REALIZATIONS OF GROUPS OF PIECEWISE CONTINUOUS TRANSFORMATIONS OF THE CIRCLE

YVES CORNULIER

Abstract. We study the near action of the group PC of piecewise continuous self-transformations of the circle. Elements of this group are only defined modulo indeterminacy on a finite subset, which raises the question of realizability: a subgroup of PC is said to be realizable if it can be lifted to group of permutations of the circle.

We show that every finitely generated abelian subgroup of PC is realizable.

We show that this is not true for arbitrary subgroups, by exhibiting a non-realizable finitely generated subgroup of the group of interval exchanges with flips.

The group of (oriented) interval exchanges is obviously realizable (choosing the unique left-continuous representative). We show that it has only two realizations (up to conjugation by a finitely supported permutation): the left and right-continuous ones.

1. Introduction

1.1. **Context.** We deal with various groups of piecewise continuous transformations in dimension 1. The best known is the group of piecewise translations, better known as group of interval exchange transformations. Interval exchanges were introduced by Keane [Ke]; they have mostly been studied individually. Its study as a group notably starts in the determination by Arnoux-Fathi and Sah of its abelianization [Ar]. Its study has been recently pursued, notably in work by C. Novak (e.g., [Nov09]), Dahmani-Fujiwara-Guirardel [DFG, DFG2], and Boshernitzan [Bo]. Two outstanding problems about this group is whether it admits non-abelian free subgroups, a question attributed to A. Katok, and whether it is amenable [Cor1]. Recent progress on this latter question is due to Juschenko-Monod [JM] and then the paper of Juschenko, la Salle, Matte Bon and Monod [JMMS]. If we allow flips, we obtain a larger group, which is seldom studied, and usually not precisely defined. The questions of realizability, which we consider here, do not seem to have been considered, notably because defining interval exchanges with flips as a group is usually swept under the carpet.

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1.2. **Set-up.** Let $\mathbf{S}$ be the circle $\mathbf{R}/\mathbf{Z}$. We say that $f : \mathbf{S} \to \mathbf{S}$ is piecewise continuous if it is continuous outside a finite subset.

Let $\widehat{\mathrm{PC}}(\mathbf{S})$ be the group of piecewise continuous permutations of $\mathbf{S}$ that are continuous outside a finite subset (it is indeed stable under inversion, by an easy argument). It includes the group of finitely supported permutations $\mathfrak{S}_{\mathrm{fin}}(\mathbf{S})$, which is a countable normal subgroup, and we define $\mathrm{PC}(\mathbf{S})$ as the quotient $\widehat{\mathrm{PC}}(\mathbf{S})/\mathfrak{S}_{\mathrm{fin}}(\mathbf{S})$. Denote by π the quotient homomorphism $\widehat{\mathrm{PC}}(\mathbf{S}) \to \mathrm{PC}(\mathbf{S})$.

Thus, $\mathrm{PC}(\mathbf{S})$ (which we also denote $\mathrm{PC}^{\bowtie}(\mathbf{S})$) is the group of all piecewise continuous permutations of $\mathbf{S}$, up to finite indeterminacy. Let $\mathrm{PC}^{+}(\mathbf{S})$ be its subgroup of piecewise orientation-preserving elements. Let $\mathrm{IET}^{\bowtie}(\mathbf{S})$ be the subgroup of $\mathrm{PC}(\mathbf{S})$ consisting of piecewise isometric elements (also called group of interval exchanges with flips), and $\mathrm{IET}^{+}(\mathbf{S}) = \mathrm{PC}^{+}(\mathbf{S})) \cap \mathrm{IET}^{\bowtie}(\mathbf{S})$, the subgroup of piecewise translations, usually called group of interval exchanges.

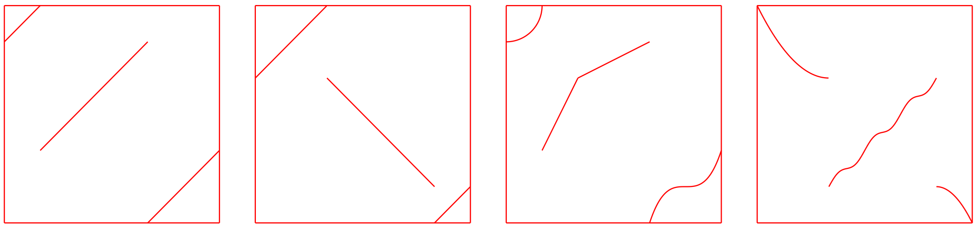

FIGURE 1. Parameterizing the circle as an interval, examples of elements of $\mathrm{PC}(\mathbf{S})$. The first belongs to IET; the second belongs to $\mathrm{IET}^{\bowtie}$; the third to PC^{+}, and the fourth belongs to none of these subgroups. The value at breakpoints is not prescribed, as elements are considered up to finite indeterminacy.

Given a group Γ, we call **piecewise continuous circle near action** on $\mathbf{S}$ a homomorphism $\Gamma \to \mathrm{PC}(\mathbf{S})$.

Definition 1.1. We say that a subgroup $\Gamma \subset \mathrm{PC}(\mathbf{S})$ is **realizable** if it can be lifted to $\widehat{\mathrm{PC}}(\mathbf{S})$, i.e., if there exists a subgroup Λ of $\widehat{\mathrm{PC}}(\mathbf{S})$ such that $\pi|_{\Lambda}$ is bijective.

More generally, a piecewise continuous near action of a group Γ on $\mathbf{S}$ means a homomorphism $\Gamma \to \mathrm{PC}(\mathbf{S})$. We call it **realizable** if it can lifted to a homomorphism $\Gamma \to \widehat{\mathrm{PC}}(\mathbf{S})$.

Let $\mathrm{PC}^{+}(\mathbf{S})$ the subgroup of $\mathrm{PC}(\mathbf{S})$ consisting of those piecewise orientation-preserving transformations. Note that unlike in the continuous analogues, it does not have index two, and actually $\mathrm{PC}^{+}(\mathbf{S})$ has infinite index and is not normal in $\mathrm{PC}(\mathbf{S})$.

Then $\mathrm{PC}^{+}(\mathbf{S})$ is realizable. Indeed, we can lift f to its unique left-continuous representative. While taking the unique left-continuous representative makes sense for all $f \in \mathrm{PC}^{+}(\mathbf{S})$, this unique lift is bijective when $f \in \mathrm{PC}^{+}(\mathbf{S})$, but not in general. Actually, it has a realizable overgroup $\mathrm{PC}^{\pm}(\mathbf{S}) = \mathrm{PC}^{+}(\mathbf{S}) \sqcup \mathrm{PC}^{-}(\mathbf{S})$ of index 2, where the non-trivial coset $\mathrm{PC}^{-}(\mathbf{S})$ consists of those piecewise orientation-reversing homeomorphisms; any f in this coset should then be lifted to its unique right-continuous representative.

That $\mathrm{PC}^{+}(\mathbf{S})$ is realizable makes it (and its subgroups) easier to define, since one can refer to piecewise continuous, left-continuous permutations of the circle. This artifact makes the definition shorter (since one does not have to mod out finitely supported permutations), and often explains the restriction to the piecewise orientation-preserving case, in many settings where this is not really used.

1.3. **Non-realizability and restriction results.** The first main result of this paper is a non-realizability theorem.

Theorem 1.2. *The group* $\mathrm{PC}(\mathbf{S})$ *is not realizable. More precisely, its subgroup of interval exchanges with flips (=piecewise isometries) is not realizable, and even has a finitely generated subgroup that is not realizable.*

Actually, the result also holds, with some technical cost, with the stronger conclusion "not stably realizable", which is a more natural notion, see §2; rather than defining it in this introduction, let us pinpoint that it is equivalent to the assertion that, for every nonempty open interval, the group of interval exchanges with flips that induce the identity on the complement of I, is not realizable. The latter can be viewed as the group of interval exchanges with flips of I. So the failure of stable realizability means that even making use of those additional points in the complement, does not allow to realize the action.

Our approach also provides, with further work, a result in the piecewise orientation-preserving case. For a subgroup Γ of $\mathrm{PC}^{+}(\mathbf{S})$, we denote by Γ_{left} (respectively Γ_{right}) the group of left-continuous (resp. right-continuous) representatives of elements of Γ.

Theorem 1.3. *Modulo conjugation by finitely supported permutations, the only subgroups lifting* $\Gamma = \mathrm{IET}^{+}$ *are* Γ_{left} *and* Γ_{right}. *The same conclusion holds when* Γ *is:*

- *any subgroup of* $\mathrm{PC}^{+}(\mathbf{S})$ *that includes* IET^{+}*;*
- *for any subgroup of rotations of* $\mathbf{Q}$*-rank* ≥ 2, *the subgroup* $\mathrm{IET}^{+}_{\Lambda}$ *of interval exchanges with singularities and translation lengths in* Λ.

1.4. **Realizability results.** Realizability makes sense in a much more general setting (near actions on sets), see §2. Finite groups are always realizable, by a very general and easy argument (see [Cor3]). This is also true for free subgroups of the quotient by finitely supported permutations, obviously. On the other hand, this is not true for general near actions of $\mathbf{Z}^2$, as a variety of examples in [Cor3] show. More precisely, it is easy to find two permutations of a set that commute as near permutations (i.e., their commutator is finitely supported), but they cannot be perturbed (i.e., multiplied by finitely supported permutations) so that the resulting permutations commute. Nevertheless, we show here that such phenomena cannot arise in the context of piecewise continuous near actions on the circle.

Theorem 1.4. *Any finitely generated abelian subgroup of* $\mathrm{PC}(\mathbf{S})$ *is realizable.*

The proof of Theorem 1.4 is not direct. It makes use the fact that the near action on the circle can be viewed as a projection of a genuine action, namely obtained by doubling all points (the Denjoy blow-up).

Let us also mention that the near action of $\mathrm{PC}(\mathbf{S})$ is completable, in the sense that there exists an action on a set X (some huge non-Hausdorff connected compact 1-dimensional manifold), in which $\mathbf{S}$ sits as a commensurated subset, so that the induced near action on $\mathbf{S}$ is the given one. This observation comes from [Cor2].

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2. Preliminaries on general near actions

Convention (contain/include): on the one hand $a \in X$ is written: a **belongs** to X, or X **contains** a; on the other hand $X \subset Y$ is written X is **included** in Y, or Y **includes** X.

2.1. **Basic facts.** By "cofinite subset" we mean "subset with cofinite complement". Essentially following Wagoner [Wa, §7] (see also [Cor3] for a detailed historical account), a cofinite-partial bijection of a set X is a bijection between two cofinite subsets of X. If we identify any two such cofinite-partial bijections when they coincide on a cofinite subset, we obtain **near symmetric group** of X, denoted by $\mathfrak{S}^\star(X)$. Its elements, namely cofinite-partial bijections modulo cofinite coincidence, are called **near permutations** of X. There is an canonical homomorphism from the group $\mathfrak{S}(X)$ of permutations of X to $\mathfrak{S}^\star(X)$. Its kernel is the subgroup of finitely supported permutations. Its image $\mathfrak{S}^\star_0(X)$ consists by definition of **balanced near permutations** and is called the **balanced symmetric group** of X. Given a cofinite-partial bijection $f : X \smallsetminus F_1 \simeq X \smallsetminus F_2$, the number $|F_2|-|F_1|$ is called the **index** $\phi_X(f)$ of f. The index map factors through a group homomorphism $\phi_X : \mathfrak{S}^\star(X) \to \mathbf{Z}$, called **index homomorphism**, whose kernel is precisely $\mathfrak{S}^\star_0(X)$. If X is infinite, the index homomorphism ϕ_X is surjective, so that the cokernel $\mathfrak{S}^\star(X)/\mathfrak{S}^\star_0(X)$ is infinite cyclic.

Definition 2.1 ([Cor3])**.** A **near action** of a group G on a set X is the datum of a homomorphism $\alpha : G \to \mathfrak{S}^\star(X)$; then X is called a **near G-set**. The near action is said to be **balanced** if it is valued in $\mathfrak{S}^\star_0(X)$, or equivalently if the **index homomorphism** $\phi_X \circ \alpha \in \mathrm{Hom}(G, \mathbf{Z})$ of the near action α is zero.

While the notion of invariant subset for a group action does not pass to near actions, we have a notion of commensurated subset. Namely, given a near action of G on X, a subset $Y \subset X$ is G-**commensurated** if for every $g \in G$, there exists a representative of g (a cofinite-partial bijection) mapping Y into Y; then the group G near acts, by restriction, on Y. Thus, G-commensurated subsets are the same as near G-subactions.

For subsets U, V of a set X, we write $U \stackrel{\star}{=} V$ and say that U and V are near equal if $U \triangle V$ is finite. We write $U \subset^\star V$ and say that U is near included in V if $U \smallsetminus V$ is finite: thus $U \stackrel{\star}{=} V$ if and only if both $U \subset^\star V$ and $V \subset^\star U$. For maps $f, g : X \to Y$, we write $f \stackrel{\star}{=} g$ if f and g coincide outside a finite subset: this means that among subsets of $X \times Y$, the graphs of f and g are near equal.

A map $f : X \to Y$ (with cofinite domain of definition) between near G-sets is said to be **near G-equivariant** if for every $g \in G$, the set of $x \in X$ such that $f(gx) = gf(x)$ is cofinite in X. Note that we choose here, for given g, self-maps $x \mapsto gx$ and $y \mapsto gy$ of X and Y; the condition does not depend on these choices. The map f is said to be a **near isomorphism** if there exists another such near G-equivariant map $f' : Y \to X$ such that $f \circ f' \stackrel{\star}{=} \mathrm{Id}_Y$ and $f' \circ f \stackrel{\star}{=} \mathrm{Id}_X$; in this case X and Y are said to be **near isomorphic** near G-sets.

Definition 2.2. A near action of a group G on a set X is:

- **realizable** if the homomorphism $G \to \mathfrak{S}^\star(X)$ comes from a homomorphism $G \to \mathfrak{S}(X)$;

- **[finitely] stably realizable** if there exists a [finite] set Y with trivial near action such that the near action on $X \sqcup Y$ is realizable;
- **completable** if there exists a near G-set Y such that the near action on $X \sqcup Y$ is realizable.

Obviously, realizable implies finitely stably realizable, which implies stably realizable, which implies completable. Examples in [Cor3] show that none of the converse implications holds, and that there exist non-completable near actions. All these examples are taken with $G = \mathbf{Z}^2$, except for the difference between stably realizable and finitely stably realizable: indeed it is established in [Cor3] that these two notions are equivalent for finitely generated groups.

Let G be a finitely generated group and X a near G-set. Fix a finite generating subset S of G, and lift each $s \in S$ to cofinite-partial bijection $x \mapsto sx$ of X. The corresponding **near Schreier graph** consists in joining x to sx for all $s \in S$. The near action is said to be **of finite type** if the near Schreier graph has finitely many components. Routine arguments in [Cor3] show that this does not depend on the choices.

The following is established in [Cor3].

Theorem 2.3. *Let A be a finitely generated abelian group. Let X be a near A-set. Equivalences:*

- *X is finitely stably realizable;*
- *X is completable, and the index homomorphism of the near G-subset X^B of B-fixed poins is zero for every subgroup B of A.*

Note that the A-commensurated subset X^B is defined up to $\stackrel{\star}{=}$, and hence whether it is balanced is a well-defined notion.

Proof. This is proved in [Cor3]; for completeness we include the proof. The forward implication is clear; suppose that the condition holds. Since every near action of A is disjoint union of a realizable one and a finite type one, we can suppose that X has finite type. This latter fact, holding for near actions of arbitrary finitely presented group is proved in [Cor3]; however we will only use Theorem 2.3 in a case where X is already assumed to be of finite type (in the proof of Theorem 2.12, when applying Lemma 2.4).

So, suppose that X has finite type and satisfies the second condition. Since every transitive A-set is finitely-ended, and X is completable, X is finitely ended.

For each subgroup B of A such that A/B has $\mathbf{Q}$-rank 1, the number of homomorphisms of A/B onto $\mathbf{Z}$ is 2, and we choose one of the two as u_B. Then the classification of 1-ended completable near A-sets up to near isomorphism is as follows: it consists of those $E_B = A/B$ when B ranges over subgroups of A such that A/B has $\mathbf{Q}$-rank ≥ 2, those $E_B^+ = u_B^{-1}(\mathbf{N})$ and $E_B^- = u_B^{-1}(-\mathbf{N})$ when B ranges over subgroups of A such that A/B has $\mathbf{Q}$-rank 1.

Let B be maximal among subgroups such that X^B is infinite. Then X^B satisfies the balanced assumption as well as X, and hence so does $X \smallsetminus X^B$. Hence, if

$X \smallsetminus X^B$ is infinite, we can argue by induction on the number of ends to deduce that X is stably realizable.

Otherwise, X^B is cofinite in X, so X is a near free near A/B-set. If A/B has $\mathbf{Q}$-rank ≥ 2, this implies that X is near isomorphic to some disjoint union of copies of X_B, and hence is stably realizable. If A/B has $\mathbf{Q}$-rank 1, this implies that X is near isomorphic to the disjoint union of k copies of E_B^+ and ℓ copies of E_B^-. Note that the index homomorphism is additive under disjoint unions, and is opposite and nonzero for E_B^+ and E_B^-. That the index homomorphism of X vanishes then implies that $k = \ell$. So X is near isomorphic to the disjoint union of k copies of A/B, so is finitely stably realizable. □

Lemma 2.4. *Let A be a finitely generated abelian group, and k a positive integer. Let X be a near A-set. If the disjoint union kX of k copies of X is finitely stably realizable, then so is X.*

Proof. The assumption implies that X is completable. Then the condition that X^C is balanced is equivalent to the condition that $(kX)^C$ is balanced. Hence the criterion of Theorem 2.3 holds for X if and only if it holds for kX. Thus, kX stably realizable implies that X is stably realizable. □

Remark 2.5. In contrast, in [Cor3], an example of a finitely generated group is given (some amalgam of two finite groups) for which there exists a near G-set X that is not stably realizable, such that $X \sqcup X$ is realizable.

2.2. **Facts involving near equivariant maps.** The purpose of this part is Theorem 2.12, which gives a general realizability results for near actions of finitely generated abelian groups, which will be applied to Theorem 1.4 in §3.

The following is a particular case of [Cor3, Corol. 7B3].

Theorem 2.6. *Let A be a finitely generated abelian group of $\mathbf{Q}$-rank ≥ 2. Let B be a subgroup of G. Let $f : A \to A/B$ be a map such that for every $g \in A$, the set of $g' \in A$ such that $f(gg') \neq gf(g')$ is finite. Then there exists a unique G-equivariant map $f' : A \to A/B$ such that $f \overset{\star}{=} f'$.*

Proof. This is proved in [Cor3], but let us provide the easy proof. Since an equivariant map f' has the form $g \mapsto gx$ with $x = f'(1)$, the uniqueness is clear. Now let us show the existence. Write $u(g) = g^{-1}f(g)$. Then for every $s \in A$, the set of $g \in A$ such that $u(sg) \neq u(g)$ is finite. Fix a finite generating subset S of A, and consider the left Cayley graph, joining g to sg for $s \in S$ and $g \in A$. For $s \in S$, erase those edges joining g to sg whenever $u(g) \neq u(sg)$: this erases finitely many edges, and u is constant on each component of the resulting graph. Since A is 1-ended, the resulting graph has a single infinite component: this is a cofinite subset on which u is constant, say equal to x. Thus $u(g) = x$ for all but finitely many $g \in G$. This means that $f(g) = gx$ for all but finitely many $g \in G$. □

Let G be a group and X a near G-set. For $g \in G$, the subset X^g of fixed points of g is defined modulo $\stackrel{\star}{=}$. By extension, for a finitely generated subgroup H of G, the subset X^H of fixed points of H is defined modulo near equality $\stackrel{\star}{=}$.

Recall that a map between sets is **proper** if it has finite fibers.

Lemma 2.7. *Let G be a finitely generated group, and let X, Y be near G-sets. Let $f : X \to Y$ be a proper, near equivariant map. Then $X^G \subset^\star f^{-1}(Y^G)$. If moreover, for every g, the set of x such that $f(gx) = f(x)$ and $gx \neq x$ is finite, then $X^G \stackrel{\star}{=} f^{-1}(Y^G)$.*

Note that because of properness of f, the subset $f^{-1}(Y^G)$ is well-defined up to near equality $\stackrel{\star}{=}$.

Proof. The first inclusion, which by properness of f is equivalent to $f(X^G) \subset^\star Y^G$, is immediate.

Let S be a finite generating subset of G and lift each $s \in S$ to a self-map of X and Y, written s by abuse of notation, and thus choose the representative X^G as the set of x such that $sx = x$ for all s, and similarly define Y^G. Let F be the set of x such that $sf(x) \neq f(sx)$ for some s; it is finite by assumption. Let F' be the set of x such that, for some s, one has $f(sx) = f(x)$ and $sx \neq x$; it is also finite by assumption. Let x belong to $f^{-1}(Y^G)$. This means that $sf(x) = f(x)$ for all x. If moreover $x \notin F$, it follows that $f(sx) = f(x)$ for all $s \in S$. If moreover $x \notin F'$, it follows that $sx = x$ for all $s \in S$. This shows that $f^{-1}(Y^G) \subset X^G \cup (F \cup F')$. $\square$

Lemma 2.8. *Let G be a finitely generated group. Let X be a G-set and let Y be a near G-set. Let $f : X \to Y$ be a proper, surjective, near equivariant map. Then there exists a partition $Y = Y_1 \sqcup Y_2$ by G-commensurated subsets and a realization of the near action as G-action on Y_1, such that $f^{-1}(Y_1)$ is G-invariant, $f^{-1}(Y_1) \to Y_1$ is G-equivariant, and $f^{-1}(Y_2)$ has finitely many G-orbits.*

Proof. Fix a finite symmetric generating subset S of G, and for each $s \in S$, choose a lift $y \mapsto sy$ on Y that is a cofinite-partial bijection. Make Y a graph by joining y to sy, for $s \in S$.

Let $\mathcal{X}$ be the set of G-orbits on X. Let $\mathcal{X}_1 \subset \mathcal{X}$ be the set of orbits on which f is G-equivariant, and $\mathcal{X}_2$ its finite complement. Write $X_2 = \bigcup \mathcal{X}_2$, and $X_1 = \bigcup \mathcal{X}_1$ its complement. Make X a graph by joining x to sx, for $s \in S$, if $sf(x) = f(sx)$. This is the Schreier graph minus finitely edges (the removed edges are in X_2), thus X_2 consists of finitely many components for this graph. Then f maps each component of X into some component of Y. Let $\mathcal{Y}_2$ be the set of components of the graph Y that meet $f(X_2)$ or on which some $s \in S$ fails to be defined everywhere; it is finite. Let $\mathcal{Y}_1$ be its complement, and $Y_1 = \bigcup \mathcal{Y}_1$. Thus $f^{-1}(Y_1) \subset X_1$, and it follows that $f^{-1}(Y_1)$ is G-invariant and $f^{-1}(Y_1) \to Y_1$ is F_S-equivariant, where F_S is the free group over S and Y_1 is endowed with the F_S-action induced the permutations $y \mapsto sy$ on Y_1. Since f is surjective, so is

$f : f^{-1}(Y_1) \to Y_1$, and since the F_S-action on $f^{-1}(Y_1)$ factors through G, the F_S-action on Y_1 also factors through G. Now Define $Y_2 = Y \smallsetminus Y_1$.

We claim that $f^{-1}(Y_2)$ consists of finitely many components. Indeed, otherwise, it includes infinitely many components that are G-orbits belonging to $\mathcal{X}_1$, and mapping to a single component. Using that S is symmetric, each element of $\mathcal{X}_1$ is mapped onto a component of Y. We contradict properness of f. □

Let X, Y be sets and fix a function $k : Y \to \mathbf{N}$. A subset ϕ of $X \times Y$ is said to be **k-onto-1** if it is the graph of a function in which every $y \in Y$ has exactly $k(y)$ preimages. For $x \in X$, write $\phi[x] = \{y \in Y : (x, y) \in \phi\}$ and for $y \in Y$, write $\phi^{-1}[y] = \{x \in X : (x, y) \in \phi\}$. We say that a subset ϕ of $X \times Y$ (typically, the graph of a function) is **near k-onto-1** if for every x, $\phi[x]$ is finite; for every y, $\phi^{-1}[y]$ is finite; for all but finitely many x, $\phi[x]$ is a singleton, and for all but finitely many y, $\phi^{-1}[y]$ has cardinal $k(y)$. Assuming that ϕ is near k-onto-1, define the **k-index** as the finitely supported sum

$$\mathrm{ind}_k^X(\phi) = \sum_{x \in X} (\#\phi[x] - 1) - \sum_{y \in Y} (\#\phi^{-1}[y] - k(y)).$$

The set of near k-onto-1 subsets of $X \times Y$ is saturated under the near equality relation $\stackrel{\star}{=}$. When adding or removing a singleton, clearly the k-index does not change (since one adds or removes 1 to both sums). Hence, the index is invariant under $\stackrel{\star}{=}$. A near k-onto-1 map $X \to Y$ is a map ϕ whose graph is near k-to-1, or equivalently a proper map such that for $\phi^{-1}(\{y\})$ has cardinal $k(y)$ for all but finitely many $y \in Y$.

Lemma 2.9. *Let A be a finitely generated abelian group. Let X, Y be A-sets, with finitely many orbits, and with no infinite orbit of linear growth. Let f be a proper, near A-equivariant map. Suppose that the cardinal of fibers is constant on each infinite A-orbit in Y. Then there exists a map f' with $f \stackrel{\star}{=} f'$, such that f' is A-equivariant on each infinite A-orbit of X, and f' maps finite orbits of X into finite orbits of Y.*

Proof. Let $X_1, \dots, X_k$ be the infinite orbits in X, and $Y_1, \dots, Y_\ell$ the infinite orbits in Y. Write $F = X \smallsetminus \bigcup X_i$. Since each X_i is 1-ended, for every i there exists a unique j_i such that $f(X_i) \subset^\star Y_{j_i}$. Hence there exists a map q with $q \stackrel{\star}{=} f$ such that $f(X_i) \subset Y_{j_i}$ for all i. Moreover, if A_i is the stabilizer of elements of X_i, then A_i acts trivially on Y_{j_i}. Thus by Theorem 2.6 (applied to $q|_{X_i} : X_i \to Y_{j_i}$, map between A/A_i-sets), there exists a unique equivariant map $f_i : X_i \to Y_{j_i}$ such that $f_i \stackrel{\star}{=} f|_{X_i}$. Define f' as equal to f_i on X_i and equal to some constant on F; then $f' \stackrel{\star}{=} f$. If $Y \neq \bigcup Y_j$, we can choose this constant to belong to the complement of $\bigcup Y_j$.

Assume otherwise, i.e., $Y = \bigcup Y_j$. For each j, the cardinal fiber $f^{-1}(\{y\})$, by assumption does not depend on $y \in Y_j$, call it $k(j)$. Then since $f' \stackrel{\star}{=} f$, all but finitely many $y \in Y_j$ have exactly $k(j)$ preimages. Applying this to the restriction

g of f to $\bigcup X_i$, which is equivariant, we deduce that $g^{-1}(\{y\})$ has cardinal $k(j)$ for all j and $y \in Y_j$. It follows that the k-index of f' is $-\#(F)$. Since the k-index of f is 0 and $f \overset{\star}{=} f'$, we deduce that F is empty. □

Let G be a finitely generated group, S a finite symmetric generating subset. Let X be a G-set and Y a subset of X. We say that the pair (Y, S) satisfies Property (P) if for every finite subset F of Y there exists a finite subset F' of Y, such that for any two $x, x' \in Y \smallsetminus F'$, there exists a sequence (s_i) in S, such that, defining $g_0 = 1$ and $g_i = s_i g_{i-1}$, we have $g_i x, g_i x' \in Y \smallsetminus F$ for all i, and (g_i) tends to infinity in G.

Lemma 2.10. *Let A be a finitely generated abelian group. Then A admits a finite symmetric generating subset S such that, for every 1-ended left-commensurated subset Y of $A(= X)$, the pair (Y, S) satisfies Property (P). More precisely, for every finite subset F of Y, there exists a finite subset F' of Y including F, such that for any two $x, x' \in Y \smallsetminus F'$, there exists $s \in S$, of infinite order, such that $s^n x, s^n x' \in Y \smallsetminus F'$ for all $n \in \mathbf{N}$.*

Proof. When $A = \mathbf{Z}$, this is immediate with $S = \{\pm 1\}$. Next suppose that $A = \mathbf{Z}^d$, $d \geq 2$, with the standard generating subset. Take $F' = \mathbf{Z}^d \cap B$ with $B = [-N+1, N-1]^d$, for N large enough so that F' includes F. Take x, x'. We can suppose, up to an isometry of $\mathbf{Z}^d$, that $x_1 \geq N$. If the result works with $s = e_1$, we are done. Otherwise, it means that $x' + ne_1 \in B$ for some $n \in \mathbf{N}_{\geq 1}$. This holds if and only if $x'_1 \leq -N$ and $|x'_i| \leq N-1$ for all $i \geq 2$. In this case, we choose $s = e_2$, which works.

Next, if A is arbitrary, let p be the projection modulo its finite torsion subgroup T. Choose $S = p^{-1}(S' \cup \{0\})$ where S' is a free generating subset of A/T. Then choose $F' = p^{-1}(B)$ with B as above (with N large enough). It is easy to check that this works. □

Lemma 2.11. *Let A be a finitely generated abelian group and Y a 1-ended left-commensurated subset of A. Let Z be a $\star$-faithful near G-set and $f : Y \to Z$ a surjective, proper near A-equivariant map. Then f is near bijective (i.e., all but finitely elements of Z have a single pre-image). In particular, Z is a completable near A-set.*

Proof. Fix a finite symmetric generating subset S satisfying the conclusion of Lemma 2.10. For every $s \in S$, choose a lift of s as a self-map of Z. Define F as the set of $y \in Y$ such that $f(sy) \neq sf(y)$ for some $s \in S$ (including those for which $sf(y)$ is not defined). Let F' be given by the lemma. We claim that f is injective on $Y \smallsetminus F'$. Indeed, suppose that $x, x' \in Y \smallsetminus F'$ with $f(x) = f(x')$. By the lemma, there exists $s \in S$, of infinite order, such that $s^n x, s^n x' \in Y \smallsetminus F'$ for all $n \in \mathbf{N}$. We claim that $f(s^n x) = f(s^n x')$ for all $n \in \mathbf{N}$: this immediately follows by induction from the hypotheses. Write $g = x^{-1}x'$ and $h_n = s^n x$. Then $f(gh_n) = f(h_n)$ for all n. Choose a lift of g as a self-map of Z. Since (h_n) is injective, this implies that $gf(h_n) = f(h_n)$ for all large n. Since f is proper, $\{f(h_n) : n \geq 0\}$ is proper. Hence, using that Z is a $\star$-faithful near G-set, we deduce $g = 1$. Hence $x = x'$. □

Theorem 2.12. *Let A be a finitely generated abelian group. Let X be an A-set and Y a near A-set. Fix a positive integer k. Let f be a near equivariant map $X \to Y$, all of whose fibers have cardinal k. Suppose that $(*)$ for every $g \in A$, the set of $x \in X$ such that $f(gx) = f(x)$ and $gx \neq x$ is finite. Then Y is a realizable near A-set.*

Proof. Lemma 2.8 allows to assume that X has finitely many A-orbits. Then X is finitely-ended. Decompose X as $\bigsqcup_{i=1}^n X_i$, where each X_i is 1-ended. Let A_i be the stabilizer of all but finitely many points in X_i; each X_i is near included in a unique infinite orbit X'_i. Define $Y_i = f(X_i)$. Then Y_i is a A-commensurated subset of Y; by Lemma 2.7, Y_i is a $\star$-faithful near A/A_i-set. By Lemma 2.11, $f : X_i \to Y_i$ is a near isomorphism of near A/A_i-sets. In particular Y_i is a completable near G-set. Since the Y_i, being 1-ended, are pairwise either near equal or near disjoint, and since they cover Y, it follows that Y is a completable near G-set.

For every i, the inverse image of $Y_i = f(X_i)$ is commensurated, hence is a finite perturbation of $\bigcup_{j \in J_i} X_i$ for some subset J_i of $\{1, \dots, n\}$. Clearly, we have $j \in J_i$ if and only if $j \in J_j$. Hence, any two J_i are equal or disjoint, and they cover $\{1, \dots, n\}$ since $i \in J_i$. Moreover, all but finitely many points in Y_i have $|J_i|$ preimages, and hence the cardinal of J_i is k. Since, again by Lemma 2.11, $f^{-1}(Y_i)$ is near isomorphic to the disjoint copy of $k = |J_i|$ copies of Y_i, we deduce that the near A-set X is near isomorphic to the disjoint copy of k copies of Y. By Lemma 2.4, we deduce that Y is a finitely stably realizable near A-set.

Finally, let us prove that Y is realizable. If by contradiction it is not realizable, since it is finitely stably realizable, there exists a positive integer ℓ such that Y union ℓ points (written $Y \sqcup \ell$) is realizable as an action with only orbits of at least quadratic growth; endow $Y' = Y \sqcup \ell$ with this action. Write $X' = X \sqcup k\ell$; endow it extending the action as the trivial action on the $k\ell$ additional points. Extend f to a k-onto-1 map f' onto Y'. Apply Lemma 2.9: there exists $f'' \overset{\star}{=} f$ that maps finite orbits into finite orbits. Since there is no finite orbit in Y', this implies that there is no finite orbit in X'. Hence $k\ell = 0$, contradiction. □

Remark 2.13. Here is a counterexample to the statement of Theorem 2.12 with $(*)$ removed. Let K be the Klein group of order 4, and u, v two distinct elements of order 2 in K, and F the subgroup generated by u. Let A be the group $F \times \mathbf{Z}$, and $X = K \times \mathbf{Z}$, which is thus a free A-set with two orbits. Define a permutation of order 2 of X by $\sigma(g, n) = (ug, n)$ for $n < 0$ and $\sigma(g, n) = (vg, n)$ for $n \geq 0$. So σ commutes with the action of u and near commutes with the action of $\mathbf{Z}$, in the sense that the commutator with the generator $(g, n) \mapsto (g, n + 1)$ has finite support. Hence, the quotient by the u-action is naturally a near A-set. It can be identified to $F \times \mathbf{Z}$, where $\mathbf{Z}$ acts by shifting, while u acts by $(g,n) \mapsto (g,n)$ for $n < 0$ and $\mapsto (ug,n)$ for $n \geq 0$. This near action is not stably realizable (this is the very first example in [Cor3]).

2.3. **Perturbation of actions.** Given two actions α, α' of a group on a set, we say that they are finite perturbations of each other if they induce the same near action. In other words, this mean that for every g, the permutations $\alpha(g)$ and $\alpha'(g)$ coincide on a cofinite subset (depending on g).

The following is established in [Cor3, Theorem 7.C.1] and will be used in §4.

Theorem 2.14. *Let G be a 1-ended group that is not locally finite, and X a G-set. Then*

(1) Suppose that G acts freely on X. Then any finite perturbation of the action is conjugate, by a unique finitely supported permutation, to the original action.
(2) Suppose instead that G acts freely on $Y = X \smallsetminus X^G$ (where X^G is the set of points fixed by all of G). Then any finite perturbation of the action is conjugate, by a finitely supported permutation, to an action that is unchanged on Y. □

3. Realizability of piecewise continuous near actions of finitely generated abelian groups

We now use the results of §2.2 (namely Theorem 2.12) to prove Theorem 1.4.

3.1. **The "true" definition of** $\mathrm{PC}(\mathbf{S})$**.** Let X be a Hausdorff topological space. The group $\mathrm{PC}(X)$ of **near self-homeomorphisms** of X consists of those elements of $\mathfrak{S}^\star(X)$ that have a representative that is a homeomorphism between two cofinite subsets.

Let $\widehat{\mathrm{PC}_0}(X)$ be the subgroup of permutations f of X such that both f and f^{-1} are continuous outside a finite subset. There is a canonical homomorphism $\widehat{\mathrm{PC}_0}(X) \to \mathrm{PC}(X)$; its image $\mathrm{PC}_0(X)$ equals $\mathrm{PC}(X) \cap \mathfrak{S}^\star_0(X)$, and its kernel consists of finitely supported permutations of X.

A basis remark is that $\mathrm{PC}_0(X)$ is a proper subgroup of $\mathrm{PC}(X)$ if and only if there exist two finite subsets F, F' of X with $|F| < |F'|$ such that $X \smallsetminus F$ and $X \smallsetminus F'$ are homeomorphic.

For instance, this holds when X is infinite discrete, or when X is a Cantor space. Nevertheless, it does not hold when $X = \mathbf{S}$: indeed, $\mathbf{S}$ minus n points is homeomorphic to $\mathbf{S}$ for $n = 0$ and to the disjoint union of n copies of $\mathbf{R}$ when $n \geq 1$, so its topological type retains n. Hence, $\mathrm{PC}_0(\mathbf{S}) = \mathrm{PC}(\mathbf{S})$ and this is in accordance with the definition in the introduction, which defined $\mathrm{PC}(\mathbf{S})$ as what is properly defined as $\mathrm{PC}_0(\mathbf{S})$.

Remark 3.1. In [Cor2], it is proved that the near action of $\mathrm{PC}(X)$ on X is completable as soon as X has no isolated point. This notably applies to $X = \mathbf{S}$.

3.2. **The Denjoy blow-up.** Let $\mathbf{S}$ denote the circle $\mathbf{R}/\mathbf{Z}$. Let $\mathbf{S}^{\pm}$ denote the "Denjoy blow-up" of $\mathbf{S}$ at all points. As a set, it can simply be defined as the Cartesian product $\mathbf{S} \times \{\pm 1\}$, where we write x^+ and x^- for the elements $(x, 1)$ and $(x, -1)$. For $y = (x, \varepsilon) \in \mathbf{S}$, we write $\hat{y} = (x, -\varepsilon)$ and $\bar{y} = x$.

It turns out that there are two natural compact Hausdorff topologies on this blow-up. The first is the product topology. The second, called circular topology, is the topology of the cyclic ordering, where, whenever $x < y < z$ in $\mathbf{S}$, we prescribe $x^- < x^+ < y^- < y^+ < z^- < z^+$. (Here, in a cyclic ordering, by $x_1 < x_2 < \cdots < x_n$ we mean that $x_i < x_j < x_k$ for all $1 \leq i < j < k \leq n$.) The circular topology is compact Hausdorff, totally disconnected, but not metrizable (since the set of clopen subsets is uncountable).

The interest is that the group $\mathrm{PC}(\mathbf{S})$ naturally acts on $\mathbf{S}$, using one-sided limits in the obvious way. This makes the projection map $\mathbf{S}^{\pm} \to \mathbf{S}$, $y \mapsto \bar{y}$, near $\mathrm{PC}(\mathbf{S})$-equivariant.

3.3. **Proof of realizability.** We need the following fact about the Denjoy blow-up.

Proposition 3.2. *For every $g \in \mathrm{PC}(\mathbf{S})$, the set of $x \in \mathbf{S}^{\pm}$ such that $g(x) = \hat{x}$ is finite.*

Proof. If by contradiction it is infinite, it has an accumulation point; conjugating by a suitable element of $\mathrm{O}(2)$, we can suppose that this accumulation point is 0^+. Hence, there is an injective sequence (x_n) tending to 0^+ such that $g(x_n) = \hat{x}_n$ for every n. There exists $\varepsilon \in]0, 1[$ such that g induces a continuous (necessarily strictly monotone) function $\bar{g}$ on $]0, \varepsilon[$, valued in $]0, 1[$. Extracting, we can suppose that $0 < \bar{x}_{n+1} < \bar{x}_n < \varepsilon$ for all n.

On the one hand, since $g(x_n) = \hat{x}_n$ for every n, $\bar{g}$ is necessarily decreasing on $]0, \varepsilon[$. On the other hand, since $g(x_n) = \hat{x}_n$, we have $\bar{g}(\bar{x}_n) = \bar{x}_n$ for all n, which implies that $\bar{g}$ is increasing on $]0, \varepsilon[$. Contradiction. □

Theorem 3.3. *Every finitely generated abelian subgroup A of $\mathrm{PC}(\mathbf{S})$ is realizable (for its near action on $\mathbf{S}$).*

Proof. We use the Denjoy blow-up map $\mathbf{S}^{\pm} \to \mathbf{S}$. Here $\mathbf{S}^{\pm}$ is an A-set, the map is 2-onto-1 and is near equivariant; moreover it satisfies the additional assumption $(*)$ of Theorem 2.12, by Proposition 3.2. Hence, Theorem 2.12 applies and $\mathbf{S}$ is a realizable A-set. □

Remark 3.4. One step to the theorem was to show that A is stably realizable. This step is much easier when $A \subset \mathrm{IET}^{\bowtie}$, or more generally when the near action of A is piecewise analytic. Indeed, in this case, the criterion of Theorem 2.3 can be checked directly, as the set of fixed points of any finitely generated subgroup is then a Boolean combination of intervals.

4. Non-realizability of groups of interval exchanges with flips

4.1. Non-realizability. In $\mathrm{IET}^+(\mathbf{S})$, we have the subgroup of genuine rotations (translations of the group $\mathbf{S}=\mathbf{R}/\mathbf{Z}$). We endow $\mathbf{S}$ with its geodesic distance. Let r_t be the rotation $x\mapsto x+t$.

Given any interval I in $\mathbf{S}$ (of measure in $]0,1[$), we have a corresponding subgroup of partial rotations, acting trivially outside I, and acting as genuine rotations on I when we "close it" by identifying endpoints of $\bar{I}$.

Given $f\in\mathrm{PC}^{\bowtie}(\mathbf{S})$, define its essential support $\mathrm{essupp}(f)\subset\mathbf{S}$ as the closure of the set of x such that $(f(x^-),f(x^+))\neq(x^-,x^+)$. It is empty if and only if f is the identity. When $f\in\mathrm{IET}^{\bowtie}$ and $\tilde{f}$ is a lift, note that $\mathrm{essupp}(f)$ has finite symmetric difference with $\{x:\tilde{f}(x)\neq x\}$.

For $f\in\mathrm{PC}^{\bowtie}(\mathbf{S})$, define $\underline{\mathrm{sing}}(f)=\{x\in\mathbf{S}^{\pm}:f(\hat{x})\neq\widehat{f(x)}\}$. It is finite, and obviously invariant under $x\mapsto\hat{x}$; let $\mathrm{sing}(f)$ be its image in $\mathbf{S}$. These are the set of points at which every lift of f is discontinuous.

For $f\in\mathrm{IET}^+(\mathbf{S})$, chose a representative $g(x)=x+s_g(x)$. Note that while s_g can depend on the choice of g, the values $s_g(x^-)$ and $s_g(x^+)$ only depend on f, and we denote them as $s_f(x^-)$ and $s_f(x^+)$. Thus $\mathrm{sing}(f)=\{x:s_f(x^-)\neq s_f(x^+)\}$. Write $\nu_f(x)=s_f(x^+)-s_f(x^-)$, so $\mathrm{sing}(f)=\{x:\nu_f(x)\neq 0\}$.

Let $E(f)\in\,]0,1/2]\cup\{\infty\}$ be the minimal distance between any two points of the finite subset $\mathrm{sing}(f)$ (where $E(f)=\infty$ if $\mathrm{sing}(f)$ is empty, i.e., if f is a rotation).

When we consider the action of $\mathrm{PC}(\mathbf{S})$ on subsets of $\mathbf{S}$, it is only well-defined modulo finite symmetric difference with finite subsets. We then talk of near subset, near disjoint (= finite intersection), near partition, etc.

Lemma 4.1. *For every $f\in\mathrm{IET}^+$ and $t\in\,]0,E(f)[$, the "commutator" $c=f^{-1}r_tfr_t^{-1}$ permutes the near intervals $[x,x+t]$, $x\in\mathrm{sing}(f)$, by translations, without preserving any of them. These intervals are pairwise near disjoint. Moreover, c near acts as the identity outside the union of these intervals. In particular, c has finite order, and is not the identity.*

Proof. In this proof, "generic" means "with finitely many exceptions", and we freely choose representatives.

If f is a rotation then c is the identity. Otherwise, f has at least two singularities. Let a,b be consecutive singularities of f. Then the representative of $b-a$ in $]0,1[$ is $\geq t$. So we can view the interval $[a,b]$ as concatenations of intervals $[a,a+t]$ and $[a+t,b]$, and for $x\in[a+t,b]$, we have $f(x)=f(x-t)+t$. For a generic $x\in[a+t,b]$, we have $c(x)=f^{-1}r_tf(x-t)=f^{-1}r_t(f(x)-t)=f^{-1}(f(x))=x$.

For generic $x\in[a,a+t]$, we have $c(x)=f^{-1}r_tf(x-t)=f^{-1}r_t(x-t+s_f(a^-))=f^{-1}(x+s_f(a^-))$. Observe that $f(a^-)=a+s_f(a^-)$ belongs to $\mathrm{sing}(f^{-1})$. Since $E(f)=E(f^{-1})$, this implies that for $x\in\,]a,a+t[$, $x+s_f(a^-)$ meets no singularity of f^{-1}. Hence c has no singularity in $]a,a+t[$.

Thus the image by c of this interval is (essentially) an interval of length t, included in the union of the intervals $[x, x+t]$, for $x \in \mathrm{sing}(f)$. Therefore, it is exactly one of these intervals. □

The diameter of a metric space is the supremum of distances between any two points.

Lemma 4.2. *Let Γ be a subgroup of $\mathrm{IET}^+(\mathbf{S})$. Suppose that Γ includes a dense subgroup of rotations, and, for some proper sub-interval, some dense subgroup of the corresponding partial rotations. Then Γ contains non-identity elements whose essential support has arbitrary small diameter.*

Proof. Fix $\varepsilon > 0$ small enough (see below), and let us produce non-identity element whose essential support has diameter $\leq 5\varepsilon$.

Up to conjugating by a rotation, we can suppose that Γ includes a dense subgroup of rotations of the interval $[0, r]$ for some $r \in]0, 1[$. Namely, we assume that $\varepsilon < \min(r, 1-r)/4$.

Choose such a partial rotation f of length -2ε. Hence s_f generically equals $r - 2\varepsilon$ on $[0, 2\varepsilon]$, -2ε on $[2\varepsilon, r]$, and 0 on $[r, 1]$. Then the essential support of $c = f^{-1} r_\varepsilon f r_\varepsilon$ is, by Lemma 4.1, equal to $[0, \varepsilon] \cup [2\varepsilon, 3\varepsilon] \cup [r, r+\varepsilon]$. Consider in Γ a rotation of length $\lambda \in]3\varepsilon, 4\varepsilon[$. Conjugate the given group of partial rotations by this rotation to obtain a dense group of rotations of $[\lambda, \lambda + r]$. (By the condition on ε, we have $\lambda < r$ and $\lambda + r < 1$.) Then there exists q in this dense group of rotations of $[\lambda, \lambda+r]$ (essentially) mapping $[r, r+\varepsilon]$ into $]\lambda, 5\varepsilon[$, say $[\lambda', \lambda'+\varepsilon]$ with $3\varepsilon < \lambda' < 4\varepsilon$. Then the essential support of $q^{-1} c q$ is $[0, \varepsilon] \cup [2\varepsilon, 3\varepsilon] \cup [\lambda', \lambda' + \varepsilon]$, and thus has diameter $\leq 5\varepsilon$. □

Definition 4.3. We say that a subgroup of $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S})$ is

- clean if its intersection with the group of finitely supported permutations is trivial;
- hyper-clean if for every g in the subgroup, the graph of g (viewed as subset of $\mathbf{S} \times \mathbf{S}$) has no isolated point; equivalently if, at every point, g is either left or right-continuous.

Lemma 4.4. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S})$, and Γ its image in $\mathrm{PC}^{\bowtie}(\mathbf{S})$. Suppose that $\tilde{\Gamma}$ includes a dense subgroup $\tilde{\Lambda}$ of rotations. Suppose that Γ admits non-identity elements with essential support of arbitrary small diameter. Then $\tilde{\Gamma}$ is hyper-clean.*

Proof. For $h \in \widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S})$, define its interior support as the intersection of the support $\{x : hx \neq x\}$ with the set of continuity points of h. It is open, and is included and cofinite in the h-invariant subset $\{x : hx \neq x\}$, and it is also included and dense in the essential support of its image $\bar{h}$ in $\mathrm{PC}^{\bowtie}(\mathbf{S})$.

In a first step, we show that $\tilde{\Gamma}$ is locally clean, in the sense that for each element g, the support $\{x : gx \neq x\}$ has no isolated point.

By contradiction, let x be an isolated non-fixed point of g. For some $\varepsilon > 0$, all other points in $]x-\varepsilon, x+\varepsilon[$ are fixed by g. There exists $h_0 \in \tilde{\Gamma}$ whose essential support has diameter $< \varepsilon$. Hence there exists some conjugate h of h_0 by some element of $\tilde{\Lambda}$ such that both x and $h(x)$ belong to the interior support of h (indeed, letting I be the interior support of h_0 and $J = h_0^{-1}I \cap I$, which is cofinite in I, it is enough to find $s \in \tilde{\Lambda}$ such that $x \in sI$ and define $h = sh_0s^{-1}$). Hence the essential support of h is included in $]x-\varepsilon, x+\varepsilon[$; in particular, $\bar{h}$ and $\bar{g}$ commute. Since $\tilde{\Gamma}$ is clean, it follows that h and g commute. Since $g(x) \neq x$, it follows that $g(h(x)) \neq h(x)$. But $h(x) \neq x$, and $h(x)$ belongs to the interior support, which is included in $]x-\varepsilon, x+\varepsilon[$, hence $h(x)$ is fixed by g. This is a contradiction, concluding the first step.

Now let us prove that $\tilde{\Gamma}$ is hyper-clean. Suppose by contradiction that $\{x, g(x)\}$ is an isolated point in the graph of g. Up to post-compose g with a nontrivial rotation, we can suppose that $g(x) \neq x$. So there exists $\varepsilon > 0$ such that none of x, $g(x^+)$, $g(x^-)$ belongs to $]g(x)-2\varepsilon, g(x)+2\varepsilon[$.

As in the proof of the first step (using the dense subgroup of rotations), let $h \in \tilde{\Gamma}$ have essential support of diameter $< \varepsilon$, with both $g(x)$ and $h(g(x))$ in the interior support of h; we can also require that $h(x) = x$.

Then for $t \in]x-\varepsilon, x+\varepsilon[$, $t \neq x$, we have $t \notin]g(x)-\varepsilon, g(x)+\varepsilon[$ and $g(t) \notin]g(x)-\varepsilon, g(x)+\varepsilon[$. Hence, with finitely many exceptions on t, we have $g(h(t)) = g(t)$ and $h(g(t)) = g(t)$. Also, we have $g(h(x)) = g(x)$ and $h(g(x)) \neq g(x)$. Hence, $g^{-1}h^{-1}gh$ has an isolated non-fixed point at x. This contradicts the assumption that $\tilde{\Gamma}$ is locally clean. ☐

Definition 4.5. Call an element f of $\mathrm{PC}^{\bowtie}(\mathbf{S})$ a 132-flip if it satisfies: there are three nonzero consecutive intervals I_1, I_2, I_3 near partioning $\mathbf{S}$, such that

(1) f has no singularity in the interior of I_j for each $j = 1, 2, 3$;
(2) $f(I_1)$, $f(I_3)$, $f(I_2)$ are consecutive intervals;
(3) $f : I_j \to f(I_j)$ is orientation-reversing for $j = 1$ and orientation-preserving for $j = 2, 3$.

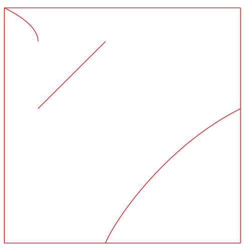

Figure 2. Graph of a 132-flip.

Lemma 4.6. *Let $f \in \mathrm{PC}^{\bowtie}(\mathbf{S})$ be a 132-flip. Then f has no hyper-clean lift.*

Proof. The assumptions implies that the singularities of f are precisely the three breaking points. Let by contradiction $\tilde{f}$ be a hyper-clean lift of f. Then $\tilde{f}$ is continuous on three intervals I'_1, I'_2, I'_3, where I_j and I'_j have the same interior for all $j = 1, 2, 3$. Since $-I'_1$ and $f(I'_1)$ have the same type, I'_1 is either a segment or open on both sides. Since I'_1 and I'_2, resp. I'_1 and $f^{-1}(I'_3)$, are consecutive, the left bounds of I'_2 and I'_3 have the same type (opposite to the right bound of I'_1), and so do their right bound by a symmetric argument. Hence I'_2 and I'_3 have the same type. Since I'_2 and I'_3 are consecutive, it follows that they are both open on a single same side. Hence I'_1 also has the same type (open on a single side). We reach a contradiction. □

Theorem 4.7. *Let* Γ *be a subgroup of* $\mathrm{PC}^{\bowtie}(\mathbf{S})$. *Suppose that*

(1) Γ *includes a subgroup* Λ *of rotations of* $\mathbf{Q}$*-rank* ≥ 2, *or infinitely generated and of* $\mathbf{Q}$*-rank 1;*
(2) Γ *includes, for some proper nonzero interval, some dense subgroup of the corresponding partial rotations;*
(3) Γ *contains an 132-flip (Definition 4.5).*

Then Γ *is not realizable.*

Proof. By contradiction, let $\tilde{\Gamma}$ be a lift; then $\tilde{\Gamma}$ is clean. The assumption on Λ implies that it is 1-ended and not locally finite. By Theorem 2.14(1), we can, after conjugation, assume that Λ lifts as a subgroup of genuine rotations. By Lemma 4.2, $\Gamma \cap \mathrm{IET}^+$ contains non-identity elements of arbitrary small essential diameter. By Lemma 4.4, $\tilde{\Gamma}$ is hyper-clean. Finally Lemma 4.6 yields a contradiction. □

Corollary 4.8. $\mathrm{IET}^{\bowtie}$ *is not realizable (for its near action on* $\mathbf{S}$*), and admits a non-realizable finitely generated subgroup.*

Proof. To fulfill the assumptions of Theorem 4.7, consider a pair (u, v) of rotations generating a subgroup of $\mathbf{R}/\mathbf{Z}$ of $\mathbf{Q}$-rank 2; a partial rotation w generating a dense subgroup in a proper interval, and an element $z \in \mathrm{IET}^{\bowtie}$ as in Lemma 4.6. Then $\{u, v, w, z\}$ generates a non-realizable subgroup, by Theorem 4.7. □

For a subgroup Λ of $\mathbf{R}/\mathbf{Z}$, let $\mathrm{IET}^{\bowtie}_{\Lambda}$ be the subgroup of $\mathrm{IET}^{\bowtie}$ of elements with discontinuities in Λ, and local isometries of the form $x \mapsto \pm x + \lambda$ with $\lambda \in \Lambda$.

Corollary 4.9. *Let* Λ *be a subgroup of* $\mathbf{R}/\mathbf{Z}$, *of* $\mathbf{Q}$*-rank* ≥ 2, *or infinitely generated of* $\mathbf{Q}$*-rank* 1. *Then* $\mathrm{IET}^{\bowtie}_{\Lambda}$ *is not realizable.* □

4.2. **Restricted realizability.** Recall that a partial rotation is an element of IET^+ which, for a partition of the circle into three consecutive (possibly empty) intervals, exchanges two of them and (pointwise) fixes the third one.

Lemma 4.10. *Let* r *be a hyper-clean lift of a partial rotation of the circle. Then* r *is either left or right-continuous.*

Proof. The circle is then concatenation of three intervals I_1, I_2, I_3, where r consists in exchanging I_2 and I_3. We have to discuss on the possible type of interval (segment, etc). Then I_2 and I_3 have to be homeomorphic; they cannot be both open or both segments. If they are both closed on the left and open on the right. It follows that I_1 is also of the same type, and hence r is right-continuous. Otherwise, this argument holds after conjugating by a reflection, and hence r is left-continuous. □

Lemma 4.11. *Let $\tilde{\Gamma}$ be a hyper-clean subgroup of $\widehat{\mathrm{IET}^+}$. Suppose that each translation length of every element of Γ is achieved by a rotation belonging to $\tilde{\Gamma}$. Then either all partial rotations in $\tilde{\Gamma}$ are left-continuous, or all are right-continuous.*

Proof. Suppose the contrary. After conjugation and using Lemma 4.10, we can suppose that we have two partial rotations r, s with left endpoint 0, with r right but not left-continuous, and s left but not right-continuous.

Then $s(0) = s(0^-) = 0$ and $s(0^+) \neq 0$. So $r(s(0)) = r(0) \neq 0$, while $r(s(0^-)) = 0$. Hence rs is not left-continuous at 0. Since $\tilde{\Gamma}$ is hyper-clean, this implies that rs is right-continuous at 0. Hence $rs(0^+) = rs(0)$. So $rs(0^+) = r(0) = r(0^+)$. Hence $s(0^+) = 0$; this is a contradiction. □

For $\Gamma \leq \mathrm{PC}^+(\mathbf{S})$, denote by Γ_{left} (respectively Γ_{right}) the group of left-continuous (resp. right-continuous) representatives of elements of Γ.

Theorem 4.12. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\mathrm{IET}^+}$ and Γ its image in IET^+. Suppose that the group of rotations in Γ achieves all translation lengths of Γ, and has $\mathbf{Q}$-rank ≥ 2, or is infinitely generated of $\mathbf{Q}$-rank ≥ 1. Suppose that Γ is generated by its partial rotations. Suppose that Γ includes a dense subgroup of partial rotations (for some proper sub-interval). Then $\tilde{\Gamma}$ is conjugate, by a (unique) finitely supported permutation, to either Γ_{left} or Γ_{right}.*

Proof. Using the $\mathbf{Q}$-rank assumption, by Theorem 2.14(1), we can conjugate by a finitely supported permutation, to ensure that rotations indeed act by rotations. In this case, we will prove that $\tilde{\Gamma}$ is then *equal* to either Γ_{left} or Γ_{right}.

By the assumption of existence of both a dense subgroup of rotations and a dense partial subgroup of partial rotations, Lemma 4.2 implies that Γ admits elements of arbitrary small essential diameter. In turn, again using a dense subgroup of rotations, and the existence of elements of arbitrary small essential diameter, Lemma 4.4 ensures that $\tilde{\Gamma}$ is hyper-clean. Since $\tilde{\Gamma}$ is hyper-clean and all its translations length are achieved by rotations, we apply Lemma 4.11 to ensure that all partial rotations are, say, left-continuous (the right-continuous case is equivalent up to conjugate by a reflection). We conclude, since Γ is generated by its partial rotations. □

Let Λ be a subgroup of $\mathbf{R}/\mathbf{Z}$. Let IET^+_Λ be the subgroup of IET consisting of elements whose singularities and translation lengths belong to Λ.

Corollary 4.13. *Suppose that Λ has $\mathbf{Q}$-rank ≥ 2, or is infinitely generated of $\mathbf{Q}$-rank 1. Then IET^+_Λ has only two lifts to $\widehat{\mathrm{IET}^+}(\mathbf{S})$ up to conjugation by finitely supported permutations, namely the left-continuous and the right-continuous lift.*

Proof. We have to check that the assumptions of Theorem 4.12 are fulfilled. The group of both translations lengths and of rotations of IET^+_Λ equals Λ. Also, for every $x < x_0 \in]0,1[$ representing elements of Λ, the corresponding partial rotation (exchanging $[0,x[$ and $[x,x_0[$ belongs to IET^+_Λ; hence they achieve (x_0 fixed, x varying) a dense subgroup of partial rotations (here we only use that Λ is dense). Finally, that IET^+_Λ is generated by its partial rotations is proved in Lemma 4.14. □

Lemma 4.14. *For every subgroup Λ of $\mathbf{R}/\mathbf{Z}$, the group IET^+_Λ is generated by its partial rotations.*

Proof. Every element of IET^+ can be written as $T = [u,\sigma]$ where, for some n, $u \in \{(u_1,\dots,u_n) \in \mathbf{R}^n_+ : \sum u_i = 1\}$ and $\sigma \in \mathfrak{S}_n$ (where $\mathbf{R}_+$ denotes non-negative reals). Recall that, denoting $X_i = [\sum_{j=1}^{i-1} u_j, \sum_{j=1}^{i} u_j[$, the left-continuous representative of the transformation T consists in rearranging the intervals $X_1,\dots,X_n$, moving X_i in position $\sigma(i)$. The precise formula [Ke] is given by

$$Tx = x - \sum_{j=1}^{i-1} u_j + \sum_{j=1}^{\sigma(i)-1} u_{\sigma^{-1}(j)}.$$

In particular, the singularities belong to $\{u_1, u_1+u_2, \dots, u_1+\dots u_{n-1}\}$, which is included in the subgroup generated by the u_i. Say that σ is admissible if $\sigma(i+1) \neq \sigma(i)+1$ for all $i<n$. Moreover if we assume that n is minimal (which amounts to that σ is admissible), all these are singularities (in the interval model, i.e., we always consider 0 and $\sigma^{-1}(0)$ as singularities), so the subgroup generated by singularities equals the subgroup Ξ_T generated by the u_i, and hence achieves all translation lengths.

The composition formula is given by

$$[\psi \circ \sigma, u] = [\psi, \sigma \cdot u] \circ [\sigma, u], \quad \text{where } (\sigma \cdot u)_i = u_{\sigma^{-1}(i)},$$

which, iterating (and omitting signs $\circ$), yields

$$[\sigma_m \dots \sigma_1, u] = [\sigma_m, \sigma_{m-1} \dots \sigma_1 u] \dots [\sigma_2, \sigma_1 u][\sigma_1, u].$$

Note that each $[\sigma_i, \sigma_{i-1} \dots \sigma_1 u]$ has all its singularities and translation lengths in Ξ_u.

Now consider $T \in \mathrm{IET}^+_\Lambda$. Write $T = [\sigma, u]$ with σ admissible. Then $\Xi_u \subset \Lambda$. Write $\sigma = \psi_m \dots \psi_1$ where each ψ_i is a transposition (n_i, n_i+1). Since $\Xi_u \subset \Lambda$, we deduce that $\psi_i \in \mathrm{IET}^+_\Lambda$. Since ψ_i is a transposition of two consecutive elements, for every v, $[\psi_i, v]$ is a partial rotation. We deduce that IET^+_Λ is generated by its partial rotations. □

Corollary 4.15. *Let Γ be a subgroup of $\mathrm{PC}^+(\mathbf{S})$ including IET^+ (e.g., $\mathrm{PC}^+(\mathbf{S})$, or its piecewise analytic subgroup). Then Γ has, up to conjugation by a (unique) finitely supported permutation, only the two lifts Γ_{left} and Γ_{right}.*

Proof. The uniqueness is clear. Let $\tilde{\Gamma}$ be a lift. By Theorem 4.12, we can suppose (up to conjugate by a finitely supported permutation and possibly by a reflection, that in restriction to IET^+, we have the left-continuous lift. By Lemma 4.4, $\tilde{\Gamma}$ is hyper-clean.

Let g be an element of $\tilde{\Gamma}$ and $x \in \mathbf{S}$. Then there exists an interval exchange h such that $h(x^-) = g(x^-)$ and $h(x^+) = g(x^+)$. We can view h as an element of $\tilde{\Gamma}$, and thus h is left-continuous, so $h(x) = h(x^-)$. We have $h^{-1}g(x^-) = x^-$ and $h^{-1}g(x^+) = x^+$. Since $\tilde{\Gamma}$ is hyper-clean, we deduce that $h^{-1}g(x) = x$. So $g(x) = h(x) = h(x^-) = g(x^-)$, showing that g is left-continuous at x. □

4.3. **Stable non-realizability.** Let X be a set. Let $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S} \sqcup X, \mathbf{S})$ be the subgroup of permutations of $\mathbf{S} \sqcup X$ that are identity on a cofinite subset of X, and that induce elements of $\mathrm{PC}^{\bowtie}(\mathbf{S})$ on $\mathbf{S}$. So there is a canonical projection $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S} \sqcup X, \mathbf{S}) \to \mathrm{PC}^{\bowtie}(\mathbf{S})$, whose kernel consists of finitely supported permutations of $\mathbf{S} \sqcup X$.

Here is an adaptation of Lemma 4.4.

For $f \in \widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S} \sqcup X), \mathbf{S}$, we call essential support the closure of the set of $x \in \mathbf{S}$ such that $f(x) \notin \{f(x^+), f(x^-)\}$.

Lemma 4.16. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S} \sqcup X, \mathbf{S})$, and Γ its image in $\mathrm{PC}^{\bowtie}(\mathbf{S})$. Suppose that $\tilde{\Gamma}$ includes a dense subgroup $\tilde{\Lambda}$ of rotations acting as the identity on X. Suppose that Γ admits non-identity elements with essential support of arbitrary small diameter. Then*

(1) for every $g \in \tilde{\Gamma}$ and $x \in \mathbf{S}$, $g(x)$ belongs to $\{g(x^+), g(x^-)\} \cup X$.

(2) for every $g \in \tilde{\Gamma}$, the subset $\{x \in \mathbf{S} : gx \neq x\}$ has no isolated point.

Proof. (1) This is an adaptation of the proof of Lemma 4.4 and we skip details; note that when X is empty this is precisely the same statement. The first step consists in proving that for $x \in \mathbf{S}$ if $g(x^-) = g(x^+) = x$, then $g(x) \in \{x\} \cup X$. The second step assumes that $g(x) \in \mathbf{S}$ and $g(x) \notin \{g(x^-), g(x^+)\}$ and reaches a contradiction.

(2) Suppose by contradiction that there exists $x \in \mathbf{S}$ such that $f(x') = x'$ for all $x' \neq x$ close enough to x, but $f(x) \neq x$. By (1), we have $f(x) \in X$. Let $r \in \tilde{\Lambda}$ be a small enough non-trivial rotation and $x' \neq x$ close enough to x ("close enough" may depend on r). Then $r^{-1}f^{-1}rf(x') = x'$, while $r^{-1}f^{-1}rf(x) = r^{-1}f^{-1}f(x) = r^{-1}x \neq x$. Since $r^{-1}x \in \mathbf{S}$, this contradicts (1) □

Say that an element g of $\mathrm{PC}^{\bowtie}(\mathbf{S})$ has small support if there exists a rotation r such that $rS \cap S = \emptyset$, where S is the essential support of g.

Lemma 4.17. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\mathrm{PC}^{\bowtie}}(\mathbf{S}\sqcup X,\mathbf{S})$ such that for every $g\in\tilde{\Gamma}$, the subset $\{x\in\mathbf{S}: gx\neq x\}$ has no isolated point.*

Suppose that $\tilde{\Gamma}$ includes a dense subgroup of rotations (acting as the identity on X). Let $f\in\Gamma$ be a an element with small support, and $\tilde{f}$ its lift in $\tilde{\Gamma}$. Then X is $\tilde{f}$-invariant.

Proof. Let T be the essential support of $\tilde{f}$; by the assumption, it includes $\{x\in\mathbf{S}: \tilde{f}x\neq x\}$. There exists a non-empty open interval of rotations mapping T to a disjoint subset; fix one in $\tilde{\Lambda}$. Let T' be the support of $\tilde{f}$. Then $T'\cap\mathbf{S}\subset S$. Hence $r(T')\cap T'$ is equal to $T'\cap X$, which is finite. Note that $\tilde{r}(T')$ is the support of $\tilde{r}\tilde{f}\tilde{r}^{-1}$. Since r and rfr^{-1} have essentially disjoint support, they commute, and hence $\tilde{f}$ and $\tilde{r}\tilde{f}\tilde{r}^{-1}$ commute. Thus $T\cap X$ is $\tilde{f}$-invariant, and hence X is $\tilde{f}$-invariant. ☐

Theorem 4.18. *The near action of* $\mathrm{IET}^{\bowtie}$ *on* $\mathbf{S}$ *is not stably realizable. Moreover, there exists a finitely generated subgroup of* $\mathrm{IET}^{\bowtie}$ *whose near action on* $\mathbf{S}$ *is not stably realizable.*

Proof. For two essentially disjoint intervals I,J of the same nonzero size, let $u_{I,J}$ be the element of order 2 in $\mathrm{IET}^{\bowtie}$ exchanging I and J by an orientation-reversing isometry.

Consider $b=u_{[0,1/6],[3/6,4/6]}$, $c=u_{[2/6,3/6],[3/6,4/6]}$. Let d be a nontrivial partial rotation of $[4/6,1]$. Note that bc acts as a reflection on $[0,4/6]$, and bcd is an element as in Lemma 4.6.

Let Γ be a subgroup of $\mathrm{IET}^{\bowtie}$ containing b,c,d, including a subgroup Λ of rotations of $\mathbf{Q}$-rank ≥ 2. By Lemma 4.2, $\Gamma\cap\mathrm{IET}^{+}$ contains non-identity elements with arbitrary small essential diameter. We claim that Γ is not stably realizable.

By contradiction, consider a lift $\tilde{\Gamma}\subset\mathrm{IET}^{\bowtie}(\mathbf{S}\sqcup X)$. After conjugation by a finitely supported permutation, we can assume, by Theorem 2.14(1), that the lift $\tilde{\Lambda}$ acts by rotations on $\mathbf{S}$. Then Lemma 4.16 applies and ensures that the assumptions of Lemma 4.17 are fulfilled. Hence, each small element in $\tilde{\Gamma}$ preserves $\mathbf{S}$. This applies to b,c,d, and hence $\langle b,c,d,\tilde{\Lambda}\rangle$ preserves $\mathbf{S}$. For elements of this subgroup, the conclusion of Lemma 4.16(1) says that the intersection of their graph with $\mathbf{S}\times\mathbf{S}$ has no isolated point. In particular, $bcd|_{\mathbf{S}}$ is hyper-clean. This contradicts Lemma 4.6. ☐

Question 4.19. Let Λ be a infinite cyclic subgroup of $\mathbf{R}/\mathbf{Z}$. Is the near action of IET_{Λ} on $\mathbf{R}/\mathbf{Z}$ realizable? stably realizable?

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CNRS and Univ Lyon, Univ Claude Bernard Lyon 1, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne

E-mail address: `cornulier@math.univ-lyon1.fr`