KAZHDAN PROPERTY FOR SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We combine results of Vaserstein and Shalom to prove Kazhdan's Property (T) for various topological groups of the form $\mathrm{SL}(n, \mathcal{C}(X, \mathbf{R}))$ or $\mathrm{SL}(n, \mathcal{C}(X, \mathbf{C}))$, for $n \geq 3$ and X a topological subspace of a Euclidean space.

All the rings here are unitary and commutative. If R is a ring, let $\mathrm{E}(n, R)$ denote the subgroup of $\mathrm{SL}(n, R)$ generated by elementary matrices. If $\mathrm{E}(n, R)$ is normal in $\mathrm{SL}(n, R)$, the quotient is denoted by $SK_{1,n}(R)$.

If R is a topological ring, then $\mathrm{E}(n, R)$ and $\mathrm{SL}(n, R)$ are topological groups for the topology induced by the inclusion in R^{n^2}. We say that a topological ring is *topologically finitely generated* it it has a finitely generated dense subring.

For any topological spaces X, Y, we denote by $\mathcal{C}(X, Y)$ the set of all continuous functions $X \to Y$. If $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$, $\mathcal{C}(X, \mathbf{K})$ is a topological ring for the compact-open topology, which coincides with the topology of uniform convergence on compact subsets.

It is known [Vas86] that, if $n \geq 3$,

$$\mathrm{E}(n, \mathcal{C}(X, \mathbf{K})) = \{u : X \to \mathrm{SL}(n, \mathbf{K}) \text{ homotopically trivial}\}$$

(this is immediate if X is compact, for all $n \geq 2$).

If G is a topological group, a continuous unitary representation of G on a Hilbert space $\mathcal{H}$ *almost has invariant vectors* if, for every compact (not necessarily Hausdorff) subset $K \subset G$ and every $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$, of norm one, such that $\sup_{g \in K} \|g \cdot \xi - \xi\| \leq \varepsilon$. The topological group G has *Kazhdan's Property (T)* or *Property (T)* if every continuous unitary representation of G almost having invariant vectors actually has nonzero invariant vectors; see [BHV05] for more about Property (T).

Theorem 1. *Let $n \geq 3$, and $X \subset \mathbf{R}^d$ be a topological subspace of a Euclidean space. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Endow $\mathcal{C}(X, \mathbf{K})$ with the topology of uniform convergence on compact subsets. Then $\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ has Kazhdan's Property (T).*

Corollary 2. *Let $n \geq 3$, and $X \subset \mathbf{R}^d$ be a compact subset. Endow X with the topology of uniform convergence. Then $\mathrm{SL}(n, \mathcal{C}(X, \mathbf{K}))$ has Kazhdan's Property (T) if and only if the discrete group $SK_{1,n}(\mathcal{C}(X, \mathbf{K})) = \mathrm{SL}(n, \mathcal{C}(X, \mathbf{K}))/\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ does.*

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Proof: Since X is compact[1], $\mathrm{E}(n,\mathcal{C}(X,\mathbf{K}))$, is open (hence closed) in $\mathrm{SL}(n,\mathcal{C}(X,\mathbf{K}))$. The corollary follows from the trivial fact that Property (T) is stable under quotients and extensions. ■

Example 3. Fix $k \geq 1$ and $n \geq 3$. Then $SK_{1,n}(\mathcal{C}(S^k,\mathbf{K})) = \pi_k(\mathrm{SL}(n,\mathbf{K}))$, which is an abelian group. It follows that $\mathrm{SL}(n,\mathcal{C}(S^k,\mathbf{K}))$ has Kazhdan's Property (T) if and only if $\pi_k(\mathrm{SL}(n,\mathbf{K}))$ is finite; it is known (see [MiTo91]) that it is infinite if and only if:

- $\mathbf{K} = \mathbf{C}$, k is odd, and $3 \leq k \leq 2n-1$, or
- $\mathbf{K} = \mathbf{R}$, ($k \equiv -1 \pmod 4$ and $3 \leq k \leq 2n-1$) or (n is even and $k = n-1$)

In particular, $\pi_k(\mathrm{SL}(n,\mathbf{K}))$ is finite for $k=1$, k even, or $k \geq 2n$.

Example 4. Let W denote a Cantor set. It is straightforward to show that, for every connected manifold M, all maps $W \to M$ are homotopic. Thus, $SK_{1,n}(\mathcal{C}(W,\mathbf{K}))$ is trivial, and accordingly, for all $n \geq 3$, $\mathrm{SL}(n,\mathcal{C}(W,\mathbf{K}))$ has Kazhdan's Property (T).

Theorem 1 rests on two results: a K-theoretic result of Vaserstein (Theorem 8), and the work of Shalom on Kazhdan's Property (T) (Theorem 7).

In [Sha99], Shalom introduces new methods to establish Kazhdan's Property (T) for the special linear groups over certain rings. This leads, for instance, to the first proof that $\Gamma = \mathrm{SL}(n,\mathbf{Z})$, $n \geq 3$, has Property (T), that does not use the embedding of Γ into $\mathrm{SL}(n,\mathbf{R})$ as a lattice.

Before stating his main result, let us begin with a definition.

Definition 5. If G is a group and $S \subset G$ is a subset, we say that G is boundedly generated by S if there exist $m < \infty$ such that every $g \in G$ is a product of at most m elements in S.

Theorem 6 (Shalom, [Sha99])**.** *Let $n \geq 3$, and let R be a topologically finitely generated ring. Suppose that $\mathrm{SL}(n,R)$ is boundedly generated by elementary matrices. Then $\mathrm{SL}(n,R)$ has Kazhdan's Property (T).*

As an application, Shalom proves bounded elementary generation for the loop group $\mathrm{SL}(n,\mathcal{C}(S^1,\mathbf{C}))$ ($n \geq 3$), and deduces that it has Property (T). He asks if the same holds for $\mathbf{R}$ instead of $\mathbf{C}$, noting that $\mathrm{SL}(n,\mathcal{C}(S^1,\mathbf{R}))$ is not generated by elementary matrices (since $\pi_1(\mathrm{SL}(n,\mathbf{R})) \neq 1$). This is answered positively by Theorem 1; see Example 3 above.

Actually, without modification, the proof of Theorem 6 ([Sha99], see also [BHV05]) gives a stronger statement.

Theorem 7. *Let $n \geq 3$, let R be a topologically finitely generated commutative ring, and suppose that $\mathrm{E}(n,R)$ is boundedly generated by elementary matrices. Then $\mathrm{E}(n,R)$ has Kazhdan's Property (T).* ■

Theorem 6 is the particular case of Theorem 7 when $\mathrm{E}(n,R) = \mathrm{SL}(n,R)$.

Theorem 7 is a strong motivation for studying bounded elementary generation for the group $\mathrm{E}(n,R)$. For instance, this is an open question for $R = \mathbf{F}_p[X,Y]$ or

[1]If X is not compact, we do not know whether $\mathrm{E}(n,\mathcal{C}(X,\mathbf{K}))$ is closed in $\mathrm{SL}(n,\mathcal{C}(X,\mathbf{K}))$ for the topology of uniform convergence on compact subsets. However, we can restate the corollary as follows: $\mathrm{SL}(n,\mathcal{C}(X,\mathbf{K}))$ has Kazhdan's Property (T) if and only if the group $\overline{SK}_{1,n}(\mathcal{C}(X,\mathbf{K})) = \mathrm{SL}(n,\mathcal{C}(X,\mathbf{K}))/\overline{\mathrm{E}(n,\mathcal{C}(X,\mathbf{K}))}$ does.

$R = \mathbf{Z}[X]$, for all $n \geq 3$. We now focus on the case when $R = \mathcal{C}(X, \mathbf{K})$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

The notion of dimension of a topological space involved here is defined in [Vas71], and it will be sufficient for our purposes to know that $\dim(X)$ is finite for every topological subspace of a Euclidean space.

Theorem 8 (Vaserstein, [Vas88])**.** *Let X be a finite dimensional topological space, and let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and fix $n \geq 3$. Then $\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ is boundedly generated by elementary matrices.*

Now we show how Theorems 7 and 8 imply Theorem 1. Since $X \subset \mathbf{R}^d$ for some d, X is finite dimensional, so that Theorem 8 applies: $\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ is boundedly generated by elementary matrices.

It remains to show that Theorem 7 applies, that is, $\mathcal{C}(X, \mathbf{K})$ is topologically finitely generated for the compact-open topology (that is, the topology of uniform convergence on compact subsets).

Let $p_1, \ldots, p_d$ be the projections of X on the d coordinates of $\mathbf{R}^d$, and let A be the (unital) $\mathbf{K}$-subalgebra of $\mathcal{C}(X, \mathbf{K})$ generated by $p_1, \ldots, p_d$. By the Stone-Weierstrass Theorem, A is dense in $\mathcal{C}(X, \mathbf{K})$ for the topology of uniform convergence on compact subsets. Then the finite family $\{(p_j), \sqrt{2}\}$ (resp. $\{(p_j), \sqrt{2}, i\}$) generates a dense subring in $\mathcal{C}(X, \mathbf{K})$ if $\mathbf{K} = \mathbf{R}$ (resp. if $\mathbf{K} = \mathbf{C}$). ■

Remark 9.

(1) The hypothesis in Theorem 1 that X is homeomorphic to a subset of an Euclidean space is close to being necessary in order to apply Theorem 7. Indeed, suppose that $\mathcal{C}(X, \mathbf{K})$ is endowed with a topology such that the evaluation functions $f \mapsto f(x)$ are continuous. Besides, suppose that $\mathcal{C}(X, \mathbf{K})$ is topologically finitely generated as a ring by $p_1, \ldots p_d$. Then there exists a continuous injection of X in some Euclidean space, given by $x \mapsto (p_1(x), \ldots, p_d(x))$.

(2) If X is metrizable and non-compact, and $\mathcal{C}(X, \mathbf{K})$ is endowed with the uniform convergence topology, then an easy growth argument shows that $\mathcal{C}(X, \mathbf{K})$ is not topologically finitely generated.

It would be interesting to generalize Theorem 8 to general semisimple Lie groups without compact factors, and Theorem 7 to semisimple groups without compact factors and with Property (T), or at least to higher rank ones. Theorem 6 is extended to symplectic groups in [Neu03]. On the other hand, if G is a connected compact simple Lie group, $\mathcal{C}(S^1, G)$ does not have Kazhdan's Property (T) [BHV05, Exercise 4.4.5].

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