite subsets $Y \subset X$ and $Y' \subset Y$, and a piecewise projective G-biequivariant homeomorphism $h : Y \to Y'$.*

Moreover, if G acts continuously on X with no finite orbit, we can choose $Y = X$, $Y' = X'$. If G acts by $\mathcal{C}^1$ diffeomorphisms, we can choose h to be of class $\mathcal{C}^1$. If G both acts by $\mathcal{C}^1$ diffeomorphisms without finite orbit, we can choose $Y = X$, $Y' = X$, and h a $\mathcal{C}^1$-diffeomorphism.

Given the classification of projectively modeled curves and their automorphisms, we obtain as corollaries:

Corollary 4.18. *Under the same hypotheses, G has a subgroup of finite index H with a finite normal subgroup Z such that H/Z can be embedded as a subgroup of $\mathrm{PSL}_2(\mathbf{R})^k$ for some k. If moreover X is connected and G acts continuously, we can choose $k = 1$.*

Proof. By Corollary 4.17, we can suppose that $G \subset \mathrm{Aut}_{\mathbf{Proj}}(X)$. Let (X_i) be the connected components of X; by Appendix A, the automorphism group of each X_i has finitely many components. There exists a subgroup H of finite index in G stabilizing each component, and mapping into $\mathrm{Aut}(X_i)^\circ$. By Appendix A, $\mathrm{Aut}(X_i)$ is isomorphic to either $\mathbf{R}$, $\mathbf{R}/\mathbf{Z}$, $\mathbf{R} \rtimes \mathbf{R}$, or $\mathrm{PSL}_2(\mathbf{R})_m$ (the m-fold connected covering of $\mathrm{PSL}_2(\mathbf{R})$). The first three, as well as the quotient of the latter by its center, embed as subgroups into $\mathrm{PSL}_2(\mathbf{R})$. We thus obtain the conclusion. □

Corollary 4.19. *For every finitely-charted curve X, the group $\mathrm{PC}_{\mathbf{Proj}}(X)$ has no infinite subgroup with Kazhdan's Property T.*

Proof. If we suppose so, using Corollary 4.18 and the fact that Kazhdan's Property T passes to finite index subgroups (and obviously to quotients), we would deduce the existence of an infinite subgroup with Property T in $\mathrm{PSL}_2(\mathbf{R})$. But it is well-known that there is no such group. Indeed, by Faraut-Harzallah [FH], it would be conjugate into the maximal compact subgroup $\mathrm{PSO}_2(\mathbf{R})$, which is abelian, a contradiction. □

This is new even in the continuous case. See also 1.9.

5. More on piecewise projective self-homeomorphisms

5.A. **Construction of actions.** We now provide more explicit actions for the group of piecewise projective self-homeomorphisms of the circle.

For a standard curve X, denote $X^\pm = X \times \{1, -1\}$. We usually write x^+ and x^- for $(x, 1)$ and $(x, -1)$. Define $\mathcal{L}^2(X) = X^\pm \times \mathbf{R}_{>0} \times \mathbf{R}$.

Let X, Y be standard curves. Define $\mathrm{CPD}_2(X, Y)$ as the set of continuous, piecewise-$\mathcal{C}^2$ functions, whose one-sided derivatives do not vanish. For $f \in \mathrm{CPD}_2(X, Y)$, define $f_{(2)} : \mathcal{L}^2(X) \to \mathcal{L}^2(Y)$ by

$$f_{(2)}(x,t,u) = \left(f(x),\ \frac{f'(x)}{f'(\hat{x})}t,\ \frac{1}{f'(x)}u + \frac{f''(x)}{2f'(x)^2} - \frac{f''(\hat{x})}{2f'(x)f'(\hat{x})}t^{-1} \right).$$

A simple computation shows if Z is another standard curve and $g : Y \to Z$ is continuous and piecewise of class $\mathcal{C}^2$, then that $(g \circ f)_{(2)}$ and $g_{(2)} \circ f_{(2)}$ are both equal: indeed, they are equal to

$$(x,t,u) \mapsto \left(g(f(x)), \frac{f'(x)g'(f(x)}{f'(\hat{x})g'(f(\hat{x}))}t,\ a(x)u + b(x) - c(x)t^{-1} \right),$$

where

$$a(x) = \frac{1}{f'(x)g'(f(x))},\ b(x) = \frac{f''(x)}{2g'(f(x))f'(x)^2} + \frac{g''(f(x))}{2g'(f(x))^2};$$

$$c(x) = \frac{1}{2g'(f(x))f'(x)} \left(\frac{f''(\hat{x})}{f'(\hat{x})} + \frac{g''(f(\hat{x}))f'(\hat{x})}{g'(f(\hat{x}))} \right).$$

Let X, Y be standard curves. In the following proposition, we use this action to define a natural pull-back for functions $X^{\pm} \to \mathbf{R}_{>0} \times \mathbf{R}$, which is used throughout the sequel.

Proposition 5.1. *Let $\mu = (\mu_1, \mu_2)$ be a function $Y^{\pm} \to \mathbf{R}_{>0} \times \mathbf{R}$ (with $\mu_1 : Y^{\pm} \to \mathbf{R}_{>0}$ and $\mu_2 : Y^{\pm} \to \mathbf{R}$), and $P_\mu \subset \mathcal{L}^2(X)$ its graph. Then $f_{(2)}^{-1}(P_\mu) = P_{f^*\mu}$, where for all $x \in X$ we have*

$$(f^*\mu)_1(x) = \frac{f'(\hat{x})}{f'(x)}\mu_1(f(x))$$

and

$$(f^*\mu)_2(x) = f'(x)\mu_2(f(x)) - \frac{f''(x)}{2f'(x)} + \frac{f''(\hat{x})f'(x)}{2f'(\hat{x})^2}\mu_1(f(x))^{-1}.$$

Proof. We have $(x,t,u) \in f_{(2)}^{-1}(P_\mu)$ if and only if $(x',t',u') := f_{(2)}(x,t,u) \in P_\mu$. This means that $\mu_1(x') = t'$ and $\mu_2(x') = u'$. This means that $\mu_1(f(x)) = \frac{f'(x)}{f'(\hat{x})}t$ and $\mu_2(f(x)) = \frac{1}{f'(x)}u + \frac{f''(x)}{2f'(x)^2} - \frac{f''(\hat{x})}{2f'(x)f'(\hat{x})}t^{-1}$. In turn, this means that $t = \frac{f'(\hat{x})}{f'(x)}\mu_1(f(x))$ and $u = f'(x)\mu_2(f(x)) - \frac{f''(x)}{2f'(x)} + \frac{f''(\hat{x})}{2f'(\hat{x})}t^{-1}$. Thus $f^{-1}(P_\mu)$ is indeed the graph of the given function. □

From the covariant functoriality of $f \mapsto f_{(2)}$, we immediately deduce the contravariant functoriality of $f \mapsto f^*$, acting on all $(\mathbf{R}_{>0} \times \mathbf{R})$-valued functions.

In addition, define an involution $\tau_X : \mathcal{L}^2(X) \to \mathcal{L}^2(X)$ by $\tau(x,t,u) = (\hat{x}, t^{-1}, -tu)$. Denote $\tau(t,u) = (t^{-1}, -tu)$. Then a computation shows that $\tau_Y \circ f_{(2)} = f_{(2)} \circ \tau_X$.

Let $\mathcal{A}^2(X)$ be the set of functions $f : X^{\pm} \to \mathbf{R}_{>0} \times \mathbf{R}$, that take the value $(1,0)$ outside a finite subset, and such that $f(\hat{x}) = \tau(f(x))$ for all x in the domain of definition. Equivalently, this symmetry condition means that the graph of f is

τ_X-invariant. Hence any proper function $f \in \mathrm{CPD}_2(X,Y)$ induces $f^* : \mathcal{A}^2(Y) \to \mathcal{A}^2(X)$.

On a standard curve, denote by $\nu_0^X \in \mathcal{A}^2(X)$ the "trivial" constant function $(1,0)$.

Lemma 5.2. *Let V, V' be standard curves and f a piecewise projective homeomorphism $V \to V'$. Then f is projective if and only $f^*\nu_0^{V'} = \nu_0^V$.*

Proof. By definition, for any $f \in \mathrm{CPD}^2(V,V')$, we have

$$\big(f^*\nu_0^{V'}\big)(x) = \left(\frac{f'(\hat{x})}{f'(x)}, -\frac{f''(x)}{2f'(x)} + \frac{f''(\hat{x})f'(x)}{2f'(\hat{x})^2}\right)$$
$$= \left(\frac{f'(\hat{x})}{f'(x)}, \frac{f''(\hat{x}) - f''(x) + \frac{f''(\hat{x})}{f'(\hat{x})^2}(f'(x)^2 - f'(\hat{x})^2)}{2f'(x)}\right).$$

It immediately that f is of class $\mathcal{C}^2$ at $x \in V$ if and only if $f^*\nu_0^{V'}(x) = (1,0)$, which equals $\nu_0^V(x)$. Therefore, f is of class $\mathcal{C}^2$ on V if and only if $f^*\nu_0^{V'} = \nu_0^V$.

In particular, since we assume that f is a piecewise projective homeomorphism, f is projective on V if and only if $f^*\nu_0^{V'} = \nu_0^V$. □

Lemma 5.3. *Let X be a standard curve, $x \in X$ and $\nu \in \mathcal{A}^2(X)$. Suppose that, at the neighborhood of x, the function ν is invariant by homographies that are close enough to the identity. Then $\nu = \nu_0^X$ around x.*

Proof. More explicitly, since this is local at x, we can suppose that $X \subset \mathbf{R}$. For all intervals I, J around x such that the closure in $\mathbf{R}$ of I is contained in J, we can consider the subset of homographies $f \in \mathrm{PSL}_2(\mathbf{R})$ mapping I into J. This is a neighborhood $W_{I,J}$ of the identity in $\mathrm{PSL}_2(\mathbf{R})$. Precisely, the assumption is that we assume that for some such I, J, the identity element has a subneighborhood W contained in $W_{I,J}$ such that $f^*\nu = \nu$ on I for all $f \in W$.

We can suppose that $x = 0$. Then the translation $T_a : x \mapsto x + a$ belongs to W for a small enough, say $|a| \le a_0$. The local condition $T_a^*\nu = \nu$, read at the first coordinate, implies that $\nu_1(t^\pm) = \nu_1(t^\pm + a)$ for every t small enough (say, $|t| \le t_0$: note that this does not depend on a). Then, since $\nu_1 - 1$ is finitely supported, we deduce that $\nu_1 - 1$ vanishes at the neighborhood of $0^\pm$.

Let us now rewrite the definition of $f^*\mu$ when f is of class $\mathcal{C}^2$, for each t such that $\mu_1(f(t)) = 1$. Namely it simplifies to $(f^*\mu)(t) = \big(1, f'(t)\mu_2(f(t))\big)$. We now choose a homothety $L_c(t) = ct$ with $c > 1$ close enough to c to ensure $L_c^*\nu = \nu$ at the neighborhood of 0, and in particular at zero. Thus we have $\nu_2(0^\pm) = c\nu_2(0^\pm)$, which implies $\nu_2(0^\pm)$.

Hence (ν_1, ν_2) takes the value $(1,0)$ at 0. Since ν and ν_0^X coincide outside a finite subset, they thus coincide at the neighborhood of 0. □

Definition 5.4. Denote by π_0^X the standard projective structure on a standard curve. On a standard curve with standard projective structure π_0, we say that a projective structure is compatible if the identity $(\mathbf{R}/\mathbf{Z}, \pi_0) \to (\mathbf{R}/\mathbf{Z}, \pi)$ is piecewise projective. We say that it is $\mathcal{C}^1$-compatible if in addition this identity map is of class $\mathcal{C}^1$.

We reach the goal of this preparatory work: encoding a compatible projective structure in an element of $\mathcal{A}^2(X)$:

Proposition 5.5. *Let X be a standard curve and $\mu \in \mathcal{A}^2(X)$. Define an atlas on X by considering piecewise projective homeomorphisms $h : U \to V$, with U open in X and V a standard curve, such that $h^*\nu_0^V = \mu$. Then this defines a projective structure $\pi = \pi_\mu$ on X, such that the identity map is a piecewise projective homeomorphism $(X, \pi_0^X) \to (X, \pi)$.*

Given $f \in \mathrm{PC}^0_{\mathbf{Proj}}(X, Y)$ with Y another standard curve, given $\mu \in \mathcal{A}^2(X)$ and $\nu \in \mathcal{A}^2(Y)$, f is a projective isomorphism $(X, \pi_\mu) \to (X, \pi_\nu)$ if and only if $f^*\nu = \mu$.

Conversely, every projective structure π on X, such that the identity map $(X, \pi_0^X) \xrightarrow{i} (X, \pi)$ is a piecewise projective homeomorphism, has the form π_μ for a unique $\mu \in \mathcal{A}^2(X)$.

Proof. To check that it defines an atlas, the compatibility is clear from functoriality of $f \mapsto f^*$. The only thing to check is that X is covered by charts. This will of course make use of the "τ-condition" saying that $\nu(\hat{x}) = \tau(\nu(x))$. If $\mu(x^+) = (1, 0)$ then $\mu(x^-) = (1, 0)$ by the τ-condition, and hence the identity map at a small neighborhood does the job. Otherwise, we can choose a small interval around x on which μ equals $(1, 0)$ except at x. Then first composing locally with a piecewise affine local homeomorphism fixing x with a singular point at x, boils down to the case when $\mu(x^+) = (1, u)$ for some $u \in \mathbf{R}$. By the τ-condition, $\mu(x^-) = (1, -u)$. Then consider the formula in the proof of Lemma 5.2, assuming in addition that f is of class $\mathcal{C}^1$: for every t

$$\left(f^*\nu_0^{V'}\right)(t) = \left(1, \frac{f''(\hat{t}) - f''(t)}{2f'(t)}\right).$$

We can suppose (by an orientation-preserving isometric change of standard coordinates) that $x = 0$. Then we can indeed find f of class $\mathcal{C}^1$ and piecewise projective and $\mathcal{C}^1$ around 0 with $f'(0) = 1$ and $f''(0^+) - f''(0^-)$ arbitrary. Namely, choose $f_c(t) = t$ for $t \le 0$, and $f_c(t) = t/(1 - ct)$ for some $c \in \mathbf{R}$. Then $f'(0) = 1$, $f''(0^-) = 0$ and $f''(0^+) = c/2$. Hence locally composing with such a map with c well-chosen, we can ensure that $\mu(0^+) = (1, 0)$; the τ-condition ensures that $\mu(0^-) = (1, 0)$, and we have found our projective chart.

The second assertion is straightforward from the definition and functoriality of $f \mapsto f^*$.

As regards the third statement, let us start with uniqueness: consider μ, μ' defining the same projective structure. Since this is a local assertion, we can use the first statement to assume that $\mu' = \nu_0^X$. Then the result follows from Lemma 5.3. The existence is immediate: just define $\mu = (i^{-1})^*\nu_0^X$. □

Proposition 5.6. *Let X be a standard curve. Let G be a subgroup of $\mathrm{PC}^0_{\mathbf{Proj}}(X)$ with no finite orbit on X. Then there is at most one compatible G-invariant projective structure π on X. If moreover G acts by $\mathcal{C}^1$-diffeomorphisms, then π has to be $\mathcal{C}^1$-compatible.*

Proof. By Proposition 5.5, this is equivalent to showing that there is at most one G-invariant element $\nu \in \mathcal{A}^2(X)$.

For the first coordinate, the G-invariance of ν says that $\nu_1(x) = \frac{f'(\hat{x})}{f'(x)}\nu_1(f(x))$. If μ is also G-invariant, it satisfies the same formula, and we deduce, taking the quotient $\eta_1 = \nu_1/\mu_1$, that $\eta(x) = \eta(f(x))$ for all $x \in X^\pm$ and all $f \in G$. Hence $\{x \in X^\pm : \eta_1(x) \neq 1\}$ is G-invariant. Since G has no finite orbit and $\eta_1 = 1$ outside a finite subset, we deduce $\eta = 1$ and hence $\mu_1 = \nu_1$.

Next, the G-invariance of ν, read at the second coordinate, says that, for all $x \in X^\pm$ and $f \in G$,

$$\nu_2(x) = f'(x)\nu_2(f(x)) - \frac{f''(x)}{2f'(x)} + \frac{f''(\hat{x})f'(x)}{2f'(\hat{x})^2}\nu_1(f(x))^{-1}.$$

Given that μ satisfies the same invariance, and that $\mu_1 = \nu_1$, we deduce, substracting and setting $\eta_2 = \nu_2 - \mu_2$, that for all $x \in X^\pm$ and $f \in G$,

$$\eta_2(x) = f'(x)\eta_2(f(x)).$$

In particular, the subset $\{x \in X^\pm : \eta_2(x) \neq 0\}$ is G-invariant; since it is finite and G has no finite orbit, we deduce $\eta_2 = 0$ and hence $\nu = \mu$.

For the last statement, observe that if G consists of $\mathcal{C}^1$-diffeomorphisms, then the G-invariance of ν implies that the finite subset $\{x \in X^\pm : \nu(x) \neq 1\}$ is G-invariant. Hence it is empty. This means that π is $\mathcal{C}^1$-compatible. ☐

Corollary 5.7. *Let X be a standard curve. Consider two conjugate subgroups G_1, G_2 in $\mathrm{PC}^0_{\mathbf{Proj}}(X)$, that have no finite orbit on X, and preserve compatible projective structures π_1, π_2. Then (X, π_1) and (X, π_2) are isomorphic as projectively modeled curves.*

If moreover G_1, G_2 consist of $\mathcal{C}^1$-diffeomorphisms, then they are conjugate by some $\mathcal{C}^1$-diffeomorphism.

Proof. When $G_1 = G_2$, we deduce from Proposition 5.6 that π_1 and π_2 are equal. When $G_2 = fG_1f^{-1}$ with $f \in \mathrm{PC}^0_{\mathbf{Proj}}(X)$, this implies that $\pi_1 = f^*\pi_2$ is the pull-out by f of π_2, and hence f induces an isomorphism from (X, π_1) to (X, π_2).

In the $\mathcal{C}^1$-case, Proposition 5.6 implies that π_1 and π_2 are $\mathcal{C}^1$-compatible. Then the condition $\pi_1 = f^*\pi_2$ forces f to be of class $\mathcal{C}^1$. ☐

5.B. **Classification of exotic circles.** It is easy and standard that every closed subgroup of the group of self-homeomorphisms of the circle $X = \mathbf{R}/\mathbf{Z}$, if topologically isomorphic to $\mathbf{R}/\mathbf{Z}$, is conjugate by an orientation-preserving self-homeomorphism to the group $\mathbf{R}/\mathbf{Z}$ of translations. Roughly speaking, given a subgroup W of Homeo($\mathbf{R}/\mathbf{Z}$) (typically, the automorphism group of some enriching structure), an "exotic circle" is a subgroup conjugate to the group $\mathbf{R}/\mathbf{Z}$ of translations in Homeo($\mathbf{R}/\mathbf{Z}$), but not in W. In the context of piecewise affine self-homeomorphisms, exotic circles were defined and classified by Minakawa [Mi].

Denote by $\mathrm{PC}^1_{\mathbf{Proj}}(X)$ the group of $\mathcal{C}^1$-diffeomorphisms in $\mathrm{PC}^0_{\mathbf{Proj}}(X)$. Also use the notation of the appendix. Roughly speaking, below, Θ_1 is the standard circle, Θ_t for $t > 1$ are the non-standard affine circles, Ξ_n is the connected n-fold covering of the projective line, and are interpolated by the "metaelliptic circles" Ξ_r (which come from lifts of elliptic elements in the universal covering of $\mathrm{SL}_2(\mathbf{R})$).

Theorem 5.8. *Let X be the standard circle $\mathbf{R}/\mathbf{Z}$ with its subgroup K of isometries and its subgroup $K^+ \simeq \mathrm{SO}(2)$ of orientation-preserving isometries. Fix $H \in \{K, K^+\}$. Consider a faithful continuous action of H on X, whose image G is contained in $\mathrm{PC}^0_{\mathbf{Proj}}(X)$ Then*

1. *G preserves a unique compatible (Definition 5.4) projective structure π on X;*
2. *π is $\mathcal{C}^1$-compatible if G consists of $\mathcal{C}^1$-diffeomorphisms.*
3. *the projectively modeled curve (X, π) is isomorphic to Ξ_r for some $r > 0$, or Θ_t for some $t \geq 1$.*
4. *the conjugacy class of G among subgroups of $\mathrm{PC}^0_{\mathbf{Proj}}(X)$ is characterized by the isomorphy type of the projectively modeled curve (X, π), thus by either the case Θ_t for $t \geq 1$ ("affine case") or Ξ_r for $r > 0$ ("metaelliptic case"). We thus say that G is of type Θ_t, or of type Ξ_r. This also characterizes G modulo conjugation by $\mathrm{PC}^{0,+}_{\mathbf{Proj}}(X)$.*
5. *if $G \subset \mathrm{PC}^1_{\mathbf{Proj}}(X)$, then this also characterizes the conjugacy class modulo the conjugation action of $\mathrm{PC}^1_{\mathbf{Proj}}(X)$, or also $\mathrm{PC}^{1,+}_{\mathbf{Proj}}(X)$.*
6. *G preserves an affine structure on X if and only if it is of type Θ_1, or if $H = K^+$ and G is of type Θ_t for some $t > 1$.*
7. *Consider the corresponding conjugacy classification for the corresponding action of H, the classification is the same when considered modulo conjugation by $\mathrm{PC}^0_{\mathbf{Proj}}(X)$, or $\mathrm{PC}^1_{\mathbf{Proj}}(X)$ in the $\mathcal{C}^1$-case.*

Proof. Write G^+ as the image of K^+ in G. First observe that G has no finite orbit. Otherwise G^+ would fix a point and it is easy to check and well-known that K^+ has no nontrivial continuous action on an interval.

By Lemma 2.4, there is a dense subgroup Γ of G such that (G, Γ) has relative Property FW (where G is considered as discrete group). By Corollary 4.17, there exists a finitely charted projectively modeled curve X', and a piecewise projective homeomorphism $f : X \to X'$ such that the conjugate Γ-action preserves the projective structure. Pulling back to X, we obtain a compatible Γ-invariant projective structure π. By Proposition A.6, the automorphism group of (X,π) is closed in $\mathrm{Homeo}(X)$, and hence G preserves π.

When G_i consists of $\mathcal{C}^1$-diffeomorphisms, Proposition 5.6 ensures that π is $\mathcal{C}^1$-compatible.

So (1) and (2) are proved. (3) follows from the classification of projective structures on the circle and their maximal compact subgroups of automorphisms, established in the appendix.

For (4), one direction is provided by Corollary 5.7: the conjugacy class of G determines the isomorphism type of (X,π). In the other direction, suppose that the isomorphism type of (X,π) is given. Proposition A.7, based on classification provides the converse: once G preserves π, it is uniquely determined modulo conjugation by the orientation-preserving automorphism group of (X,π).

For (5), the only nontrivial improvement with respect to (4) lies in the $\mathcal{C}^1$ assertion of Corollary 5.7.

(6) if G preserves an affine structure ξ, it preserves the corresponding projective structure π_ξ, which is then equal to π. By the affine classification (see the appendix), (X,ξ) is isomorphic as affinely modeled curve to Θ_t for some $t\ge 1$. In the case of $H=K$, the affine action does not preserve the orientation, which excludes Θ_t if $t>1$.

(7) As a topological group, the outer automorphism group of K is trivial, and hence the result is immediate. For K^+, the classification implies that every action of K^+ on a projectively modeled curve extends to K, and then we obtain the equivalences of both conjugacy notions. (Of course this does not extend to conjugacy modulo orientation-preserving elements. This comes from the plain topological setting: the only action of K^+ on the circle is not orientation-reversing isomorphic to itself.) □

Theorem 5.9. *Fix $n\in\mathbf{N}_{\ge 1}$. Let X be the connected n-covering of the projective line $\mathbf{P}^1_{\mathbf{R}}$ (with respect to the basepoint ∞) and let H be either the connected n-fold covering of $\mathrm{PSL}_2^{(n)}(\mathbf{R})$, or the corresponding overgroup $\mathrm{PGL}_2^{(n)}(\mathbf{R})$ of index two. Let G be the image of H in a faithful continuous action α of H on X, valued in $\mathrm{PC}^0_{\mathbf{Proj}}(X)$. Then:*

(1) G is conjugate to H by some element of $\mathrm{PC}^{0,+}_{\mathbf{Proj}}(X)$, which can be chosen in $\mathrm{PC}^{1,+}_{\mathbf{Proj}}(X)$ if $G\subset \mathrm{PC}^1_{\mathbf{Proj}}(X)$

(2) the action α itself of H is conjugate to the inclusion action by some element of $\mathrm{PC}^0_{\mathbf{Proj}}(X)$, which can be chosen in $\mathrm{PC}^{1,+}_{\mathbf{Proj}}(X)$ if $G=\alpha(H)\subset \mathrm{PC}^1_{\mathbf{Proj}}(X)$.

Proof. As in the proof of Theorem 5.8, we first need to have some Property FW phenomenon: actually we find a dense subgroup with Property FW: If $H=\mathrm{PSL}_2$ or PGL_2, we consider $\mathrm{PSL}_2(\mathbf{Z}[\sqrt{2}])$ or $\mathrm{PGL}_2(\mathbf{Z}[\sqrt{2}])$, and for their finite coverings we consider their inverse images.

Clearly the G-action has no finite orbit, and as in the proof of Theorem 5.8, using Property FW we find a compatible Γ-invariant projective structure π ($\mathcal{C}^1$-compatible if G acts by $\mathcal{C}^1$-diffeomorphisms), and again using Proposition A.6, the automorphism group of (X,π) is closed, and hence G preserves π.

By the classification in the appendix, the only possibility is that (X,π) is isomorphic to Ξ_n, the n-fold covering of the projective line, that is, X itself (chosen by anticipation!). This precisely means that G is conjugate, by some element of $\mathrm{PC}^{0,+}_{\mathbf{Proj}}(X)$, to a subgroup of H, and hence to H itself (since clearly any injective continuous homomorphism $G \to H$ is an isomorphism). Also in the $\mathcal{C}^1$-case, since π is $\mathcal{C}^1$-compatible, the conjugating element has to be $\mathcal{C}^1$ as well.

By post-conjugation by an orientation-reversing element of $\mathrm{PGL}_2^{(n)}(\mathbf{R})$ if necessary, we can arrange also the conjugation to be orientation-preserving. □

The following answers a question in an early version of [LMT].

Corollary 5.10. *Fix $n \in \mathbf{N}_{>0}$. Let H be either equal to* $\mathrm{PSL}_2^{(n)}(\mathbf{R})$ *or* $\mathrm{PSL}_2^{(n)}(\mathbf{R})$. *Let G be the image of a faithful continuous action of H on $\mathbf{R}/\mathbf{Z}$. Suppose that $G \subset \mathrm{PC}^0_{\mathbf{Proj}}(\mathbf{R}/\mathbf{Z})$.*

Then G is closed in $\mathrm{Homeo}(\mathbf{R}/\mathbf{Z})$, *and is uniquely defined up to conjugation in* $\mathrm{PC}^0_{\mathbf{Proj}}(\mathbf{R}/\mathbf{Z})$. *If moreover $G \subset \mathrm{PC}^1_{\mathbf{Proj}}(\mathbf{R}/\mathbf{Z})$, then it is uniquely defined up to conjugation in* $\mathrm{PC}^1_{\mathbf{Proj}}(\mathbf{R}/\mathbf{Z})$.

Proof. Choose a piecewise projective $\mathcal{C}^1$-diffeomorphism f_n from $\mathbf{R}/\mathbf{Z}$ to the n-fold covering of $\mathbf{P}^1_{\mathbf{R}}$ to inherit the result from Theorem 5.9. □

Let us now provide the affine version of Theorem 5.8. In Minakawa's original formulation [Mi], it consists of the classification of subgroups of $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(\mathbf{R}/\mathbf{Z})$ modulo conjugation, among those conjugate to the group $\mathbf{R}/\mathbf{Z}$ of translations within $\mathrm{Homeo}(\mathbf{R}/\mathbf{Z})$.

In analogy with Definition 5.4, an affine structure ξ on a standard curve X is compatible if, ξ_0 being the standard structure, the identity map from X to (X,ξ_0) to (X,ξ) is piecewise affine.

Theorem 5.11. *Let X be the standard circle $\mathbf{R}/\mathbf{Z}$ with its subgroup K of isometries and its subgroup $K^+ \simeq$* $\mathrm{SO}(2)$ *of orientation-preserving isometries. Consider a faithful continuous action of K^+ on X, whose image G is contained in $\mathrm{PC}^0_{\mathbf{Aff}}(X)$. Then*

1. *(1) G preserves a unique compatible affine structure ξ on X;*
2. *(2) the affinely modeled curve (X,ξ) is isomorphic to Θ_t for some $t \geq 1$. We then say that G is of type Θ_t.*
3. *(3) the conjugacy class of G among subgroups of $\mathrm{PC}^0_{\mathbf{Aff}}(X)$ is characterized by the isomorphy type of the affinely modeled curve (X,ξ), thus by the number $t \geq 1$*
4. *(4) define $t' = 1$ if $t = 1$ and t' as equal to t or t^{-1} according to whether the universal covering of (X,ξ), endowed with its orientation inherited from*

$\mathbf{R}/\mathbf{Z}$, is isomorphic as an affinely modeled curve to $\mathbf{R}_{>0}$ or $\mathbf{R}_{<0}$. Then the conjugacy class of G modulo conjugacy by $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(X)$ is characterized by the number $t' \in \mathbf{R}_{>0}$

(5) *Consider the corresponding conjugacy classification for the corresponding action of K^+ modulo conjugation by $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(X)$: then t' is a full invariant.*
(6) *Given the K^+-action, define t'' as equal to t' or $-t'$ according to whether the orientation determined by the action of small positive element and the given orientation on X coincide. Then $t'' \in \mathbf{R}^*$ is a full invariant for such faithful K^+-actions, modulo conjugation by $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(X)$.*

In contrast, if G is the image of K for such an action, then G is conjugate to K by some element of $\mathrm{PC}^{0,+}_{\mathbf{Aff}}(X)$, and similarly the action of K is conjugate to its canonical isometric action by some element of $\mathrm{PC}^{0}_{\mathbf{Aff}}(X)$.

Sketch of proof. This can be done similarly as Theorem 5.8, but with the significant simplification of "erasing all order 2 terms" in the preliminary work. Namely, we work with $\mathcal{L}^1(X) = X^{\pm} \times \mathbf{R}_{>0}$, and $\mathcal{A}^1(X)$ defined as those functions $\nu : X^{\pm} \to \mathbf{R}_{>0}$ such that $\nu(\hat{x}) = \nu(x)^{-1}$ for all x and equal to 1 outside a finite subset. Then one can encode compatible affine structures by elements of $\mathcal{A}^1(X)$; here compatible is in the affine sense, meaning that the identity map is piecewise affine.

The sequel is similar, with some specific points we now emphasize. The affinely modeled curves Θ_t for $t > 1$ have no orientation-reversing automorphism. This has no such analogue in the projective setting. Since the oriented isomorphism type of (X, ξ) is determined by the $\mathrm{PC}^{0}_{\mathbf{Aff}}(X)$-conjugacy class of G, this conjugacy classification corresponds to oriented affine structures, as described.

This phenomenon reappears if one considers classification of actions, since the action of "small positive" elements of K^+ determines an orientation. □

5.C. **Explicit formulas for commensurating actions.** Let us provide explicit commensurating actions of groups of piecewise $\mathcal{C}^k$-transformations for $k = 0, 1, 2$. I obtained them I started this work, before I realized that the formalism of partial actions could get around such computations: while for $k = 0$ it is easy and practical, for $k = 2$ it becomes quite cumbersome and I did no attempt beyond. Still, it may be instructive to mention these formulas, notably to show what the universal globalization allows to conceal. I will provide no proof.

5.D. **The continuous case.** Let X be an oriented standard curve. Define $X^{\pm} = X \times \{\pm 1\}$. We let $\mathrm{PC}(X)$ act on $X^{\pm}$ exactly as in the proof of Lemma 2.28. For $y = (x, \varepsilon) \in X^{\pm}$, we write $\hat{y} = (x, -\varepsilon)$.

Define $\mathcal{L}^0_X = (X^{\pm})^2$, and let $\mathrm{PC}(X)$ act diagonally. Define $M = \mathcal{M}^0_X \subset \mathcal{L}^0_X$ as the set of pairs $(y, \hat{y})$ when y ranges over $X^{\pm}$. Then $M \smallsetminus \sigma^{-1}M$ is the set of $(y, \pm 1)$ when y ranges over outer discontinuity points of σ. For most applications this is enough; for precise counting results it can be convenient to rather work in the set of unordered pairs of distinct elements. It is instructive to interpret what

M being transfixed means and to thus prove the conjugacy results (for PC, PC^+, $\mathrm{IET}^{\pm}$, IET^+).

5.E. **The derivable/affine case.** Define $\mathcal{L}^1_X = (X^{\pm})^2 \times \mathbf{R}_{>0}$. Recall that $\mathrm{PC}_{\mathcal{C}^1\sharp}(X)$ denotes the set of piecewise-$\mathcal{C}^1$ self-diffeomorphisms of X. For $\sigma \in \mathrm{PC}_{\mathcal{C}^1\sharp}(X)$, we define

$$\sigma \cdot (u, v, y) = \left(f(u), f(v), \frac{f'(u)}{f'(v)} t \right).$$

A simple computation (or a computation-free interpretation of the formula) shows that this is an action.

Here $f'(u)$ for $u = (x, \varepsilon)$ is defined in the only natural way: choosing a local orientation preserving around the first coordinate $f(u)_1$ of $f(u)$, we have $f'(u) = \lim_{t\to 0^+}(f(x+\varepsilon t) - f(u)_1)/t$.

Define $M = \mathcal{M}^1_X = \{(x, \hat{x}, 1) : x \in X\}$. Then $M \smallsetminus \sigma M$ is the set of $(x, \hat{x}, 1)$ where σ cannot be assigned a value at x for which it is of class $\mathcal{C}^1$ at x.

Then the transfixing property implies the transfixing property of the coarser quotient action on $\mathcal{L}^0_X$, and then the piecewise continuous setting yields a first (piecewise affine) conjugacy to a continuous action. This allows, in a second step, to work in the continuous setting (the conjugacy preserves the transfixing property in the new model).

Assume now that we are working with piecewise affine self-homeomorphisms and the invariant subset $\mathcal{L}^1_X = (X^{\pm})^2 \times \mathbf{R}_{>0}$, with the induced action. Consider the set $\mathcal{A}^1_X$ of functions ν defined on a cofinite subset $X^{\pm}$ to $\mathbf{R}_{>0}$, taking the value 1 outside a finite subset, and satisfying the condition $\nu(\hat{y}) = 1/\nu(y)$ for all $y \in X^{\pm}$ at which it is defined, and encode an affine structure on X such that the identity map becomes piecewise affine, with such a function.

Since the set of functions from $X^{\pm}$ to $\mathbf{R}_{>0}$ can be thought of as a subset of $\mathcal{L}^{1,0}(X)$, it inherits an action of $\mathrm{PC}^0_{\mathcal{C}^1\sharp}(X)$. Then given such ν, defined on the complement of a finite subset F_ν, the condition of preserving ν, for $\sigma \in \mathrm{PC}^0_{\mathcal{C}^1\sharp}(X)$, is equivalent to leaving F_ν invariant and acting as $\mathcal{C}^1$-diffeomorphisms on the complement of F_ν. In particular, if σ is piecewise affine, it means leaving F_ν invariant and acting as affine automorphisms on the complement of F_ν. Finally, one checks that for a subgroup of $\mathrm{PC}^0_{\mathcal{C}^1\sharp}(X)$, the condition of transfixing $\mathcal{M}^1_X$ is equivalent to the existence of a Γ-invariant element $\nu \in \mathcal{A}^{1,0}_X$.

The above commensurating action (modulo a minor nuance), or rather the affine isometric Hilbertian action it induces, appears in [LMT, §5] in the continuous case, but without the interpretation in terms of affine structures.

5.F. **The doubly derivable/projective case.** To simplify, we stick to the continuous case. Indeed, the set has already been defined in §5.A, namely $\mathcal{L}^2(X) = X^{\pm} \times \mathbf{R}_{>0} \times \mathbf{R}$, with a somewhat complicated action.

A commensurated subset $\mathcal{M}^2$ is given by the set of triples $(x,1,0)$ when x ranges over $X^{\pm}$. This illustrates how it is frequent in commensurating actions that the commensurated subset is simpler to define than the whole set, and the benefit of using partial actions.

Now consider the set $\mathcal{A}^2_X$ of functions μ defined on a cofinite subset of $X^{\pm}$, valued in $\mathbf{R}_{>0}\times\mathbf{R}$, taking the value $(1,0)$ outside a finite subset, and satisfying the condition $\mu(\hat{x}) = \tau(\mu(x))$. This is a little generalization of $\mathcal{A}^2(X)$ which only considers everywhere defined functions. Such functions define a compatible projective structure outside a finite subset.

The point is that the transfixing property implies the existence of such an invariant partially defined function (which, in the absence of finite orbit, implies a the existence of globally defined one), defined outside an invariant finite subset F and then, the conjugation to an action of class $\mathcal{C}^2$ outside F, and projective outside F if we started from a piecewise projective action.

Appendix A. Affinely, projectively modeled curves and their automorphisms

This section classifies affinely and projectively modeled curves and describe their automorphism groups.

The easy affine classification was obtained by Kuiper [Ku1]: the complete case is trivial: there are only $\mathbf{R}$ and $\mathbf{R}/\mathbf{Z}$, while the non-complete case puts forward the non-standard circles $\mathbf{R}_{>0}/\langle t\rangle$. Paradoxically, while in the affine case the complete case is the most trivial and less surprising part, in the projective case the non-complete case has essentially nothing new (the compact ones come from the affine world) while the complete case is richer. The classification was established by Kuiper (1954) [Ku2], with an inaccuracy (namely, in the notation below, Kuiper did not distinguish $\Xi_{n,+}$ and $\Xi_{n,-}$). With a similar approach, Goldman [Go1] claims to also obtain this classification, but actually rather establishes a correspondence to the classification of conjugacy classes in the universal covering of $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$, without providing details on the latter. In the analytic context of Hill equations, an equivalent classification to Kuiper's was obtained by Lazutkin and Pantrakova (1975) [LP] which fixes Kuiper's error in another language (Kuiper is not quoted). Later, G. Segal (1981) [Seg], also not quoting Kuiper, claims to correct an error in [LP], but instead resurrects Kuiper's error, based on the same incorrect classification of $\widetilde{\mathrm{PGL}}_2(\mathbf{R})$-conjugacy classes in $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$!

Kuiper's error reappeared at various places (often "rediscovered"), and was fixed at other places, often not even noticing that there is an error, or by authors quoting several contradictory results without noticing the difference. The above error is fixed in [LP], and a careful classification appears in [BFP, Gor]; Gorinov [Gor] is the first to explicitly mention the error, and also the first to fix it in the language of geometric structure (used in [Ku2] and also here). Most other references are concerned with the Hill equation, which amounts to classifying orbits of the Bott-Virasoro extension of the diffeomorphism group on its Lie algebra, and this approach is much more complicated.

Below, the classification and the computation of automorphism groups are done in the same impetus (carrying out the classification allowing to introduce notation). A description of orientation-preserving automorphism groups of projectively modeled curves is claimed in [Gui], but the result is incorrect (and the proof way too long). Indeed, for the projectively modeled curves $\Xi_{n,\pm}$ and $\Xi_{n,t}$, he obtains that the orientation-preserving automorphism group is isomorphic to $(\mathbf{R}/\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$, while it is actually isomorphic to $\mathbf{R} \times (\mathbf{Z}/n\mathbf{Z})$. Since the classification of maximal compact subgroup of automorphisms is needed here, this is an essential difference. So I am not aware of a reference for the classification of automorphism groups of projectively modeled curves.

A.1. **Generalities.** We consider the pseudogroups **Aff** and **Proj** of Example 4.5. By affinely and projectively modeled curves we mean **Aff**-modeled and **Proj**-modeled curves.

Denote by Σ_∞ the universal covering of $\mathbb{P}^1(\mathbf{R})$. We identify it with $]-\infty, +\infty] \times \mathbf{Z}$ with the lexicographic order. Restricting the universal covering map $(t, n) \mapsto t$ to open subsets on which it is injective, we obtain charts for a **Proj**-structure on Σ_∞ (note that it is not finitely-charted).

Lemma A.1. *Any* **Aff***-modeled Hausdorff simply connected curve X is isomorphic to an open interval in* $\mathbf{R}$*.*

Any **Proj***-modeled Hausdorff simply connected curve X is isomorphic to an open interval in Σ_∞.*

Proof. In the affine case, by [Go2, Prop. 4.5], there is a locally projective immersion $X \to \mathbf{R}$. Since a locally injective continuous map between intervals is injective, the latter map has to be injective.

In the projective case, by [Go2, Prop. 4.5], there is an locally projective immersion $X \to \mathbf{P}^1_{\mathbf{R}}$, which therefore lifts to a locally projective immersion $X \to \Sigma_\infty$. We deduce injectivity by the same argument. □

A.2. **The affine case.** This is especially an apéritif to the projective case, otherwise we would just give a list without proof.

The group of affine self-transformations of $\mathbf{R}$ is 2-transitive and contains reversing elements. Hence there are only three open intervals up to affine automorphism: $\mathbf{R}$, $\mathbf{R}_{>0}$, and $]-1, 1[$.

Let us list in each case the fixed-point-free automorphisms up to conjugation.

- For $]-1, 1[$ the automorphism group is reduced to a group of order 2, and hence there is no fixed-point-free automorphism.
- For $\mathbf{R}_{>0}$, the automorphism group is reduced to positive homotheties $u_t = x \mapsto tx$, $t > 0$; note that it preserves an orientation: indeed, this interval $\mathbf{R}_{>0}$ has one "complete" end $(+\infty)$ and a non-complete one (0).

They are pairwise non-conjugate, and fixed-point-free for $t \neq 1$. We have $u_t^{-1} = u_{t^{-1}}$. We endow the quotient $R_{>0}/\langle u_t \rangle$ with the orientation inherited from $\mathbf{R}_{>0}$ if $t > 1$, and with the reverse orientation if $t < 1$. Thus the oriented affinely modeled curves Θ_t for $t \in \mathbf{R}_{>0}$ are pairwise non-isomorphic. In the non-oriented setting, for $t > 1$, they are pairwise non-isomorphic, while Θ_t and $\Theta_{t^{-1}}$ are isomorphic (indeed equal!).

The affine automorphism group of Θ_t is the normalizer (hence the whole automorphism group here) modulo $\langle u_t \rangle$, hence is naturally isomorphic to $\mathbf{R}_{>0}/\langle t \rangle$ and non-canonically isomorphic to $\mathbf{R}/\mathbf{Z}$; it coincides with the oriented automorphism group.

- For $\mathbf{R}$, the automorphism group consists of all affine automorphisms, and the fixed-point-free ones are translations, and are all conjugate; actually the cyclic subgroups generated by translations are all conjugate by orientation-preserving affine automorphisms, and the quotient we obtain is unique up to affine isomorphism: this is the standard circle $\mathbf{R}/\mathbf{Z}$. The normalizer of the given cyclic subgroup is the group of isometries of $\mathbf{R}$ and hence the affine automorphism group is isomorphic to $(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$, and can be identified to the group of isometries of $\mathbf{R}/\mathbf{Z}$. We denote it by Θ_1.

We conclude the following classification of affinely modeled curves and their automorphism groups. In the left column, $\bullet$ and $\circ$ mean complete vs non-complete.

$\bullet/\circ$	affine curve	univ. cover	$\mathrm{Aut}^+ \simeq$	$\mathrm{Aut} \simeq$
$\bullet$	$\mathbf{R}$	itself	$\mathbf{R} \rtimes \mathbf{R}_{>0}$	$\mathbf{R} \rtimes \mathbf{R}^*$
$\circ$	$\mathbf{R}_{>0}$	itself	$\mathbf{R}_{>0} \simeq \mathbf{R}$	$= \mathrm{Aut}^+$
$\circ$	$]-1,1[$	itself	$\{1\}$	$\{\pm 1\}$
$\bullet$	Θ_1	$\mathbf{R}$	$\mathbf{R}/\mathbf{Z}$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
$\circ$	$\Theta_t \quad t>1$	$\mathbf{R}_{>0}$	$\mathbf{R}_{>0}/\langle t \rangle \simeq \mathbf{R}/\mathbf{Z}$	$= \mathrm{Aut}^+$

Note that we have several ways of defining affinely modeled curves using a pseudogroup. The chosen way does not affect the classification as above. However, it affects the notion of being finitely charted (i.e. having a finite atlas in the model space). For the most standard way to define affinely modeled curves, one uses $\mathbf{R}$ as model space and all the above curves are finitely-charted. However, in our work and applications, the natural model space is rather the standard affine structure on $\mathbf{R}/\mathbf{Z}$. Then $\mathbf{R}$ and $\mathbf{R}_{>0}$ are not finitely-charted and only $]-1,1[$, Θ_1 and Θ_t remain; in particular the orientation-preserving automorphism group of any finitely-charted affine manifold (modeled on $\mathbf{R}/\mathbf{Z}$) is always abelian.

A.3. **The projective case.** As reminded in §A.1, we have to classify open intervals in Σ_∞, and then classify their fixed-point-free automorphisms to obtain the compact projective manifolds.

The automorphism group of Σ_∞ contains the automorphism group as a subgroup of index 2, and the latter can be identified with the universal covering $\widetilde{\mathrm{SL}_2}$

of $\mathrm{SL}_2(\mathbf{R})$. Since the conjugation by a reflection yields a automorphism of order 2, we can view the automorphism group as a group $\widetilde{\mathrm{PGL}_2}$, projecting to $\mathrm{PGL}_2(\mathbf{R})$ with an infinite cyclic kernel (which is the center of $\widetilde{\mathrm{SL}_2}$). The action of $\widetilde{\mathrm{PGL}_2}$ on $\mathbf{R}$ is transitive, but not transitive on ordered pairs. Namely, the stabilizer of $(\infty, 0) \in \Sigma_\infty$ acts on Σ_∞ with countably many orbits: the singletons (∞, n), $n \in \mathbf{Z}$, and the intervals in between. Therefore, we can classify open intervals in Σ_∞ up to automorphism.

- Σ_∞ itself (not finitely-charted)
- the interval Σ_∞^+ of elements $> (\infty, 0)$ in Σ_∞ (not finitely-charted)
- the interval $\Sigma_n =](\infty, 0), (\infty, n)[$ in Σ_∞ ($n \geq 1$ integer).
- the interval $\Sigma_{n-\frac{1}{2}} =](0, 1), (\infty, n)[$ in Σ_∞ ($n \geq 1$ integer).

This yields the classification of simply connected projectively modeled curves. It remains to obtain the classification of compact connected projectively modeled curves, so we have to consider, among the above curves, which ones have a fixed-point-free orientation-preserving automorphism, and classify these up to conjugation by an automorphism. We start with the easier non-complete case, that is, those for which the universal covering is properly contained in Σ_∞.

In Σ_∞^+, in Σ_n or $\Sigma_{n-\frac{1}{2}}$ for $n \geq 2$, the element $(\infty, 1)$ is fixed by every orientation-preserving automorphism. The remaining intervals are $\Sigma_1 \simeq \mathbf{R}$ and $\Sigma_{\frac{1}{2}} \simeq \mathbf{R}_{>0}$ (and Σ_∞, which we consider afterwards). Here $\mathbf{R}$ and $\mathbf{R}_{>0}$ can be viewed as subsets of the projective line.

- In $\Sigma_1 \simeq \mathbf{R}$, the automorphism group is the affine group. Fixed-point-free elements are nonzero translations, and are all conjugate. Let Θ_1 be the corresponding curve; call it the round circle.
- In $\Sigma_{\frac{1}{2}} \simeq \mathbf{R}_{>0}$, the orientation-preserving automorphism group consists of the positive homotheties $u_t : x \mapsto tx$; for $t \neq 1$ these are fixed-point-free. The automorphism group also contains $t \mapsto t^{-1}$; thus u_t is conjugate to u_s if and only $s \in \{t, t^{-1}\}$. Let Θ_t be the corresponding curve ($t \in \mathbf{R}_{>0} \smallsetminus \{1\}$); thus Θ_t are pairwise non-isomorphic as oriented projectively modeled curves, and Θ_t and $\Theta_{t^{-1}}$ are (orientation-reversing) isomorphic as projectively modeled curves.

Remark A.2. Note that the Θ_t already appear in the affine classification. But there are a few little differences: as affinely modeled curve, Θ_1 is complete, while Θ_t is non-complete if $t \neq 1$. The other difference is that, for $t \neq 1$, Θ_t and $\Theta_{t^{-1}}$ are isomorphic as oriented projectively modeled curves, an isomorphism being induced by $x \mapsto x^{-1}$. This difference is also reflected in the fact that Θ_t admits orientation-reversing projective automorphisms.

Now we have classified the non-complete projectively modeled curves, we need to compute their automorphism groups (not anymore for classification, but because we obtain results of conjugation into the automorphism group of a projectively modeled curve). In the simply connected case we already did the job.

In the compact case, where it was obtained as quotient $X = \langle r\rangle C$ of a simply connected projectively modeled curve by a cyclic group of automorphisms acting freely, we have to compute the normalizer N_r of this cyclic subgroup $\langle r\rangle$; then the automorphism group A of X is $N_r/\langle r\rangle$.

- For $\Sigma_1 \simeq \mathbf{R}$, and $r(x) = x+1$, the normalizer N_r is the group of isometries of $\mathbf{R}$, and A can be viewed as the group of of isometries of $X \simeq \mathbf{R}/\mathbf{Z}$.
- For $\Sigma_{\frac{1}{2}} \simeq \mathbf{R}_{>0}$ and $r(x) = tx$ with $t > 1$, the cyclic subgroup $\langle r\rangle$ is normal in the whole automorphism group $\mathbf{R}_{>0} \rtimes \langle\tau\rangle$ (where $\tau(x) = x^{-1}$). The quotient $(\mathbf{R}_{>0}/\langle t\rangle) \rtimes \langle\tau\rangle$ is also (non-canonically) isomorphic to the group of isometries of $\mathbf{R}/\mathbf{Z}$.

We summarize the classification of non-complete finitely-charted projectively modeled curves up to isomorphism, along with their orientation-preserving automorphism group. Note that the only infinitely-charted one is Σ_∞^+.

projective curve	universal cover	$\mathrm{Aut}^+ \simeq$	$\mathrm{Aut} \simeq$
Σ_n $\quad n \in \mathbf{N}_{>0}$	itself	$\mathbf{R} \rtimes \mathbf{R}_{>0}$	$\mathbf{R} \rtimes \mathbf{R}^*$
$\Sigma_{n-\frac{1}{2}}$ $\quad n \in \mathbf{N}_{>0}$	itself	$\mathbf{R}$	$\mathbf{R} \rtimes (\mathbf{Z}/2\mathbf{Z})$
Θ_1	$\Sigma_1 \simeq \mathbf{R}$	$\mathbf{R}/\mathbf{Z}$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
Θ_t $\quad t > 1$	$\Sigma_{1/2} \simeq \mathbf{R}_{>0}$	$\mathbf{R}/\mathbf{Z}$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$

Let us now deal with complete curves. In Σ_∞, the positive automorphism group consists of $\widetilde{\mathrm{SL}}_2$. We call elements of $\widetilde{\mathrm{SL}}_2$ metaelliptic, metaparabolic, or metahyperbolic according to the corresponding behavior of the projection on $\mathrm{PSL}_2(\mathbf{R})$, with the convention that elements mapping to 1 are metaelliptic. We add "meta" because the wording "elliptic", etc, does not reflect the dynamical behavior on Σ_∞: for instance non-identity metaelliptic elements rather behave as loxodromic elements on Σ_∞.

- Non-identity metaelliptic elements are fixed-point-free on Σ_∞. Every metaelliptic element is conjugate to an element in the inverse image $\widetilde{\mathrm{SO}}_2$ of SO_2. Write this 1-parameter subgroup as $(\xi_r)_{r\in\mathbf{R}}$, so that the center of $\widetilde{\mathrm{SL}}_2$ consists of the ξ_r for $r \in \mathbf{Z}$. Lifting conjugation by the lift of an orthogonal reflection conjugates ξ_r to ξ_{-r}. These are the only conjugacies among the ξ_r since the rotation number of ξ_r is r, and rotation number is conjugacy-invariant up to sign in the whole automorphism group $\widetilde{\mathrm{PGL}}_2$ (and is conjugacy-invariant in $\widetilde{\mathrm{SL}}_2$). Since we are interested by the cyclic subgroup generated by ξ_r, we can restrict to $r > 0$. We define Ξ_r ($r > 0$) as the corresponding projectively modeled curve (quotient of Σ_∞ by $\langle\xi_r\rangle$).
- Metahyperbolic and metaparabolic elements have a fixed point on $\mathbf{P}^1$; up to conjugating in $\widetilde{\mathrm{SL}}_2$, we can suppose that ∞ is fixed. Its action on $\mathbf{P}^1$ is therefore given by some affine map $A : x \mapsto tx + b$ with $t > 0$ and $(t, b) \neq (1, 0)$. Conjugation inside the positive affine group reduces to either $b = 0$ or ($t = 1$ and $b \in \{1, -1\}$), which we therefore assume. For

its action on Σ_∞, the element ξ preserves the subset $\{\infty\} \times \mathbf{Z}$, on which it acts as a translation, say by some $n \in \mathbf{Z}$. Then $\xi = \xi_{n,A}$ is determined by A and n, namely $\xi_{n,A}(x,m) = (Ax, n+m)$ in the previous coordinates. It is fixed-point-free if and only if $n \neq 0$. Write $\xi_{n,t}$ (resp. $\xi_{n,\pm}$) for $\xi_{n,A}$ when $A(x) = tx$ (resp. $A(x) = x \pm 1$)

We have to classify the cyclic subgroups $\langle \xi_{n,A} \rangle$ up to conjugation in the automorphism group $\widetilde{\mathrm{PGL}_2}$ of Σ_∞. Let us start with the elements $\xi_{n,A}$ themselves. Since n is the rotation number, it is determined by conjugation up to sign; also $\{t, t^{-1}\}$ is determined by conjugation (by conjugacy classification in $\mathrm{PSL}_2(\mathbf{R})$). Given this, the elements we have not yet distinguished are (for a given $n \in \mathbf{N}_{>0}$ and $t \geq 1$):

– on the one hand, the four elements $\xi_{n,t}$, $\xi_{n,t^{-1}}$, $\xi_{-n,t} (= \xi_{n,t^{-1}}^{-1})$, $\xi_{-n,t^{-1}} (= \xi_{n,t}^{-1})$;

– on the other hand, the four elements $\xi_{n,+}$, $\xi_{n,-}$, $\xi_{-n,+} (= \xi_{n,-}^{-1})$, $\xi_{-n,-} (= \xi_{n,+}^{-1})$.

Define two particular automorphisms of Σ_∞ as follows: $s(x,m) = (-x, -m)$ for $x \neq \infty$ and $s(\infty, m) = (\infty, -m-1)$; $w(x,m) = (-x^{-1}, m+1)$ for $x > 0$ and $w(x,m) = (-x^{-1}, m)$ for $x \leq 0$. Note that w is orientation-preserving.

Then $w^{-1}\xi_{n,t}w = \xi_{n,t^{-1}}$, $s^{-1}\xi_{n,t}s = \xi_{-n,t}$, $s^{-1}\xi_{n,\pm}s = \xi_{n,\pm}^{-1}$. This shows that the first four elements $\xi_{\pm n, t^{\pm 1}}$ are conjugate in $\widetilde{\mathrm{PGL}_2}$, and the two corresponding cyclic subgroups $\langle \xi_{n,t} \rangle$, $\langle \xi_{n,t^{-1}} \rangle$ are conjugate within $\widetilde{\mathrm{SL}_2}$. On the other hand, this leaves the possibility that the subgroups $\langle \xi_{n,+} \rangle$ and $\langle \xi_{n,-} \rangle$ are not conjugate at all. It is indeed the case that they are not conjugate: indeed, a conjugating element should fix the unique fixed point in $\mathbf{P}^1_{\mathbf{R}}$, and thus preserve $\{\infty\} \times \mathbf{Z}$, and these are precisely the elements we have tested. We write $\Xi_{n,t} = \Sigma_\infty / \langle \xi_{n,t} \rangle$ for $t > 1$, and $\Xi_{n,\pm} = \Sigma_\infty / \langle \xi_{n,\pm} \rangle$.

Remark A.3. 1) Recall that $\xi_{n,\pm}(x,m) = (x \pm 1, n+m)$ for $(x,m) \in \Sigma_\infty$ (identified as above to $]-\infty, +\infty] \times \mathbf{Z}$ with the lexicographic ordering). We have observed that, for $n \in \mathbf{Z} \smallsetminus \{0\}$ the cyclic subgroups $\langle \xi_{n,+} \rangle$ and $\langle \xi_{n,-} \rangle$ are not conjugate in $\widetilde{\mathrm{PGL}_2}$. The error in Kuiper's classification [Ku2], reproduced at many other places (but fixed in [LP, BFP, Gor], and consciously only in [Gor]) is to not distinguish those elements (essentially, the error amounts to deducing their conjugacy from the fact that their images in $\mathrm{PSL}_2(\mathbf{R})$ are conjugate).

2) That $w^{-1}\xi_{n,t}w = \xi_{n,t^{-1}}$ is quite a subtle point (the subtlety is to consider w). This is another gap in [Ku2], which takes for granted that we immediately boil down to homotheties $x \mapsto tx$ for $t > 1$; this gap does not result in another error thanks to this possibly unexpected conjugacy. This point is carefully taken care of in [Gor].

Let us mention, as a digression of independent interest, that this lack of conjugacy exists at a purely topological level:

Proposition A.4. *For every $n \in \mathbf{N}_{\geq 1}$, the cyclic subgroups $\langle \xi_{n,+} \rangle$ and $\langle \xi_{n,-} \rangle$ are not conjugate in the group $\widetilde{\mathrm{Homeo}}(\mathbf{P}^1_{\mathbf{R}})$, namely the normalizer in* $\mathrm{Homeo}(\Sigma_\infty)$ *of the cyclic subgroup $\langle \xi_1 \rangle$. More precisely, for any $k \in \mathbf{N}_{\geq 3} \cup \{\infty\}$ not dividing $2n$, the cyclic subgroups $\langle \xi_{n,+} \rangle$ and $\langle \xi_{n,-} \rangle$ are not conjugate in the normalizer $\widetilde{\mathrm{Homeo}}^{(k)}(\mathbf{P}^1_{\mathbf{R}})$ of $\langle \xi_1 \rangle$ in the group of self-homeomorphisms of Σ_k (which contains $\mathrm{PGL}_2^{(k)}(\mathbf{R})$).*

Proof. Indeed, suppose by contradiction that they are conjugate in E_k. It means that the generators of these cyclic subgroups are conjugate, or that one is conjugate to the other's inverse. Let us show that the latter case implies the former case. In the latter case, there exists $b \in E_k$ such that $b\xi_{n,+}b^{-1} = \xi_{n,-}^{-1}$. This changes the rotation number n to $-n$ (which lives in $\mathbf{R}/k\mathbf{Z}$). By assumption, $n \neq -n$ in $\mathbf{R}/k\mathbf{Z}$. Hence necessarily b is orientation-reversing and thus $c = bs$ is orientation-preserving, that is, commutes with ξ_1, and $c\xi_{n,+}c^{-1} = \xi_{n,-}$. Projecting on $\mathrm{Homeo}^+(\mathbf{P}^1_{\mathbf{R}})$, we obtain an orientation-preserving self-homeomorphism γ conjugating u to u^{-1}, with $u(x) = x + 1$. Hence γ fixed the unique fixed point ∞ of u and thus this is a conjugacy in $\mathrm{Homeo}(\mathbf{R})$. But $u(x) > x$ for all x, which is clearly an obstruction for u to be conjugate to its inverse within orientation-preserving self-homeomorphisms of $\mathbf{R}$. This is a contradiction.

(Note that conversely, whenever $2n = 0$ [mod k], the elements $\xi_{n,+}$ and $\xi_{n,-}$ are inverse to each other in $\mathrm{PSL}_2^{(k)}(\mathbf{R})$, and thus the cyclic subgroups they generate are not only conjugate, but equal.) ☐

The classification of complete projectively modeled curves being completed, it remains to compute their automorphism groups. In each case $\Sigma_\infty/\langle \xi \rangle$, the automorphism group is $N_\xi/\langle \xi \rangle$, where N_ξ is the normalizer of $\langle \xi \rangle$ in the automorphism group.

To determine N_ξ, define some auxiliary subgroups, easier to determine. Namely, define B_ξ as the stabilizer in $\mathrm{PGL}_2(\mathbf{R})$ for conjugation of the image of $\{\xi, \xi^{-1}\}$ in $\mathrm{PSL}_2(\mathbf{R})$; let B_ξ be its unit component. Let M_ξ as the inverse image in $\widetilde{\mathrm{PGL}}_2$ of the stabilizer B_ξ, and M_ξ^0 the inverse image of B_ξ° (note that M_ξ^0 is not necessarily connected). Clearly, $N_\xi \subset M_\xi$. Moreover, the unit component M_ξ° is included in N_ξ (it is even included in the centralizer of ξ); since the center of $\widetilde{\mathrm{PGL}}_2$ is also included in the normalizer, it follows that M_ξ^0 is included in N_ξ. Thus $M_\xi^0 \subset N_\xi \subset M_\xi$. We can deduce the automorphism group, in each case:

- For $\Xi_r = \Sigma_\infty/\langle \xi_r \rangle$, we have to discuss on whether $r \in \mathbf{Z}$. If $r = n \in \mathbf{Z}$, the subgroup $\langle \xi_r \rangle$ is normal in the whole automorphism group. Hence the orientation-preserving automorphism group is the quotient $\mathrm{PSL}_2^{(n)}(\mathbf{R})$, the n-fold connected covering of $\mathrm{PSL}_2(\mathbf{R})$; the full automorphism group $\mathrm{PGL}_2^{(n)}(\mathbf{R})$ is obtained as semidirect product with the automorphism induced by a reflection.

- When $r \notin \mathbf{Z}$, the normalizer of $\langle \xi_r \rangle$ is reduced to the inverse image of the orthogonal group, which has two components then s is contained in the nontrivial component and normalizes $\langle \xi_r \rangle$, so in this case $N_\xi = M_\xi$. Thus the automorphism group is isomorphic to $(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$ with action by sign, and the orientation-preserving automorphism group is isomorphic to $\mathbf{R}/\mathbf{Z}$.
- For $\Xi_{n,\pm}$, the normalizer is the inverse image of the group of upper triangular matrices that are either scalar or trace-zero. In $\mathrm{PSL}_2(\mathbf{R})$, the latter group has two connected components; the nontrivial component corresponding to trace zero matrices. A subset of representatives for M_ξ modulo M_ξ^0 is given by $\{1, s\}$. Since $s\xi_{n,\pm}s^{-1} = \xi_{n,\pm}^{-1}$, we deduce that $s \in N_\xi$ and hence $N_\xi = M_\xi$. thus the automorphism group is isomorphic to $(\mathbf{R} \rtimes \mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$, with action by sign multiplication, and the orientation-preserving automorphism group is isomorphic to $\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}$.
- For $\Xi_{n,t}$ ($t > 1$, $n \in \mathbf{N}_{>0}$) and $\xi = \xi_{n,t}$, the subgroup B_ξ consists of monomial matrices in $\mathrm{PGL}_2(\mathbf{R})$, which has 4 components. A subset of representatives of B_ξ modulo B_ξ° is $\{1, s, w, sw\}$.

  The element w, which belongs to M_ξ, does not normalize $\langle \xi \rangle$ (because $w^{-1}\xi_{n,t}w = \xi_{n,t^{-1}} \notin \langle \xi_{n,t} \rangle$). For the same reason, s does not normalize. However, $q = sw$ normalizes (and does not centralize) $\langle \xi_{n,t} \rangle$. Note that $q^2 = 1$.

  So the normalizer of $\langle \xi \rangle$ in $\widetilde{\mathrm{SL}}_2$, which is also its centralizer in $\widetilde{\mathrm{PGL}}_2$, is the direct product $(\xi_{0,\eta})_{\eta>0} \times (\xi_m)_{m\in\mathbf{Z}}$. Its quotient by $\langle \xi \rangle$ can be described as the direct product $(\xi_{0,\eta})_{\eta>0} \times (\xi_{m,t^{m/n}})_{m\in\mathbf{Z}/n\mathbf{Z}}$, which is isomorphic to $\mathbf{R} \times (\mathbf{Z}/n\mathbf{Z})$. This is the orientation-preserving automorphism group of $\Xi_{n,t}$.

  The normalizer in $\widetilde{\mathrm{PGL}}_2$ is the semidirect product $((\xi_{0,\eta})_{\eta>0} \times (\xi_m)_{m\in\mathbf{Z}}) \rtimes \langle q \rangle$, where q acts by sign. Its quotient by $\langle \xi \rangle$ can be described as the semidirect product $((\xi_{0,\eta})_{\eta>0} \times (\xi_{m,t^{m/n}})_{m\in\mathbf{Z}/n\mathbf{Z}}) \rtimes \langle q \rangle$, which is isomorphic to $(\mathbf{R} \times (\mathbf{Z}/n\mathbf{Z})) \rtimes (\mathbf{Z}/2\mathbf{Z})$, again with action by sign.

Let us summarize the classification up to isomorphism of complete finitely-charted projectively modeled curves (in each case the universal cover is Σ_∞, which is the only infinitely-charted complete projectively modeled curve up to isomorphism):

projective curve		$\mathrm{Aut}^+ \simeq$	$\mathrm{Aut} \simeq$
Ξ_n	$n \in \mathbf{N}_{>0}$	$\mathrm{PSL}_2^{(n)}(\mathbf{R})$	$\mathrm{PGL}_2^{(n)}(\mathbf{R})$
$\Xi_{n,t}$	$t > 1$, $n \in \mathbf{N}_{>0}$	$\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}$	$(\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
$\Xi_{n,\varepsilon}$	$\varepsilon \in \{+,-\}$, $n \in \mathbf{N}_{>0}$	$\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}$	$(\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
Ξ_r	$r > 0, r \notin \mathbf{Z}$	$\mathbf{R}/\mathbf{Z}$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$

Note that in all finitely-charted cases (non-complete and complete), the whole automorphism group is a semidirect product of its orientation-preserving normal subgroup by $\mathbf{Z}/2\mathbf{Z}$. This also holds for the infinitely-charted Σ_∞, while Σ_∞^+ has no orientation-reversing automorphism.

Remark A.5. The non-complete structures Θ_t and Θ_1 can be thought of as the $n = 0$ case of $\Xi_{n,t}$ and $\Xi_{n,\pm}$ (note the collapse of the $\pm$ distinction when $n = 0$, which is not always well-reflected in the literature). We thus have a canonical bijection, as observed and used by Kuiper [Ku2], between the orbit space of $\widetilde{\mathrm{PSL}_2} \smallsetminus \{1\}$ modulo the conjugation action of $\widetilde{\mathrm{PGL}_2}$, and the set of projective structures on the circle, modulo diffeomorphism; in addition, it also corresponds to the set of projective structures modulo oriented diffeomorphism, because (unlike in the affine case) all such structures admit an orientation-reversing automorphism.

Let us mention that Ghys [Ghy, §4.2] provides a direct argument of the following alternative: for every projectively modeled curve homeomorphic to the circle, either it is projectively isomorphic to a finite covering of the projective line, or its oriented automorphism group is abelian.

We now use the classification to derive the following consequences. For the following one, it is possible that there is a more elegant, classification-free argument.

Proposition A.6. *For every projectively modeled curve X (with finitely many components), the automorphism group A of X is closed in* $\mathrm{Homeo}(X)$*.*

Proof. Since the componentwise preserving subgroup of A is open in A, we can restrict to this one, and thus boil down to the case when X is connected, and thus use the classification. Note that in each case, the description of the automorphism group as a Lie group makes it clear that it acts continuously on the given curve. Since in each case A has finitely many components, it is enough to check that A° is closed.

When A° is compact, since it acts continuously it is closed. When X is Σ_∞^+, or Σ_n for $n \in \mathbf{N}_{>0}$, the subgroup A° can be viewed as the subgroup of element pointwise stabilizing some closed discrete subset and acting as orientation-preserving affine transformation in each interval of its complement, and such that all these affine maps (on all intervals) are equal. This is easily checked to be a closed condition.

For Σ_∞ and Ξ_n ($n \in \mathbf{N}_{>0}$), we start with Ξ_1: this is PGL_2, and is the stabilizer of the cross-ratio, and thus is closed. For others, we start observing that the centralizer of the deck transformation is closed in the whole homeomorphism group, so it is enough to show that A° is closed in this centralizer, and then it consists of the preimage of PSL_2 for the projection of this centralizer to $\mathrm{Homeo}^+(\mathbf{P}^1_{\mathbf{R}})$. So it is closed.

The remaining cases are $\Xi_{n,t}$, $\Xi_{n,\varepsilon}$, and $\Sigma_{n-1/2}$. Then we observe that A° preserves a finite subset on which it acts trivially, and that it acts properly on its complement. This implies that it is closed. □

Let us now deal with maximal compact subgroups. Let us recall a classical result of Mostow [Mos] that in a virtually connected Lie group, all maximal compact subgroups are conjugate by some element of the unit component, and that moreover they have as many components as the whole group. We thus draw a table indicating, in the right column, one isomorphic copy in each case of a maximal compact subgroup (we write no quantifiers on the left column: the indices n, t, ε, r are meant to be the same as in the previous two tables). We then derive a conjugacy result, which is used in the paper.

projective curve	Aut $\simeq$	Maximal compact $\simeq$
Σ_∞	$\mathrm{PGL}_2^{(\infty)}(\mathbf{R})$	$\mathbf{Z}/2\mathbf{Z}$
Σ_∞^+, Σ_n	$\mathbf{R} \rtimes \mathbf{R}^*$	$\mathbf{Z}/2\mathbf{Z}$
$\Sigma_{n-\frac{1}{2}}$	$\mathbf{R} \rtimes (\mathbf{Z}/2\mathbf{Z})$	$\mathbf{Z}/2\mathbf{Z}$
$\Xi_{n,t}$, $\Xi_{n,\varepsilon}$	$(\mathbf{R} \times \mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$	$(\mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
Θ_t, Θ_1, Ξ_r	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$
Ξ_n	$\mathrm{PGL}_2^{(n)}(\mathbf{R})$	$(\mathbf{R}/\mathbf{Z}) \rtimes (\mathbf{Z}/2\mathbf{Z})$

As an immediate consequence of the classification and Mostow's conjugacy result, we derive:

Proposition A.7. *Let A be the automorphism group of a connected projectively modeled curve. Then two closed subgroups of G that are topologically isomorphic to* $\mathrm{SO}(2)$ *are conjugate by some element of the identity component A°. The same holds with* $\mathrm{SO}(2)$ *replaced with* $\mathrm{O}(2)$. $\square$

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