

REALIZATIONS OF GROUPS OF PIECEWISE CONTINUOUS TRANSFORMATIONS OF THE CIRCLE

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ABSTRACT. We study the near action of the group PC of piecewise continuous self-transformations of the circle. Elements of this group are only defined modulo indeterminacy on a finite subset, which raises the question of realizability: a subgroup of PC is said to be realizable if it can be lifted to a group of permutations of the circle.

We prove that every finitely generated abelian subgroup of PC is realizable. We show that this is not true for arbitrary subgroups, by exhibiting a non-realizable finitely generated subgroup of the group of interval exchanges with flips.

The group of (oriented) interval exchanges is obviously realizable (choosing the unique left-continuous representative). We show that it has only two realizations (up to conjugation by a finitely supported permutation): the left and right-continuous ones.

1. INTRODUCTION

1.1. **Context.** We deal with various groups of piecewise continuous transformations in dimension 1. The best known is the group of piecewise translations, better known as group of interval exchange transformations. Interval exchanges were introduced by Keane [Ke]; they have mostly been studied in classical dynamics (iteration of a single transformation). Its study as a group notably starts in the determination by Arnoux-Fathi and Sah of its abelianization [Ar1]. This study has been recently pursued, notably in work by C. Novak (e.g., [Nov09]), Dahmani-Fujiwara-Guirardel [DFG, DFG2], and Boshernitzan [Bo], see also [Cor2]. Two outstanding problems about this group is whether it admits non-abelian free subgroups, a question attributed to A. Katok, and whether it is amenable [Cor1]. Recent progress on this latter question is due to Juschenko-Monod [JM], subsequently improved by these two authors along with Matte Bon and de la Salle [JMMS]. If we allow flips, we obtain a larger group, which is seldom studied, and usually not precisely defined; it apparently first appears in the thesis [Ar1], where it is shown to be simple. The questions of realizability, which we consider here, do not seem to have been considered, notably because defining interval exchanges with flips as a group is usually swept under the carpet.

Date: October 23, 2019.

2010 *Mathematics Subject Classification.* 37E05 (primary); 20B27, 20F65, 22F05, 37C85 (secondary).

1.2. **Set-up.** Let \mathbf{S} be the circle \mathbf{R}/\mathbf{Z} .

Definition 1.1. Let $\widehat{\text{PC}}^{\bowtie}(\mathbf{S})$ be the group of permutations of \mathbf{S} that are continuous outside a finite subset (it is indeed stable under inversion, by an easy argument).

The group $\widehat{\text{PC}}^{\bowtie}(\mathbf{S})$ includes the normal subgroup of finitely supported permutations $\mathfrak{S}_{\text{fin}}(\mathbf{S})$ of \mathbf{S} .

Definition 1.2. We define $\text{PC}^{\bowtie}(\mathbf{S})$ as the quotient group $\widehat{\text{PC}}^{\bowtie}(\mathbf{S})/\mathfrak{S}_{\text{fin}}(\mathbf{S})$.

Thus, $\text{PC}^{\bowtie}(\mathbf{S})$ is the group of all piecewise continuous permutations of \mathbf{S} , up to finite indeterminacy.

Definition 1.3. Let $\text{PC}^+(\mathbf{S})$ be its subgroup of piecewise orientation-preserving transformations. Let $\text{PC}^-(\mathbf{S})$ be the subset of $\text{PC}^{\bowtie}(\mathbf{S})$ consisting of those piecewise-reversing transformations. Let $\text{PC}^{\pm}(\mathbf{S}) \subset \text{PC}^{\bowtie}(\mathbf{S})$ be the disjoint union $\text{PC}^+(\mathbf{S}) \sqcup \text{PC}^-(\mathbf{S})$.

Thus $\text{PC}^+(\mathbf{S})$ is a subgroup of index two in $\text{PC}^{\pm}(\mathbf{S})$; it first appears in [Ar2], where it is shown to be simple. Note that in contrast to what happens in self-homeomorphism groups, $\text{PC}^+(\mathbf{S})$ has infinite index in $\text{PC}^{\bowtie}(\mathbf{S})$, which we have to distinguish from $\text{PC}^{\pm}(\mathbf{S})$.

Definition 1.4. Let $\text{IET}^{\bowtie}(\mathbf{S})$ be the subgroup of $\text{PC}^{\bowtie}(\mathbf{S})$ consisting of piecewise isometric elements (also called group of interval exchanges with flips). Define $\text{IET}^+(\mathbf{S}) = \text{PC}^+(\mathbf{S}) \cap \text{IET}^{\bowtie}(\mathbf{S})$, the subgroup of piecewise translations, usually called group of interval exchanges. Also define $\text{IET}^{\pm}(\mathbf{S}) = \text{PC}^{\pm}(\mathbf{S}) \cap \text{IET}^{\bowtie}(\mathbf{S})$.

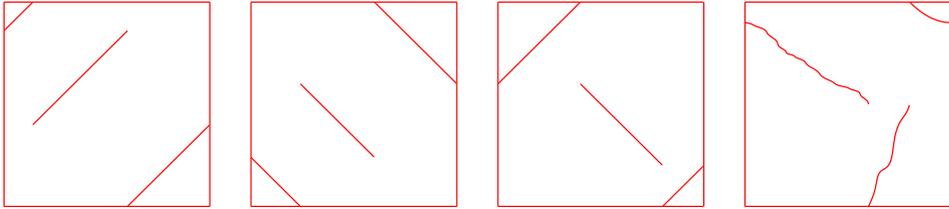


FIGURE 1. Parameterizing the circle as an interval, examples of graphs of elements of $\text{PC}^{\bowtie}(\mathbf{S})$. The first belongs to IET^+ ; the second belongs to IET^- , the third to $\text{IET}^{\bowtie} \setminus \text{IET}^{\pm}$. The fourth is a more “typical” element of $\text{PC}^{\bowtie}(\mathbf{S})$. The value at breakpoints is not prescribed, as we consider their group elements as defined up to finite indeterminacy.

We denote by π the quotient group homomorphism $\widehat{\text{PC}}^{\bowtie}(\mathbf{S}) \rightarrow \text{PC}^{\bowtie}(\mathbf{S})$.

Definition 1.5. We say that a subgroup $\Gamma \subset \text{PC}^{\bowtie}(\mathbf{S})$ is **realizable** if it can be lifted to $\widehat{\text{PC}^{\bowtie}(\mathbf{S})}$, i.e., if there exists a subgroup Λ of $\widehat{\text{PC}^{\bowtie}(\mathbf{S})}$ such that $\pi|_{\Lambda}$ is bijective.

More generally, a **piecewise continuous near action** of a group Γ on \mathbf{S} means a homomorphism $\Gamma \rightarrow \text{PC}^{\bowtie}(\mathbf{S})$. We call it **realizable** if it can be lifted to a homomorphism $\Gamma \rightarrow \widehat{\text{PC}^{\bowtie}(\mathbf{S})}$.

Note that every subgroup of a realizable subgroup is realizable.

Example 1.6. Every finite subgroup of $\text{PC}^+(\mathbf{S})$ is realizable. For instance, every element of order 2 has a lift of order 2. This easy fact is true in the broader context of near actions, see Remark 2.3.

Example 1.7. The subgroup $\text{PC}^+(\mathbf{S})$ is realizable. Indeed, we can lift every $f \in \text{PC}^{\bowtie}(\mathbf{S})$ to its unique left-continuous representative, and in restriction to $\text{PC}^+(\mathbf{S})$, this lift is bijective and taking this lift is a group homomorphism.

Note that the left-continuous representative of f is bijective if and only if $f \in \text{PC}^{\pm}(\mathbf{S})$. However this does not make $\text{PC}^{\pm}(\mathbf{S})$ realizable: it is easy to find in IET^- a transformation of order 2 that has no lift of order 2 that is either left-continuous or right-continuous (see Lemma 4.6).

That $\text{PC}^+(\mathbf{S})$ is realizable makes it (and its subgroups) easier to define, since one can refer to piecewise continuous, left-continuous permutations of the circle. This artifact makes the definition shorter (since one does not have to mod out finitely supported permutations), and often explains the restriction to the piecewise orientation-preserving case, in many settings where this is not really used.

1.3. Non-realizability and restriction results. The first main result of this paper is a non-realizability theorem.

Theorem 1.8 (Theorem 4.7). *The group $\text{PC}^{\bowtie}(\mathbf{S})$ is not realizable. More precisely, IET^{\pm} is not realizable, and even has a finitely generated subgroup that is not realizable.*

Actually, the result also holds, with some technical cost, with the stronger conclusion “not stably realizable” (Theorem 4.17). The latter is a more natural notion, see §2; rather than defining it in this introduction, let us pinpoint that it is equivalent to the assertion that, for every nonempty open interval, the group of interval exchanges with flips that induce the identity on the complement of I , is not realizable. The latter can be viewed as the group of interval exchanges with flips of I . So the failure of stable realizability means that even making use of those additional points in the complement, does not allow to realize the action.

Our approach also provides, with further work, a result in the piecewise orientation-preserving case. For a subgroup Γ of $\text{PC}^+(\mathbf{S})$, we denote by Γ_{left} (respectively Γ_{right}) the group of left-continuous (resp. right-continuous) representatives of elements of Γ .

Recall that the **Q-rank** of an abelian group A is the dimension of $A \otimes_{\mathbf{Z}} \mathbf{Q}$; for a finitely generated abelian group $A \simeq \mathbf{Z}^d \times (\text{finite})$, this is just d .

Theorem 1.9 (Theorem 4.11 and Corollary 4.13). *Modulo conjugation by finitely supported permutations, the only subgroups lifting $\Gamma = \text{IET}^+$ are Γ_{left} and Γ_{right} . The same conclusion holds when Γ is:*

- *any subgroup of $\text{PC}^+(\mathbf{S})$ that includes IET^+ (e.g., the piecewise affine orientation-preserving subgroup);*
- *for any subgroup Λ of rotations of **Q-rank** ≥ 2 , the subgroup IET_{Λ}^+ of interval exchanges with singularities and translation lengths in Λ .*

1.4. Realizability results. Realizability makes sense in a much more general setting (near actions on sets), see §2. As already mentioned, finite groups are always realizable, see Remark 2.3. This is also true for free subgroups of the quotient by finitely supported permutations, obviously. On the other hand, this is not true for general near actions of \mathbf{Z}^2 , as a variety of examples in [Cor3] show. More precisely, it is easy to find two permutations of a set that commute as near permutations (i.e., their commutator is finitely supported), but they cannot be perturbed (i.e., multiplied by finitely supported permutations) so that the resulting permutations commute. Nevertheless, we show here that such phenomena cannot arise in the context of piecewise continuous near actions on the circle.

Theorem 1.10. *Any finitely generated abelian subgroup of $\text{PC}^{\boxtimes}(\mathbf{S})$ is realizable.*

The proof of Theorem 1.10 is not direct. It makes use the fact that the near action on the circle can be viewed as a projection of a genuine action, namely obtained by doubling all points (the Denjoy blow-up).

Remark 1.11. The near action of $\text{PC}^{\boxtimes}(\mathbf{S})$ is completable, in the sense that there exists an action on a set X (some huge non-Hausdorff connected compact 1-dimensional manifold), in which \mathbf{S} sits as a commensurated subset, so that the induced near action on \mathbf{S} is the given one. This observation comes from [Cor2].

1.5. Outline. In §2, we recall the language of near actions introduced in [Cor3], and prove some preliminary results for which the 1-dimensional context does not play any role. In §3, we conclude the proof of the positive results, on piecewise continuous near actions of finitely generated abelian groups. In §4, we prove the non-realizability results.

1.6. Open questions.

Question 1.12. Let Λ be a (virtually) infinite cyclic subgroup of \mathbf{S} . Is the near action of $\text{IET}_{\Lambda}^{\boxtimes}$ on \mathbf{S} realizable? stably realizable?

Question 1.13. Is there a subgroup of $\text{PC}^{\boxtimes}(\mathbf{S})$ that is stably realizable but not realizable?

Question 1.14. Is every solvable subgroup of $\text{PC}^{\bowtie}(\mathbf{S})$ realizable? stably realizable? Same question for a finitely generated subgroup, and/or assuming it is included in $\text{IET}^{\bowtie}(\mathbf{S})$.

Question 1.15. Does the near action of $G = \text{PC}^{\bowtie}(\mathbf{S})$ (or IET^{\bowtie}) on \mathbf{S} have a zero Kapoudjian class? This vanishing would mean that G can be lifted to the quotient of $\widehat{\text{PC}^{\bowtie}(\mathbf{S})}$ by its subgroup of *alternating* finitely supported permutations. (See [Cor3, §8.C] for more on the Kapoudjian class.)

Question 1.16. Is IET^{\bowtie} realizable in the measurable setting (in the sense of Mackey [Mack])? Namely, denote by \hat{G} the group of permutations f of \mathbf{S} such that both f and f^{-1} are Lebesgue-measurable and such that f is measure-preserving. Let G be its quotient by the subgroup of those f that are identity on a subset of measure 1. Is the surjective homomorphism $\hat{G} \rightarrow G$ split in restriction to $\text{IET}^{\bowtie} \subset G$?

Acknowledgement. I am very grateful to the referee for a careful reading and pointing out serious mistakes in a first version.

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2. PRELIMINARIES ON GENERAL NEAR ACTIONS

Convention (contain/include): on the one hand, the assertion “ $a \in X$ ” is written: “ a **belongs** to X ”, or “ X **contains** a ”; on the other hand “ $X \subset Y$ ” is written “ X is **included** in Y ”, or “ Y **includes** X ”.

2.1. Basic definitions and facts. By “cofinite subset” we mean “subset with finite complement”. Essentially following Wagoner [Wa, §7] (see also [Cor3] for a detailed historical account), a cofinite-partial bijection of a set X is a bijection between two cofinite subsets of X . If we identify any two such cofinite-partial bijections when they coincide on a cofinite subset, we obtain the **near symmetric group** of X , denoted by $\mathfrak{S}^*(X)$. Its elements, namely cofinite-partial bijections modulo cofinite coincidence, are called **near permutations** of X . There is a canonical homomorphism from the group $\mathfrak{S}(X)$ of permutations of X to $\mathfrak{S}^*(X)$. Its kernel is the subgroup $\mathfrak{S}_{\text{fin}}(X)$ of finitely supported permutations. Its image $\mathfrak{S}_0^*(X)$ consists by definition of **balanced near permutations** and is called the **balanced symmetric group** of X . Given a cofinite-partial bijection $f : X \setminus F_1 \simeq X \setminus F_2$, the number $|F_2| - |F_1|$ is called the **index** $\phi_X(f)$ of f . The index map factors through a group homomorphism $\phi_X : \mathfrak{S}^*(X) \rightarrow \mathbf{Z}$, called **index homomorphism**, whose kernel is precisely $\mathfrak{S}_0^*(X)$. If X is infinite, the index homomorphism ϕ_X is surjective, so that the cokernel $\mathfrak{S}^*(X)/\mathfrak{S}_0^*(X)$ is infinite cyclic.

Definition 2.1 ([Cor3]). A **near action** of a group G on a set X is the datum of a homomorphism $\alpha : G \rightarrow \mathfrak{S}^*(X)$; then X is called a **near G -set**. The near action is said to be **balanced** if α is valued in $\mathfrak{S}_0^*(X)$, or equivalently if the **index homomorphism** $\phi_X \circ \alpha \in \text{Hom}(G, \mathbf{Z})$ of the near action α is zero.

A near G -set X as above (or the near action of G on X) is said to be \star -faithful if α is injective. It is said to be near free if for every $g \in G \setminus \{1\}$, the set of points fixed by g (which is well-defined modulo symmetric difference with finite subsets) is finite.

For subsets U, V of a set X , we write $U \stackrel{\star}{=} V$ and say that U and V are near equal if $U \Delta V$ is finite. We write $U \subset^{\star} V$ and say that U is near included in V if $U \setminus V$ is finite: thus $U \stackrel{\star}{=} V$ if and only if both $U \subset^{\star} V$ and $V \subset^{\star} U$. For maps $f, g : X \rightarrow Y$, we write $f \stackrel{\star}{=} g$ if f and g coincide outside a finite subset: this means that among subsets of $X \times Y$, the graphs of f and g are near equal.

While the notion of invariant subset for a group action does not pass to near actions, we have a notion of commensurated subset. Namely, given a near action of G on X , a subset $Y \subset X$ is **G -commensurated** if for every $g \in G$, we have $gY \stackrel{\star}{=} Y$. (Note that gY is well defined modulo near equality.) Thus, G -commensurated subsets are the same as near G -subactions.

A near G -set X is said to be **1-ended** if X is infinite and its only commensurated subsets are finite or have finite complement. That is, X is not decomposable

as disjoint union of two infinite near G -subactions. A near G -set is said to be **finitely-ended** if it had only finitely many commensurated subset modulo near equality. This means that it decomposes as a finite disjoint union of 1-ended commensurated subsets (which are well-defined modulo near equality), their number is called the number of ends of the near G -set X .

Recall that a map between sets is said to be **proper** if it has finite fibers. A map $f : X \rightarrow Y$ (with cofinite domain of definition) between near G -sets (either assuming f to be proper, or Y to be a given G -set) is said to be **near G -equivariant** if for every $g \in G$, the set of $x \in X$ such that $f(gx) = gf(x)$ is cofinite in X . Note that we choose here, for given g , self-maps $x \mapsto gx$ and $y \mapsto gy$ of X and Y ; the condition does not depend on these choices when f is proper, or when the G -action on Y is fixed (with neither assumption the notion of near equivariant map is ill-defined). The map f is said to be a **near isomorphism** if there exists another such near G -equivariant map $f' : Y \rightarrow X$ such that $f \circ f' \stackrel{*}{=} \text{Id}_Y$ and $f' \circ f \stackrel{*}{=} \text{Id}_X$; in this case X and Y are said to be **near isomorphic** near G -sets. If moreover f can be chosen to be bijective, we say that X and Y are **balanceably near isomorphic**.

Definition 2.2. A near action of a group G on a set X is:

- **realizable** if the homomorphism $G \rightarrow \mathfrak{S}^*(X)$ comes from a homomorphism $G \rightarrow \mathfrak{S}(X)$;
- **[finitely] stably realizable** if there exists a [finite] set Y with trivial near action such that the near action on $X \sqcup Y$ is realizable;
- **completable** if there exists a near G -set Y such that the near action on $X \sqcup Y$ is realizable.

Obviously, realizable implies finitely stably realizable, which implies stably realizable, which implies completable. Examples in [Cor3] show that none of the converse implications holds, and that there exist non-completable near actions. All these examples are taken with $G = \mathbf{Z}^2$, except for the difference between stably realizable and finitely stably realizable: indeed it is established in [Cor3] that these two notions are equivalent for finitely generated groups.

Remark 2.3. Every near action of a finite group is realizable [Cor3, Proposition 5.A.1]. For completeness let us sketch the argument. It is enough to check that every finite subgroup $F \subset \mathfrak{S}^*(X)$ can be lifted. Since $\text{Hom}(F, \mathbf{Z}) = \{0\}$, we have $F \subset \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}(X)$. Hence we can find a finitely generated subgroup $F' \subset \mathfrak{S}(X)$ projecting onto F , with kernel N of finite index. Since N is finitely generated and acts by finitely supported permutations, the union Y of supports of its elements is finite; it is also F' -invariant. Changing the F' -action to be trivial on Y , we obtain another subgroup, which is a lift of F .

Let G be a finitely generated group and X a near G -set. Fix a finite generating subset S of G , and lift each $s \in S$ to cofinite-partial bijection $x \mapsto sx$ of X . The corresponding **near Schreier graph** consists in joining x to sx for all $s \in S$.

The near action is said to be **of finite type** if the near Schreier graph has finitely many components. Routine arguments in [Cor3] show that this does not depend on the choices.

2.2. Rigidity of 1-ended actions. The following rigidity results are used both in §3 and §4. They are borrowed from [Cor3, §7]; we include most proofs for completeness and for convenience.

We first recall, given an infinite finitely generated abelian group A , that the 1-ended commensurated subsets of A are, up to finite symmetric difference

- A itself if A has \mathbf{Q} -rank ≥ 2 ;
- $f^{-1}(\mathbf{N})$ and $f^{-1}(-\mathbf{N})$ if A has \mathbf{Q} -rank 1, where f is a homomorphism of A onto \mathbf{Z} .

Lemma 2.4. *Let G be a group and Y a 1-ended commensurated subset (for the left action of G on itself). Let $f : Y \rightarrow G$ be a near equivariant map, in the sense that for every g , for all but finitely many h we have $f(gh) = gf(h)$. Then there exists a unique $s \in G$ such that for all but finitely many $g \in Y$ we have $f(g) = gs$.*

Proof. Uniqueness holds because Y is infinite. Let us prove the existence. Define $u(g) = g^{-1}f(g)$. By assumption, for every $g \in G$, for all but finitely many $h \in Y$ we have $u(gh) = u(h)$. If we check that u is constant outside a finite subset of Y , we obtain the conclusion. Otherwise, there exists a partition of G into two subsets Z, Z' such that both $u^{-1}(Z)$ and $u^{-1}(Z')$ are infinite; since these form a partition of Y into two infinite commensurated subsets, we contradict the assumption that Y is 1-ended. \square

Given two actions α, α' of a group on a set, we say that they are finite perturbations of each other if they induce the same near action. In other words, this means that for every g , the permutations $\alpha(g)$ and $\alpha'(g)$ coincide on a cofinite subset (depending on g).

Proposition 2.5. *Let A be a finitely generated abelian group. Let X be an A -set, and let $X_{\geq 2}$ be the union of A -orbits of at least quadratic growth. Then any finite perturbation of the action is conjugate, by a finitely supported permutation, to an action that is unchanged on $X_{\geq 2}$.*

Proof. We call “orbits” those orbits of the original action α and “new orbits” those of the perturbed action α' .

First assuming that the number n of G -orbits of at least quadratic growth is finite, we argue by induction on n . The case $n = 0$ is trivial; assume $n \geq 1$. Let Y be an orbit of at least quadratic growth. Let $B \subset A$ be point stabilizer of Y , that is, the stabilizer of some/any element of Y . Then Y is a 1-ended commensurated subset of at least quadratic growth (for either action), and hence has finite symmetric difference with a new orbit Y' , also of at least quadratic growth, with point stabilizer B' ; choose $y \in Y \cap Y'$. Since every element of B

acts with finite support on Y' (for the original, and hence for the new action), and since the new action of A/B' is free on Y' and Y' is infinite, we have $B \subset B'$. Arguing the other way round, we have $B' \subset B$, and hence $B = B'$. Extend the identity map of $Y \cap Y'$ to a map $f : Y \rightarrow Y'$. Then f is near equivariant from the original to the new action. The map $g \mapsto \alpha(g)y$ induces an equivariant bijection $u : A/B \rightarrow Y$, and similarly the map $g \mapsto \alpha'(g)y$ induces an equivariant (for the new action) bijection $u' : A/B \rightarrow Y'$. Define $f' = u'^{-1} \circ f \circ u$. Then f' is a near equivariant (proper) self-map of A/B . By Lemma 2.4, f' coincides outside a finite subset with a translation $g \mapsto gh_0$. In particular, f' has index zero, and hence f has index zero. This means that $Y \setminus Y'$ and $Y' \setminus Y$ have the same cardinal. Hence, after conjugating the new action by a finitely supported permutation, we can suppose that $Y = Y'$.

Now $Y = Y'$. Since for all but finitely many $g \in A/B$ we have $f'(g) = gh_0$, setting we have, for all but finitely $g \in A/B$, the equality $u(g) = u'(gh_0)$. Setting $y' = \alpha'(h_0)y$, this rewrites as: for all but finitely many $g \in A/B$, we have $\alpha(g)y = \alpha'(g)y'$. We define a self-map w of Y by $w(\alpha(g)y) = \alpha'(g)y'$. So, w is injective and equals the identity outside a finite subset, hence it is surjective. Thus w is a finitely supported permutation. Then for all $g \in G$ and $x \in X$, choosing $h \in G$ such that $x = \alpha(h)y$, we have

$$w(\alpha(g)x) = w(\alpha(gh)y) = \alpha'(gh)y' = \alpha'(g)w(x),$$

thus $w^{-1} \circ \alpha'(g) \circ w = \alpha(g)$. Hence conjugating by w yields the original action on Y . Now removing Y we can continue by induction.

Finally, when X has infinitely many orbits, the original and the new action differ only on finitely many orbits, so the result follows from the case of an action with finitely many orbits. \square

The following is established in [Cor3, Theorem 7.C.1] and will be used in §4. The case when G is a finitely generated abelian group of \mathbf{Q} -rank ≥ 2 is covered by Proposition 2.5 and is enough for most of the purposes, so we do not prove the general case, which relies on the same ideas. Recall that a group G (not necessarily finitely generated) is said to be **1-ended** if it is a 1-ended G -set (under the left action).

Proposition 2.6. *Let G be a 1-ended group that is not locally finite, and X a G -set. Then*

- (1) *Suppose that G acts freely on X . Then any finite perturbation of the action is conjugate, by a unique finitely supported permutation, to the original action.*
- (2) *Suppose instead that G acts freely on $Y = X \setminus X^G$ (where X^G is the set of points fixed by all of G). Then any finite perturbation of the action is conjugate, by a finitely supported permutation, to an action that is unchanged on Y .* \square

It is an old result of W. Scott and L. Sonneborn [ScS, Theorems 1 and 2] that an abelian group A is 1-ended if and only if it satisfies one of the following conditions: A is uncountable, or A has \mathbf{Q} -rank ≥ 2 , or A is infinitely generated of \mathbf{Q} -rank 1. (The remaining abelian groups are countable locally finite or virtually cyclic.)

2.3. Facts on realizability for finitely generated abelian groups. The next two propositions are borrowed from [Cor3]. They are used to prove Corollary 2.9, which is used in §3. Again, we include the proofs for completeness.

Proposition 2.7. *Let A be a finitely generated abelian group. Let X be a near A -set. Equivalences:*

- X is finitely stably realizable;
- X is completable, and the index homomorphism of the near A -subset X^B of B -fixed points is zero for every subgroup B of A .

Note that the A -commensurated subset X^B is defined up to $\stackrel{*}{\cong}$, and hence whether it is balanced is a well-defined notion.

Proof. This is proved in [Cor3]; for completeness we include the proof. The forward implication is clear; suppose that the condition holds. Complete X to an A -set $X \sqcup Y$, so that X meets every orbit. Since X has finite boundary in the Schreier graph, all but finitely many orbits are included in X . We can then restrict to those orbits not including X , and thus assume that X has finite type.

So, suppose that X has finite type and satisfies the second condition. Since every transitive A -set is finitely-ended, and X is completable, X is finitely-ended.

For each subgroup B of A such that A/B has \mathbf{Q} -rank 1, the number of homomorphisms of A/B onto \mathbf{Z} is 2, and we choose one of the two as u_B . Then the classification of 1-ended completable near A -sets up to near isomorphism is as follows: it consists of those $E_B = A/B$ when B ranges over subgroups of A such that A/B has \mathbf{Q} -rank ≥ 2 , those $E_B^+ = u_B^{-1}(\mathbf{N})$ and $E_B^- = u_B^{-1}(-\mathbf{N})$ when B ranges over subgroups of A such that A/B has \mathbf{Q} -rank 1.

Let B be maximal among subgroups such that X^B is infinite. Then X^B satisfies the balanced assumption: indeed, by maximality of B , for every subgroup B' of A , we have $(X^B)^{B'} \stackrel{*}{\cong} X^B$ if $B' \subseteq B$, and $(X^B)^{B'} \stackrel{*}{\cong} \emptyset$ otherwise. Since both X^B and X satisfy the balanced assumption, so does $X \setminus X^B$. Hence, if $X \setminus X^B$ is infinite, we can argue by induction on the number of ends to deduce that X is finitely stably realizable.

Now we assume that X^B is cofinite in X . So, using the maximality of B , the near A -set X is a completable near free near A/B -set of finite type. Hence, it is near isomorphic to a commensurated subset of some free A/B -action with finitely many orbits. If A/B has \mathbf{Q} -rank ≥ 2 , this implies that X is near isomorphic to some disjoint union of copies of E_B , and hence is stably realizable. If A/B has \mathbf{Q} -rank 1, this implies that X is near isomorphic to the disjoint union of k copies of E_B^+ and ℓ copies of E_B^- . Note that the index homomorphism is additive under

disjoint unions, and is opposite and nonzero for E_B^+ and E_B^- . That the index homomorphism of X vanishes then implies that $k = \ell$. So X is near isomorphic to the disjoint union of k copies of A/B , so is finitely stably realizable. \square

Proposition 2.8. *Let A be a finitely generated abelian group. A near A -set X is finitely stably realizable but not realizable if and only if for some nonempty finite set F , the disjoint union $X \sqcup F$ is realizable as an action with only orbits of at least quadratic growth.*

Proof. \Rightarrow Suppose that $X \sqcup F$ is realizable for some nonempty finite set F , with F of minimal cardinal, and assume X not realizable, so F is not empty. Fix a realization. Then it has no finite orbit, because this would contradict the minimality of F . Also it has no orbit of linear growth: indeed, an orbit of linear growth, say isomorphic to a quotient of A isomorphic to $\mathbf{Z} \times K$ with K finite, is balanceably isomorphic to itself minus $|K|$ points. Again this contradicts the minimality of F , and hence the realization has only orbits of at least quadratic growth.

Conversely, for an A -set X with only orbits of at least quadratic growth, let us show that X minus any nonempty finite set is not realizable. Equivalently, let us show that every realization of the near action on X has no finite orbit. This follows from Proposition 2.5 (with $X = X_{\geq 2}$). \square

Corollary 2.9. *Let A be a finitely generated abelian group. Let X be a near A -set. Suppose that for some $n \geq 1$, the disjoint union nX of n copies of X is realizable (resp. finitely stably realizable). Then so is X .*

Proof. The assumption immediately implies that X is completable.

Then the result for finite stable realizability immediately follows from the criterion of Proposition 2.7. Now assume that nX is realizable. Hence X is finitely stably realizable. Assuming by contradiction that X is not realizable, there exists, by the forward implication in Lemma 2.8, a nonempty finite set F such that $X \sqcup F$ is realizable as an action with only orbits of quadratic growth. Hence $nX \sqcup nF$ has the same property. By the reverse implication in Lemma 2.8, we deduce that nX is not realizable, a contradiction. \square

Remark 2.10. In contrast, in [Cor3], examples of groups G are given for which there exists a near G -set X that is not stably realizable, such that $X \sqcup X$ is realizable. Such groups can both be chosen to be finitely generated (some well-chosen amalgam of two finite groups), or abelian (the quasi-cyclic group $\mathbf{Z}[1/2]/\mathbf{Z}$).

2.4. A realizability criterion. Applying results of the previous subsections, we obtain the following results (which are not in [Cor3]), and are used in §3 to obtain the realizability result.

Lemma 2.11. *Let A be a finitely generated abelian group and X an A -set with finitely many orbits. Let τ be a fixed-point-free permutation of X , of order 2. Suppose that τ is near A -equivariant, and suppose that $(*)$ for every $g \in A$, the*

set of $x \in X$ such that $gx = \tau x$ is finite. Then there exist $n \geq 1$ and subsets X_0, \dots, X_n of X such that we have the partition

$$X = X_0 \sqcup \tau(X_0) \sqcup X_1 \sqcup \tau(X_1) \sqcup \dots \sqcup X_n \sqcup \tau(X_n),$$

such that X_0 is finite, and such that each X_i for $i \geq 1$ is A -commensurated and 1-ended.

Proof. Decompose X as finite union of 1-ended subsets. By near equivariance of τ , the permutation τ permutes these subsets up to finite symmetric difference. Removing finite subsets, we can find $m \geq 0$ and a partition $X = Z_0 \sqcup \dots \sqcup Z_m$ such that Z_0 is finite, each Z_i is 1-ended for $m \geq 1$, and τ permutes the Z_i . We claim that $\tau(Z_i) \neq Z_i$ for each $i \geq 1$.

First assume that Z_i has at least quadratic growth. Then there exists a 1-ended A -orbit Ax_0 having finite symmetric difference with Z_i ; let B be the stabilizer of x_0 . Define $\tau' : Ax_0 \rightarrow Ax_0$ coinciding with τ on $\tau^{-1}(Ax_0)$. Then τ' can be viewed as a near equivariant self-map of A/B . Hence, by Lemma 2.4, it coincides outside a finite subset with a translation. This translation cannot be trivial because τ is fixed-point-free. Hence, for some $s \in A \setminus B$, we have $\tau(x) = sx \neq x$ for all but finitely many $x \in Ax_0$. This contradicts (*).

The other case is when Z_i has linear growth. Then there exists $x_0 \in X$, with stabilizer B , such that A/B has \mathbf{Q} -rank 1, and a homomorphism f from A/B onto \mathbf{Z} , such that, denoting $A^+ = f^{-1}(\mathbf{N})$, Z_i has finite symmetric difference with A^+x_0 . Again by Lemma 2.4, we see that τ coincides outside a finite subset of Z_i with a translation and get a contradiction.

We conclude by defining $X_0 \subset Z_0$ so that $Z_0 = X_0 \sqcup \tau(X_0)$, defining I as a maximal subset of $\{1, \dots, m\}$ such that $\bigcup_{i \in I} Z_i$ is disjoint to its image by τ , and enumerate the Z_i for $i \in I$ as X_1, \dots, X_n . Since each Z_i for $i \geq 1$ is disjoint to its image by τ , the X_i and their images by τ indeed cover X . \square

Theorem 2.12. *Let A be a finitely generated abelian group. Let X be an A -set, and Y a near A -set. Let τ be a fixed-point-free near A -equivariant self-map of X , with $\tau^2 = \text{id}_X$. Suppose (*) that for every $g \in A$, the set of $x \in X$ such that $gx = \tau(x)$ is finite. Let $f : X \rightarrow Y$ be a surjective near equivariant map such that for every $x \in X$ we have $f^{-1}(\{f(x)\}) = \{x, \tau(x)\}$. Then*

- (1) Y is a realizable near A -set;
- (2) if X has finitely many A -orbits, then X is balanceably isomorphic, as near A -set, to $Y \sqcup Y$.

Proof. The assertion (2) is immediate from Lemma 2.11: indeed, in restriction to $U = X_0 \sqcup \dots \sqcup X_n$, f is near equivariant and bijective, and also in restriction to $\tau(U)$.

Now let us deduce (1). Let X' be the union of all orbits Z such that f and τ are equivariant on both Z and $\tau(Z)$; so X' is τ -invariant. So $f(X')$ is isomorphic as near A -set to the quotient of an A -set, and hence is realizable.

If X'' is the complement of X' , then $Y'' = f(X'')$ is the complement of $f(X')$, and X'' consists of finitely many A -orbits. Hence we have to prove that Y'' is realizable. By (2), $Y'' \sqcup Y''$ is realizable. Finally by Corollary 2.9, Y'' is realizable, and hence so is $Y = Y'' \sqcup f(X')$. \square

Remark 2.13. Here is a counterexample to the statement of Theorem 2.12 with (*) removed. Let K be the Klein group of order 4, and u, v two distinct elements of order 2 in K , and F the subgroup generated by u . Let A be the group $F \times \mathbf{Z}$, and $X = K \times \mathbf{Z}$, which is thus a free A -set with two orbits. Define a permutation of order 2 of X by $\sigma(g, n) = (ug, n)$ for $n < 0$ and $\sigma(g, n) = (vg, n)$ for $n \geq 0$. So σ commutes with the action of u and near commutes with the action of \mathbf{Z} , in the sense that the commutator of σ with the generator $(g, n) \mapsto (g, n + 1)$ has finite support. Hence, the quotient by the u -action is naturally a near A -set. It can be identified to $F \times \mathbf{Z}$, where \mathbf{Z} acts by shifting, while u acts by $(g, n) \mapsto (g, n)$ for $n < 0$ and $\mapsto (ug, n)$ for $n \geq 0$. This near action is not stably realizable (this is the very first example in [Cor3]).

3. REALIZABILITY OF PIECEWISE CONTINUOUS NEAR ACTIONS OF FINITELY GENERATED ABELIAN GROUPS

We now use the results of §2.4 (namely Theorem 2.12) to prove Theorem 1.10.

3.1. The “true” definition of $\text{PC}^{\bowtie}(\mathbf{S})$. Let X be a Hausdorff topological space. The group $\text{PC}(X)$ of **near self-homeomorphisms** of X consists of those elements of $\mathfrak{S}^*(X)$ that have a representative that is a homeomorphism between two cofinite subsets.

Let $\widehat{\text{PC}}_0(X)$ be the subgroup of permutations f of X such that both f and f^{-1} are continuous outside a finite subset. There is a canonical homomorphism $\widehat{\text{PC}}_0(X) \rightarrow \text{PC}(X)$; its image $\text{PC}_0(X)$ equals $\text{PC}(X) \cap \mathfrak{S}_0^*(X)$, and its kernel consists of finitely supported permutations of X .

A basis remark is that $\text{PC}_0(X)$ is a proper subgroup of $\text{PC}(X)$ if and only if there exist two finite subsets F, F' of X with $|F| < |F'|$ such that $X \setminus F$ and $X \setminus F'$ are homeomorphic.

For instance, this holds when X is infinite discrete, or when X is a Cantor space. Nevertheless, we have $\text{PC}_0(X) = \text{PC}(X)$ for $X = \mathbf{S}$: indeed, \mathbf{S} minus n points is homeomorphic to \mathbf{S} for $n = 0$ and to the disjoint union of n copies of \mathbf{R} when $n \geq 1$, so its topological type retains n . Hence, $\text{PC}(\mathbf{S}) = \text{PC}_0(\mathbf{S}) = \text{PC}^{\bowtie}(\mathbf{S})$.

Remark 3.1. In [Cor2], it is proved that the near action of $\text{PC}(X)$ on X is completable as soon as X has no isolated point. This notably applies to $X = \mathbf{S}$.

3.2. The Denjoy blow-up. Let \mathbf{S} denote the circle \mathbf{R}/\mathbf{Z} . Let \mathbf{S}^{\pm} denote the “Denjoy blow-up” of \mathbf{S} at all points. As a set, it can simply be defined as the Cartesian product $\mathbf{S} \times \{\pm 1\}$, where we write x^+ and x^- for the elements $(x, 1)$ and $(x, -1)$. For $y = (x, \varepsilon) \in \mathbf{S}^{\pm}$, we write $\hat{y} = (x, -\varepsilon)$ and $\bar{y} = x$.

It turns out that there are two natural compact Hausdorff topologies on this blow-up. The first is the product topology. The second, called circular topology, is the topology of the cyclic ordering, where, whenever $x < y < z$ in \mathbf{S} , we prescribe $x^- < x^+ < y^- < y^+ < z^- < z^+$. (Here, in a cyclic ordering, by $x_1 < x_2 < \dots < x_n$ we mean that $x_i < x_j < x_k$ for all $1 \leq i < j < k \leq n$.) The circular topology is compact Hausdorff, totally disconnected, but not metrizable (since the set of clopen subsets is uncountable).

The interest is that the group $\text{PC}^\times(\mathbf{S})$ naturally acts on \mathbf{S}^\pm , using one-sided limits in the obvious way, and this action preserves the circular topology. This makes the projection map (the Denjoy blow-up map) $\mathbf{S}^\pm \rightarrow \mathbf{S}$, $y \mapsto \bar{y}$, near $\text{PC}^\times(\mathbf{S})$ -equivariant.

3.3. Proof of realizability. We need the following fact about the Denjoy blow-up.

Proposition 3.2. *For every $g \in \text{PC}^\times(\mathbf{S})$, the set of $x \in \mathbf{S}^\pm$ such that $g(x) = \hat{x}$ is finite.*

Proof. If by contradiction it is infinite, it has an accumulation point; conjugating by a suitable element of the isometry group $\text{O}(2)$, we can suppose that this accumulation point is 0^+ . Hence, there is an injective sequence (x_n) tending to 0^+ such that $g(x_n) = \hat{x}_n$ for every n . There exists $\varepsilon \in]0, 1[$ such that g induces a continuous (necessarily strictly monotone) function \bar{g} on $]0, \varepsilon[$, valued in $]0, 1[$. Extracting, we can suppose that $0 < \bar{x}_{n+1} < \bar{x}_n < \varepsilon$ for all n .

On the one hand, since $g(x_n) = \hat{x}_n$ for every n , \bar{g} is necessarily decreasing on $]0, \varepsilon[$. On the other hand, since $g(x_n) = \hat{x}_n$, we have $\bar{g}(\bar{x}_n) = \bar{x}_n$ for all n , which implies that \bar{g} is increasing on $]0, \varepsilon[$. Contradiction. \square

Theorem 3.3. *Every finitely generated abelian subgroup A of $\text{PC}^\times(\mathbf{S})$ is realizable (for its near action on \mathbf{S}).*

Proof. We use the Denjoy blow-up map $\mathbf{S}^\pm \rightarrow \mathbf{S}$. Here \mathbf{S}^\pm is an A -set, this map has fibers $\{x, \hat{x}\}$ of cardinal 2 and is near equivariant as well as the involution $x \mapsto \hat{x}$; moreover it satisfies the additional assumption (*) of Theorem 2.12, by Proposition 3.2. Hence, Theorem 2.12 applies and \mathbf{S} is a realizable A -set. \square

Remark 3.4. One step to the theorem was to show that A is stably realizable. This step is much easier when $A \subset \text{IET}^\times$, or more generally when the near action of A is piecewise analytic. Indeed, in this case, the criterion of Proposition 2.7 can be checked directly, as the set of fixed points of any finitely generated subgroup is then a Boolean combination of intervals.

4. NON-REALIZABILITY OF GROUPS OF INTERVAL EXCHANGES WITH FLIPS

4.1. Non-realizability. Recall that the projection $\mathbf{S}^\pm \rightarrow \mathbf{S}$, mapping x^+ and x^- to x , is denoted by $z \mapsto \bar{z}$.

For a subgroup Λ of \mathbf{R}/\mathbf{Z} , let $\text{IET}_\Lambda^\boxtimes$ be the subgroup of IET^\boxtimes of elements with discontinuities in Λ , and local isometries of the form $x \mapsto \pm x + \lambda$ with $\lambda \in \Lambda$. Define $\text{IET}_\Lambda^+ = \text{IET}_\Lambda^\boxtimes \cap \text{IET}^+$.

In IET^+ , we have the subgroup of genuine **rotations** $r_t : x \mapsto x+t$ (translations of the group $\mathbf{S} = \mathbf{R}/\mathbf{Z}$). We endow \mathbf{S} with its geodesic distance.

Given any interval I in \mathbf{S} (of measure in $]0, 1[$), we have a corresponding subgroup of **partial rotations**, acting trivially outside I , and acting as genuine rotations on I when we “close it” by identifying endpoints of \bar{I} .

Given $f \in \text{PC}^\boxtimes(\mathbf{S})$, define its **essential support** $\text{essupp}(f) \subset \mathbf{S}$ as the closure of the set of x such that $(f(x^-), f(x^+)) \neq (x^-, x^+)$. It is empty if and only if f is the identity. When $f \in \text{IET}^\boxtimes$ and \tilde{f} is a lift, note that $\text{essupp}(f)$ has finite symmetric difference with $\{x : \tilde{f}(x) \neq x\}$.

For $f \in \text{PC}^\boxtimes(\mathbf{S})$, define $\text{sing}(f) = \{x \in \mathbf{S}^\pm : f(\hat{x}) \neq \widehat{f(x)}\}$. It is finite, and obviously invariant under $x \mapsto \hat{x}$; let $\text{sing}(f)$ be its image in \mathbf{S} . This is the set of points at which every lift of f is discontinuous.

For $f \in \text{IET}^+(\mathbf{S})$, choose a representative $g(x) = x + \tau_g(x)$. Note that while τ_g depends on the choice of g , the values $\tau_g(x^-)$ and $\tau_g(x^+)$ (in \mathbf{S}^\pm) only depend on f . We write $\tau_f^-(x) = \overline{\tau_g(x^-)}$ and $\tau_f^+(x) = \overline{\tau_g(x^+)}$ (these elements of \mathbf{S} are just the one-sided limits at x). Thus $\text{sing}(f) = \{x : \tau_f^-(x) \neq \tau_f^+(x)\}$. Write $\nu_f(x) = \tau_f^+(x) - \tau_f^-(x)$, so $\text{sing}(f) = \{x : \nu_f(x) \neq 0\}$.

Let $E(f) \in]0, 1/2] \cup \{\infty\}$ be the minimal distance between any two points of the finite subset $\text{sing}(f)$ (where $E(f) = \infty$ if $\text{sing}(f)$ is empty, i.e., if f is a rotation).

When we consider the action of $\text{PC}^\boxtimes(\mathbf{S})$ on subsets of \mathbf{S} , it is only well-defined modulo finite symmetric difference with finite subsets. We then talk of near subset, near disjoint (= finite intersection), near partition, etc.

Lemma 4.1. *For every $f \in \text{IET}^+$ and $t \in]0, E(f)[$, the “commutator” $c = f^{-1}r_t f r_t^{-1}$ permutes the near intervals $[x, x+t]$, $x \in \text{sing}(f)$, by translations, without preserving any of them. These intervals are pairwise near disjoint, and the essential support is $\bigsqcup_{x \in \text{sing}(f)} [x, x+t]$.*

Proof. In this proof, “generic” means “with finitely many exceptions”, and we freely choose representatives.

If f is a rotation then c is the identity. Otherwise, f has at least two singularities. Let a, b be consecutive singularities of f . Then the representative of $b - a$ in $]0, 1[$ is $\geq t$. So we can view the interval $[a, b]$ as concatenations of intervals $[a, a+t]$ and $[a+t, b]$, and for generic $x \in [a+t, b]$, we have $f(x) = f(x-t) + t$. For a generic $x \in [a+t, b]$, we have $c(x) = f^{-1}r_t f(x-t) = f^{-1}r_t(f(x-t)) = f^{-1}(f(x)) = x$.

For generic $x \in [a, a+t]$, we have $c(x) = f^{-1}r_t f(x-t) = f^{-1}r_t(x-t + s_f^-(a)) = f^{-1}(x + s_f^-(a))$. Observe that $\overline{f(a^-)} = a + \tau_f^-(a)$ belongs to $\text{sing}(f^{-1})$. Since $E(f) = E(f^{-1})$, this implies that for $x \in]a, a+t[$, $x + \tau_f^-(a)$ meets no singularity

of f^{-1} . Hence c has no singularity in $]a, a+t[$. In addition, for generic $x \in [a, a+t]$, we have $f(x-t) \neq f(x)-t$, and hence $c(x) = f^{-1}(t+f(x-t)) \neq f^{-1}(f(x)) = x$.

Thus the image by c of $[a, a+t]$ is (essentially) an interval of length t , included in the union of the intervals $[x, x+t]$, for $x \in \text{sing}(f)$. Therefore, it is exactly one of these intervals (and not $[a, a+t]$). \square

The diameter of a metric space is the supremum of distances between any two points.

Lemma 4.2. *Let Γ be a subgroup of $\text{IET}^+(\mathbf{S})$. Suppose that Γ includes a dense subgroup of rotations, and, for some proper sub-interval, some dense subgroup of the corresponding partial rotations. Then Γ contains non-identity elements whose essential support has arbitrary small diameter.*

Proof. Up to conjugating by a rotation, we can suppose that Γ includes a dense subgroup of partial rotations, namely of the interval $[0, \rho]$ for some $\rho \in]0, 1[$.

Fix $\varepsilon_0 > 0$ small enough (see below), and let us produce non-identity element whose essential support has diameter $\leq 5\varepsilon_0$. We choose some $\varepsilon \in]0, \varepsilon_0[$ with the two additional conditions:

- $\varepsilon < \min(\rho, 1 - \rho)/5$;
- defining the partial rotation f of $[0, \rho]$, of length -2ε , the value of ε has been chosen so that $f \in \Gamma$.

Hence s_f generically equals $\rho - 2\varepsilon$ on $[0, 2\varepsilon]$, -2ε on $[2\varepsilon, \rho]$, and 0 on $[\rho, 1]$. Choose η with $0 < \eta \leq \varepsilon$, such that $r_\eta \in \Gamma$. Then the essential support of $c = f^{-1}r_\eta f r_\eta^{-1}$ is, by Lemma 4.1, equal to $[0, \eta] \cup [2\varepsilon, 2\varepsilon + \eta] \cup [\rho, \rho + \eta]$. Consider in Γ a rotation of length $\lambda \in]3\varepsilon, 4\varepsilon[$. Conjugate the given group of partial rotations by this rotation to obtain a dense group of partial rotations on $[\lambda, \lambda + \rho]$. (By the condition on ε , we have $\lambda < \rho$ and $\lambda + \rho < 1$.) Then there exists q in this dense group of rotations of $[\lambda, \lambda + \rho]$ (essentially) mapping $[\rho, \rho + \eta]$ into $] \lambda, 5\varepsilon[$, say $[\lambda', \lambda' + \eta]$ with $3\varepsilon < \lambda' < 4\varepsilon$. Then the essential support of $q^{-1}c q$ is $[0, \eta] \cup [2\varepsilon, 2\varepsilon + \eta] \cup [\lambda', \lambda' + \eta]$, and thus has diameter $\leq 5\varepsilon \leq 5\varepsilon_0$. \square

Definition 4.3. We say that a subgroup of $\widehat{\text{PC}}^{\bowtie}(\mathbf{S})$ is

- clean if its intersection with the group of finitely supported permutations is trivial;
- hyper-clean if for every g in the subgroup, the graph of g (viewed as subset of $\mathbf{S} \times \mathbf{S}$) has no isolated point; equivalently if, at every point, g is either left or right-continuous.

Lemma 4.4. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\text{PC}}^{\bowtie}(\mathbf{S})$, and Γ its image in $\text{PC}^{\bowtie}(\mathbf{S})$. Suppose that $\tilde{\Gamma}$ includes a dense subgroup $\tilde{\Lambda}$ of rotations. Suppose that Γ admits non-identity elements with essential support of arbitrary small diameter. Then $\tilde{\Gamma}$ is hyper-clean.*

Proof. For $h \in \widehat{\text{PC}^\infty(\mathbf{S})}$, define its interior support as the intersection of the support $\{x : hx \neq x\}$ with the set of continuity points of h . It is open, and is included and cofinite in the h -invariant subset $\{x : hx \neq x\}$, and it is also included and dense in the essential support of its image \bar{h} in $\text{PC}^\infty(\mathbf{S})$.

In a first step, we show that $\tilde{\Gamma}$ is locally clean, in the sense that for each element g , the support $\{x : gx \neq x\}$ has no isolated point.

By contradiction, let x be an isolated non-fixed point of g . For some $\varepsilon > 0$, all other points in $]x - \varepsilon, x + \varepsilon[$ are fixed by g . There exists $h_0 \in \tilde{\Gamma}$ whose essential support has diameter $< \varepsilon$. Hence there exists some conjugate h of h_0 by some element of $\tilde{\Lambda}$ such that both x and $h(x)$ belong to the interior support of h (indeed, letting I be the interior support of h_0 and $J = h_0^{-1}I \cap I$, which is cofinite in I , it is enough to find $s \in \tilde{\Lambda}$ such that $x \in sI$ and define $h = sh_0s^{-1}$). Hence the essential support of h is included in $]x - \varepsilon, x + \varepsilon[$; in particular, \bar{h} and \bar{g} commute. Since $\tilde{\Gamma}$ is clean, it follows that h and g commute. Since $g(x) \neq x$, it follows that $g(h(x)) \neq h(x)$. But $h(x) \neq x$, and $h(x)$ belongs to the interior support, which is included in $]x - \varepsilon, x + \varepsilon[$, hence $h(x)$ is fixed by g . This is a contradiction, concluding the first step.

Now let us prove that $\tilde{\Gamma}$ is hyper-clean. Suppose by contradiction that $\{x, g(x)\}$ is an isolated point in the graph of g . Up to post-compose g with a nontrivial rotation, we can suppose that $g(x) \neq x$. So there exists $\varepsilon > 0$ such that none of $x, g(x^+), g(x^-)$ belongs to $]g(x) - 2\varepsilon, g(x) + 2\varepsilon[$.

As in the proof of the first step (using the dense subgroup of rotations), let $h \in \tilde{\Gamma}$ have essential support of diameter $< \varepsilon$, with both $g(x)$ and $h(g(x))$ in the interior support of h ; we can also require that $h(x) = x$.

Then for $t \in]x - \varepsilon, x + \varepsilon[$, $t \neq x$, we have $t \notin]g(x) - \varepsilon, g(x) + \varepsilon[$ and $g(t) \notin]g(x) - \varepsilon, g(x) + \varepsilon[$. Hence, with finitely many exceptions on t , we have $g(h(t)) = g(t)$ and $h(g(t)) = g(t)$. Also, we have $g(h(x)) = g(x)$ and $h(g(x)) \neq g(x)$. Hence, $g^{-1}h^{-1}gh$ has an isolated non-fixed point at x . This contradicts the assumption that $\tilde{\Gamma}$ is locally clean. \square

Definition 4.5. Call an element f of $\text{PC}^\infty(\mathbf{S})$ a 132-flip if it satisfies: there are three nonzero consecutive intervals I_1, I_2, I_3 near partitioning \mathbf{S} , such that

- (1) f has no singularity in the interior of I_j for each $j = 1, 2, 3$;
- (2) $f(I_1) = I_1, f(I_2) = I_3, f(I_3) = I_2$;
- (3) $f : I_j \rightarrow f(I_j)$ is orientation-reversing for $j = 1$ and orientation-preserving for $j = 2, 3$;
- (4) f^2 is the identity.

Say that f is a triple flip if f^2 is the identity, and there are three nonzero consecutive intervals I_1, I_2, I_3 near partitioning \mathbf{S} , such that f essentially preserves I_j for each j , and is orientation-reversing on it.

Lemma 4.6. *Let $f \in \text{PC}^\infty(\mathbf{S})$ be a 132-flip, or a triple flip. Then f has no hyper-clean lift squaring to the identity.*

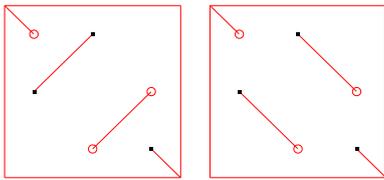


FIGURE 2. Graphs of a 132-flip and a triple flip: in each case there are two hyper-clean lifts, choosing either the endpoints denoted as circles or dots.

Proof. For a 132-flip, conjugating by a rotation, we can suppose that for some $0 < a < b < 1$, f preserves and is orientation-reversing on $[0, a]$, exchanges $[a, b]$ and $[b, 1]$ in an orientation-preserving way. If q is a hyper-clean lift of f , then, viewing q as subset of $\mathbf{S} \times \mathbf{S}$, we have $(0, a) \in q \Leftrightarrow (b, a) \notin q \Leftrightarrow (b, 0) \in q \Leftrightarrow (a, 0) \notin q \Leftrightarrow (a, b) \in q \Leftrightarrow (0, b) \notin q$. Hence q acts on $\{0, a, b\}$ as a 3-cycle, and in particular q^2 is not the identity.

The case of a triple flip is similar; again, we obtain that there are exactly two hyper-clean lifts, both of order 6. This is illustrated in Figure 2. \square

Theorem 4.7. *Let Γ be a subgroup of $\text{PC}^{\times}(\mathbf{S})$. Suppose that*

- (1) Γ includes a subgroup Λ of rotations of \mathbf{Q} -rank ≥ 2 , or infinitely generated and of \mathbf{Q} -rank 1;
- (2) Γ includes, for some proper nonzero interval, some dense subgroup of the corresponding partial rotations;
- (3) Γ contains an 132-flip or a triple flip (Definition 4.5).

Then Γ is not realizable.

Proof. By contradiction, let $\tilde{\Gamma}$ be a lift; then $\tilde{\Gamma}$ is clean. The assumption on Λ implies that it is 1-ended and not locally finite. By Proposition 2.6(1), we can, after conjugation, assume that Λ lifts as a subgroup of genuine rotations. By Lemma 4.2, $\Gamma \cap \text{IET}^+$ contains non-identity elements of arbitrary small essential diameter. By Lemma 4.4, $\tilde{\Gamma}$ is hyper-clean. Finally Lemma 4.6 yields a contradiction. \square

Corollary 4.8. *IET^{\times} is not realizable (for its near action on \mathbf{S}), as well as its subgroup IET^{\pm} . Moreover, the latter admits a non-realizable finitely generated subgroup.*

Proof. To fulfill the assumptions of Theorem 4.7, consider a pair (u, v) of rotations generating a subgroup of \mathbf{R}/\mathbf{Z} of \mathbf{Q} -rank 2; a partial rotation w generating a dense subgroup in a proper interval, and a triple flip $z \in \text{IET}^{\pm}$ as in Lemma 4.6. Then $\{u, v, w, z\}$ generates a non-realizable subgroup, by Theorem 4.7. \square

Corollary 4.9. *Let Λ be a subgroup of \mathbf{R}/\mathbf{Z} , of \mathbf{Q} -rank ≥ 2 , or infinitely generated of \mathbf{Q} -rank 1. Then $\text{IET}_{\Lambda}^{\times}$ is not realizable, and neither is its subgroup $\text{IET}_{\Lambda}^{\pm} = \text{IET}_{\Lambda}^{\times} \cap \text{IET}^{\pm}$.* \square

4.2. Restricted realizability.

Lemma 4.10. *Let Λ be a subgroup of \mathbf{R}/\mathbf{Z} . Let $\tilde{\Gamma}$ be a hyper-clean subgroup of $\widehat{\text{IET}}^+$ and Γ its image in IET^+ . Suppose that*

- each translation length of every element of Γ is in Λ ;
- each singularity of every element of Γ belongs to Λ ;
- for each element of Λ , the corresponding rotation belongs to $\tilde{\Gamma}$.

Then either all elements of $\tilde{\Gamma}$ are left-continuous, or all are right-continuous.

Proof. Otherwise, there are two elements r, s in $\tilde{\Gamma}$ and points x, y such that r is not left-continuous at x and s is not right-continuous at y . Using that x, y belong to Λ and that the corresponding rotations are in $\tilde{\Gamma}$, we can conjugate so as to choose $x, y = 0$. Moreover, using that translation lengths are in Λ and are achieved by rotations in $\tilde{\Gamma}$, we can post-compose to ensure that $r(0) = s(0) = 0$.

So there exist nonzero $\alpha, \beta \in \mathbf{R}/\mathbf{Z}$ and $\varepsilon > 0$ such that r is the identity on $[0, \varepsilon]$, s is the identity on $[-\varepsilon, 0]$, $r(t) = t + \alpha$ on $[-\varepsilon, 0[$ and $s(t) = t + \beta$ on $]0, \varepsilon]$. Hence $s \circ r(t) = s(t)$ for $t \in [0, \varepsilon]$ (namely $t + \beta$ for $t > 0$ and 0 for $t = 0$), and thus $s \circ r$ is not right-continuous at 0 . Hence $s \circ r$ is left-continuous at 0 . Hence $s \circ r(t) = t = s(t)$ for small enough negative t , and hence $r(t) = t$ for small negative t , a contradiction. \square

For $\Gamma \leq \text{PC}^+(\mathbf{S})$, denote by Γ_{left} (respectively Γ_{right}) the group of left-continuous (resp. right-continuous) representatives of elements of Γ .

Theorem 4.11. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\text{IET}}^+$ and Γ its image in IET^+ . Suppose that*

- (1) *the group of rotations Λ in Γ achieves all translation lengths of Γ ;*
- (2) *Λ has \mathbf{Q} -rank ≥ 2 , or is infinitely generated of \mathbf{Q} -rank ≥ 1 ;*
- (3) *each singularity of every element of Γ belongs to Λ ;*
- (4) *for some proper sub-interval, Γ includes a dense subgroup of partial rotations.*

Then $\tilde{\Gamma}$ is conjugate, by a (unique) finitely supported permutation, to either Γ_{left} or Γ_{right} .

Proof. Using the \mathbf{Q} -rank assumption, by Proposition 2.6(1), we can conjugate by a finitely supported permutation, to ensure that rotations indeed act by rotations. In this case, we will prove that $\tilde{\Gamma}$ is then *equal* to either Γ_{left} or Γ_{right} .

By the assumption of existence of both a dense subgroup of rotations and a dense partial subgroup of partial rotations, Lemma 4.2 implies that Γ admits elements of arbitrary small essential diameter. In turn, again using a dense subgroup of rotations, and the existence of elements of arbitrary small essential diameter, Lemma 4.4 ensures that $\tilde{\Gamma}$ is hyper-clean. Since $\tilde{\Gamma}$ is hyper-clean and all its translations length are achieved by rotations, we apply Lemma 4.10 to ensure that all elements of $\tilde{\Gamma}$ are, say, left-continuous (the right-continuous case is equivalent, as we can conjugate by a reflection). \square

Corollary 4.12. *Suppose that Λ has \mathbf{Q} -rank ≥ 2 , or is infinitely generated of \mathbf{Q} -rank 1. Then IET_Λ^+ has only two lifts to $\widehat{\text{IET}}^+$ up to conjugation by finitely supported permutations, namely the left-continuous and the right-continuous lift.*

Proof. We have to check that the assumptions of Theorem 4.11 are fulfilled. The group of translations lengths in IET_Λ^+ and the group of rotations of IET_Λ^+ are both equal to Λ . Also, for every $x < x_0 \in]0, 1[$ representing elements of Λ , the corresponding partial rotation (exchanging $]0, x[$ and $[x, x_0[$ belongs to IET_Λ^+ ; hence they achieve (x_0 fixed, x varying) a dense subgroup of partial rotations (here we only use that Λ is dense). \square

Corollary 4.13. *Let Γ be a subgroup of $\text{PC}^+(\mathbf{S})$ including IET^+ (e.g., $\text{PC}^+(\mathbf{S})$, or its piecewise analytic subgroup). Then Γ has, up to conjugation by a (unique) finitely supported permutation, only the two lifts Γ_{left} and Γ_{right} .*

Proof. The uniqueness is clear. Let $\tilde{\Gamma}$ be a lift. By Theorem 4.11, we can suppose (up to conjugate by a finitely supported permutation and possibly by a reflection, that in restriction to IET^+ , we have the left-continuous lift. By Lemma 4.4, $\tilde{\Gamma}$ is hyper-clean.

Let g be an element of $\tilde{\Gamma}$ and $x \in \mathbf{S}$. We claim that g is continuous at x . Since $g \in \widehat{\text{PC}^+(\mathbf{S})}$, there exists an interval exchange h such that $h(x^-) = g(x^-)$ and $h(x^+) = g(x^+)$. We can view h as an element of $\tilde{\Gamma}$, and thus h is left-continuous, so $h(x) = \overline{h(x^-)}$. We have $h^{-1}g(x^-) = x^-$ and $h^{-1}g(x^+) = x^+$. Since $\tilde{\Gamma}$ is hyper-clean, we deduce that $h^{-1}g(x) = x$. So $g(x) = h(x) = \overline{h(x^-)} = \overline{g(x^-)}$, showing that g is left-continuous at x . \square

Remark 4.14. Let Λ be a nontrivial proper subgroup of \mathbf{R}/\mathbf{Z} , and let Γ be the subgroup of IET^+ consisting of those elements all of whose translations lengths are in Λ (but with no restriction on singularities). Partition \mathbf{R}/\mathbf{Z} into two non-empty Λ -invariant subsets. For every $g \in \Gamma$, we can choose the unique lift \tilde{g} of g that is left-continuous on X and right-continuous on Y . Then $g \mapsto \tilde{g}$ is a group homomorphism, defining a lift contradicting the conclusion of Theorem 4.11.

Assuming that Λ has \mathbf{Q} -rank ≥ 2 , or is infinitely generated of \mathbf{Q} -rank ≥ 1 , all assumptions of Theorem 4.11, except that on singularities, are fulfilled. This shows the importance of that hypothesis. We thank the referee for suggesting this example, pointing out that this hypothesis was missing in stating Lemma 4.10 and Theorem 4.11 in the previous versions.

4.3. Stable non-realizability. Let X be a set. Let $\widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S})$ be the subgroup of permutations of $\mathbf{S} \sqcup X$ that are identity on a cofinite subset of X , and that induce elements of $\text{PC}^{\boxtimes}(\mathbf{S})$ on \mathbf{S} . So there is a canonical projection $\widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S}) \rightarrow \text{PC}^{\boxtimes}(\mathbf{S})$, whose kernel $\mathfrak{S}_{\text{fin}}(\mathbf{S} \sqcup X)$ consists of finitely supported permutations of $\mathbf{S} \sqcup X$. We say that a subgroup Γ of $\widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S})$ is **clean** if it has trivial intersection with $\mathfrak{S}_{\text{fin}}(\mathbf{S} \sqcup X)$.

For $f \in \widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S})$, we call **essential support** of f the closure of the set of $x \in \mathbf{S}$ such that $f(x) \notin \{\overline{f(x^+)}, \overline{f(x^-)}\}$. (Recall that $y \mapsto \bar{y}$ denotes the projection $\mathbf{S}^\pm \rightarrow \mathbf{S}$.)

Here is an adaptation of Lemma 4.4.

Lemma 4.15. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S})$, and Γ its image in $\text{PC}^{\boxtimes}(\mathbf{S})$. Suppose that $\tilde{\Gamma}$ includes a dense subgroup $\tilde{\Lambda}$ of rotations acting as the identity on X . Suppose that Γ admits non-identity elements with essential support of arbitrary small diameter. Then*

- (1) *for every $g \in \tilde{\Gamma}$ and $x \in \mathbf{S}$, $g(x)$ belongs to $\{\overline{g(x^+)}, \overline{g(x^-)}\} \cup X$.*
- (2) *for every $g \in \tilde{\Gamma}$, the subset $\{x \in \mathbf{S} : gx \neq x\}$ has no isolated point.*

Proof. (1) This is an adaptation of the proof of Lemma 4.4 and we skip details; note that when X is empty this is precisely the same statement. The first step consists in proving that for $x \in \mathbf{S}$ if $\overline{g(x^-)} = \overline{g(x^+)} = x$, then $g(x) \in \{x\} \cup X$. The second step assumes that $g(x) \in \mathbf{S}$ and $g(x) \notin \{g(x^-), g(x^+)\}$ and reaches a contradiction.

(2) Suppose by contradiction that there exists $x \in \mathbf{S}$ such that $f(x') = x'$ for all $x' \neq x$ close enough to x , but $f(x) \neq x$. By (1), we have $f(x) \in X$. Let $r \in \tilde{\Lambda}$ be a small enough non-trivial rotation and $x' \neq x$ close enough to x (“close enough” may depend on r). Then $r^{-1}f^{-1}rf(x') = x'$, while $r^{-1}f^{-1}rf(x) = r^{-1}f^{-1}f(x) = r^{-1}x \neq x$. Since $r^{-1}x \in \mathbf{S}$, this contradicts (1). \square

Say that an element g of $\text{PC}^{\boxtimes}(\mathbf{S})$ has small support if there exists a rotation r such that $rS \cap S = \emptyset$, where S is the essential support of g .

Lemma 4.16. *Let $\tilde{\Gamma}$ be a clean subgroup of $\widehat{\text{PC}}^{\boxtimes}(\mathbf{S} \sqcup X, \mathbf{S})$ such that for every $g \in \tilde{\Gamma}$, the subset $\{x \in \mathbf{S} : gx \neq x\}$ has no isolated point. Let Γ be its projection to $\text{PC}^{\boxtimes}(\mathbf{S})$.*

Suppose that $\tilde{\Gamma}$ includes a dense subgroup $\tilde{\Lambda}$ of rotations (acting as the identity on X). Let $f \in \Gamma$ be an element with small support, and \tilde{f} its lift in $\tilde{\Gamma}$. Then X is \tilde{f} -invariant.

Proof. Let $T \subset \mathbf{S}$ be the essential support of f , and let $T' = \{x \in \mathbf{S} \sqcup X : f(x) \neq x\}$ be the support of \tilde{f} (so $T \cap X$ is finite). By the assumption on isolated points, we have $T' \cap \mathbf{S} \subset T$.

There exists a non-empty open interval of rotations mapping T to a disjoint subset; fix one, say \tilde{r} , in $\tilde{\Lambda}$. Let r denote its image in Γ . Hence $\tilde{r}(T') \cap T'$ is equal to $T' \cap X$, which is finite.

Note that $\tilde{r}(T')$ is the support of $\tilde{r}\tilde{f}\tilde{r}^{-1}$. Since r and rfr^{-1} have essentially disjoint support, they commute, and hence \tilde{f} and $\tilde{r}\tilde{f}\tilde{r}^{-1}$ commute. Thus $T' \cap X$ is \tilde{f} -invariant, and hence X is \tilde{f} -invariant. \square

Theorem 4.17. *The near action of IET^{\pm} (and hence of IET^{\boxtimes}) on \mathbf{S} is not stably realizable. Moreover,*

- (1) *there exists a finitely generated subgroup of IET^{\pm} whose near action on \mathbf{S} is not stably realizable;*

- (2) for every subgroup Λ of \mathbf{R}/\mathbf{Z} that has \mathbf{Q} -rank ≥ 2 , or infinitely generated of \mathbf{Q} -rank 1, the near action of IET_Λ^\pm (and hence of $\text{IET}_\Lambda^\infty$) on \mathbf{S} is not stably realizable.

Proof. For two essentially disjoint intervals I, J of the same nonzero size, let $\varphi(I, J)$ be the element of order 2 in IET^+ exchanging I and J by an orientation-preserving isometry.

Consider $a \in]\frac{1}{8}, \frac{1}{4}[$ and $b = 1 - 4a$, so $0 < b < \frac{1}{2}$ and $4a + b = 1$. Define $u, v \in \text{IET}^+$ by $u = \varphi([0, a], [2a + b, 3a + b])$ and $v = \varphi([a, 2a], [3a + b, 1])$. Let w be a partial rotation on the interval $[0, 2a]$, of infinite order. Let s be the orientation-reversing isometry flipping all $[0, 1]$.

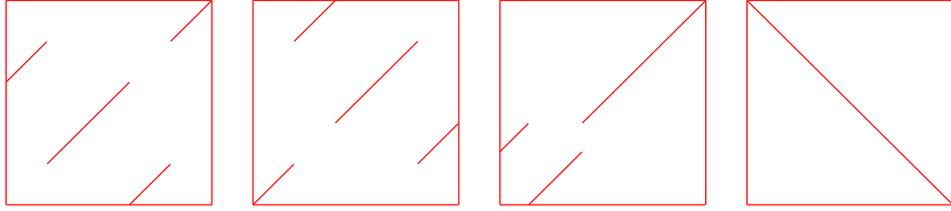


FIGURE 3. Graphs of u, v, w and s .

Let Γ be a subgroup of IET^∞ including a dense subgroup Λ of rotations, either of \mathbf{Q} -rank ≥ 2 , or infinitely generated of \mathbf{Q} -rank 1. Furthermore we assume that u, v, w and s all belong to Γ . Let us show that the near action of Γ on \mathbf{S} is not stably realizable.

Arguing by contradiction, we assume it is realizable on $\mathbf{S} \sqcup X$ for some set X , and denote by $\tilde{\Gamma}$ a lift. Let Λ be the group of rotations in Γ . After conjugation by a finitely supported permutation, we can assume, by Proposition 2.6(1), that the lift $\tilde{\Lambda}$ acts by rotations on \mathbf{S} (so it acts on X by finitely supported permutations). Fix an element r of infinite order in Λ . Since $sr s^{-1} = r^{-1}$, we also have $\tilde{s}\tilde{r}\tilde{s}^{-1} = \tilde{r}^{-1}$. Hence \tilde{s} normalizes $\langle \tilde{r} \rangle$, and thus \tilde{s} preserves the union of all infinite $\langle \tilde{r} \rangle$ -orbits, which is exactly \mathbf{S} .

By Lemma 4.16 (which applies using Lemma 4.15(2)), the elements $\tilde{u}, \tilde{v}, \tilde{w}$ also preserve \mathbf{S} . Let Γ_1 be generated by $\{u, v, w, s\} \cup \Lambda$. Then $\tilde{\Gamma}_1$ preserves \mathbf{S} , so X plays no longer any role: indeed since Γ is clean, the action of $\tilde{\Gamma}_1$ on \mathbf{S} is faithful. Since w is a partial rotation (on $[1 - 2c, 1]$) of infinite order, Lemma 4.2 ensures that Γ has non-identity elements with essential support of arbitrary small diameter. Then Theorem 4.4 implies that $\tilde{\Gamma}_1$, viewed as subgroup of $\widehat{\text{PC}}^\pm(\mathbf{S})$, is hyper-clean. The product $su v \in \Gamma_1$ is a triple flip on \mathbf{S} , but has no hyper-clean lift squaring to the identity (Lemma 4.6). We reach a contradiction.

Since Λ can be chosen 2-generated, we obtain by construction a 6-generator subgroup that is not stably realizable. Moreover, if we restrict to a given Λ as in the theorem, a and then w can be chosen to belong to IET_Λ^\pm . \square

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