FIXED POINTS AND ALMOST FIXED POINTS, AFTER GROMOV AND V. LAFFORGUE

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Lemma 1. *Let X be a complete metric space. Let $\rho : X \to \mathbf{R}_+$ be a function satisfying the following condition, implied by lower semi-continuity:*

$$\forall (x_n), x \in X, \quad (x_n) \to x, \rho(x_n) \to 0 \Rightarrow \rho(x) = 0.$$

Suppose that $0 \notin \rho(X)$. Then for every $r < \infty$ and every $\varepsilon > 0$ there exists $v \in X$ such that

$$\forall v' \in B'(v, r\rho(v)),\ \rho(v') > (1-\varepsilon)\rho(v).$$

Proof. Suppose that the conclusion is false: for some $r < \infty, \varepsilon > 0$, we have

$$\forall v \in X, \exists v' \in B'(v, r\rho(v)),\ \rho(v') \leq (1-\varepsilon)\rho(v).$$

Define by induction a sequence (w_n) in X by picking any w_0 in X, and apply this hypothesis to $v = w_n$, defining $w_{n+1} = v'$. We thus have $\rho(w_{n+1}) \leq (1-\varepsilon)\rho(w_n)$, so that $\rho(w_n) \leq (1-\varepsilon)^n \rho(w_0)$, and $d(w_n, w_{n+1}) \leq 2r\rho(w_n)$, showing that (w_n) is a Cauchy sequence; let w be its limit. As $\rho(w_n) \to 0$, our assumption on ρ yields $\rho(w) = 0$, a contradiction. □

Remark 2. Let a semigroup generated by a finite subset S act by homeomorphisms on a complete metric space X, without fixed points. Then the function

$$\rho(x) = \sup_{s \in S} d(x, sx)$$

satisfies all the hypotheses of Lemma 1. Such a statement is used in the proof [L, lemme 2] and this was a starting point for the present note.

Lemma 1 allows to give a proof of the following theorem claimed as "obvious" by Gromov [G, 3.8.D p.117].

Theorem 3. *Let G be a finitely generated semigroup acting by Lipschitz transformations on a complete metric space (X, d) with no fixed point. Then there exists a metric space Y that is a scaling Hausdorff limit of a sequence $(X, x_n, \lambda_n d)$ on which G acts without almost fixed points.*

Corollary 4. *Let $\mathcal{X}$ be a class of complete metric spaces that is stable under scaling Hausdorff limits (e.g. Hilbert spaces, $\mathbf{R}$-trees, $CAT(0)$-spaces, L^p-spaces, but neither $CAT(-1)$-spaces nor simplicial trees). Suppose that G is a finitely generated semigroup such that every action by Lipschitz transformations (resp. isometries) on a metric space in the class $\mathcal{X}$ almost has a fixed point. Then every such action actually has a fixed point.*

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Proof of Theorem 3. Let G act on the complete metric space X without fixed points. Fix $0 < \varepsilon < 1$. Lemma 1 provides a sequence (v_n) in X such that

$$\forall n, \forall v \in B'(v_n, n\rho(v_n)),\ \rho(v) > (1-\varepsilon)\rho(v_n), \tag{1}$$

where ρ is defined as in Remark 2.

Consider the ultraproduct Y of the sequence of pointed metric spaces

$$\left(X, v_n, \frac{1}{\rho(v_n)}d\right).$$

We have to check that G acts by Lipschitz transformations (resp. isometries) on Y. First it must be checked that the action is well-defined. Namely, we have to show that the sequence $(\rho(v_n)^{-1}d(v_n, sw_n))$ is bounded for every $s \in S$ whenever $(\rho(v_n)^{-1}d(v_n, w_n))$ is bounded. By the triangular inequality, it suffices to check that $(\rho(v_n)^{-1}d(v_n, sv_n))$ and $(\rho(v_n)^{-1}d(sv_n, sw_n))$ are both bounded. The first one is bounded by 1 by definition of ρ. For the second one, let m be an upper bound for the Lipschitz ratio of all Lipschitz transformations $x \mapsto sx$ on X. Then the sequence $(\rho(v_n)^{-1}d(sv_n, sw_n))$ is bounded by $m(\rho(v_n)^{-1}d(v_n, w_n))$, which is bounded by assumption. Therefore the action is well-defined; it is now straightforward that the action of every $s \in S$ on the ultraproduct is m-Lipschitz.

Now let us check that the action of G on Y has no almost fixed point.

Consider an element of Y, given by a sequence (w_n) such that the sequence $(\rho(v_n)^{-1}d(v_n, w_n))$ is bounded. Then

$$d((w_n), s(w_n)) = \lim \frac{d(w_n, sw_n)}{\rho(v_n)}.$$

Now there exists $s_0 \in S$ such that $\rho(w_n) = d(w_n, s_0w_n)$ for ω-almost all n (i.e. for all n belonging to some element of ω). So

$$d((w_n), s_0(w_n)) = \lim \frac{\rho(w_n)}{\rho(v_n)}.$$

Now as $d(w_n, v_n)/\rho(v_n)$ is bounded, eventually $w_n \in B(v_n, n\rho(v_n))$ and therefore by (1) we have $\rho(w_n) \geq (1-\varepsilon)\rho(v_n)$ for large n. Thus $d((w_n), s_0(w_n)) \geq 1-\varepsilon$. So we just proved that for every $w \in Y$, we have $\sup_{s\in S} d(w, sw) \geq 1-\varepsilon$, and therefore the action does not have almost fixed points. ☐

Theorem 5. *Let G be a finitely generated semigroup, and Q_n a family of quotients of G, converging to a quotient Q. Suppose that each Q_n has an action by Lipschitz transformations on a metric space X_n with no fixed point. Then there exists a complete metric space Y that is a scaling Hausdorff limit of a sequence $(X_n, x_n, \lambda_n d)$ on which Q acts without almost fixed points.*

Proof. Lemma 1 and Remark 2 provide a sequence (v_n) where $v_n \in X_n$ satisfies

$$\forall n, \forall v \in B'(v_n, N\rho(v_n)),\ \rho(v) > (1-\varepsilon)\rho(v_n).$$

The proof goes on as that of Theorem 3. It is clear that the limit action factors through Q. ☐

Corollary 6. *Let $\mathcal{X}$ be a class of complete metric spaces that is stable under scaling Hausdorff limits. Suppose that G is a finitely generated (semi)group such that every action by isometries on a metric space in the class $\mathcal{X}$ has a fixed point. Then G is a quotient of a finitely presented semi(group) with the same property.*

References

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