Isometric group actions on Hilbert spaces: structure of orbits

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Abstract

Our main result is that a finitely generated nilpotent group has no isometric action on an infinite dimensional Hilbert space with dense orbits. In contrast, we construct such an action with a finitely generated metabelian group.

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1 Introduction

The study of isometric actions of groups on affine Hilbert spaces has, in recent years, found applications ranging from the K-theory of C^*-algebras [HiKa], to rigidity theory [Sh2] and geometric group theory [Sh3, CTV]. This renewed interest motivates the following general problem: *How can a given group act by isometries on an affine Hilbert space?*

This paper is a sequel to [CTV], but can be read independently. In [CTV], given an an isometric action of a finitely generated group G on a Hilbert space $\alpha : G \to \mathrm{Isom}(\mathcal{H})$, we focused on the growth of the function $g \mapsto \alpha(g)(0)$. Here the emphasis is on the structure of orbits.

We will mainly focus on actions of nilpotent groups. Let us begin by a simple example: every isometric action of $\mathbf{Z}$ on a Euclidean space is the direct sum of an action with a fixed point and an action by translations. This actually remains true for general locally compact nilpotent groups. The situation becomes more subtle when we study action on infinite-dimensional Hilbert spaces. However, something remains from the finite-dimensional case.

We say that a convex subset of a Hilbert space is *locally bounded* if its intersection with any finite dimensional subspace is bounded. The main result of the paper is the following theorem.

Theorem 1. (see Corollary 3.9 and Theorem 4.3) *Let G be a locally compact, second countable, nilpotent group. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a subspace T of $\mathcal{H}$ (the "translation part"), contained in the invariant vectors of π, and*
- *a closed, locally bounded convex subset U of the orthogonal subspace $T^{\perp}$,*

such that $\mathcal{O}$ is contained in $T \times U$.

We owe the following general question to A. Navas: which locally compact groups have an isometric action on a infinite-dimensional separable Hilbert space with dense orbits (i.e. minimal)?

Theorem 1 allows us to provide a negative answer in the case of finitely generated nilpotent groups.

Theorem 2. (see Corollary 4.6) *A compactly generated, nilpotent-by-compact locally compact group does not admit any affine isometric action with dense orbits on an infinite-dimensional Hilbert space.*

Actually, for compactly generated nilpotent groups, one can describe all affine isometric actions with dense orbits; see Corollary 4.5.

In the course of our proof, we introduce the following new definition: a unitary or orthogonal representation π of a group is *strongly cohomological* if it satisfies: for every nonzero subrepresentation $\rho \leq \pi$, we have $H^1(G, \rho) \neq 0$. It is easy to observe that the linear part of a affine isometric action with dense orbits is strongly cohomological. The non-trivial step in the proof of the main theorem is the following result.

Proposition 3. (see Corollary 3.9) *Let π be an orthogonal or unitary representation of a second countable, nilpotent locally compact group G. Suppose that π is strongly cohomological. Then π is a trivial representation.*

Another case for which we have a negative answer is the following.

Theorem 4. (see Theorem 4.7) *Let G be a connected semisimple Lie group. Then G has no isometric action on a nonzero Hilbert space with dense orbits.*

It is not clear how the main theorem can be generalized, in view of the following example.

Proposition 5. (see Proposition 2.1) *There exists a finitely generated metabelian group admitting an affine isometric action with dense orbits on* $\ell^2_{\mathbf{R}}(\mathbf{Z})$.

Another construction provides

Proposition 6. (see Proposition 2.3) *There exists a countable group admitting an affine isometric action with dense orbits on an infinite dimensional Hilbert space, in such a way that every finitely generated subgroup has a fixed point.*

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2 Existence results

Here is a first positive result regarding Navas' question.

Proposition 2.1. *There exists an isometric action of a metabelian 3-generator group on a infinite-dimensional separable Hilbert space, all of whose orbits are dense.*

Proof. Observe that $\mathbf{Z}[\sqrt{2}]$ acts by translations, with dense orbits, on $\mathbf{R}$; so the free abelian group of countable rank $\mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})}$ acts by translations, with dense orbits, on $\ell^2_{\mathbf{R}}(\mathbf{Z})$. Observe now that the latter action extends to the wreath product $\mathbf{Z}[\sqrt{2}] \wr \mathbf{Z} = \mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})} \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts on $\ell^2_{\mathbf{R}}(\mathbf{Z})$ by the shift. That wreath product is metabelian, with 3 generators. ☐

Corollary 2.2. *There exists an isometric action of a free group of finite rank on a Hilbert space, with dense orbits.* ☐

Recall that an isometric action $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ *almost has fixed points* if for every $\varepsilon > 0$ and every compact subset $K \subset G$ there exists $v \in \mathcal{H}$ such that $\sup_{g\in K} \|v - \alpha(g)v\| \leq \varepsilon$.

In the example given by Proposition 2.1, the given isometric action clearly does not almost have fixed points, i.e. it defines a non-zero element in reduced 1-cohomology. The next result shows that this is not always the case.

Proposition 2.3. *There exists a countable group* Γ *with an affine isometric action* α *on a Hilbert space, such that* α *has dense orbits, and every finitely generated subgroup of* Γ *has a fixed point. In particular, the action almost has fixed points.*

Proof. We first construct an uncountable group G and an affine isometric action having dense orbits and almost having fixed points.

In $\mathcal{H} = \ell^2_{\mathbf{R}}(\mathbf{N})$, let A_n be the affine subspace defined by the equations

$$x_0 = 1,\, x_1 = 1, ..., \, x_n = 1,$$

and let G_n be the pointwise stabilizer of A_n in the isometry group of $\mathcal{H}$. Let G be the union of the G_n's. View G as a discrete group.

It is clear that G almost has fixed points in $\mathcal{H}$, since any finite subset of G has a fixed point. Let us prove that G has dense orbits.

Claim 1. For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} |d(x, A_n) - d(y, A_n)| = 0$.

By density, it is enough to prove Claim 1 when x, y are finitely supported in $\ell^2_{\mathbf{R}}(\mathbf{N})$. Take $x = (x_0, x_1, ..., x_k, 0, 0, ...)$ and choose $n > k$. Then

$$d(x, A_n)^2 \;=\; \sum_{j=0}^{k}(x_j - 1)^2 + \sum_{j=k+1}^{n} 1^2 \;=\; n + 1 - 2\sum_{j=0}^{k} x_j + \sum_{j=0}^{k} x_j^2,$$

so that $d(x, A_n) = \sqrt{n} + O(\frac{1}{\sqrt{n}})$, which proves Claim 1.

Denote by p_n the projection on the closed convex set A_n, namely $p_n(x_0, x_1, \dots) = (1, 1, \dots, x_{n+1}, x_{n+2}, \dots)$.

Claim 2. For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} \|p_n(x) - p_n(y)\| = 0$.

This is a straightforward computation.

Claim 3. G has dense orbits in $\mathcal{H}$.

Observe that two points $x, y \in \mathcal{H}$ are in the same G_n-orbit if and only if $d(x, A_n) = d(y, A_n)$ and $p_n(x) = p_n(y)$. Fix $x_0, z \in \mathcal{H}$. We want to show that $\lim_{n\to\infty} d(G_n x_0, z) = 0$. So fix $\varepsilon > 0$. By the second claim, for some n_0, $\|p_n(x_0) - p_n(z)\| \le \varepsilon/2$ whenever $n \ge n_0$. Set

$$W = \{x \in \mathcal{H} |\, p_n(x) = p_n(z)\};$$

this is the orthogonal affine subspace of A_n passing through z. Then $y_0 = x_0 + (p_n(z) - p_n(x_0)) \in W$. By the first claim, there exists $n_1 \ge n_0$ such that $|d(y_0, A_n) - d(z, A_n)| \le \varepsilon/2$ for every $n \ge n_1$. Therefore there exists $y \in W$ such that $\|y - z\| \le \varepsilon/2$ and $d(y, A_n) = d(y_0, A_n) = d(x_0, A_n)$. By the previous observation, there exists $g \in G_n$ such that $y = gy_0$. Then

$$d(gx_0, z) \le d(gx_0, gy_0) + d(gy_0, z) \le \varepsilon,$$

so that $d(G_n x_0, z) \le \varepsilon$ for every $n \ge n_1$, proving the last claim.

Using separability of $\mathcal{H}$, it is now easy to construct a countable subgroup Γ of G also having dense orbits on $\mathcal{H}$. □

Question 1. Does there exist an affine isometric action of a *finitely generated* group on a Hilbert space, having dense orbits and almost having fixed points?

3 Cohomology of unitary representations of nilpotent groups

Our non-existence results concerning nilpotent locally compact groups will be based on the following study of their unitary representations.

Definition 3.1. If G is a topological group and π a unitary representation, we say that π is *strongly cohomological* if every nonzero subrepresentation of π has nonzero first cohomology.

The following Lemma is Proposition 3.1 in Chapitre III of [Gu2].

Lemma 3.2. *Let π be a unitary representation of G that does not contain the trivial representation. Let z be a central element of G. Suppose that $1-\pi(z)$ has a bounded inverse (equivalently, 1 does not belong to the spectrum of $\pi(z)$). Then $H^1(G,\pi)=0$.* □

Proof. If $g \in G$, expanding the equality $b(gz) = b(zg)$, we obtain that $(1-\pi(z))b(g)$ is bounded by $2\|b(z)\|$, so that b is bounded by $2\|(1-\pi(z))^{-1}\|\|b(z)\|$. □

Lemma 3.3. *Let G be a locally compact, second countable group, and π a strongly cohomological representation. Then π is trivial on the centre $Z(G)$.*

Proof. Fix $z \in Z(G)$. As G is second countable, we may write $\pi = \int_{\hat{G}}^{\oplus} \rho \, d\mu(\rho)$, a disintegration of π as a direct integral of irreducible representations. Let $\chi : \hat{G} \to S^1 : \rho \mapsto \rho(z)$ be the continuous map given by the value of the central character of ρ on z. For $\varepsilon > 0$, set $X_\varepsilon = \{\rho \in \hat{G} : |\chi(\rho) - 1| > \varepsilon\}$ and $\pi_\varepsilon = \int_{X_\varepsilon}^{\oplus} \rho \, d\mu(\rho)$, so that π_ε is a subrepresentation of π. Since $|\rho(z)-1|^{-1} < \varepsilon^{-1}$ for $\rho \in X_\varepsilon$, the operator

$$(\pi_\varepsilon(z) - 1)^{-1} = \int_{X_\varepsilon}^{\oplus} (\rho(z)-1)^{-1} \, d\mu(\rho)$$

is bounded. We are now in position to apply Lemma 3.2, to conclude that $H^1(G, \pi_\varepsilon) = 0$. By definition, this means that π_ε is the zero subrepresentation, meaning that the measure μ is supported in $\hat{G} - X_\varepsilon$. As this holds for every $\varepsilon > 0$, we see that μ is supported in $\{\rho \in \hat{G} : \rho(z) = 1\}$, to the effect that $\pi(z) = 1$. □

Proposition 3.4. *Let G be a topological group, and π a unitary representation of G. Suppose that $\overline{H^1}(G,\pi) \neq 0$. Then π has a nonzero subrepresentation that is strongly cohomological.*

Proof. Suppose the contrary. Then, by an standard application of Zorn's Lemma, π decomposes as a direct sum $\pi = \bigoplus_{i \in I} \pi_i$, where $H^1(G, \pi_i) = 0$ for every $i \in I$, so that $\overline{H^1}(G, \pi) = 0$ by Proposition 2.6 in Chapitre III of [Gu2]. □

Remark 3.5. The converse is false, even for finitely generated groups: indeed, it is easy to check (see [Gu1]) that every nonzero representation of the free group F_2 has non-vanishing H^1, so that every unitary representation of F_2 is strongly cohomological. But it turns out that F_2 has an irreducible representation π such that $\overline{H^1}(F_2, \pi) = 0$ (see Proposition 2.4 in [MaVa]).

Corollary 3.6. *Let G be a locally compact, second countable group, and let π be a unitary representation of G without invariant vectors. Write $\pi = \pi_0 \oplus \pi_1$, where π_1 consists of the $Z(G)$-invariant vectors. Then*

(1) π_0 does not contain any strongly cohomological subrepresentation (in particular, $\overline{H^1}(G, \pi_0) = 0$);

(2) every 1-cocycle of π_1 vanishes on $Z(G)$, so that $H^1(G, \pi_1) \simeq H^1(G/Z(G), \pi_1)$.

Proof. (1) follows by combining Lemma 3.3 and Proposition 3.4. For (2), we use the idea of proof of Theorem 3.1 in [Sh2]: if $b \in Z^1(G, \pi_1)$, then for every $g \in G$, $z \in Z(G)$,

$$\pi_1(g)b(z) + b(g) \;=\; b(gz) \;=\; b(zg) \;=\; b(g) + b(z)$$

as $\pi_1(z) = 1$. So $\pi_1(g)b(z) = b(z)$; this forces $b(z) = 0$ as π has no G-invariant vector. So b factors through $G/Z(G)$. □

Observe that Corollary 3.6 provides a new proof of Shalom's Corollary 3.7 in [Sh2]: under the same assumptions, every cocycle in $Z^1(G, \pi)$ is almost cohomologous to a cocycle factoring through $G/Z(G)$ and taking values in a subrepresentation factoring through $G/Z(G)$.

From Corollary 3.6 we immediately deduce

Corollary 3.7. *Let G be a locally compact, second countable, nilpotent group, and let π be a representation of G without invariant vectors. Let (Z_i) be the ascending central series of G ($Z_0 = \{1\}$, and Z_i is the centre modulo Z_{i-1}). Let σ_i denote the subrepresentation of G on the space of Z_i-invariant vectors, and finally let π_i be the orthogonal of σ_{i+1} in σ_i, so that $\pi = \bigoplus \pi_i$.*

Then $H^1(G, \pi_i) \simeq H^1(G/Z_i, \pi_i)$ for all i, and π is not a strongly cohomological subrepresentation. In particular, $\overline{H^1}(G, \pi) = 0$. □

Note that the latter statement is a result of Guichardet [Gu1, Théorème 7], which can be stated as: G has Property H_T (i.e. every unitary representation with non-vanishing reduced cohomology contains the trivial representation).

Definition 3.8. We say that a locally compact group G has Property H_{CT} if every strongly cohomological unitary representation of G is trivial.

It is a straightforward verification that this is equivalent to: every strongly cohomological *orthogonal* representation of G is trivial. This will be useful in the next paragraph since we will deal with orthogonal rather than unitary representations.

As a corollary of Proposition 3.4, Property H_{CT} implies Property H_T. We have proved

Proposition 3.9. *If G is a locally compact, second countable nilpotent group, then G has Property H_{CT}.* □

4 Non-existence results

Definition 4.1. 1) We say that subset Y of a metric space (X, d) is *coarsely dense* if there exists $C \geq 0$ such that, for every $x, y \in X$,

$$d(x, G.y) \leq C.$$

2) We say that a subset Y of a Hilbert space $\mathcal{H}$ is *enveloping* if its closed convex hull is all of $\mathcal{H}$.

Observe that every dense subset of a metric space is coarsely dense. Besides, in a Hilbert space $\mathcal{H}$, every coarsely dense subset Y is enveloping. Indeed, suppose that Y is contained in a closed, convex proper subset X of $\mathcal{H}$. Consider $v \notin X$ and let y denote its projection on X (excluding the trivial case $Y = \emptyset$). Then, for every $\lambda \geq 0$, we have $d(y + \lambda(v - y), Y) \geq d(y + \lambda(v - y), X) = \lambda$, which is unbounded, so that Y is not coarsely dense.

Example 1. In $\ell^2_{\mathbf{R}}(\mathbf{Z})$, let X denote the elements with integer coefficients. Then X is enveloping: indeed, its intersection with the subspace $F_n = \ell^2_{\mathbf{R}}(\{-n, \ldots, n\})$ is coarsely dense, hence enveloping in F_n, and the increasing union $\bigcup F_n$ is dense in $\ell^2_{\mathbf{R}}(\mathbf{Z})$. But X is not coarsely dense: indeed, for every $n \geq 0$, the element $\frac{1}{2}\mathbf{1}_{\{1,\ldots,4n\}}$ is at distance $\sqrt{n}$ to X.

Note that X is the orbit of 0 for the natural action of the wreath product $\mathbf{Z} \wr \mathbf{Z} = \mathbf{Z}^{(\mathbf{Z})} \rtimes \mathbf{Z}$ on $\ell^2_{\mathbf{R}}(\mathbf{Z})$, where $\mathbf{Z}^{(\mathbf{Z})}$ acts by translations and the factor $\mathbf{Z}$ acts by shifting (compare the example in the proof of Proposition 2.1).

Lemma 4.2. *Let G be a topological group and π an orthogonal representation, admitting a 1-cocycle b with enveloping orbits. Then π is strongly cohomological.*

Proof. If σ is a nonzero subrepresentation of π, let b_σ be the orthogonal projection of b on $\mathcal{H}_\sigma$, so that $b_\sigma \in Z^1(G, \sigma)$. Then $b_\sigma(G)$ is enveloping in $\mathcal{H}_\sigma$, in particular b_σ is unbounded. So b_σ defines a non-zero class in $H^1(G, \sigma)$. □

Theorem 4.3. *Let G be a locally compact group with Property H_{CT}. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a subspace T of $\mathcal{H}$, contained in $\mathcal{H}^{\pi(G)}$, and*
- *a closed, locally bounded convex subset U of $T^\perp$,*

such that $\mathcal{O}$ is contained in $T \times U$.

Proof. We immediately reduce to the case when π has no invariant vectors, so that we must prove that the closed convex hull U of $\mathcal{O}$ is locally bounded.

Observe that a convex subset of a Hilbert space is locally bounded if and only if it contains no affine half-line. Thus denote by $\mathcal{D}$ the set of affine half-lines contained in U, and suppose by contradiction that $\mathcal{D} \neq \emptyset$. Denote by $\mathcal{D}_0$ the corresponding set of linear half-lines (where the linear half-line corresponding to a half-line $x + \mathbf{R}_+ v$ is simply $\mathbf{R}_+ v$). Then $\mathcal{D}_0$ is invariant under the linear action π of G. Let W be the closed subspace of $\mathcal{H}$ generated by all the half-lines in $\mathcal{D}_0$, and denote by σ the corresponding subrepresentation. By assumption, σ is non-zero.

We claim that σ is strongly cohomological, contradicting that π has no invariant vectors along with the H_{CT} assumption. Let ρ be a non-zero subrepresentation of σ. Then by the definition of W, there exists an half-line of U which projects injectively into the subspace of ρ. Thus $H^1(G, \rho) \neq 0$, proving the claim, and ending the proof. □

Corollary 4.4. *Let G be a locally compact group with Property H_{CT}. Let $\mathcal{H}$ be a Hilbert space on which G acts with with enveloping (respectively coarsely dense, resp. dense) image. Then the action is by translations, defined by a continuous morphism: $u : G \to (\mathcal{H}, +)$ with enveloping (resp. coarsely dense, resp. dense) image.* □

Corollary 4.5. *Let G be a locally compact, compactly generated group with Property H_{CT}, and let $\mathcal{H}$ be a (real) Hilbert space. Then*

- *G has a isometric action on $\mathcal{H}$ with coarsely dense (respectively enveloping) orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $n + m \geq k$.*

- *G has a isometric action on $\mathcal{H}$ with dense orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $\max(n + m - 1, n) \geq k$.*

Proof. Let α be an affine isometric action of G with enveloping orbits (this encompasses all possible assumptions). By Corollary 4.4, the action is by translations; let u be the morphism $G \to (\mathcal{H}, +)$; its image generates $\mathcal{H}$ as a topological vector space. Let W denote the kernel of u.

Then $A = G/W$ is a locally compact, compactly generated abelian group, which embeds continuously, in a Hilbert space. By standard structural results, A has a compact subgroup K such that A/K is a Lie group. Since K embeds in a Hilbert space, it is necessarily trivial, so that A is an abelian Lie group without compact subgroup. Accordingly, A is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$ for some integers n, m; the embedding of A into $\mathcal{H}$ extends canonically to a linear mapping of $\mathbf{R}^{n+m}$ into $\mathcal{H}$. In particular $\mathcal{H}$ is finite-dimensional, of dimension $k \leq n + m$.

If the action has dense orbits, then either $m = 0$ and $n \geq k$, or $m \geq 1$ and $m \geq k - n + 1$; this means that $k \leq \max(n + m - 1, n)$. Conversely, if $k \leq n + m - 1$, then, since $\mathbf{Z}$ has a dense embedding in the torus $\mathbf{R}^k/\mathbf{Z}^k$, $\mathbf{Z}^{k+1}$ has a dense embedding in $\mathbf{R}^k$, and this embedding can be extended to $\mathbf{R}^n \times \mathbf{Z}^m$. □

From Proposition 3.9 and Corollary 4.5, we deduce

Corollary 4.6. *A compactly generated, nilpotent-by-compact group does not admit any isometric action with enveloping (e.g. dense) orbits on an infinite-dimensional Hilbert space.*

Proof. The only thing we have to care now is that the group G is not necessarily second countable. So let α be an isometric action with enveloping orbits on a Hilbert space $\mathcal{H}$. By the Kakutani-Kodaira Theorem [Com, Theorem 3.7], there exists a compact normal subgroup N such that G/N is second countable. Since N is compact, the affine subspace α^N of N-fixed points is non-empty. Since it is G-invariant and the orbits are enveloping, necessarily $\alpha^N = \mathcal{H}$, so we are reduced to the case when G is second countable, allowing us to use Proposition 3.9 to conclude. □

Proposition 2.1 on the one hand, and Corollary 4.6 on the other, isolate the first test-case for Navas' question:

Question 2. Does there exist a polycyclic group admitting an affine isometric action with dense orbits on an infinite-dimensional Hilbert space?

Let us prove a related result for semisimple groups.

Theorem 4.7. *Let G be a connected, semisimple Lie group. Then G cannot act on a Hilbert space $\mathcal{H} \neq 0$ with coarsely dense (e.g. dense) orbits.*

Proof. Suppose by contradiction the existence of such an action α, and let π denote its linear part. Then π is strongly cohomological. By Lemma 3.3, π is trivial on the centre of G. Thus the centre acts by translations, generating a finite-dimensional subspace V of $\mathcal{H}$. The action induces a map $p : G \to \mathrm{O}(V) \ltimes V$. Since G is semisimple, the kernel of p contains the sum G_{nc} of all noncompact factors of G, and thus factors though the compact group G/G_{nc}. Thus $H^1(G, V) = 0$, and since π is strongly cohomological, this implies that $V = 0$.

It follows that α is trivial on the centre of G, so that we can suppose that G has trivial centre. Then G is a direct product of simple Lie groups with trivial centre. We can write $G = H \times K$ where K denotes the sum of all simple factors S of G such that $\alpha(S)(0)$ is bounded (in other words, $H^1(S, \pi|_S) = 0$). Then the restriction of α to H also has coarsely dense orbits. Moreover, every simple factor of H acts in an unbounded way, so that, by a result of Shalom [Sh1, Theorem 3.4][1], the action of H is proper. That is, the map $i : H \to \mathcal{H}$ given by $i(h) = \alpha(h)(0)$ is metrically proper and its image is coarsely dense. By metric properness, the subset $X = i(H) \subset \mathcal{H}$ satisfies: X is coarsely dense, and every ball in X (for the metric induced by $\mathcal{H}$) is compact.

Suppose that $\mathcal{H}$ is infinite dimensional and let us deduce a contradiction. For some $d > 0$, we have $d(x, X) \leq d$ for every $x \in \mathcal{H}$. If $\mathcal{H}$ is infinite dimensional, there exists, in a fixed ball of radius $7d$, infinitely many pairwise disjoint balls $B(x_n, 3d)$ of radius $3d$. Taking a point in $X \cap B(x_n, 2d)$ for every n, we obtain a closed, infinite and bounded discrete subset of X, a contradiction.

Thus $\mathcal{H}$ is finite dimensional; since every simple factor of H is non-compact, it has no non-trivial finite dimensional orthogonal representation, so that the action is by translations, and hence is trivial, so that finally $\mathcal{H} = \{0\}$. $\square$

Remark 4.8. 1) The same argument shows that a semisimple, linear algebraic group over any local field, cannot act with coarsely dense orbits on a Hilbert space.

2) The argument fails to work with enveloping orbits: indeed, in $\ell^2_{\mathbf{R}}(\mathbf{N})$, let X denote the set sequences (x_n) such that $x_n \in 2^n\mathbf{Z}$ for every $n \in \mathbf{N}$. Then X

[1] Shalom only states the result for a simple group, but the proof generalizes immediately. See for instance [CLTV] for another proof, based on the Howe-Moore Property.

is coarsely dense in $\ell^2_{\mathbf{R}}(\mathbf{N})$, but, for the metric induced by $\mathcal{H}$, every ball in X is finite, hence compact. We do not know if a semisimple Lie group (e.g. $\mathrm{SL}_2(\mathbf{R})$) can act isometrically on a non-zero Hilbert space with enveloping orbits.

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