NONLINEARITY OF SOME SUBGROUPS OF THE PLANAR CREMONA GROUP

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Abstract. We give some examples of nonlinear subgroups of the planar Cremona group.

1. Introduction

Let K be a field. The planar *Cremona group* $\mathrm{Cr}_2(K)$ of K in dimension d is defined as the group of birational transformations of the 2-dimensional K-affine space. It can also be described as the group of K-automorphisms of the field of rational functions $K(t_1, t_2)$.

We provide here two observations about the planar Cremona group. The first is an example of a non-linear finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$. The existence of such a subgroup is not new, for instance it follows from an unpublished construction of S. Cantat; our example has the additional feature of being 3-solvable. Its non-linearity follows from the fact it contains nilpotent subgroups of arbitrary large nilpotency length. We also show there that $\mathrm{Cr}_2(K)$ has no nontrivial linear representation over any field, extending a result of Cerveau and Déserti.

We end this short introduction by a few questions, which I heard from Serge Cantat.

(1) for $d \geq 2$, and any field K, is $\mathrm{Cr}_d(K)$ locally residually finite (i.e. is every finitely generated subgroup residually finite)?
(2) Does there exist a finitely generated subgroup of $\mathrm{Aut}(\mathbf{C}^2)$ with no faithful linear representation (see Remark 2.3)?

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2. A nonlinear subgroup of the Cremona group

We provide in this section an example of a finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$ that is not linear over any field. It is 3-solvable and actually lies in the Jonquières subgroup, that is, the group of birational transformations preserving the partition of $\mathbf{C}^2$ by horizontal lines.

If $f \in K(X)$ and $g \in K(X)^\times$, define $\alpha_f, \mu_g \in \mathrm{Cr}_2(K)$ by

$$\alpha_f(x, y) = (x, y + f(x)); \quad \mu_g(x, y) = (x, yg(x)).$$

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We have

$$\alpha_{f+f'} = \alpha_f \alpha_{f'}; \quad \mu_{gg'} = \mu_g \mu_{g'}; \quad \mu_g \alpha_f \mu_g^{-1} = \alpha_{fg}.$$

Also for $t \in K$, define $s_t \in \mathrm{Cr}_2(K)$ by $(x, y) = (x + t, y)$, so that

$$s_t \alpha_{f(X)} s_t^{-1} = \alpha_{f(X-t)}; \quad s_t \mu_{g(X)} s_t^{-1} = \mu_{g(X-t)}.$$

Consider the subgroup Γ_n of $\mathrm{Cr}_2(K)$ generated by s_1 and α_{X^n} $(n \geq 0)$.

Lemma 2.1. *The group Γ_n is nilpotent of class at most $n + 1$; moreover if K has characteristic zero the nilpotency length of Γ_n is exactly $n + 1$, and Γ_n is torsion-free.*

Proof. Consider the largest group R_n, generated by s_1 and by the abelian subgroup A_n consisting of all α_P, where P ranges over polynomials of degree at most n. Then A_n is normalized by s_1 and $[s_1, A_n] \subset A_{n-1}$ for all $n \geq 1$, while $A_0 = \{1\}$. Therefore R_n is nilpotent of class at most $n+1$, and therefore so is Γ_n. Conversely, the n-iterated group commutator $[s_1, [s_1, \cdots, [s_1, \alpha_{X^n}] \cdots]]$ is equal to $\alpha_{\Delta^n X^n}$, where Δ is the discrete differential operator $\Delta P(X) = -P(X) + P(X - 1)$. So if K has characteristic zero (or $p > n$) then $\Delta^n X^n \neq 0$ and Γ_n is not n-nilpotent. In this case it is also clear that R_n is torsion-free. □

Now assume that K has characteristic zero and consider the group $G \subset \mathrm{Cr}_2(\mathbf{Q}) \subset \mathrm{Cr}_2(K)$ generated by $\{s_1, \alpha_1, \mu_X\}$.

Proposition 2.2. *The group $G \subset \mathrm{Cr}_2(\mathbf{Q})$ is solvable of length three; it is not linear over field.*

Proof. From the conjugation relations above it is clear that the subgroup generated by s_1, all α_f and μ_g, is solvable of length at most three. If we restrict to those g of the form $\prod_{n \in \mathbf{Z}} (X - n)^{k_n}$ (where (k_n) is finitely supported), we obtain a subgroup containing Γ, that is clearly torsion-free.

Since $\mu_X^n \alpha_1 \mu_X^{-n} = \alpha_{X^n}$, we see that G contains Γ_n for all n, which is nilpotent of length exactly $n + 1$. Therefore it has no linear representation over any field.

[Sketch of proof of the latter (well-known) result: in characteristic $p > 0$, any torsion-free nilpotent subgroup is abelian, so this discards this case. Otherwise in characteristic zero, since any finite index subgroup of a torsion-free nilpotent group of nilpotency length $n+1$ still has nilpotency length $n+1$, the existence of a linear representation of G into $\mathrm{GL}_d(\mathbf{C})$ implies the existence of a Lie subalgebra of $\mathfrak{gl}_d(\mathbf{C})$ of nilpotency length $n + 1$ for all n; this necessarily implies $n + 1 \leq d^2$, and since n is unbounded this is a contradiction.]

The fact that G is not 2-solvable (=metabelian) can be checked by hand, but also follows from the fact that every torsion-free finitely generated metabelian group is linear over a field of characteristic zero [Re]. □

With little further effort, it actually follows from the same argument that G is not linear over any finite product of fields (and therefore over any reduced commutative ring): indeed at least one of the projections should contain torsion-free nilpotent subgroups of arbitrary large nilpotency length.

Remark 2.3. It is unknown whether there exists a finitely generated subgroup of the group $\mathrm{Aut}(\mathbf{C}^2)$ of *polynomial automorphisms* of $\mathbf{C}^2$, that is not linear in characteristic zero. A construction in the same fashion does not work: indeed let E be the group of elementary automorphisms, namely of the form $(x, y) \mapsto (\alpha x + P(y), \beta y + c)$ for $(\alpha, \beta, c, P) \in \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}[X]$. Then, although E is not linear (since by the argument above, it contains all Γ_n), every finitely generated subgroup of E is linear over $\mathbf{C}$.

To see this, write E as a semidirect product $(\mathbf{C}^* \times (\mathbf{C}^* \ltimes \mathbf{C})) \ltimes \mathbf{C}[X]$, where the action on $\mathbf{C}[X]$ is by $(\alpha, \beta, c) \cdot P(X) = \alpha P(\beta X + c)$. In particular, this action stabilizes the subgroup $\mathbf{C}_n[X]$ of polynomials of degree at most n. Therefore any finitely generated subgroup of E is contained in the subgroup $(\mathbf{C}^* \times (\mathbf{C}^* \ltimes \mathbf{C})) \ltimes \mathbf{C}_n[X]$ for some $n \geq 1$. This is a (finite-dimensional) complex Lie group whose center is easily shown to be trivial, so its adjoint representation is a faithful complex linear representation.

A nice observation by Cerveau and Déserti [CD, Lemme 5.2] is that the Cremona group has no faithful linear representation in characteristic zero. Actually, an easy refinement of the same argument provides a stronger result.

Proposition 2.4. *If K is an algebraically closed field, there is no nontrivial finite-dimensional linear representation of* $\mathrm{Cr}_2(K)$ *over any field.*

(Note that since the Cremona group is not simple by a recent difficult result of Cantat and Lamy [CL], the non-existence of a faithful representation does not formally imply the non-existence of a nontrivial representation.)

Proof of Proposition 2.4. In $\mathrm{Cr}_2(K)$, there is a natural copy of $G = (K^\times)^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by the automorphism $\sigma(x, y) = (x, xy)$ of $(K^\times)^2$. Here, it corresponds, in affine coordinates, to the group of transformations of the form

$$(x_1, x_2) \mapsto (\lambda_1 x_1, x_1^n \lambda_2 x_2) \quad \text{for} \quad (\lambda_1, \lambda_2, n) \in (K^\times)^2 \times \mathbf{Z}.$$

Consider an linear representation $\rho : G \to \mathrm{GL}_n(F)$, where F is any field (here G is viewed as a discrete group). If p is a prime which is nonzero in K and if $\omega_p \in K$ is a primitive p-root of unity, set $\alpha_p(x_1, x_2) = (\omega_p x_1, \omega_p x_2)$ and $\beta_p(x_1, x_2) = (x_1, \omega_p x_2)$. Then $\sigma \alpha_p \sigma^{-1} \alpha_p^{-1} = \beta_p$ and commutes with both σ and α_p. An argument of Birkhoff [Bi, Lemma 1] shows that if $\rho(\alpha_p) \neq 1$ then $n > p$ (the short argument given in the proof of [CD, Lemme 5.2] for F of characteristic zero works if it is assumed that p is not the characteristic of F).

Picking p to be greater than n and the characteristics of K and F, this shows that if we have an arbitrary representation $\pi : \mathrm{Cr}_2(K) \to \mathrm{GL}_n(F)$, the restriction of π to $\mathrm{PGL}_3(K)$ is not faithful; since $\mathrm{PGL}_3(K)$ is simple, this implies that π is trivial on $\mathrm{PGL}_3(K)$; since $\mathrm{Cr}_2(K)$ is generated by $\mathrm{PGL}_3(K)$ as a normal subgroup, this yields the conclusion. □

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