# DIMENSION OF ASYMPTOTIC CONES OF LIE GROUPS

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Abstract. We compute the covering dimension the asymptotic cone of a connected Lie group. For simply connected solvable Lie groups, this is the codimension of the exponential radical.

As an application of the proof, we give a characterization of connected Lie groups that quasi-isometrically embed into a non-positively curved metric space.

## 1. Introduction

There are many kinds of dimensions associated to a connected Lie group. The most naive one is the usual dimension; for the point of view of coarse geometry it is not interesting as it is not a quasi-isometry invariant, since for instance it would have to vanish for all compact Lie groups. However, the dimension of $G/K$, where $K$ is a maximal compact subgroup of $G$, is a quasi-isometry invariant of $G$ by a theorem of J. Roe [Roe, Proposition 3.33 and Corollary 3.35]. Other kinds of dimensions that provide quasi-isometry invariants are, among others, asymptotic dimension [Gro, BeDr], subexponential corank [BuSc1], hyperbolic dimension [BuSc2]. Here we focus on the covering dimension of the asymptotic cone, which was considered in Gromov's book [Gro, Chap. 2], in Burillo's paper [Bur] and in [LePi] in the context of solvable groups, in [BeMi] in the case of the mapping class group, in [DrSm] in connection with asymptotic Assouad-Nagata dimension, and in [Kl] in the context of non-positively curved manifolds.

The covering dimension $\mathrm{covdim}(X)$ of a topological space X is the minimal $d$ such that every open covering has a refinement with multiplicity at most $d+1$. This is a topological invariant, and as the asymptotic cone (with respect to a given ultrafilter) is a bilipschitz invariant of quasi-isometry classes of metric spaces, the cone dimension is a quasi-isometry invariant.

Let us recall the definition of asymptotic cones. For more details see for instance [Dru]. Fix a non-principal ultrafilter $\omega$ on $\mathbf{N}$. Let $X$ be a metric space. Define its asymptotic precone $\mathrm{Precone}(X,\omega)$ as the following semi-metric space. As a set, $\mathrm{Precone}(X,\omega)$ is the set of sequences $(x_n)$ in $X$ such that $(d(x_n,x_0)/n)_{n\ge 1}$ is bounded. The semi-metric is defined as $d((x_n),(y_n)) = \lim_\omega d(x_n,y_n)/n$. The asymptotic cone $\mathrm{Cone}(X,\omega)$ is the metric space obtained from $\mathrm{Precone}(X,\omega)$ by identifying points at distance 0. Define the cone dimension $\mathrm{conedim}(X,\omega)$ as the covering dimension of $\mathrm{Cone}(X,\omega)$.

Besides, recall that, in a simply connected solvable Lie group $G$, the exponential radical is a closed normal subgroup $\mathrm{R}_{\exp}G$ defined by saying that $G/\mathrm{R}_{\exp}G$ is the biggest quotient of $G$ with polynomial growth. It was introduced and studied by Guivarc'h in [Gui2] and subsequently by Osin [Osin].

*Date*: December 23, 2007.

2000 *Mathematics Subject Classification.* 22E15 (Primary); 20F69, 22E25, 54F45 (Secondary).

*Key words and phrases.* Asymptotic cone; Exponential radical; Covering dimension.

**Theorem 1.1.** *Let $G$ be a simply connected solvable Lie group. Then*

$$\mathrm{conedim}(G, \omega) = \dim(G/\mathrm{R}_{\exp}G)$$

*for every non-principal ultrafilter $\omega$ on $\mathbf{N}$.*

The inequality $\leq$ is a slight generalization of a result of Burillo [Bur] that we establish in Section 3, along with the estimates on the distortion of the exponential radical, due to Guivarc'h.

When the map $G \to G/\mathrm{R}_{\exp}G$ is split, $G/\mathrm{R}_{\exp}G$ embeds quasi-isometrically into $G$ so that the other inequality is immediate, as observed in [LePi, Proposition 4.4]. However in general this map is not split, even in groups that do not deserve to be considered as "pathological". Example 4.1 gives a simply connected solvable Lie group $G$ in which this map is not split, and which moreover is an algebraic group defined over the rationals, and having an arithmetic lattice.

**Remark 1.2.** The fact that, for a connected Lie group, the cone dimension is independent of the ultrafilter $\omega$ is anything but surprising. However, it is not known if a connected Lie group can have two non-homeomorphic asymptotic cones; nevertheless in [KSTT, KrTe] it is proved that an absolutely simple group of real rank at least two has a unique asymptotic cone up to homeomorphism if and only if the continuum hypothesis is true.

Nevertheless, there are many connected Lie groups for which the asympotic cone is known to be unique up to (bilipschitz) homeomorphism. The best-known examples are groups with polynomial growth, and Gromov hyperbolic Lie groups. There are other examples, such as the group $\mathrm{SOL}(\mathbf{R})$, for which every asymptotic cone is a "Diestel-Leader $\mathbf{R}$-graph", namely the "hypersurface" of equation $b(x)+b(y) = 0$ in the product $T \times T$, where $T$ is a complete universal homogeneous $\mathbf{R}$-tree of degree $2^{\aleph_0}$ and $b$ is a Busemann function on $T$ (see Section 9). It is easy to extend such a description to every semidirect product $\mathbf{R}^m \ltimes \mathbf{R}^n$, where the action of $\mathbf{R}^n$ on $\mathbf{R}^m$ is semisimple.

On the other hand it was recently established [OOS, Remark 5.13] that there exists a finitely generated group for which the dimension of the asymptotic cone does depend on the ultrafilter. More precisely, they construct a finitely generated (not finitely presented) group having one asymptotic cone homeomorphic to an $\mathbf{R}$-tree $T$ and therefore of covering dimension one, and one other homeomorphic to $T \times \mathbf{R}/\mathbf{Z}$ and therefore of covering dimension two.

Before sketching the proof of Theorem 1.1, let us give some corollaries and reformulations.

A first straightforward corollary is the following.

**Corollary 1.3.** *The dimension of $G/\mathrm{R}_{\exp}G$ is a quasi-isometry invariant of the simply connected solvable Lie group $G$.* □

This corollary might be improved by proving that the *quasi-isometry type* of $G/\mathrm{R}_{\exp}G$ only depends on the quasi-isometry type of the simply connected solvable Lie group $G$; we leave this as an open problem.

Turning back to Theorem 1.1, although it is well-known that the study of the asymptotic cones of general connected Lie groups reduces to simply connected solvable Lie groups (see Section 6), it is convenient to have a formula giving the covering dimension, not appealing to a solvable cocompact subgroup. To write such a formula, we extend the notion of exponential radical as follows.

First recall [Cor, Definition 1.8] that a connected Lie group is called *M-decomposed* if every non-compact simple Lie subgroup centralizes the radical, or equivalently if it is locally isomorphic to the direct product of a semisimple group with no compact factors and a solvable-by-compact group, called its amenable radical. If moreover the amenable radical has polynomial growth, we call the group *P-decomposed*. This property is already proved to be relevant in the geomeric study of Lie groups; for instance it is proved in [CPS] that a connected Lie group has the *Rapid Decay (RD) Property* if and only if it is P-decomposed.

**Definition 1.4.** If $G$ is a connected Lie group, its exponential radical $\mathrm{R}_{\exp}G$ is the smallest kernel of a map of $G$ onto a P-decomposed connected Lie group.

In Section 6 we check that this definition, which extends that of Guivarc'h makes sense, and we give general results concerning the exponential radical (see Theorem 6.3).

**Definition 1.5.** 1) If $G$ is a connected Lie group, define its geometric dimension $\mathrm{geodim}(G)$ as $\dim(G) - \dim(K)$, where $K$ is a maximal compact subgroup: this is the dimension of a contractible Riemannian manifold on which $G$ acts properly cocompactly by isometries.

2) If $S$ is a connected semisimple Lie group with centre $Z$, define its geometric rank $\mathrm{georank}(S)$ as $r + z$, where $r$ is the $\mathbf{R}$-rank of $S/Z$, and $z$ is the $\mathbf{Q}$-rank of $Z$ (i.e. the dimension of the vector space $Z \otimes_{\mathbf{Z}} \mathbf{R}$).

**Corollary 1.6.** *Let $G$ be a connected Lie group with radical $R$. We have, for every non-principal ultrafilter $\omega$*

$$\mathrm{conedim}(G, \omega) = \mathrm{georank}(G/R) + \mathrm{geodim}(R/\mathrm{R}_{\exp}G).$$

**Remark 1.7.** It follows from Corollary 1.6 that if $G$ is a connected Lie group and $H$ a quotient of $G$, then $\mathrm{conedim}(G) \geq \mathrm{conedim}(H)$. This is not true for general finitely generated groups, as we see by taking $G = F_2$ (free group) and $H = \mathbf{Z}^2$, where $\mathrm{conedim}(G) = 1$ and $\mathrm{conedim}(H) = 2$. Note also that there exist finitely generated groups with infinite cone dimension: for instance, $\mathbf{Z} \wr \mathbf{Z}$ has infinite cone dimension: indeed, it contains quasi-isometrically embedded free abelian subgroups of arbitrary large rank.

The proof of Theorem 1.1 splits into three steps. Let $G$ denote a simply connected solvable Lie group.

The first paramount step is the case of polynomial growth, and is already settled: we have to know that

$$\mathrm{conedim}(G/\mathrm{R}_{\exp}G, \omega) = \dim(G/\mathrm{R}_{\exp}G).$$

Indeed, if $H$ is a $d$-dimensional simply connected Lie group of polynomial growth, then its asymptotic cone is homeomorphic to $\mathbf{R}^d$: in the essential case where $P$ is nilpotent, this is due to Pansu [Pan], and in general this follows from the nilshadow construction (see for instance Lemma 2.2 or [Bre]). Finally, we have to know the basic but non-trivial fact that $\mathbf{R}^d$ has covering dimension $d$ (see [Nag, Theorem IV.4]).

The second step is the inequality

$$\mathrm{conedim}(G, \omega) \leq \mathrm{conedim}(G/\mathrm{R}_{\exp}G, \omega).$$

It relies on Guivarc'h's result that the exponential radical is strictly exponentially distorted, along with a variant of theorem of Burillo [Bur, Theorem 16] about the behaviour of the covering dimension under a map with ultrametric fibers. See the discussion following Corollary 3.5.

The last step is the inequality

$$\mathrm{conedim}(G,\omega) \geq \mathrm{conedim}(G/\mathrm{R}_{\exp}G,\omega).$$

It is obvious when the extension

$$1 \to \mathrm{R}_{\exp}G \to G \to G/\mathrm{R}_{\exp}G \to 1$$

is a semidirect product. However, this is not always the case, see Examples 4.1 and 4.2. In the general case, it relies on the following result.

**Theorem 1.8.** *Let $G$ be a connected solvable Lie group. Fix any non-principal ultrafilter $\omega$ on $\mathbf{N}$. Then there exists a bilipschitz embedding*

$$\mathrm{Cone}(G/\mathrm{R}_{\exp}G,\omega) \to \mathrm{Cone}(G,\omega).$$

*In particular,* $\mathrm{conedim}(G/\mathrm{R}_{\exp}G,\omega) \leq \mathrm{conedim}(G,\omega)$.

We construct explicitly such an embedding in Section 5 when $G/\mathrm{R}_{\exp}G$ is simply connected nilpotent. However in general $G/\mathrm{R}_{\exp}G$ is only of polynomial growth; nevertheless a *trigshadow* construction, carried out in Section 2, allows to reduce to the case when $G/\mathrm{R}_{\exp}G$ is nilpotent.

Let us now give an analogous result for $p$-adic algebraic groups. If $G$ is a linear algebraic group over an ultrametric local field of characteristic zero (equivalently, some finite extension of $\mathbf{Q}_p$), the assumption that $G$ is compactly generated is strong enough to ensure that the role of the exponential radical is played by the unipotent radical (see Proposition 7.1). In this case the Burillo dimension Theorem [Bur, Theorem 16] applies directly, and moreover as the unipotent radical is always split, our lifting construction (Theorem 1.8) is not needed. See the detailed discussion in Section 7, whose conclusion is the following.

**Theorem 1.9.** *Let $G$ be a linear algebraic group over an ultrametric local field of characteristic zero. Suppose that $G$ is compactly generated and let $d$ be the dimension of a maximal split torus. Then, for every non-principal ultrafilter $\omega$,*

$$\mathrm{conedim}(G,\omega) = d.$$

Note that it follows from Corollary 1.6 that if $G$ is a linear algebraic group over $\mathbf{R}$ or $\mathbf{C}$, with unipotent radical $U$, and $d$ is the dimension of a maximal split torus, then $\mathrm{R}_{\exp}G \subset U$ and

$$\mathrm{conedim}(G,\omega) = d + \dim_{\mathbf{R}}(U/\mathrm{R}_{\exp}G),$$

where $\dim_{\mathbf{R}}$ means twice the dimension when the ground field is $\mathbf{C}$.

Let us make a few remarks about other kinds of dimension.

- For every topological space $X$, one can define its (large) inductive dimension, $\mathrm{Ind}(X)$, which is a non-negative integer (see [Nag, Section I.4]). If $X$ is metrizable (as all asymptotic cones as considered in the paper), then it is known that it coincides with the covering dimension.

- As the asymptotic cone is defined up to bilipschitz homeomorphism, its Hausdorff dimension, which is a non-negative real number, is well-defined. If $G$ has polynomial growth, it is known to have growth of degree $d$ for some (explicit) non-negative integer $d$ [Gui1], and work of Pansu [Pan] shows that the Hausdorff dimension of the asymptotic cone is exactly $d$.

  On the other hand, if $G$ is a Lie group or a compactly generated linear algebraic group over a local field of characteristic zero and if $G$ has non-polynomial growth, then by [CoTe] it contains a quasi-isometrically embedded regular trivalent tree, and therefore all its asymptotic cone contain a bilispchitz embedded complete universal $\mathbf{R}$-tree, everywhere branched of degree $2^{\aleph_0}$. This $\mathbf{R}$-tree has the property that every $r$-ball contains uncountably many disjoint open $r/2$-balls, and it immediately follows from the definition of Hausdorff dimension that the Hausdorff dimension of this $\mathbf{R}$-tree, and therefore of all asymptotic cones of $G$, is infinite.

Finally, here is an application of our results to embeddings into non-positively curved metric spaces.

Recall that a geodesic metric space is called *non-positively curved* or *CAT(0)* if for every geodesic triangle $abc$ and every $t \in [0,1]$, the distance of $(1-t)b+tc$ (which makes sense on the given geodesic segment from $b$ to $c$) to $a$ is less or equal to the same distance computed from a triangle $a, b, c$ inside the Euclidean plane, with the same edge lengths. For instance, every simply connected Riemannian manifold with non-positive sectional curvature is a CAT(0) metric space.

**Theorem 1.10.** *Let $G$ be a simply connected solvable Lie group endowed with a left-invariant Riemannian metric. Suppose that $G$ is triangulable (i.e. embeds as a subgroup of real upper triangulable matrices, see Lemma 2.5). Then the following are equivalent*

(1) *$G$ quasi-isometrically embeds into a CAT(0) space;*
(2) *$G$ quasi-isometrically embeds into a non-positively curved simply connected symmetric space;*
(2') *$G$ bilipschitz embeds into a non-positively curved simply connected symmetric space;*
(3) *$G/\mathrm{R}_{\exp}G$ is abelian.*

This is proved in Section 8, where we formulate a statement for all connected Lie groups (although the general case is based, as always, on the case of triangulable Lie groups).

It is in the implication (1)⇒(3) that we make use of Theorem 1.8, combined with the following theorem of Pauls [Pau]: if $G$ is a non-abelian simply connected nilpotent Lie group, then $G$ has no quasi-isometric embedding into any CAT(0) space.

**Remark 1.11.** If $G$ is any linear algebraic group over a local field $\mathbf{K}$ of characteristic zero, then any algebraic embedding $G \to \mathrm{SL}_n(\mathbf{K})$ is quasi-isometric by [Mus] and thus induces a quasi-isometric embedding of $G$ into a CAT(0) space, namely the Euclidean Bruhat-Tits building [BT] of $\mathrm{SL}_n(\mathbf{K})$.

**Question 1.12.** Which non-positively curved simply connected symmetric spaces quasi-isometrically embed into a finite product of trees (resp. binary trees)?

It was proved in [BDS] that every word hyperbolic group quasi-isometrically embeds into a finite product of binary trees, and therefore the question has a positive answer for products of rank one non-positively curved simply connected symmetric space. But I do not know the answer for any irreducible non-positively curved simply connected symmetric space of higher rank, e.g. $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. Of course it makes sense to extend the question to general connected Lie groups.

## 2. Trigshadow

**Lemma 2.1.** *Let $G$ be a connected Lie subgroup of $\mathrm{GL}_n(\mathbf{R})$ and $H$ its Zariski closure. Then if $G$ has polynomial growth, so does $H$.*

*Proof.* Write $H = DKU$, where $U$ is the unipotent radical, $K$ is reductive anisotropic, $D$ is an isotropic torus, $[D, K] = 1$ and $K \cap D$ is finite.

Consider the adjoint action of $\mathrm{GL}_n(\mathbf{R})$ on its Lie algebra. Then $G$ fixes its Lie algebra $\mathfrak{g}$, on which every element of $G$ acts with eigenvalues of modulus one. Therefore all of $H$ fixes $\mathfrak{g}$, acting with eigenvalues of modulus one. On the other hand, $D$ acts diagonally with real eigenvalues, therefore acts trivially on $\mathfrak{g}$. It follows that $D$ is, up to a finite central kernel, a direct factor of $KU$ in $H$, so that $H$ has polynomial growth. □

**Lemma 2.2.** *Let $G$ be a connected Lie group with polynomial growth. Then there exists a connected linear algebraic group $H$ such that*

- *$H(\mathbf{R})$ has polynomial growth;*
- *$H$ contains no nontrivial isotropic torus (equivalently: the unipotent radical $U$ of $H$ is cocompact in $H(\mathbf{R})$)*
- *$G$ maps properly (i.e. with compact kernel) into $H(\mathbf{R})$, with cocompact, Zariski dense image.*
- *If $K$ is a maximal compact subgroup of $G$, then $\dim(U) = \dim(G) - \dim(K)$.*

The unipotent radical $U$ is known as the *nilshadow* of $G$. The construction given here is close to the original one [AuGr, Theorem 4.1].

*Proof.* By Lemma 2.3, $G$ has a proper mapping into some $\mathrm{GL}_n(\mathbf{R})$. Therefore we can suppose that $G$ is a closed subgroup in $\mathrm{GL}_n(\mathbf{R})$. Let $B$ be its Zariski closure. Then $B$ also has polynomial growth by Lemma 2.1. It follows that $B$ has a unique maximal split torus $D$, which is central.

Now consider a vector space $V$, viewed as a unipotent algebraic $\mathbf{R}$-group, with a proper surjective map $G/[G, G] \to V$. Consider the diagonal mapping $G \to H = B/D \times V$. Then this map is proper.

We have obtained that $G$ maps properly into an algebraic group $H = KW$ with $K$ compact and $W$ unipotent. View $G$ as a closed subgroup. Then $GK$ contains $G$ as a cocompact subgroup, and it is the semidirect product of $K$ by $G \cap W$; in particular the latter is connected. Accordingly $G \cap W$, being a connected subgroup of $W$, is an algebraic group. Therefore $G$ is cocompact in the algebraic $\mathbf{R}$-group $KG$.

To prove the last equality, first observe that we take the quotient by $(G \cap U)_0$ and therefore suppose that $G \cap U$ is discrete. Thus $G$ maps with discrete kernel, into $H/U$, which is abelian. So the Lie algebra of $G$ is abelian, and by connectedness this implies that $G$ is abelian. Since $G$ is cocompact in $H$, the desired formula is immediate. □

The following lemma is a particular case of [Mos2, Lemma 5.2].

**Lemma 2.3.** *Let $G$ be a solvable, connected Lie group $G$ whose derived subgroup is simply connected (e.g. $G$ itself is simply connected). Then $G$ has a faithful linear proper (i.e. with closed image) linear representation.* □

**Lemma 2.4** (trigshadow)**.** *Let $G$ be a solvable, connected Lie group $G$ whose derived subgroup is simply connected. Then $G$ embeds cocompactly into a group $H$ containing a cocompact connected subgroup $T$, containing $[G,G]$, that has a faithful proper embedding into some group of real upper triangular matrices.*

*Proof.* Using Lemma 2.3, fix a faithful proper linear real representation of $G$. Consider its Zariski closure $DKU$, where $U$ is the unipotent radical, $DK$ a maximal torus, $D$ its isotropic part and $K$ its anisotropic part. Then set $H = GK$ and $T_1 = H \cap DU$. As $G$ normalizes its Lie algebra $\mathfrak{g}$, so does $K$; therefore $K$ normalizes $G$, so that $H = KG$ is a subgroup; as $K$ is compact and $G$ closed, $H$ is closed. The restriction to $H$ of the natural map $DKU \to DKU/DU \simeq K/(D \cap K)$ being surjective, its kernel $T_1$ is cocompact in $H$. Finally, as the Zariski closure of $T_1$ has no anisotropic torus, it is triangulable. Note that $[G,G] \subset U$ and therefore $[G,G] \subset T_1$. We claim that $T_1$ has finitely many components. Indeed, $H = T_1K$ is connected and $T_1 \cap K = D \cap K$ is finite. We conclude by defining $T$ as the unit component of $T_1$. □

**Lemma 2.5.** *Let $G$ be a connected Lie group. The following are equivalent*

(i) *$G$ embeds into a group of triangular matrices;*
(i') *$G$ embeds properly (i.e. as a closed subgroup in the real topology) into a group of triangular matrices;*
(ii) *$G$ embeds into a solvable linear algebraic group with no anisotropic torus;*

*Besides, if these conditions are fulfilled, then $G/\mathrm{R}_{\exp}G$ is nilpotent, i.e. the inclusion of $\mathrm{R}_{\exp}G$ in the stable term of the descending central series of $G$ is an equality.*

*Proof.* (i')⇒(i)⇒(ii) is trivial. The implication (ii)⇒(i) follows from the fact that every split torus is conjugate to a group of diagonal matrices. To get (i)⇒(i') it suffices to observe that every connected Lie subgroup of the group of triangular matrices is closed.

Finally suppose that the conditions are fulfilled: more precisely suppose (ii) and embed $G$ as a Zariski dense subgroup of a solvable linear algebraic group $H$ with no anisotropic torus, and with unipotent radical $U$. Note that the exponential radical $W$ of $G$ is contained in $U$; therefore it is Zariski closed in $H$. The group $G/W$ is Zariski dense in $H/W$ and has polynomial growth; therefore $H/W$ has polynomial growth by Lemma 2.1. As $H/W$ has no anisotropic torus, it is therefore the direct product of its unipotent radical by a split torus. In particular it is nilpotent, hence $G/W$ is nilpotent. □

**Definition 2.6.** We call a simply connected solvable Lie group $G$ *weakly triangulable* if $G/\mathrm{R}_{\exp}G$ is nilpotent.

**Remark 2.7.** Let $\mathbf{R}$ act on $\mathbf{R}^2$ by the one-parameter subgroup

$$\begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix};\ t \in \mathbf{R}.$$

Then $G = \mathbf{R} \ltimes \mathbf{R}^2$ is weakly triangulable but not triangulable.

## 3. Around Burillo's dimension theorem

**Definition 3.1.** A map $f : X \to Y$ between metric spaces has *metrically parallel fibers* if for every $x, y \in X$ there exists $z \in X$ such that

$$d(x, z) = d(f(x), f(z)) \text{ and } f(y) = f(z).$$

**Example 3.2.** Let $G$ be a locally compact group generated by a compact subset $K$, and $G/N$ a quotient of $G$, generated by the image of $K$. Endow both with the corresponding word metrics. Then the quotient map $G \to G/N$ has metrically parallel fibers, and so does the induced map

$$\mathrm{Cone}(G, \omega) \to \mathrm{Cone}(G/N, \omega).$$

**Lemma 3.3.** *The fibers of the map*

$$\mathrm{Cone}(G, \omega) \to \mathrm{Cone}(G/N, \omega)$$

*of Example 3.2 are all isometric to* $\mathrm{Cone}(N, \omega)$*, $N$ being endowed with the restriction of the word metric of $G$.*

*Proof.* If $(g_n) \in \mathrm{Cone}(G, \omega)$ belongs to some fiber, the mapping $(h_n) \mapsto (g_n h_n)$ maps isometrically the fiber of (1) to the fiber of $(g_n)$.

Now we have an obvious isometric inclusion of $\mathrm{Cone}(N, \omega)$ ($N$ being endowed with the restriction of the word metric of $G$) into the fiber of (1) in $\mathrm{Cone}(G, \omega)$. This isometry is surjective: indeed, let $(g_n)$ belong to the fiber of (1). Write $g_n$ as a word of length $|g_n|_{G/N}$ in $G/N$; lifting to $G$, we can write $g_n = h_n \nu_n$ with $|h_n| = |g_n|_{G/N}$ and $\nu_n \in N$. Then, since $(g_n)$ belongs to the fiber of (1), we see that $(g_n)$ and $(\nu_n)$ coincide in $\mathrm{Cone}(G, \omega)$. This proves surjectivity. □

**Theorem 3.4.** *Let $X \to Y$ be a map between metric spaces, with metrically parallel fibers. Suppose that all fibers are bilipschitz to ultrametric spaces, with bilipschitz constants independent of the fiber. Then* $\dim(X) \le \dim(Y)$*.*

This theorem is obtained in [Bur, Theorem 16] with the slightly stronger assumption that all fibers are ultrametric (that version was claim by Gromov in [Gro, Chap. 2]); this is not sufficient for our purposes. However, the proof follows the same lines, so we omit some details.

Recall [Nag] that a metric space $Y$ has covering dimension at most $n$ if and only if there exists a sequence of positive numbers $(r_k) \to 0$, and a sequence of coverings $(\mathcal{U}_k)$ by open subsets of diameter at most $r_k$, such that every $x \in Y$ belongs to at most $n+1$ elements of $\mathcal{U}_k$, and $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$ for all $k$.

*Proof.* We can suppose that there exists $\lambda > 0$ and, for every fiber $X_y = f^{-1}(\{y\})$, a map from the fiber to an ultrametric space $\Upsilon_y$ satisfying, for all $x, x' \in X_y$

$$d(x, x) \le d(f(x), f(x')) \le \lambda d(x, x').$$

Suppose that $Y$ has covering dimension at most $n$. Consider a sequence $(\mathcal{U}_k)$ of open coverings of $Y$ as above. Extracting if necessary, we can suppose that $r_{k+1} \le r_k/(3\lambda + 1)$ for all $k$.

Write $\mathcal{U}_k = (U_k^\alpha)_\alpha$. For every $\alpha$ pick $y_\alpha \in U_k^\alpha$. As $\Upsilon_{y_\alpha}$ is ultrametric, it can be covered by pairwise disjoint open balls $B_k^{\alpha\beta}$ of radius $3r_k$. More precisely, for all $x, x'$

in $\Upsilon_{y_\alpha}$, either they are in the same ball and $d(x,x') < 3r_k$, either they are in two distinct balls and $d(x,x') \geq 3r_k$.

Consider the corresponding covering ${B'}_k^{\alpha\beta}$ of $F_{y_\alpha}$ by disjoint open subsets. For all $x, x'$ in $F_{y_\alpha}$, either they belong to the same subset of the covering and $d(x,x') < 3\lambda r_k$, or they belong to two distinct subsets and $d(x,x') \geq 3r_k$.

Define $V_k^{\alpha\beta}$ as the set of elements in $f^{-1}(U_\alpha)$ that are at distance $\leq r_k$ from ${B'}_k^{\alpha\beta}$. This is, $\alpha$ being fixed, a family of open subsets of $X$; any two elements in the same subset are at distance $< (3\lambda+2)r_k$, and any two elements in two distinct subsets in this family ($\alpha$ being fixed) are at distance $\geq r_k$, in particular for $\alpha$ fixed they are pairwise disjoint. This implies that any element of $X$ belongs to at most $n+1$ elements of the family ($\alpha$ being no longer fixed).

Moreover, this is a covering of $X$, denoted $\mathcal{V}_k$: indeed, if $x \in X$, then $x \in f^{-1}(U_\alpha)$ for some $\alpha$, and therefore there exists $x' \in F_\alpha$ such that $d(x,x') = d(f(x), y_\alpha) \leq r_k$, so $x'$ belongs to some ${B'}_k^{\alpha\beta}$, implying $x \in V_k^{\alpha\beta}$.

Finally it remains to show that $\mathcal{V}_{k+1}$ is a refinement of $\mathcal{U}_k$. If $V_{k+1}^{\alpha\beta}$ belongs to $\mathcal{V}_{k+1}$, then $U_{k+1}^{\alpha}$ is contained in an element $U_k^{\gamma}$ of $\mathcal{U}_k$. Now, we have just shown that if $x \in V_{k+1}^{\alpha\beta}$, then $x \in V_k^{\gamma\delta_x}$ for some $\delta_x$. Now we claim that all $\delta_x$ are equal to one single element $\delta$, proving $V_{k+1}^{\alpha\beta} \subset V_k^{\gamma\delta}$. Indeed, if $\delta_x \neq \delta_{x'}$, then $x$ and $x'$ are at distance at least $r_k$; on the other hand they are at distance $< (3\lambda+2)r_{k+1}$, contradicting the assumption $r_{k+1} \leq r_k/(3\lambda+2)$. □

We call a metric space $X$ quasi-ultrametric if it satisfies a quasi-ultrametric inequality: for some $C < \infty$ and for all $x, y, z \in X$,

$$d(x,z) \leq \max(d(x,y), d(y,z)) + C.$$

Note that this immediately implies that any asymptotic cone of $X$ is ultrametric. Important examples are log-metrics, that is, when $(X,d)$ is a metric space, the new metric space $(X, \log(1+d))$.

**Corollary 3.5.** *Let $G$ be a locally compact compactly generated group, and $N$ a closed normal subgroup. Suppose that $N$, endowed with the word metric of $G$, is quasi-isometric to a quasi-ultrametric space. Then, for every non-principal ultrafilter $\omega$,*

$$\operatorname{conedim}(G,\omega) \leq \operatorname{conedim}(G/N,\omega).$$

Let $G$ be a simply connected solvable Lie group. By Guivarc'h-Osin's Theorem, the restriction $d$ of the word metric (or Riemannian metric) to the exponential radical is equivalent to the metric $d'$ defined by the length $\log(|\cdot|_{R_{\exp}(G)} + 1)$.

Corollary 3.5 then implies

$$\dim(\operatorname{Cone}(G,\omega)) \leq \dim(\operatorname{Cone}(G/\mathrm{R}_{\exp}G,\omega)). \tag{3.1}$$

## 4. Splittings of the exponential radical

As we mentioned in the introduction, it is obvious that if the exponential radical is split, i.e. when the extension

$$1 \to \mathrm{R}_{\exp}G \to G \to G/\mathrm{R}_{\exp}G \to 1$$

is a semidirect product, then the converse of Inequality (3.1) holds.

However it is not true in general that the exponential radical is split. Here is an example, which is moreover $\mathbf{Q}$-algebraic with no $\mathbf{Q}$-split torus and which therefore contains an arithmetic lattice.

**Example 4.1.** Let $H_3(R)$ denote the Heisenberg group over the ring $R$. If $K$ is a field of characteristic zero, the automorphism group of the algebraic group $H_3(K)$ is easily seen (working with the Lie algebra) to be $K$-isomorphic to $\mathrm{GL}_2(K)\ltimes K^2$ (where the $K^2$ corresponds to inner automorphisms). If an automorphism corresponds to $(A, v) \in \mathrm{GL}_2(K) \ltimes K^2$, its action on the center of $H_3(K)$ is given by multiplication by $\det(A)^2$.

Think now at $K = \mathbf{R}$ and pick a one-parameter diagonalizable subgroup $(\alpha_t)$ inside $\mathrm{SL}_2(\mathbf{R})$, viewed as a subgroup of $\mathrm{Aut}H_3(\mathbf{R})$. If we want to fix a $\mathbf{Q}$-structure, choose $\alpha_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ so that this group is $\mathbf{Q}$-anisotropic (although $\mathbf{R}$-isotropic). This defines a semidirect product $\mathbf{R}^* \ltimes H_3(\mathbf{R})$. Define $G_1 = (\mathbf{R}^* \ltimes H_3(\mathbf{R})) \times H_3(\mathbf{R})$. Its center is the product of both centers and is therefore isomorphic to $\mathbf{R} \times \mathbf{R}$. Let $Z$ be the diagonal of this $\mathbf{R} \times \mathbf{R}$, and define

$$G = G_1/Z.$$

Then $\mathrm{R}_{\mathrm{exp}}G$ is the left-hand copy of $H_3(\mathbf{R})$. The quotient $G/\mathrm{R}_{\mathrm{exp}}G$ is isomorphic to $\mathbf{R}^3$. However it is not split: indeed otherwise, $G$ would have $\mathbf{R}^2$ as a direct factor, while the centre of $G$ is isomorphic to $\mathbf{R}$. Note that $G$ is triangulable.

Obviously, if $G$ is a connected solvable Lie group with $\dim(G/\mathrm{R}_{\mathrm{exp}}G) \leq 1$ (note that $\dim(G/\mathrm{R}_{\mathrm{exp}}G) = 0$ only if $G$ is compact), then the exponential radical is split.

Also, if $G$ is a connected solvable Lie group that is algebraic over $\mathbf{R}$ (that is, the unit component in the real topology of $H(\mathbf{R})$ for some linear algebraic $\mathbf{R}$-group $H$), and if $\dim(G/\mathrm{R}_{\mathrm{exp}}G) \leq 2$, then it is not hard to see that the exponential radical is split. Here is a non-algebraic example where $\dim(G/\mathrm{R}_{\mathrm{exp}}G) = 2$ and $\mathrm{R}_{\mathrm{exp}}G$ is not split.

**Example 4.2.** Keep the one-parameter subgroup in $\mathrm{Aut}(H_3)$ of Example 4.1. Denote by $H_3'$ another copy of $H_3$, and fix any homomorphism of $H_3'$ onto $\mathbf{R}$. Thus define a group $G_1 = H_3' \ltimes H_3$, where $H_3'$ acts on $H_3$ through the homomorphism onto $\mathbf{R}$ and the one-parameter subgroup fixed above. (Note that $G_1$ is definitely non-algebraic because $H_3'$ plays both a semisimple and a unipotent role.) Let $Z$ be the diagonal of the center $\mathbf{R} \times \mathbf{R}$ of $G_1$, and define $G = G_1/Z$. Then $\mathrm{R}_{\mathrm{exp}}G = H_3$ has codimension 2 in $G$ and is not split; note that $G$ is triangulable.

## 5. Liftings in weakly triangulable groups

The purpose of this section is to prove that Equality (3.1) holds without assuming that the exponential radical is split. This is a consequence of the following theorem.

**Theorem 5.1.** *Let $G$ be a connected solvable Lie group. Fix any non-principal ultrafilter $\omega$ on $\mathbf{N}$. Then there exists a bilipschitz section of the projection*

$$\mathrm{Cone}(G, \omega) \to \mathrm{Cone}(G/\mathrm{R}_{\mathrm{exp}}G, \omega).$$

*In particular,* $\mathrm{conedim}(G/\mathrm{R}_{\mathrm{exp}}G, \omega) \leq \mathrm{conedim}(G, \omega)$.

*Proof.* The second statement follows from the first as if $Y$ is a closed subset of a topological space $X$, then obviously $\mathrm{covdim}(Y) \leq \mathrm{covdim}(X)$.

By Lemma 2.4, we can suppose that the intersection $N$ of the descending central series of $G$ coincides with the exponential radical. Denote by $\mathfrak{g}$, $\mathfrak{n}$ the corresponding Lie algebras. It is known [Bou, Chapitre 7, p.19-20] that there exists a "Cartan subalgebra", which is nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}+\mathfrak{n} = \mathfrak{g}$. (Examples 4.1 and 4.2 show that we cannot always demand in addition $\mathfrak{h}\cap\mathfrak{n} = 0$.) Set $\mathfrak{w} = \mathfrak{h}\cap\mathfrak{n}$ and choose a complement subspace $\mathfrak{v}$ of $\mathfrak{w}$ in $\mathfrak{n}$. Denote by $G, H, N, W$ the corresponding simply connected Lie groups, and by $\nu$ the restriction to $\mathfrak{v}$ of the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$.

Define a map $\psi : G/N(= H/W) \to H \subset G$ as

$$\psi = \exp_H \circ \nu^{-1} \circ {\exp_{H/W}}^{-1}.$$

This obviously a continuous section of the projection $G \to G/N$, but it can be checked that this is not a homomorphism nor is large-scale Lipschitz in general. Nevertheless, we are going to show that $\psi$ induces a map $\tilde{\psi} : \mathrm{Cone}(G/N) \to \mathrm{Cone}(G)$. Obviously $\tilde{\psi}$ is expansive (i.e. increases distances), and in particular it is injective. Eventually we are going to show that $\tilde{\psi}$ is bilipschitz.

To avoid lengthy notation, it is convenient to identify every (simply connected) nilpotent Lie group with its Lie algebra. From that point of view, $\psi$ is simply the inclusion of $\mathfrak{v}$ in $\mathfrak{g}$, where $\mathfrak{g}$ is endowed with the word metric $|\cdot|_G$ of $G$ and $\mathfrak{v}$ is endowed with the word metric $|\cdot|_{G/N}$ of $G/N$, through the identification of $\mathfrak{g}/\mathfrak{n}$ and $\mathfrak{v}$ by $\nu$. Write $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w}$. For real-valued functions $f, g$ defined on $G$, write $f \preceq g$ if there exists constants $\alpha, \beta > 0$ such that $f \leq \alpha g + \beta$. Fix a norm $\|\cdot\|$ on the vector space $\mathfrak{g}$.

**Lemma 5.2.** *If* $(x, w) \in \mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w}$, *then*

$$|(x, w)|_G \preceq |x|_{G/H} + \log(1 + \|w\|)$$

*Proof.* Let $|\cdot|$ denote the word length on $G$, $|\cdot|_N$ that on $N$, and let $\pi$ denote the projection $G \to G/N$. Fix a compact symmetric generating subset $S$ of $H$. If $g \in H$, write $\pi(g)$ as an element of minimal length with respect to $\pi(S)$: $\pi(g) = \pi(s_1)\dots\pi(s_m)$. Set $h = s_1 \dots s_m$. As $h^{-1}g$ belongs to the exponential radical, the inequality $|g| \leq |h| + |h^{-1}g|$ yields, in view of [Osin, Theorem 1.1(3)],

$$|g| \preceq |\pi(g)|_{G/N} + \log(1 + |h^{-1}g|_N) \quad (g \in H)$$

and therefore

$$|g| \preceq |\pi(g)|_{G/N} + \log(1 + \|h^{-1}g\|).$$

As $\mathfrak{h}$ is nilpotent, its group law is given by a vector-valued polynomial and we have an upper bound $\|h^{-1}g\| \leq A(\|g\| + 1)^k(\|h\| + 1)^k$ for some constants $A, k \geq 1$. This implies

$$|g| \preceq |\pi(g)|_{G/N} + \log(1 + \|g\|) + \log(1 + \|h\|).$$

As $H$ is nilpotent, we have $\|h\| \leq A'|h|_H^{k'}$ for some constants $A', k' \geq 1$ (this follows from [Gui1, Lemma II.1]). Now $|h|_H = m \leq |g|_H \preceq \|g\|$. Therefore

$$|g| \preceq |\pi(g)| + \log(1 + \|g\|).$$

Take now $(x, u)$ as in the statement of the lemma. Then the above formula yields

$$|(x, u)| \preceq |x| + \log(1 + \|(x, u)\|),$$

so that

$$|(x,u)| \preceq |x| + \log(1+\|x\|) + \log(1+\|u\|).$$

As $\log(1+\|x\|) \preceq |x|_H \preceq |x|$, we get

$$|(x,u)| \preceq |x| + \log(1+\|u\|). \quad \square$$

Lemma 5.2 implies that $\psi$ preserves the word length, and therefore induces a map $\tilde{\psi} : \text{Precone}(G/N) \to \text{Cone}(G,\omega)$.

On the other hand, observe that $\psi$ is a section of the projection $G \to G/N$, which decreases distances; in particular $\psi$, and therefore $\tilde{\psi}$, increases distances. However let us show that $\tilde{\psi}$ is bilipschitz.

Denote by $\cdot$ the group laws in $\mathfrak{v}$ and $\mathfrak{h}$ inherited from $H/W$ and $H$; although they do not coincide in restriction to $\mathfrak{v}$ there will be no ambiguity as elements of $\mathfrak{h}$ will always be written as pairs $(x,w) \in \mathfrak{v} \oplus \mathfrak{w}$. Note that in both spaces, the inverse law is simply given by $x^{-1} = -x$ (this simply translates the fact that $\exp(-x) = \exp(x)^{-1}$).

We can write the group law in $\mathfrak{h}$, restricted to $\mathfrak{v}$, as

$$(x,0)\cdot(y,0) = (x\cdot y, P(x,y)),$$

where $P$ is a polynomial vector-valued function.

Then

$$(x,0)^{-1}\cdot(y,0) = (x^{-1}\cdot y, P(-x,y)), \quad \forall x,y \in \mathfrak{v}.$$

Introduce the following notation:

If $a \geq 1$ and $b \geq 0$ are constants, write $u \leq_{a,b} v$ if $u \leq av + b$. Then by Lemma 5.2

$$|(x,0)^{-1}\cdot(y,0)|_G \leq_{a,b} |x^{-1}\cdot y|_{G/N} + \log(1+\|P(-x,y)\|), \quad \forall x,y \in \mathfrak{v}.$$

Write $P = \sum P_i v_i$ where $P_i$'s are real-valued polynomials and $v_i$ are unit vectors. Then

$$|(x,0)^{-1}\cdot(y,0)|_G \leq_{a,b} |x^{-1}\cdot y|_{G/N} + \sum \log(1+|P_i(-x,y)|), \quad \forall x,y \in \mathfrak{v}.$$

Now let us work in the asymptotic cone. All limits and cones are with respect to a fixed ultrafilter $\omega$. Any linearly bounded sequences $(x_n)$ and $(y_n)$ in $\mathfrak{v}$ (i.e. $|x_n|_{H/W}, |y_n|_{H/W} = O(n)$) define elements $\underline{x}$ and $\underline{y}$ in both $\text{Cone}(H/W)$ and $\text{Cone}(G)$. We have

$$\begin{aligned}
d(\underline{x},\underline{y})_{\text{Cone}(G)} &= \lim \frac{1}{n}|(x_n,0)^{-1}(y_n,0)|_G \\
&\leq_{a,0} \lim \frac{1}{n}|{x_n}^{-1}\cdot y_n|_{G/N} + \lim \frac{1}{n}\sum \log(1+|P_i(-x_n,y_n)|) \\
&= d(\underline{x},\underline{y})_{\text{Cone}(H/W)} + \sum \lim \frac{1}{n}\log(1+|P_i(-x_n,y_n)|).
\end{aligned}$$

Now, as $(x_n)$ and $(y_n)$ are linearly bounded, $|P_i(-x_n,y_n)| = O(n^d)$ where $d$ is the the total degree of $P$. Thus $\lim \frac{1}{n}\log(1+|P_i(-x_n,y_n)|) = 0$ for all $i$, and we obtain

$$d(\underline{x},\underline{y})_{\text{Cone}(G)} \leq_{a,0} d(\underline{x},\underline{y})_{\text{Cone}(H/W)}.$$

This shows that $\tilde{\psi}$ is lipschitz, and therefore is bilipschitz. In particular, it factors through a map

$$\text{Cone}(G/N) \to \text{Cone}(G),$$

ending the proof. □

**Remark 5.3.** We have shown that the asymptotic cone of every simply connected solvable Lie group $G$ contains a bilipschitz-embedded copy of $\mathrm{Cone}(G/\mathrm{R}_{\exp}G)$. It is natural to ask if $G$ itself contains a quasi-isometrically embedded copy of $G/\mathrm{R}_{\exp}G$. For instance, if $G$ is the group of Example 4.2, does $G$ contain a flat, that is, an quasi-isometrically embedded copy of $\mathbf{R}^2$?

## 6. Exponential radical

The exponential radical was introduced by Guivarc'h in [Gui2, Définition 8, p.58] as the "unstable subgroup" as follows: the exponential radical of a connected amenable Lie group is the smallest closed normal subgroup $\mathrm{R}_{\exp}G$ such that the quotient $G/\mathrm{R}_{\exp}G$ has polynomial growth. The terminology "exponential radical" is due to Osin in [Osin], who independently introduced it.

**Theorem 6.1** (Guivarc'h, Osin)**.** *Let $G$ be a simply connected, solvable Lie group. Then*

(1) $\mathrm{R}_{\exp}G$ *is a closed, connected, nilpotent, characteristic subgroup of $G$.*
(2) *The exponential radical is* strictly exponentially distorted*, that is the restriction of the metric $|\cdot|$ to $R_{\exp}(G)$ is equivalent to the log-metric $\log(|\cdot|_{R_{\exp}(G)}+1)$, where $|\cdot|_{R_{\exp}(G)}$ denotes the intrinsic word metric of $R_{\exp}(G)$.*

An argument for (1) is given after [Gui2, Définition 8]. In (2), the upper bound follows directly from [Gui2, Propositions 4 and 5], and the lower bound is obtained in [Osin].

For general connected Lie groups, although Guivarc'h's definition would make sense, we modify it so that it is suitable for our purposes.

**Definition 6.2.** Let $G$ be a connected Lie group, and $R$ its (usual) radical. Define its exponential radical $\mathrm{R}_{\exp}G$ so that $G/\mathrm{R}_{\exp}G$ is the biggest quotient of $G$ that is P-decomposed, i.e. locally isomorphic to a direct product of a semisimple group and a group with polynomial growth.

**Theorem 6.3.** *Let $G$ be a connected Lie group,*

(1) $\mathrm{R}_{\exp}G$ *is a closed connected characteristic subgroup of $G$.*
(2) $\mathrm{R}_{\exp}G$ *is contained in the nilpotent radical of $G$.*
(3) *Denote $R$ the radical of $G$ and $S_{nc}$ a maximal semisimple subgroup with no compact factors. Then*
$$\mathrm{R}_{\exp}G=\overline{[S_{nc},R]\mathrm{R}_{\exp}R}.$$
(4) *Suppose that $G$ is algebraic, that $U$ is its unipotent radical, and $L_{nc}$ a maximal reductive purely isotropic (i.e. generated by its split subtori) subgroup. Then*
$$\mathrm{R}_{\exp}G=[L_{nc},U].$$
(5) $\mathrm{R}_{\exp}G$ *is strictly exponentially distorted in $G$.*

We need a series of lemmas.

**Lemma 6.4.** *Let $A$ be a connected abelian Lie group, $K$ its maximal torus, and $F$ be a closed subgroup of $A$ such that $A\cap K=1$. Then $F$ is contained in a direct factor of $K$ in $A$.*

*Proof.* Taking the quotient by the connected component $F_0$, we can suppose that $F$ is discrete, and thus is free abelian. Write $A = \tilde{A}/\Gamma$, and lift $F$ to a subgroup $\tilde{F}$ of $\tilde{A}$; we have $\tilde{F} \cap \Gamma = \{0\}$. As $\tilde{F} + \Gamma$ is closed and countable, it is discrete. This implies that $\tilde{F} \cap \mathrm{Vect}_{\mathbf{R}}(\Gamma) = \{0\}$. Thus $\tilde{F}$ is contained in a complement subspace of $\mathrm{Vect}_{\mathbf{R}}(\Gamma)$ and we are done. □

The following lemma is well-known but we found no reference.

**Lemma 6.5.** *Let $K$ be a field of characteristic zero, $\mathfrak{g}$ a finite dimensional Lie algebra, and $\mathfrak{n}$ its nilpotent radical. Then $\mathfrak{g}/\mathfrak{n}$ is reductive, i.e. is the direct product of a semisimple Lie subalgebra and an abelian one.*

*Proof.* Fix an embedding of $\mathfrak{g}$ into $\mathfrak{gl}_n$. Let $H \subset \mathrm{GL}_n$ be the connected normalizer of $\mathfrak{g}$, $R$ the radical of $H$ and $U$ its unipotent radical. Then $[H, R] \subset U$. Denoting by Gothic letters the corresponding Lie algebras, we have $[\mathfrak{h}, \mathfrak{r}] \subset \mathfrak{u}$. Denoting by $\mathfrak{r}'$ the radical of $\mathfrak{g}$, we have $\mathfrak{r}' \subset \mathfrak{r}$ and therefore $[\mathfrak{g}, \mathfrak{r}'] \subset \mathfrak{u} \cap \mathfrak{g}$. So $[\mathfrak{g}, \mathfrak{r}']$ is a nilpotent ideal in $\mathfrak{g}$ and is thus contained in its nilpotent radical. □

**Definition 6.6.** If $G$ is a group and $H$ a subgroup, we say that $H$ is cocentral in $G$ if $G$ is generated by $H$ and its centre $Z(G)$.

Note that if $H$ is cocentral in $G$, every normal subgroup of $H$ remains normal in $G$.

The following lemma is quite standard.

**Lemma 6.7.** *Let $G$ be a connected Lie group and $N$ its nilradical. Then $G$ has a cocompact subgroup $L$ (which can be chosen normal if $G$ is solvable), containing $N$, and embedding cocompactly cocentrally into a connected solvable Lie group $M$ whose fundamental group coincides*[1] *with that of $N$. In particular, if $G$ has no nontrivial compact normal torus, then $M$ is simply connected. Moreover, $M$ decomposes as a direct product $Q \times \mathbf{R}^n$ for some $n$ and some closed subgroup $Q$ of $L$; in particular $\mathrm{R}_{\exp}M \subset G$.*

*Proof.* Working inside $G/N$ and taking the preimage in $G$ allows us to work in $H = G/N$. Let $R$ be the radical of $H$; it is central in $H$ by Lemma 6.5. Let $S$ be the semisimple Levi factor in $H$, and $\bar{S}$ its closure. The centre of the semisimple part of $G$ has a subgroup of finite index whose action by conjugation on $N$ is trivial. Thus if $\bar{S} \cap R = \overline{Z(S)} \cap R$ contained a torus, the preimage of this torus in $G$ would be nilpotent and this would contradict the definition of $N$. Thus $\bar{S} \cap R$ has a subgroup of finite index whose intersection with the maximal torus $K$ of $R$ is trivial, which is therefore contained in a direct factor $L$ of $K$ in $R$ by Lemma 6.4. We obtain that $H_1 = L\bar{S}$ is a connected, cocompact subgroup of $H$ whose radical $R_1$ is simply connected.

The group $H_1/Z(H_1)$ is semisimple with trivial centre and thus has a connected, solvable, simply connected cocompact subgroup $B$. Let $C$ be the preimage of $B$ in $H_1/R_1$. Then the covering $H_1/R_1 \to H_1/Z(H_1)$ induces a covering $C \to B$, with fibre $Z(H_1)/R \simeq Z(H_1/R)$. As $B$ is simply connected, we deduce that $C$ is isomorphic to $B \times Z(H_1/R_1)$; in particular we identify $B$ with the unit component of $C$. The group $Z(H_1/R_1)$ has a free abelian subgroup $\Lambda$ of finite index, which

[1]If $G$ is a topological group and $H$ a subgroup, we say that $\pi_1(G)$ coincides with $\pi_1(H)$ if the natural map $\pi_1(H) \to \pi_1(G)$ is an isomorphism.

embeds as a lattice in some vector space $V$. Let $Q$ be the preimage of $B \subset H_1/R_1$ in $G$. Then $Q$ is a closed, connected, solvable subgroup of $G$, whose fundamental group coincides with that of $N$.

On the other hand, let $\tilde{S}$ be a Levi factor of $G$. Then the quotient map $G \to G/R$ maps $Z(\tilde{S})$ onto $Z(H_1/R_1)$. Now $Z(\tilde{S})$ contains a subgroup of finite index which is central in $G$; let $\Upsilon$ be its closure. Then $Q\Upsilon$ has finite index in the preimage of $C$ in $G$; in particular it is a closed, cocompact solvable subgroup. As $Q$ contains $N$, it also contains $\Upsilon_0$; so there exists a discrete, free abelian subgroup $\Gamma$ of $\Upsilon$ such that $\Gamma \cap Q = 1$ and $Q\Gamma$ has finite index in $Q\Upsilon$. Now $L = Q \times \Gamma$ embeds cocompactly cocentrally in $M = Q \times (\Gamma \otimes_{\mathbf{Z}} \mathbf{R})$. □

**Lemma 6.8.** *Let $G$ be a connected Lie group $R$ its radical, $S$ a Levi factor, and $S_{nc}$ its non-compact part. Then $[S_{nc}, R]$ is normal in $G$..*

Let us first check that $[S_{nc}, R]$ is normal.

*Proof.* First observe that $[S_{nc}, R] = [S_{nc}, [S_{nc}, R]]$: this is obtained by noting that $S_{nc} = [S_{nc}, S_{nc}]$ and using the Jacobi identity. Then

$$[G, [S_{nc}, R]] = [G, [S_{nc}, [S_{nc}, R]]] \subset [S_{nc}, [[S_{nc}, R], G]] + [[S_{nc}, R], [G, S_{nc}]]$$
$$\subset [S_{nc}, R] + [[S_{nc}, R], [S_{nc}, R]] \subset [S_{nc}, R].$$

□

*Proof of Theorem 6.3.*

(1) This is obvious; connectedness is due to the fact that P-decomposability only depends on the Lie algebra.
(2) As the quotient of $G$ by its nilpotent radical $\mathrm{R_{nil}}G$ is a reductive Lie group (see Lemma 6.5), $\mathrm{R_{exp}}G$ is contained in $\mathrm{R_{nil}}G$.
(3) By definition $\mathrm{R_{exp}}G$ is obviously the closure of the normal subgroup generated by $\mathrm{R_{exp}}R$ and $[S_{nc}, R]$. So we only have to check that $\overline{[S_{nc}, R]\mathrm{R_{exp}}R}$ is normal in $G$, and this is a consequence of the fact that $[S_{nc}, R]$ is normal (Lemma 6.8).
(4) Noting that $[L_{nc}, U]$ is closed, using (3) allows to reduce to the case when $G$ is solvable. Now for $G$ solvable, $\mathrm{R_{exp}}G$ is the kernel of the biggest quotient with polynomial growth. Therefore, denoting by $D$ a maximal split torus ($D = L_{nc}$), it follows from the Lie algebra characterisation of type R that $[D, G]$, and hence $[D, U]$, is contained in $\mathrm{R_{exp}}G$. Conversely, if $[D, U] = 1$, then it immediately follows from a Levi decomposition that $G$ is nilpotent-by-compact. Therefore all what remains to prove is that $[D, U]$ is a normal subgroup. This is checked by means of the Lie algebra. First, using that the action of $D$ on $\mathfrak{u}$ is semisimple, one checks that $[D, [D, U]] = [D, U]$. Then, using this along with the Jacobi Identity, one obtains that $[D, U]$ is normalized by $U$. Thus it is normal.
(5) Let $K$ be the maximal compact subgroup of $\mathrm{R_{nil}}G$. By Lemma 6.7, there exists a cocompact subgroup $L$ of $G/K$, containing $\mathrm{R_{nil}}(G/K)$, embedding cocompactly into a simply connected solvable Lie group $M$. We have inclusions $\mathrm{R_{exp}}G/K \subset \mathrm{R_{exp}}M \subset M$. By Guivarc'h-Osin's Theorem (Theorem 6.1), $\mathrm{R_{exp}}M$ is strictly exponentially distorted in $M$. On the other hand, as both $\mathrm{R_{exp}}G/K$ and $\mathrm{R_{exp}}M$ are nilpotent, we have polynomial distortion: for

some $a \geq 1$, we have, for $g \in \mathrm{R}_{\exp}G/K$,

$$|g|^{1/a}_{\mathrm{R}_{\exp}G/K} \preceq |g|_{\mathrm{R}_{\exp}M} \preceq |g|_{\mathrm{R}_{\exp}G/K}.$$

This implies that $\mathrm{R}_{\exp}G/K$ is strictly exponentially distorted in $G$, and therefore in $G/K$. Thus $\mathrm{R}_{\exp}G$ is strictly exponentially distorted in $G$. □

**Lemma 6.9.** *Let $G$ be a connected Lie group, and $N$ a closed normal solvable subgroup. Then*

$$\mathrm{conedim}(G,\omega) \geq \mathrm{conedim}(G/N,\omega).$$

*Proof.* By Lemma 6.7, $G/N$ has a cocompact solvable subgroup $H/N$ embedding cocompactly into a connected solvable Lie group $L/N$. Therefore we can suppose $G$ solvable.

We can suppose that $G$ and $G/N$ have no non-trivial compact normal subgroup. Now taking a cocompact simply connected normal subgroup in $G/N$ and taking the unit component of its preimage in $G$, we can suppose that $G$ and $G/N$ are both simply connected solvable Lie groups. Then the result follows from Theorem 1.1. □

*Proof of Corollary 1.6.* Set $c(G) = \mathrm{conedim}(G,\omega)$ and $\rho(G) = \mathrm{georank}(G/R) + \mathrm{geodim}(R/\mathrm{R}_{\exp}G)$.

By Corollary 3.5 and using Theorem 6.3(5), we have $c(G/\mathrm{R}_{\exp}G) \leq c(G)$; by Lemma 6.9 this is an equality. On the other hand, it is clear that $\rho(G) = \rho(G/\mathrm{R}_{\exp}G)$. Thus to prove that $\rho = c$ we are reduced to the case when $\mathrm{R}_{\exp}G = 1$.

Thus suppose $\mathrm{R}_{\exp}G = 1$, i.e. $G$ is P-decomposed. Let $M$ denote the amenable radical of $G$, namely the subgroup generated by $R$ and compact simple factors. Set $S = G/S$, and $Z$ a finite index subgroup of its centre, free abelian of rank $n$. Then $S/Z$ has a cocompact simply connected solvable subgroup, whose preimage in $S$ is denoted by $B$. As $B/Z$ is simply connected, $B$ is the direct product of its unit component $B_0$ and $Z$. Moreover, as $G$ is M-decomposed, the preimage of $Z$ in $G$ is central. So lift $Z$ to a subgroup $Z'$ of $G$. Let $H$ be the preimage of $B$ in $G$. Then $H = Z' \times H_0$. It follows from Theorem 1.1 that for connected Lie groups, $c$ is additive under direct products. Thus $c(H) = n + c(H_0)$. Now as $G$ is M-decomposed and $B_0$ has trivial centre, $H_0$ is isomorphic to the direct product of $S/Z$ and $M$. Thus $c(G) = n + c(S/Z) + c(M)$. On the other hand, it is immediate that $\rho(G) = n + \rho(S/Z) + \rho(M)$. Thus we are reduced to prove $c = \rho$ in two cases: semisimple groups with finite centre and groups with polynomial growth.

Suppose that $G$ is semisimple with finite centre. Then $G$ has a cocompact simply connected solvable group $AN$, where $D$ is isomorphic to $\mathbf{R}^r$, where $r = \mathbf{R}\text{-rank}(G)$, and acts faithfully on the Lie algebra of $N$ by diagonal matrices and no fixed points. In particular, $N$ is the exponential radical of $DN$, and by Theorem 1.1, $c(DN) = \mathbf{R}\text{-rank}(G) = \rho(G)$.

Suppose that $G$ has polynomial growth. Then by Lemma 2.2, $G$ is quasi-isometric to a simply connected nilpotent Lie group of dimension $d = \mathrm{geodim}(G) = \rho(G)$, and the asymptotic cone of $G$ is homeomorphic to $\mathbf{R}^d$ by Pansu's work [Pan]. □

## 7. The $p$-adic case

In this section we deal with groups $G$ of the form $\mathbf{G}(\mathbf{Q}_p)$ where $\mathbf{G}$ is a connected linear algebraic group defined over $\mathbf{Q}_p$ for some prime $p$. For the sake of simplicity, we call $G$ a connected linear algebraic $\mathbf{Q}_p$-group.

Fix a connected linear algebraic $\mathbf{Q}_p$-group $G$, and take a Levi decomposition $G = H \ltimes U$ where $U$ is the unipotent radical, and $H$ is a maximal reductive subgroup [Mos1]. Decompose $H$ as an almost product $T = SK$, where $S$ is contains all isotropic factors (simple or abelian) and $K$ contains all anisotropic factors.

Recall that $G$ is compactly generated if and only if $U = [S, U]$ [BoTi, Théorème 13.4]. In this case, the role of the exponential radical is played by the unipotent radical, in view of the following proposition, which is probably folklore.

**Proposition 7.1.** *Let $G$ be a compactly generated connected linear algebraic group defined over $\mathbf{Q}_p$, and $U$ its unipotent radical. Then the restriction to $U$ of the word metric of $G$ is equivalent to an ultrametric. More precisely, if $G$ is solvable with no non-trivial anisotropic torus, there exists a left-invariant metric on $G$ that is equivalent to the word metric, and which is ultrametric in restriction to $U$.*

*Proof.* If we replace $G$ by a cocompact solvable algebraic subgroup containing $U$, then the unipotent radical of the new group contains $U$. Therefore we can suppose that $G$ is solvable and without anisotropic torus. It follows that $G$ has a faithful representation into a group of upper triangular matrices in $\mathbf{Q}_p$. Now S. Mustapha proved that an algebraic embedding between compactly generated linear algebraic groups is always quasi-isometric. Therefore we are reduced to the case of the upper triangular group $G$ of size $n \times n$ over $\mathbf{Q}_p$, where $U$ is the set of upper unipotent matrices.

Fix the field norm $|\cdot|$ on $\mathbf{Q}_p$, defined as $|x| = e^{-v_p(x)}$ where $v_p$ is the $p$-valuation. Let us define a real-valued map $\|\cdot\|$ on $U$ as follows: if $A = (a_{ij}) \in U$, set $\|A\| = \max\{|a_{ij}|^{1/(j-i)} : 1 \le i < j \le n$. Then $\|\cdot\|$ is an ultralength on $U$, that is satisfies the inequality $\|AB\| \le \max(\|A\|, \|B\|)$ and the equalities $\|1\| = 0$ and $\|A^{-1}\| = A$. We omit the details.

For $x \ge 0$, set $\ell(x) = \max(0, \log(x))$; it satisfies the inequality $\ell(xy) \le \ell(x) + \ell(y)$.

Denote by $D$ the group of diagonal matrices, so that $G = D \ltimes U$. Set, for $d = \mathrm{diag}(d_1, \dots, d_n) \in D$, $\|d\| = \max(|d_1|, \dots, |d_n|)$. Note that for all $(d, u) \in D \times U$, we have $\|d^{-1}ud\| \le \|d\|^2 \|u\|$.

Define, for $g = du$, where $(d, u) \in D \times U$, $|g| = \max(2\ell(\|d\|), \ell(\|u\|))$. Then $|\cdot|$ is subadditive: indeed

$$|d_1u_1d_2u_2| = |d_1d_2.d_2^{-1}u_1d_2u_2| = \max(2\ell(\|d_1d_2\|), \ell(\|d_2^{-1}u_1d_2u_2\|))$$

$$\le \max(2\ell(\|d_1\|) + 2\ell(\|d_2\|), \ell(\|d_2^{-1}u_1d_2\|), \ell(\|u_2\|))$$

$$\le \max(2\ell(\|d_1\|) + 2\ell(\|d_2\|), 2\ell(\|d_2\|) + \ell(\|u_1\|), \ell(\|u_2\|)) \le |d_1u_1| + |d_2u_2|$$

As $|\cdot|$ is bounded on compact subsets, it follows that it is dominated by the word length. It remains to check that the word length is dominated by $|\cdot|$. On $D$, this is obvious, so it suffices to check this on $U$. Suppose that $u \in U$ satisfies $|u| \le n$. Let $w \in D$ be the matrix $\mathrm{diag}(1, p, p^2, \dots, p^{n-1})$. Then $\|w^{-n}uw\| \le 1$. As $\{u \in U : \|u\| \le 1\}$ is compact, this shows that $|\cdot|$ is equivalent to the word length on $G$. Now $|\cdot|$ is ultrametric on $U$ and this completes the proof. □

Let $G$ be a compactly generated linear algebraic group over a local field of characteristic zero as above, and $U$ its unipotent radical. Let $d$ be the dimension of a maximal split torus in $G$. Then $G$ has a cocompact subgroup $H$ containing $U$ that is solvable with no non-trivial anisotropic torus, and whose maximal split tori have dimension $d$. By Proposition 7.1, $H$ has a left-invariant metric equivalent to the word metric, for which the restriction to $U$ (and hence to every left coset of $U$)is ultrametric. Therefore [Bur, Theorem 16] applies directly to obtain

$$\dim(\mathrm{Cone}(H,\omega)) \leq \dim(\mathrm{Cone}(H/U,\omega)). \tag{7.1}$$

Of course, we can also use Corollary 3.5 to get it.

Now $H$ has a Levi decomposition: $H = D \ltimes U$ where $D$ is a split torus, isomorphic to $H/U$. Thus the inclusion of $D$ in $H$ is a quasi-isometric embedding, and therefore induces a bilipschitz embedding of $\mathrm{Cone}(H/U,\omega)$ into $\mathrm{Cone}(H,\omega)$.

Thus (7.1) is an equality. Moreover, $D$ is isomorphic to $\mathbf{K}^{*d}$, and $\mathbf{K}^*$ is isomorphic to the direct product of $\mathbf{Z}$ and of the 1-sphere in $\mathbf{K}$, which is compact. Thus the asymptotic cone of $D$ is homeomorphic to $\mathbf{R}^d$, and thus we get, recalling that $H$ is cocompact in $G$

$$\dim(\mathrm{Cone}(G,\omega)) = d.$$

## 8. Embeddings into non-positively curved metric spaces

Define a connected Lie group $G$ as quasi-abelian if it acts properly cocompactly on a Euclidean space (equivalently, if it has polynomial growth and its nilshadow is abelian). Define $G$ as quasi-reductive if its amenable radical $A$ is quasi-abelian and is, up to local isomorphism, a direct factor (note that this is stronger than P-decomposability, see just before Definition 1.4).

**Theorem 8.1.** *Let $G$ be a connected Lie group endowed with a left-invariant Riemannian metric. Then the following are equivalent*

(1) *$G$ quasi-isometrically embeds into a CAT(0) metric space;*
(2) *$G$ quasi-isometrically embeds into a non-positively curved simply connected symmetric space;*
(3) *$G/\mathrm{R}_{\exp}G$ is quasi-reductive.*

*If moreover $G$ is simply connected and solvable, these are also equivalent to:*

(2') *$G$ bilipschitz embeds into a non-positively curved simply connected symmetric space;*

In the essential case when $G$ is simply connected and triangulable, (3) is equivalent to: "$G/\mathrm{R}_{\exp}G$ is abelian" and we get the statement of Theorem 1.10.

The following lemma may be of independent interest.

**Lemma 8.2.** *Let $G$ be a simply connected solvable Lie group. Consider any embedding $i : G \to S$, with closed image, into a solvable Lie group $S$. Denote by $p$ the quotient morphism $G \to G/\mathrm{R}_{\exp}G$. Then the product morphism $i \times p : G \to S \times G/\mathrm{R}_{\exp}G$ is a bilipschitz embedding (where groups are endowed with left-invariant Riemannian metrics).*

*Proof.* As a group morphism, $i \times p$ is Lipschitz; as it is injective it is locally bilipschitz; so it suffices to prove that $i \times p$ is a quasi-isometric embedding. Endow our groups with their word length relative to suitable compact generating subsets.

Suppose that $|i \times p(g)| \leq n$. Then $|p(g)| \leq n$. Thus $p(g) = p(s_1) \dots p(s_n)$ with $s_i \in W$, our compact generating subset of $G$. Set $h = g(s_1 \dots s_n)^{-1}$. Then $|g| \leq |h| + n$, $h \in \mathrm{R}_{\exp}G$, and $|i(h)| \leq 2n$.

Now $|h| \leq C_1 \log(\| \log(h) \|) + C_1$ by Lemma 8.3. The Lie algebra map induced by $i$ is linear injective, therefore bilipschitz. Therefore, $|h| \leq C_1 \log(\| \log(i(h)) \|) + C_2$. It follows, using Lemma 8.3 again, that $|h| \leq C_3 |i(h)| + C_3 \leq C_3(2n+1)$. Thus $|g| \leq (2C_3 + 1)n + C_3$ and we are done. □

**Lemma 8.3.** *Let $G$ be a simply connected solvable Lie group and $N$ its exponential radical. Take $H \in \mathfrak{n}$, the Lie algebra of $N$, and $h = \exp(H)$. Then we have, for constants independent of $H$*

$$C \log(\|H\| + 1) - C \leq |h| \leq C' \log(\|H\| + 1) + C'.$$

*Proof.* Let $|\cdot|_N$ denote the intrinsic word length in $N$. Then by Guivarc'h-Osin's Theorem (Theorem 6.1(2)),

$$C_1 \log(|h|_N + 1) - C_1 \leq |h| \leq C_1' \log(|h|_N + 1) + C_1'.$$

On the other hand, as a consequence of [Gui1, Lemma II.1] we have,

$$C_2 \|H\|^{1/d} - C_2 \leq |h|_N \leq C_2' \|H\| + C_2' \quad \square$$

*Proof of Theorem 8.1.* (2')⇒(2)⇒(1) is trivial.

First suppose $G$ simply connected solvable. Then (3)⇒(2') follows from Lemma 8.2, where we choose $S$ as the upper triangular matrices in some $\mathrm{SL}_n(\mathbf{C})$.

Finally if (1) is satisfied, then the asymptotic cone of $G$ embeds bilipschitz into a CAT(0) space. But the asymptotic cone of $G$ contains by Theorem 1.8 a bilipschitz embedded copy of the asymptotic cone of $G/\mathrm{R}_{\exp}G$, which embeds quasi-isometrically into a CAT(0) space only if the nilshadow of $G/\mathrm{R}_{\exp}G$ is abelian by Pauls' Theorem [Pau].

The general case follows from the following observation: let $G$ be any connected Lie group. Write $G/\mathrm{R}_{\exp}G = SA$, where $A$ is its amenable radical and $S$ is semisimple without compact factors. Then $G/\mathrm{R}_{\exp}G$ has a cocompact closed subgroup of the form $R \times \mathbf{Z}^n \times A$, where $R$ is a cocompact subgroup of $S/Z(S)$ that is simply connected and solvable. As $R/\mathrm{R}_{\exp}R$ is abelian, we see that $G/\mathrm{R}_{\exp}G$ is quasi-reductive if and only if $A$ is quasi-abelian.

Therefore, if $G$ satisfies (3), $A$ is quasi-abelian and accordingly $G$ is quasi-isometric to $H = R \times \mathbf{Z}^n \times B/K$, where $B, K$ are normal closed subgroups of $A$ with $B$ solvable cocompact, $K$ compact, and $B/K$ simply connected. Now $H$ is solvable simply connected and $H/\mathrm{R}_{\exp}H$ is quasi-abelian, so $H$ and hence $G$ satisfies (2).

Conversely if $G$ satisfies (1), then so does $H$, so by the solvable simply connected case, $H/\mathrm{R}_{\exp}H$ is quasi-abelian. This means that $A$ is quasi-abelian, and therefore $G/\mathrm{R}_{\exp}G$ is quasi-reductive. □

## 9. On the cone of SOL

We prove here the observation that the asymptotic cone of SOL is bilipschitz homeomorphic to a "Diestel-Leader **R**-graph" namely the "hypersurface" of equation $b(x)+b(y)=0$ in the product of two **R**-trees with Busemann function $b$.

First note that on the group $\mathbf{R}_+^* \ltimes \mathbf{R}$ with group law $(t,x)(u,y)=(tu,ux+y)$, a Busemann function is given by the function $b(t,x)=\log(t)$. Thus SOL can be viewed as the hypersurface in $(\mathbf{R}_+^* \ltimes \mathbf{R})^2$ with equation $b(g)+b(h)=0$. It is well-known and easy to check that this embedding of SOL into $(\mathbf{R}_+^* \ltimes \mathbf{R})^2$ is quasi-isometric. The group $\mathbf{R}_+^* \ltimes \mathbf{R}$ is quasi-isometric to the hyperbolic plane and therefore all of its asymptotic cones are isometric to the "universal" homogeneous **R**-tree of degree $2^{\aleph_0}$ [DyPo].

On the other hand, $b$ defines a Busemann function $\tilde{b}$ on the asymptotic cone $\mathrm{Cone}(\mathbf{R}_+^* \ltimes \mathbf{R}, \omega)$, defined as

$$\tilde{b}((g_n)) = \lim_{\omega} b(g_n)/n.$$

So the embedding above induces a bilipschitz embedding of $\mathrm{Cone}(\mathrm{SOL},\omega)$ into $\mathrm{Cone}(\mathbf{R}_+^* \ltimes \mathbf{R}, \omega)^2$, with image in the "hypersurface" with equation $\tilde{b}(g)+\tilde{b}(h)=0$.

This embedding actually maps *onto* this "hypersurface". Indeed, let $((t_n,x_n),(t'_n,x'_n))$ belong to this hypersurface. By definition, we have $\lim_\omega \log(t_n t'_n)/n = 0$. It follows that in the asymptotic cone, $(t'_n,x'_n)$ coincides with

$$(t'_n,x'_n)(t_n t'_n,0)^{-1} = (t_n^{-1}, t_n^{-1}{t'_n}^{-1}x'_n).$$

Now the pair $((t_n,x_n),(t_n^{-1}, t_n^{-1}{t'_n}^{-1}x'_n))$ clearly belongs to the image of the embedding above.

The proof for other groups is similar. For instance, for the Baumslag-Solitar group $\mathbf{Z} \ltimes_n \mathbf{Z}[1/n]$, the proof goes as follows: let $\mathbf{K}$ be the direct product of all $\mathbf{Q}_p$, where $p$ ranges over distinct prime divisors of $n$. Then $\mathbf{Z} \ltimes_n \mathbf{Z}[1/n]$ is a cocompact lattice in $\mathbf{Z} \ltimes_n (\mathbf{R}\times\mathbf{K})$ and therefore has the same cone, and for the latter group, the description of the cone is similar, making use of the diagonal embedding into the product of two Gromov hyperbolic locally compact groups

$$(\mathbf{Z} \ltimes_n \mathbf{R}) \times (\mathbf{Z} \ltimes_n \mathbf{K}).$$

The case of the lamplighter group $\mathbf{Z}/n\mathbf{Z} \wr \mathbf{R}$ is also similar and makes use of the embedding into

$$\mathbf{Z} \ltimes_t [\mathbf{Z}/n\mathbf{Z}((t)) \times \mathbf{Z}/n\mathbf{Z}((t^{-1}))].$$

The non-unimodular deformations of SOL also have the same asymptotic cone (up to bilipschitz homeomorphism). Indeed, such a group is a semidirect product $\mathbf{R} \ltimes \mathbf{R}^2$, where $\mathbf{R}$ acts by the one-parameter subgroup of diagonal matrices with diagonal entries $e^{-\lambda t}$ and $e^{\mu t}$ with $0<\lambda<\mu$. Arguing as in the case of SOL, we obtain a natural identification of the asymptotic cone with the "hypersurface" of equation

$$\lambda b(g) + \mu b(h)$$

in the product of two **R**-trees. Now, there exists a similarity of the **R**-tree mapping $b$ to $(\mu/\lambda)b$. Therefore this hypersurface is bilipschitz homeomorphic to the hypersurface of equation $b(g)+b(h)=0$.

Observe that in the case when $\lambda<0<\mu$, we would get a Gromov hyperbolic Lie group and arguing as above we obtain that the asymptotic cone is bilipschitz homeomorphic to the hypersurface of equation $b(g) - b(h) = 0$, which is duly an **R**-tree.

For a general semidirect product $G = D \ltimes \mathbf{R}^n$ where $D \ltimes \mathbf{R}^k$ is a closed connected subgroup of the group of diagonal matrices, we can proceed as follows: removing an abelian direct factor if necessary, we can suppose that $D$ fixes no nonzero vector in $\mathbf{R}^n$. Then, let $H_i$ be the hypersurface of equation $(x_i = 0)$ in $\mathbf{R}^n$ and $D_i$ the hypersurface of equation $(d_i = 1)$ in $D$. Then $G/(D_i \ltimes H_i)$ is isomorphic to the affine group $\mathbf{R} \ltimes \mathbf{R}$. The diagonal mapping of $G$ into the product of all $G/(D_i \ltimes H_i)$ is a quasi-isometric embedding.

Now we can define $D$ by a system of linear relations between logarithms of diagonal entries: $\sum_j a_{ij} \log(d_j) = 0$, $i = 1 \dots k$. Then the asymptotic cone of $G$ can be described as the "variety" defined by the equation $\sum_j a_{ij} b(g_j) = 0$ in the product of $n$ **R**-trees. The study *per se* of such "varieties" is likely to shed light on the geometry of these groups.

**Acknowledgments.** I thank E. Breuillard, A. Eskin, P. Pansu, and especially R. Tessera for valuable discussions.

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