Distortion in transformation groups

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Abstract We exhibit rigid rotations of spheres as distortion elements in groups of diffeomorphisms, thereby answering a question of J. Franks and M. Handel. We also show that every homeomorphism of a sphere is, in a suitable sense, as distorted as possible in the group $\mathrm{Homeo}(\mathbf{S}^n)$, thought of as a discrete group.

An appendix by Y. de Cornulier shows that $\mathrm{Homeo}(\mathbf{S}^n)$ has the *strong boundedness* property, recently introduced by G. Bergman. This means that every action of the discrete group $\mathrm{Homeo}(\mathbf{S}^n)$ on a metric space by isometries has bounded orbits.

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1 Introduction

The study of abstract groups as geometric objects has a long history, but has been pursued especially vigorously since the work of Gromov (see [7],[8]). Typically the focus is on finitely presented groups; however, interesting results have also been obtained from this perspective in the theory of transformation groups — i.e. groups of homeomorphisms of manifolds.

The topic of this paper is distortion in transformation groups, especially groups of homeomorphisms of spheres. Informally, an element h in a finitely generated group G is *distorted* if the word length of h^n grows sublinearly in n. One also sometimes says that the *translation length* of h vanishes. Geometrically, this corresponds to the condition that the homomorphism from $\mathbb{Z}$ to G sending n to h^n is not a quasi-isometric embedding.

One can also make sense of the concept of distortion in infinitely generated groups. An element h in a (not necessarily finitely generated) group G is distorted if there is a finitely generated subgroup H of G containing h such that h is distorted in H as above. To show that an element is undistorted, one typically tries to define an appropriate real-valued function on G which is (almost) subadditive, and which grows linearly on h^n. For example, *quasi-morphisms* are useful in this respect, and highlight one point of contact between distortion and the theory of bounded cohomology. On the other hand, exhibiting distortion is typically done *ad hoc*, and there do not seem to be many very general or flexible constructions known.

In this paper, we study distortion in groups of homeomorphisms of spheres, especially groups consisting of transformations with a definite amount of analytic regularity (i.e. C^1 or C^∞). By contrast with [20], [4] or [6], we do not insist that our groups preserve a probability measure; the considerable additional flexibility this affords has the consequence that our results have more of an existential character than those of the papers cited above, exhibiting distortion rather than ruling it out.

1.1 Statement of results

Notation 1.1 The letters G, H will denote groups of some sort, and S a (symmetric) generating set, although $\mathbf{S}^n$ denotes the n-sphere. If G is a group, and H is a subgroup, we write $H < G$. The group G will often be a transformation group on some manifold, and a typical element h will be a homeomorphism or diffeomorphism of some analytic quality. The letters i, j, n will denote integers, r will denote a degree of smoothness, and g will denote a growth function (i.e. a function $g : \mathbb{N} \to \mathbb{N}$). c and k will usually denote (implicit) constants in some inequality. We let $\mathbb{R}_+$ denote the *non-negative* real numbers. Other notation will be introduced as needed.

In §2 and §3 we summarize some basic definitions and study examples of distorted and undistorted elements in various groups.

In §4 we exhibit rigid rotations of $\mathbf{S}^2$ as distortion elements in the group of C^∞ diffeomorphisms of the sphere.

Our main result in this section is:

Theorem A *For any angle $\theta \in [0, 2\pi)$ the rigid rotation R_θ of $\mathbf{S}^2$ is a distortion element in a finitely generated subgroup of* $\mathrm{Diff}^\infty(\mathbf{S}^2)$*. Moreover, the distortion function of R_θ can be chosen to grow faster than any given function.*

Here R_θ is a clockwise rotation about a fixed axis through angle θ. To say that the distortion function grows faster than any given function means that for any $g : \mathbb{N} \to \mathbb{N}$ we can find a finitely generated group $G < \mathrm{Diff}^\infty(\mathbf{S}^2)$ for which there are words of length $\sim n_i$ in the generating set which express powers $R_\theta^{f(n_i)}$ of R_θ for some sequence $n_i \to \infty$, where $f(n) > g(n)$ for all sufficiently large $n \in \mathbb{N}$. In this case we say that the distortion function grows *faster than g*.

This answers a question of John Franks and Michael Handel, motivated by results in their paper [4].

In §5 we go down a dimension, and study rigid rotations of $\mathbf{S}^1$. Our main result here is:

Theorem B *For any angle $\theta \in [0, 2\pi)$ the rigid rotation R_θ of $\mathbf{S}^1$ is a distortion element in a finitely generated subgroup of* $\mathrm{Diff}^1(\mathbf{S}^1)$*. Moreover, the distortion function of R_θ can be chosen to grow faster than any given function.*

The proof of Theorem B makes use of Pixton's results from [18], and the arguments should be familiar to people working in the theory of foliations. It should be remarked that our construction cannot be made C^2, and it appears to be unknown whether a rigid rotation of $\mathbf{S}^1$ is distorted in $\mathrm{Diff}^\infty(\mathbf{S}^1)$ (or even in $\mathrm{Diff}^2(\mathbf{S}^1)$).

Remark 1.2 The possibility of proving Theorem B was pointed out to the first author by Franks and Handel, after reading an early version of this paper.

In §6 we relax our analytic conditions completely, and study distortion in the full group of homeomorphisms of $\mathbf{S}^n$. Here our main result is quite general:

Theorem C *Fix $n \geq 1$. Let $h_1, h_2, \ldots$ be any countable subset of* $\mathrm{Homeo}(\mathbf{S}^n)$*, and $g_1, g_2, \cdots : \mathbb{N} \to \mathbb{N}$ any countable collection of growth functions. Then there is a finitely generated subgroup H of* $\mathrm{Homeo}(\mathbf{S}^n)$ *(depending on $\{h_i\}$ and $\{g_i\}$) such that* every *h_i is simultaneously distorted in H. Moreover, the distortion function of h_i grows faster than g_i.*

The proof of Theorem C uses the full power of the Kirby–Siebenmann theory of homeomorphisms of manifolds for a key factorization step. It is an interesting question whether one can exhibit distortion in an arbitrary homeomorphism of the sphere without recourse to such sophisticated technology.

Finally, in an appendix, Yves de Cornulier uses the proof of Theorem C to show that the group $\text{Homeo}(\mathbf{S}^n)$ is *strongly bounded*. Here an abstract group G is said to be strongly bounded if every symmetric subadditive non-negative real-valued function on G is bounded. A countable group has this property if and only if it is finite.

1.2 Acknowledgements

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2 Distortion elements

2.1 Conjugation notation

Notation 2.1 For a group G and elements $a, b \in G$, we abbreviate the conjugate $b^{-1}ab$ by

$$a^b := b^{-1}ab$$

Notice with this convention that

$$(a^b)^c = a^{bc}$$

2.2 Basic definitions

Definition 2.2 Let G be a finitely generated group, and let S be a finite generating set. By convention, we assume $S = S^{-1}$. Given $h \in G$, the *length of h with respect to S* is the minimum integer n such that h can be expressed as a product

$$h = s_1 s_2 \cdots s_n$$

where each $s_i \in S$. We write

$$\ell_S(h) = n$$

By convention, we take $\ell_S(1) = 0$.

Note that ℓ_S is a *subadditive function*; that is, for all $h_1, h_2 \in G$,

$$\ell_S(h_1 h_2) \leq \ell_S(h_1) + \ell_S(h_2)$$

Moreover, it is *non-negative* and *symmetric*; i.e. $\ell(h) = \ell(h^{-1})$. This motivates the definition of a *length function* on a group G.

Definition 2.3 Let G be a group. A *length function* on G is a function $L : G \to \mathbb{R}_+$ satisfying $L(1) = 0$ which is symmetric and subadditive.

The function ℓ_S depends on the choice of generating set S, but only up to a multiplicative constant:

Lemma 2.4 *If S_1, S_2 are two finite generating sets for G, then there is a constant $c \geq 1$ such that*

$$\frac{1}{c}\ell_{S_2}(h) \leq \ell_{S_1}(h) \leq c\ell_{S_2}(h)$$

for all $h \in G$.

Proof Each $s \in S_1$ can be expressed as a word of length $n(s)$ in the elements of S_2, and vice versa. Then take c to be the maximum of the $n(s)$ over all $s \in S_1 \cup S_2$. □

Definition 2.5 Let G be a finitely generated group, and let S be a symmetric finite generating set as above. The *translation length* of an element $h \in G$, denoted $\|h\|_S$, is the limit

$$\|h\|_S := \lim_{n\to\infty} \frac{\ell_S(h^n)}{n}$$

An element $h \in G$ is a *distortion element* if the translation length is 0.

Remark 2.6 Note that by the subadditivity property of ℓ_S, the limit exists. Moreover, by Lemma 2.4, the property of being a distortion element is independent of the choice of generating set S.

Remark 2.7 With this definition, torsion elements are distortion elements. Some authors (including [4]) explicitly require distortion elements to be *non-torsion*.

Sometimes, we shall pay attention to the growth rate of $\ell_S(h^n)$ as a function of n to make qualitative distinctions between different kinds of distortion elements. If h is not torsion, we define the *distortion function* to be the function

$$D_{S,h} : \mathbb{N} \to \mathbb{N}$$

defined by the property

$$D_{S,h}(n) = \max\{i \mid \ell_S(h^i) \leq n\}$$

We can remove the dependence of this function on S as follows. For two functions

$$f, g : \mathbb{N} \to \mathbb{N}$$

we write $f \precsim g$ if there is a constant $k \geq 1$ such that

$$f(n) \leq kg(kn + k) + k \text{ for all } n \in \mathbb{N}$$

and then write $f \sim g$ if $f \precsim g$ and $g \precsim f$. It is straightforward to see that $\precsim$ is transitive, and that $\sim$ is an equivalence relation. In case $f \sim g$, we say that f, g are *quasi-equivalent*. With this definition, the quasi-equivalence class of $D_{S,h}$ is independent of S, and may be denoted D_h.

We are also interested in comparing growth rates in a cofinal sense:

Definition 2.8 Given $g : \mathbb{N} \to \mathbb{N}$ we say that the distortion function of $h \in G$ (with respect to a generating set S) *grows faster than* g if there is a sequence $n_i \to \infty$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > g(n)$ for all sufficiently large n, and such that

$$\ell_S(h^{f(n_i)}) \leq n_i$$

We say for example that h has *quadratic distortion* if $g(n) = n^2$ or *exponential distortion* if $g(n) = e^n$ as above.

Finally, we may define a distortion element in an arbitrary group:

Definition 2.9 Let G be a group. An element $h \in G$ is a distortion element if there is a finitely generated subgroup $H < G$ with $h \in H$ such that h is a distortion element in H.

Note that for such an element h, the quasi-equivalence class of the distortion function may certainly depend on H.

2.3 Examples

Example 2.10 In $\mathbb{Z}$ only the identity element is distorted.

Example 2.11 If $\phi : G \to H$ is a homomorphism, and $\phi(h)$ is not distorted in H, then h is not distorted in G.

Example 2.12 If $L : G \to \mathbb{R}_+$ is a length function, and

$$\lim_{n\to\infty} \frac{L(h^n)}{n} > 0$$

then h is not distorted. More generally, a length function gives a lower bound for word length with respect to any finite generating set, and therefore and upper bound on distortion. E.g. if $L(h^n)$ grows like $\log(n)$ then h is no more than exponentially distorted.

The next few examples treat distortion in linear groups.

Example 2.13 Let $G = \mathrm{GL}(n, \mathbb{C})$ and define $L : G \to \mathbb{R}_+$ by

$$L(A) = \log \text{ of the max of the operator norms of } A \text{ and } A^{-1}$$

Then L is a length function. It follows that if A has an eigenvalue with absolute value $\neq 1$ then A is not distorted.

Example 2.14 Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ be a Galois automorphism of $\mathbb{C}$. Then A is distorted in $\mathrm{GL}(n, \mathbb{C})$ if and only if $\sigma(A)$ is. It follows that if A is distorted, then every eigenvalue must be algebraic, with all conjugates on the unit circle.

Example 2.15 Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated subgroup with entries in a number field K. We may construct length functions from valuations associated to finite primes in the ring of integers of K. If $x \in K$ then $v(x) = 0$ for all discrete valuations v on K if and only if x is a unit. A unit in a number field with absolute value 1 is a root of unity; c.f. [13]. Combined with Example 2.13 and Example 2.14, one can show that an arbitrary element $A \in \mathrm{GL}(n, \mathbb{C})$ is distorted if and only if every eigenvalue of A is a root of unity. Note that the distortion of a non-torsion element is at most *exponential*. See [14] for details.

Example 2.16 In the Baumslag-Solitar group $\langle a, b \mid aba^{-1} = b^2 \rangle$ the element b has exponential distortion. Similarly, in the group

$$\langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2 \rangle$$

the element c has doubly-exponential distortion. Note that as a corollary, we deduce that this second group is not linear. This example and others are mentioned in [8], Chapter 3.

Example 2.17 Let G be a group. A *quasi-morphism* is a map $\phi : G \to \mathbb{R}$ such that there is a constant $c > 0$ for which

$$|\phi(h_1) + \phi(h_2) - \phi(h_1 h_2)| \leq c$$

for all $h_1, h_2 \in G$. If $|\phi(h)| > c$ then h is not distorted.

Quasi-morphisms are intimately related to (second) bounded cohomology. See e.g. [6] for a salient discussion.

3 Distortion in transformation groups

3.1 Transformation groups

Notation 3.1 For a compact C^∞ manifold M, we denote the group of homeomorphisms of M by $\mathrm{Homeo}(M)$, and the group of C^r diffeomorphisms by $\mathrm{Diff}^r(M)$, where $r = \infty$ is possible. Here a homeomorphism h is in $\mathrm{Diff}^r(M)$ if both it and its inverse are C^r. Note that this implies dh has full rank everywhere. If we wish to restrict to orientation-preserving subgroups, we denote this by a $+$ superscript.

3.2 Distortion in Diff^1

Suppose M is a smooth compact Riemannian manifold, and $h \in \mathrm{Diff}^1(M)$. We define the following norm:

$$\|dh\| := \log \sup_{v \in UTM} |dh(v)|$$

where $|dh(v)|$ denotes the length of $dh(v)$, and the supremum is taken over all vectors v in the unit tangent bundle of M.

Note that since h is a diffeomorphism and M is compact, dh cannot be strictly contracting at every point, and therefore $\|dh\| \geq 0$. If we define

$$\|dh\|^+ = \max(\|dh\|, \|d(h^{-1})\|)$$

then it is clear that $\|d \cdot \|^+$ is a length function on $\mathrm{Diff}^1(M)$. In general, therefore, the growth rate of $\|dh^n\|^+$ as a function of n puts an upper bound on the distortion function of h in any finitely generated subgroup of $\mathrm{Diff}^1(M)$.

Note if we choose two distinct Riemannian metrics on M, the length functions $\|d \cdot \|^+$ they define will be quasi-equivalent, by compactness. On the other hand, if M is non-compact, different quasi-isometry classes of Riemannian metrics may give rise to qualitatively different length functions.

Example 3.2 Suppose h has a fixed point p and $dh|_{T_pM}$ has an eigenvalue with absolute value $\neq 1$. Then h is not distorted in $\text{Diff}^1(M)$.

Example 3.3 Oseledec's theorem (see [19], Chapter 2) says that for $h \in \text{Diff}^1(M)$ where M is a compact manifold, and for μ an ergodic h-invariant probability measure on M, there are real numbers $\lambda_1 > \cdots > \lambda_k$ called *Lyapunov exponents*, and a μ-measurable dh-invariant splitting $TM = \oplus_{i=1}^k E^i$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log |dh^n(v)| = \lambda_l$$

for almost every $v \in \oplus_{i=l}^k E^i$ but not in $\oplus_{i=l+1}^k E^i$. In particular, if $\lambda_1 > 0$, then h is undistorted in $\text{Diff}^1(M)$.

Example 3.4 Let M be a compact manifold, and suppose $h \in \text{Diff}^1(M)$ has positive topological entropy. Then there is an ergodic h-invariant probability measure μ for which h has positive μ-entropy. The Pesin–Ruelle inequality (see [19], Chapter 3) says

$$\sum_{\lambda_i > 0} \lambda_i \geq \mu\text{-entropy of } h$$

where the λ_i are the Lyapunov exponents for h with respect to the measure μ. It follows that some Lyapunov exponent λ_1 for μ is positive, and therefore, as in Example 3.3, h is undistorted in $\text{Diff}^1(M)$.

By contrast, if $\|dh^n\|^+$ is bounded independently of n, then the group $\langle h \rangle$ is equicontinuous, and is precompact in the group of Lipschitz homeomorphisms of M, by the Arzela–Ascoli theorem. By [22] (i.e. the Hilbert–Smith conjecture for Lipschitz actions), a compact group of Lipschitz homeomorphisms of a smooth manifold M is a Lie group. In our case, this group is abelian, since it contains a dense abelian subgroup $\langle h \rangle$, and is therefore (up to finite index) a finite dimensional torus. Thus the uniformly equicontinuous case reduces to that of torus actions.

A key case to understand in this context is when the torus in question is $\mathbf{S}^1$, and the simplest example is that of a rigid rotation of a sphere. It is this example which we study in the next few sections.

4 Rotations of $\mathbf{S}^2$

4.1 The group G

We describe a particular explicit group $G < \text{Diff}^\infty(\mathbf{S}^2)$ which will be important in the sequel. By stereographic projection, we may identify $\mathbf{S}^2$ conformally with $\mathbb{C} \cup \infty$.

Let T be the similarity

$$T : z \to 2z$$

Then $\langle T \rangle$ acts discretely and properly discontinuously on $\mathbb{C}^*$ with quotient a (topological) torus. A fundamental domain for the action is the annulus A defined by

$$A = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$$

We let ∂A^+ and ∂A^- denote the components $|z| = 2$ and $|z| = 1$ of ∂A respectively. We define a disk D by

$$D = \{z \in \mathbb{C} \mid |z - 3/2| \leq 1/4\}$$

We let F be a C^∞ diffeomorphism with the following properties:

- F is the identity outside the annulus $0.99 \leq |z| \leq 2.01$
- F restricted to the annulus $1.01 \leq |z| \leq 1.99$ agrees with the rotation $z \to -z$

We define $G = \langle T, F \rangle$, and think of it as a subgroup of $\text{Diff}^\infty(\mathbf{S}^2)$ fixing 0 and ∞. Notice that for every $h \in G$ either $h(D)$ is disjoint from D, or else $h|_D = \text{Id}|_D$. If G^D denotes the stabilizer of D in G, then we may identify the orbit GD with the product $D \times S$ where S is the set of (right) cosets of the subgroup G^D in G. Note that S is a set with a (left) G-action. This action determines the action of G on $D \times S$.

An explicit set of coset representatives for S is the set of elements of the form T^n and FT^n for all $n \in \mathbb{Z}$.

4.2 Wreath products

Let G, S and $D \subset \mathbf{S}^2$ be as in §4.1. Let $\zeta_t, t \in \mathbb{R}$ be a 1-parameter subgroup of diffeomorphisms of the unit disk with support contained in the interior. After conjugating by a diffeomorphism, we think of ζ_t as a 1-parameter subgroup of $\text{Diff}^\infty(\mathbf{S}^2)$ with support contained in the interior of D.

Definition 4.1 Let $\mathbb{R}^S$ denote the set of functions from S to $\mathbb{R}$, which can be thought of as an abelian group with respect to addition. The *wreath product* $G \wr_S \mathbb{R}$ is the semi-direct product

$$0 \to \mathbb{R}^S \to G \wr_S \mathbb{R} \to G \to 0$$

where G acts on $\mathbb{R}^S$ by

$$f^h(s) = f(hs)$$

for $h \in G, s \in S$.

The choice of 1-parameter group ζ_t determines a faithful homomorphism

$$\rho : G \wr_S \mathbb{R} \to \text{Homeo}(\mathbf{S}^2)$$

as follows. For $f \in \mathbb{R}^S$, define

$$\rho(f) = \prod_{s \in S} \zeta_{f(s)}^{\bar{s}}$$

where $\bar{s} \in G$ is a coset representative of $s \in S$. Together with the action of G on $\mathbf{S}^2$ (in its capacity as a transformation group) this defines a faithful homomorphism ρ. For the sake of brevity, in the sequel we will omit ρ, and think of $G \wr_S \mathbb{R}$ itself as a subgroup of $\text{Homeo}(\mathbf{S}^2)$.

4.3 Analytic quality

Given $f \in \mathbb{R}^S$, thought of as an element of $\text{Homeo}(\mathbf{S}^2)$ as in §4.2, the analytic quality of f is *a priori* only C^0. However, if we can estimate the C^r norm of $f(T^n), f(FT^n)$ as $|n| \to \infty$, we can improve this *a priori* estimate.

Notice that any $f \in \mathbb{R}^S$ is C^∞ away from $0, \infty$. In particular, any f with finite support is C^∞ on all of $\mathbf{S}^2$. Furthermore, conjugation by T preserves the C^1 norm, and blows up the C^r norm by 2^{r-1}, whereas conjugation by F preserves the C^r norm for every r. It follows that if we have an estimate

$$|f(T^n)|, |f(FT^n)| = o(2^{-|n|(r-1)})$$

as $|n| \to \infty$, then f is C^r at 0 (here our notation $|f(s)|$ just means the absolute value of $f(s)$ for $s \in S$, where we think of f as a function from S to $\mathbb{R}$). By the change of co-ordinates $z \to 1/z$ one sees that f is also C^r at ∞ under the same hypothesis, and is therefore C^r on all of $\mathbf{S}^2$.

We summarize this as a lemma:

Lemma 4.2 *Let $f \in \mathbb{R}^S$ be thought of as an element of* $\mathrm{Homeo}(\mathbf{S}^2)$ *as in §4.2. Then we have the following estimates:*

- *If $|f(s)|$ is bounded independently of $s \in S$ then f is Lipschitz*
- *If $\lim_{s\to\infty} |f(s)| = 0$ then f is C^1*
- *If $|f(T^n)|, |f(FT^n)| \to 0$ faster than any exponential (as a function of n), then f is C^∞*

4.4 Rotations of $\mathbf{S}^2$

For each $\theta \in [0, 2\pi)$ we let R_θ denote the rigid rotation of $\mathbf{S}^2$ with fixed points equal to 0 and ∞. In stereographic co-ordinates,

$$R_\theta : z \to e^{i\theta} z$$

where $z \in \mathbb{C} \cup \infty$. Notice that R_π is just multiplication by -1.

For $\theta \in \pi\mathbb{Q}$ the element R_θ is torsion in $\mathrm{Diff}^\infty(\mathbf{S}^2)$. We will show in this section that R_θ is a distortion element in $\mathrm{Diff}^\infty(\mathbf{S}^2)$ for arbitrary θ. Moreover, the distortion function can be taken to grow faster than any given function.

4.5 Factorizing rotations

We can factorize R_θ in a natural way as a product of two diffeomorphisms whose support is contained in closed subdisks of $\mathbf{S}^2$. This will be important for some later applications.

Let B (for bump) be a smooth function $B : \mathbb{R}^+ \to [0, 1]$ which satisfies the following properties:

- $B(t) = 0$ for $t < 1/2$ and $B(t) = 1$ for $t > 2$
- $B(t) + B(1/t) = 1$
- B is monotone decreasing and strictly positive on $(1/2, 2)$
- B is infinitely tangent to the constant function 1 at 2 and to the constant function 0 at $1/2$

For $\theta \in \mathbb{R}$, define R_θ^+ by

$$R_\theta^+ : z \to e^{iB(|z|)\theta} z$$

and define R_θ^- by the identity

$$R_\theta^+ R_\theta^- = R_\theta$$

Notice that as θ varies over $\mathbb{R}$, the set of transformations R^-_θ and R^+_θ form smooth subgroups of $\mathrm{Diff}^+(\mathbf{S}^2)$. Moreover, the support of the group $\{R^-_\theta \mid \theta \in \mathbb{R}\}$ is equal to the disk

$$E^- = \{z \mid |z| \leq 2\}$$

Similarly, the support of R^+_θ is the disk (in $\mathbf{S}^2$)

$$E^+ = \{z \mid |z| \geq 1/2\}$$

Notice the important fact that $z \to 1/z$ conjugates $R^+_{-\theta}$ to R^-_θ for any θ. The reason for the sign change is that a 1-parameter family of rotations which has a clockwise sense at one fixed point has an anticlockwise sense at the other fixed point.

4.6 Construction of the group

Throughout the remainder of this section we assume that θ has been fixed.

We define a diffeomorphism Z which takes care of some bookkeeping for us. Basically, the diffeomorphism Z lets us move back and forth between the 1-parameter groups $R^\pm_t$ with support in $E^\pm$ and a 1-parameter group ζ_t with support in D, as in §4.1 and §4.2. The exact details of how this is done are irrelevant, but we must make an explicit choice, which accounts for the (annoying) notational complexity below.

Let $Z \in \mathrm{Diff}^\infty(\mathbf{S}^2)$ satisfy the following properties:

- Z takes D to E^- and conjugates R^-_t to a 1-parameter subgroup ζ_t:
$$\zeta_t := (R^-_t)^Z$$
- Z takes FD to T^3E^+ (i.e. the image of the disk E^+ under the similarity $z \to 8z$) and conjugates $(R^+_t)^{T^{-3}}$ to ζ^F_{-t}:
$$\zeta^F_{-t} = (R^+_t)^{T^{-3}Z}$$

The existence of such a diffeomorphism Z follows from the disjointness of the disks E^-, T^3E^+ and the fact that the subgroups R^-_t and R^+_{-t} are abstractly conjugate, by $z \to 1/z$, as pointed out in §4.5.

Now form the group $\mathbb{R}^S$ as in §4.2 by means of the subgroup $\zeta_t = (R^-_t)^Z$.

Let $t_i \in \mathbb{R}$ be chosen for all non-negative integers i subject to the following constraints:

- $t_i = n_i\theta \mod 2\pi$ where $n_i \to \infty$ grow as fast as desired (i.e. faster than some growth function $g : \mathbb{N} \to \mathbb{N}$ we are given in advance)
- $t_i \to 0$ faster than any exponential function

Define the element $f \in \mathbb{R}^S$ by

$$f(T^i) = t_i \text{ if } i \geq 0, \ f(T^i) = 0 \text{ if } i < 0, \ f(FT^i) = 0 \text{ for all } i$$

By Lemma 4.2 the function f is in $\text{Diff}^\infty(\mathbf{S}^2)$ with respect to the identification of $\mathbb{R}^S$ with a subgroup of $\text{Homeo}(\mathbf{S}^2)$.

Now, for any i, the element

$$f_i := f^{T^i}(f^{T^iF})^{-1}$$

is contained in $\mathbb{R}^S$, and satisfies

$$f_i(s) = \begin{cases} t_i & \text{if } s = \text{Id} \\ -t_i & \text{if } s = F \\ 0 & \text{otherwise} \end{cases}$$

We conjugate the f_i back by Z^{-1}, and define

$$h_i := f_i^{Z^{-1}}$$

Then h_i agrees with $R^-_{t_i}$ on E^- and agrees with $(R^+_{t_i})^{T^{-3}}$ on T^3E^+.

Notice that h_i preserves the foliation of $\mathbf{S}^2 \backslash \{0, \infty\}$ by circles of equal latitude, and acts on each of these circles by a rotation. Let $\text{LAT} < \text{Diff}^\infty(\mathbf{S}^2)$ denote the group of diffeomorphisms with this property; i.e. informally, LAT preserves latitude, and acts as a rotation on each circle with a fixed latitude. An element of LAT can be thought of as a C^∞ function

$$\text{latitude} \ \to \ \text{rotation angle}$$

up to constant functions with values in $2\pi\mathbb{Z}$, and any element of LAT can be recovered pictorially from the graph of this function. Notice that $h_i \in \text{LAT}$. In this way, we can abbreviate h_i by a picture:

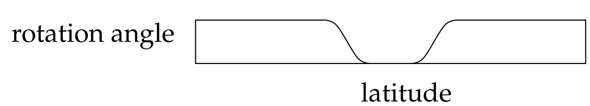

Figure 1: The element $h_i \in \text{LAT}$ represented pictorially by the graph of a function

Let LONG $<$ Diff$^\infty(\mathbf{S}^2)$ denote the group of diffeomorphisms of the form

$$z \to z \cdot u(|z|)$$

where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is infinitely tangent to the identity at 0 and at ∞. Informally, LONG is the group of diffeomorphisms which reparameterizes the set of latitudes, without changing longitudes. Then LONG is contained in the normalizer of the group LAT. The conjugation action of LONG on LAT is given pictorially by reparameterizing the base of the graph.

We claim that there are elements $M_1, M_2, M_3 \in$ LONG such that for any h_i we have an identity

$$h_i(h_i)^{M_1}(h_i)^{M_2}((h_i)^{M_3})^{-1} = R_{2t_i}$$

The proof is given graphically by figure 2:

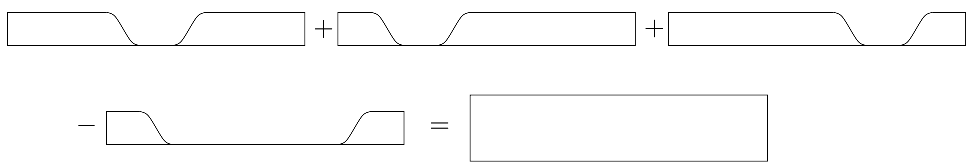

Figure 2: These figures denote the conjugates of h_i, and demonstrate how an appropriate algebraic product of these conjugates is equal to R_{2t_i}

Now, $R_{2t_i} = R_\theta^{2n_i}$. Since the n_i have been chosen to grow faster than any function given in advance, we have proved the following theorem:

Theorem A *For any angle $\theta \in [0, 2\pi)$ the rigid rotation R_θ of $\mathbf{S}^2$ is a distortion element in a finitely generated subgroup of* Diff$^\infty(\mathbf{S}^2)$. *Moreover, the distortion function of R_θ can be chosen to grow faster than any given function.*

5 Rotations of $\mathbf{S}^1$

In this section we show how to modify the construction of §4 to exhibit a rigid rotation as a distortion element in the group Diff$^1(\mathbf{S}^1)$. But first, we exhibit a rotation as a distortion element in the group of *Lipschitz* homeomorphisms of $\mathbf{S}^1$.

5.1 Rotations of $\mathbf{S}^1$

As in the previous section, we denote by R_θ the rotation of $\mathbf{S}^1$ through angle $\theta \in [0, 2\pi)$.

The first difference with §4 is that we cannot factorize a 1-parameter group of rotations as the product of two 1-parameter groups with support contained in an interval. (One way to see this is to use Poincaré's rotation number; see e.g. [25] for a definition and basic properties.)

Let θ be fixed, and we choose $t_i \to 0, n_i \to \infty$ as $i \in \mathbb{Z}$ goes from 0 to ∞, with

$$t_i = n_i\theta \mod 2\pi$$

as in §4.6.

Let $I^\pm$ be two intervals which form an open cover of $\mathbf{S}^1$. Then for t_i sufficiently close to 0, we can factorize R_{t_i} as a product of two diffeomorphisms ξ_i, ζ_i with support contained in I^+, I^- respectively. It is clear that we may choose ξ_i, ζ_i so that their support is exactly equal to an interval, and they are both conjugate to translations on these intervals.

Let J be an open interval in $\mathbf{S}^1$ which we parameterize by arclength as $[-1, 1]$. We let T be a diffeomorphism of $\mathbf{S}^1$ with support equal to J, and with no fixed points in J. Then the restriction of T to J is conjugate to a translation, and we let J_i for $i \in \mathbb{Z}$ be a tiling of J by fundamental domains for t.

Fix one such interval $J_0 \subset J$ and let F be a diffeomorphism of $\mathbf{S}^1$ with support equal to J_0, and with no fixed points in J_0. We let J_{0i} for $i \in \mathbb{Z}$ be a tiling of J_0 by fundamental domains for F.

The group $G = \langle T, F\rangle$ acts as before on the set of translates of J_{00}, and for all $h \in G$, either $h(J_{00})$ is disjoint from J_{00}, or else $h|_{J_{00}} = \mathrm{Id}|_{J_{00}}$. The interval J_{00} is the analogue of the disk D from §4, and the elements T, F are the analogues of the diffeomorphisms of the same names in that section. The difference is that if GJ_{00} denotes the set of translates of J_{00} by G, then $F|_{GJ_{00}}$ has infinite order, rather than order 2.

Now let $Z^\pm$ be diffeomorphisms of $\mathbf{S}^1$ taking $I^\pm$ respectively to the interval J_{00}. The diffeomorphisms $Z^\pm$ are the analogue of the diffeomorphism Z from §4; the reason we need two such diffeomorphisms rather than just one is that the factorization of R_{t_i} into $\xi_i\zeta_i$ is no longer canonical.

We let $f^+ \in \mathrm{Homeo}(\mathbf{S}^1)$ have support contained in J, and define it to be the product

$$f^+ = \prod_{i=0}^{\infty}\prod_{j=0}^{\infty} \xi_i^{(Z^+)^{-1}F^{-j}T^{-i}}$$

and similarly, define

$$f^- = \prod_{i=0}^{\infty}\prod_{j=0}^{\infty} \zeta_i^{(Z^-)^{-1}F^{-j}T^{-i}}$$

Notice by Lemma 4.2 that $f^{\pm}$ are Lipschitz (though *not* C^1).

Then for each i,

$$\left((f^+)^{T^i}((f^+)^{T^iF^{-1}})^{-1}\right)^{Z^+} = \xi_i$$

and

$$\left((f^-)^{T^i}((f^+)^{T^iF^{-1}})^{-1}\right)^{Z^-} = \zeta_i$$

and therefore $R_{n_i\theta}$ can be expressed as a word of length $\sim 8i$ in the group

$$\langle f^+, f^-, F, T, Z^+, Z^- \rangle$$

Notice that there is no analogue of the groups LAT and LONG, and consequently no analogue of the elements M_1, M_2, M_3.

5.2 A C^1 example

By a slight modification, using a trick of Pixton we can actually improve the Lipschitz example of §5.1 to a C^1 example.

We note that by suitable choice of factorization of R_{t_i} we can assume the following:

- The support of ξ_i, ζ_i is *exactly* equal to I^+, I^- respectively
- On I^+, each ξ_i is conjugate to a translation, and similarly for I^-

Now, the elements ξ_i for distinct i will not be contained in a fixed 1-parameter subgroup of $\text{Diff}^\infty(I^+)$, but they are all *conjugate* into a fixed 1-parameter subgroup, and similarly for the ζ_i. The final condition we insist on is:

- The conjugating maps can be taken to be C^1 and converge in the C^1 topology to the identity.

To see that this is possible, observe that for two C^∞ diffeomorphisms ϵ-close to the identity in the C^1 norm, the commutator is ϵ^2-close to the identity, also in the C^1 norm. So for diffeomorphisms ϕ_g defined by the property

$$\phi_g : \theta \to \theta + g(\theta)$$

for $g : \mathbf{S}^1 \to \mathbb{R}$, we have that

$$\phi_{g_1}\phi_{g_2} \sim \phi_{g_1+g_2}$$

with error which is comparable in size in the C^1 norm to the products of the C^1 norms of g_1, g_2. Using this fact, one can readily produce a suitable factorization.

5.3 Pixton actions

Consider an interval I on which a diffeomorphism $Y : I \to I$ acts in a manner smoothly conjugate to a translation, with fundamental domains I_i. Given another diffeomorphism $\phi : I_0 \to I_0$ we form the suspension $\Phi : I \to I$ by

$$\Phi = \prod_i \phi^{Y^i}$$

Note that $\langle Y, \Phi \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. If ϕ restricted to I_0 is smoothly conjugate to a translation, then *a priori* the action of $\langle Y, \Phi \rangle$ on I is Lipschitz. However, Pixton showed that it is *topologically* conjugate (i.e. by a homeomorphism) to a C^1 action.

For the convenience of the reader, we give an outline of the construction of a Pixton action. One chooses co-ordinates on I so that the ratio $|I_i|/|I_{i+1}|$ converges to 1 as $|i| \to \infty$. For instance, near I, the endpoints of the I_n could be the harmonic series $1/2, 1/3, \ldots$ so that the ratio of successive lengths is $i/(i+1) \to 1$. Then we require $Y : I_i \to I_{i+1}$ to expand the linear structure near the endpoints and contract it in the middle, so that the norm of the first derivative of $\Phi|_{I_{i+1}}$ is smaller than that of $\Phi|_{I_i}$ by a definite amount. Then both Y and Φ are C^1 tangent to the identity at the endpoints of I, and are therefore C^1 on the entire interval. See [18] for rigorous details of this construction. One should remark that a lemma of Kopell [12] implies that one cannot make the action C^2.

This construction has the following virtue: if ϕ is contained in a smooth 1-parameter subgroup ϕ_t, and we form the associated 1-parameter subgroup Φ_t so that

$$\langle Y, \Phi_t \rangle \cong \mathbb{Z} \oplus \mathbb{R}$$

then we can form a Pixton action of this larger group which is C^1.

5.4 T and X

Naively, one sees that by careful choice of F, one can arrange for the action of $f^+|_{J_0}$ to be C^1. However, to make f^+ C^1 on all of J requires us to modify the definition slightly.

We will construct X, a diffeomorphism of $\mathbf{S}^1$ with support equal to J, conjugate to a translation on J, and with fundamental domains J_i, just like T.

We let χ_t be a 1-parameter subgroup containing ξ_0. For each i, we require that

$$\xi_i^{(Z^+)^{-1}T^{-i}X^i} \in (\chi_t)^{(Z^+)^{-1}}$$

which is possible, by the discussion at the end of §5.2. By choosing co-ordinates on J suitably as above, we can insist that both X and T are C^1.

Now we choose co-ordinates on J_0 so that F and $(\chi_t)^{(Z^+)^{-1}}$ form a Pixton action of $\mathbb{Z} \oplus \mathbb{R}$ there, as in §5.3.

We define

$$f^+ = \prod_{i=0}^{\infty}\prod_{j=0}^{\infty} \xi_i^{(Z^+)^{-1}T^{-i}X^iF^{-j}X^{-i}}$$

Note that f^+ is actually C^1.

Moreover, we have the following formula

$$\left(\left((f^+)^{X^i}((f^+)^{X^iF^{-1}})^{-1}\right)^{X^{-i}T^i}\right)^{Z^+} = \xi_i$$

Relabelling X as X^+ and defining X^- similarly in terms of the ζ_i, one can define f^- analogously. Putting this together, we have shown

Theorem B *For any angle $\theta \in [0, 2\pi)$ the rigid rotation R_θ of $\mathbf{S}^1$ is a distortion element in a finitely generated subgroup of $\mathrm{Diff}^1(\mathbf{S}^1)$. Moreover, the distortion function of R_θ can be chosen to grow faster than any given function.*

One should remark that for a rigid rotation R of $\mathbf{S}^n$ where n is arbitrary, either R has fixed points, in which case the construction of §4 shows that R is a distortion element in $\mathrm{Diff}^\infty(\mathbf{S}^n)$, or else the construction of this section can be generalized to show that R is a distortion element in $\mathrm{Diff}^1(\mathbf{S}^n)$, in either case with distortion growing faster than any given function.

Remark 5.1 Tsuboi showed in [26] that one can construct Pixton actions which are $C^{1+\alpha}$ for every $\alpha < 1$. It is therefore likely that the construction above exhibits a rigid rotation as an arbitrarily badly distorted element in $\mathrm{Diff}^{1+\alpha}(\mathbf{S}^1)$.

By our discussion in §3.2, we make the following conjecture:

Conjecture 5.2 *Let M be a compact smooth manifold, and let $h \in \mathrm{Diff}^1(M)$. Then h is a distortion element in $\mathrm{Diff}^1(M)$ whose distortion function can be chosen to grow faster than any given function if and only if some finite power of h is contained in a C^1 action of a finite dimensional torus on M.*

Note that the "only if" direction follows from §3.2.

6 Distortion in $\mathrm{Homeo}(\mathbf{S}^n)$

The group $\mathrm{Homeo}(M)$ for an arbitrary manifold M is considerably more complicated than $\mathrm{Diff}^\infty(M)$ or even $\mathrm{Diff}^1(M)$. In this section, we first make a couple of comments about distortion in $\mathrm{Homeo}(M)$ in general, and then specialize to the case of $\mathrm{Homeo}(\mathbf{S}^n)$.

6.1 Mapping class groups

For an arbitrary compact manifold M, there is a natural homomorphism

$$\mathrm{Homeo}(M) \to \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M) =: \mathrm{MCG}(M)$$

where $\mathrm{Homeo}_0(M)$ is the normal subgroup consisting of homeomorphisms isotopic to the identity, and $\mathrm{MCG}(M)$ is the *mapping class group* of M. For reasonable M, this group is finitely presented, and quite amenable to computation. Clearly for $h \in \mathrm{Homeo}(M)$ to be a distortion element, it is necessary for the image $[h]$ of h in $\mathrm{MCG}(M)$ to be a distortion element.

Example 6.1 A pseudo-Anosov homeomorphism of a closed surface Σ of genus ≥ 2 is not a distortion element in $\mathrm{Homeo}(\Sigma)$.

6.2 Distortion in $\text{Homeo}_0(M)$

For suitable manifolds M, it is easy to find undistorted elements in $\text{Homeo}_0(M)$.

Example 6.2 Let T^2 denote the 2-torus. Let $h : T^2 \to T^2$ preserve the foliation of T^2 by meridians, and act as a rigid rotation on each meridian, where the angle of rotation is not constant. This angle of rotation defines a map $\theta : \mathbf{S}^1 \to \mathbf{S}^1$, where the first factor labels the meridian, and the second factor is the amount of rotation. If θ is homotopically trivial, h is in $\text{Homeo}_0(T^2)$. In this case, we claim h is undistorted in $\text{Homeo}_0(T^2)$. To see this, suppose to the contrary that h is distorted in some finitely generated subgroup H. Without loss of generality, we may expand H to a larger finitely generated group, where each generator h_i has support contained in a closed disk in T^2. If $\widetilde{h_i}$ denotes a lift of h_i to the universal cover $\mathbb{R}^2$, then there is a constant c such that

$$|d_{\mathbb{R}^2}(\widetilde{h_i}(p), \widetilde{h_i}(q)) - d_{\mathbb{R}^2}(p,q)| \leq c$$

for any $p, q \in \mathbb{R}^2$. Without loss of generality, we may assume that the same constant c works for all i.

Now, if I is a small transversal to the foliation of T^2 by meridians, intersecting meridians where the function θ is nonconstant, it follows that if we denote $I_n := h^n(I)$, then a lift $\widetilde{I_n}$ of I_n has the property that the endpoints are distance $\sim kn$ apart for some positive constant k. By the discussion above, this implies that any expression of h^n in the generators h_i and their inverses has word length at least $\sim nk/c$. This shows that h is undistorted, as claimed.

Example 6.3 Let M be a closed hyperbolic 3-manifold. Let $\gamma \subset M$ be a simple closed geodesic, and let N be an embedded tubular neighborhood. Let $h : M \to M$ rotate γ some distance, and be fixed outside N. Then the argument of Example 6.2 shows that h is undistorted in $\text{Homeo}_0(M)$. Since M is hyperbolic of dimension at least 3, Mostow rigidity [15] implies that $\text{MCG}(M)$ is finite. It follows that h is undistorted in the full group $\text{Homeo}(M)$.

Question 6.4 *Is h as in Example 6.2 undistorted in* $\text{Homeo}(T^2)$*?*

The method of construction in Example 6.2 produces an undistorted element of $\text{Homeo}_0(M)$ whenever $\pi_1(M)$ contains an undistorted element. Moreover, if $\text{MCG}(M)$ is finite, the element is undistorted in $\text{Homeo}(M)$. This begs the following obvious question:

Question 6.5 *Is there an infinite, finitely presented group G in which every element is distorted?*

Remark 6.6 A finitely presented infinite torsion group would answer Question 6.5 affirmatively.

Remark 6.7 It is worth observing that Ol'shanskii [17] has shown the existence of a torsion-free finitely generated group in which all elements are distorted, thereby answering a question of Gromov.

The following construction gets around Question 6.5, at a mild cost.

Example 6.8 Let M be a closed manifold with $\pi_1(M)$ infinite. Then $\widetilde{M}$ inherits a path metric pulled back from M with respect to which the diameter is infinite. It follows that $\widetilde{M}$ contains a *ray* r — that is, an isometrically embedded copy of $\mathbb{R}^+$ which realizes the minimal distance between any two points which it contains. The ray r projects to M where it might intersect itself. By abuse of notation, we refer to the projection as r. If the dimension of M is at least 3, then we can perturb r an arbitrarily small amount so that it is embedded in M (though of course not properly embedded). In fact, we can even ensure that there is an embedded tubular neighborhood N of r whose width tapers off to zero as one escapes to infinity in r in its intrinsic path metric. Let h be a homeomorphism of M, fixed outside N, which translates the core (i.e. r) by some function

$$r(t) \to r(t + f(t))$$

where $f(t)$ is positive, and goes to 0 as $t \to \infty$. Such a homeomorphism may be constructed for instance by coning this translation of r out to ∂N with respect to some radial co-ordinates. Then h might be distorted, but the distortion function can be taken to increase as slowly as desired, by making f go to 0 as slowly as desired. For example, we could ensure that the distortion function grows slower than n^α for all $\alpha > 1$.

6.3 Homeomorphisms of spheres

We now specialize to $\mathbf{S}^n$. We make use of the following seemingly innocuous lemma:

Lemma 6.9 (Kirby–Siebenmann, Quinn) *Let* $h \in \mathrm{Homeo}^+(\mathbf{S}^n)$*. Then* h *can be factorized as a product*

$$h = h_1 h_2$$

where the support of h_1 *avoids the south pole, and the support of* h_2 *avoids the north pole.*

For h sufficiently close to the identity in the compact-open topology, this can be proved by the geometric torus trick. For an arbitrary homeomorphism, it requires the full power of topological surgery theory. See [11] for details in the case $n \neq 4$ and [21] for the case $n = 4$.

Using this lemma, we can produce another factorization:

Lemma 6.10 *Let* E_1, E_2 *be two closed disks in* $\mathbf{S}^n$ *whose interiors cover* $\mathbf{S}^n$*. Then any* $h \in \mathrm{Homeo}^+(\mathbf{S}^n)$ *can be factorized as a product of at most 6 homeomorphisms, each of which has support contained in either* E_1 *or* E_2*.*

Proof Without loss of generality, we can assume that E_1 and E_2 contain collar neighborhoods of the northern and southern hemisphere respectively.

Given $h \in \mathrm{Homeo}^+(\mathbf{S}^n)$, we factorize h as $h_1 h_2$ as in Lemma 6.9. Let e_2 be a radial expansion centered at the south pole, with support contained in E_2, which takes $\mathrm{supp}(h_1) \cap E_2$ into $E_2 \cap E_1$. Then $e_2 h_1 e_2^{-1}$ has support contained in E_1. Similarly, we can find e_1 with support contained in E_1 such that $e_1 h_2 e_1^{-1}$ has support contained in E_2. Then

$$h = e_2^{-1}(e_2 h_1 e_2^{-1}) e_2 e_1^{-1}(e_1 h_2 e_1^{-1}) e_1$$

expresses h as the product of 6 homeomorphisms, each with support in either E_1 or E_2. □

Remark 6.11 Notice in the factorization in Lemma 6.10 that the homeomorphisms e_1, e_2 definitely depend on h.

Theorem C *Fix* $n \geq 1$*. Let* $h_1, h_2, \ldots$ *be any countable subset of* $\mathrm{Homeo}(\mathbf{S}^n)$*, and* $g_1, g_2, \cdots : \mathbb{N} \to \mathbb{N}$ *any countable collection of growth functions. Then there is a finitely generated subgroup* H *of* $\mathrm{Homeo}(\mathbf{S}^n)$ *(depending on* $\{h_i\}$ *and* $\{g_i\}$*) such that* every h_i *is simultaneously distorted in* H*. Moreover, the distortion function of* h_i *grows faster than* g_i*.*

Proof The subgroup $\text{Homeo}^+(\mathbf{S}^n)$ of $\text{Homeo}(\mathbf{S}^n)$ has index 2, so after replacing each h_i by h_i^2 if necessary, we can assume each $h_i \in \text{Homeo}^+(\mathbf{S}^n)$.

Fix a cover of $\mathbf{S}^n$ by disks E_1, E_2 as in Lemma 6.10. Let $n_i \to \infty$ grow sufficiently quickly, and relabel the sequence

$$h_1^{n_1}, h_1^{n_2}, h_2^{n_2}, h_1^{n_3}, h_2^{n_3}, h_3^{n_3}, h_1^{n_4}, \ldots, h_1^{n_i}, \ldots, h_i^{n_i}, h_1^{n_{i+1}} \ldots$$

as $H_0, H_1, H_2, \ldots$.

Applying Lemma 6.10, we write each H_i as a product

$$H_i = H_{i,1} H_{i,2} \ldots H_{i,6}$$

where each $H_{i,j}$ has support contained in either E_1 or E_2.

From now on, the construction proceeds as in §4 and §5, with the added simplification that we do not need to worry about the analytic quality of the construction.

We let $D_{i,j}$ be a family of disjoint balls in $\mathbf{S}^n$ for $i, j \in \mathbb{Z}$ such that there are homeomorphisms T, F for which T takes $D_{i,j}$ to $D_{i+1,j}$ for all i, j, and F takes $D_{0,j}$ to $D_{0,j+1}$, and is the identity on $D_{i,j}$ when $i \neq 0$.

Let Z_1, Z_2 be homeomorphisms taking $D_{0,0}$ to E_1 and E_2 respectively.

For each $\ell \in \{1, \ldots, 6\}$ we define f_ℓ with support contained in the closure of the union of the $D_{i,j}$ by the formula

$$f_\ell = \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} H_{i,\ell}^{Z_k F^{-j} T^{-i}}$$

where $k = 1$ if $H_{i,\ell}$ has support in E_1, and $k = 2$ if $H_{i,\ell}$ has support in E_2.

Then as before, we can write $H_{i,\ell}$ as a word of length $\sim 4i$ in $f_1, \ldots, f_6, T, F, Z_1, Z_2$ and their inverses. In detail:

$$H_{i,\ell} = Z_k y_{i,\ell} Z_k^{-1}$$

where

$$y_{i,\ell} = w_{i,\ell} F w_{i,\ell}^{-1} F^{-1}$$

and

$$w_{i,\ell} = T^{-i} f_\ell T^i.$$

Since we can do this for each i, ℓ, we can exhibit each h_i as a distortion element, whose distortion function grows as fast as desired. Note that by choosing the n_i to all be mutually coprime, we can ensure that the h_i are all actually contained in the group in question. □

Appendix: Strong boundedness of $\mathrm{Homeo}(S^n)$

by Yves de Cornulier

Definition A.1 A group G is strongly bounded[1] if is satisfies one of the following equivalent conditions:

(i) Every length function on G, i.e. function $L : G \to \mathbb{R}_+$ satisfying $L(1) = 0$, $L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$, is bounded.

(ii) Every action of G by isometries on a metric space has bounded orbits.

(iii)
- G is Cayley bounded: for every symmetric generating subset S of G, there exists n such that $G \subset S^n = \{s_1 \dots s_n \mid s_1, \dots, s_n \in S\}$, and
- G has uncountable cofinality, i.e. G cannot be expressed as the union of an increasing sequence of proper subgroups.

The definition of groups with uncountable cofinality appeared in the characterization by Serre [24, §6.1] of groups with Property (FA), meaning that every isometric action on a simplicial tree has a fixed point. For instance, a countable group has uncountable cofinality if and only if it is finitely generated.

Much later, the concept of strong boundedness was introduced by Bergman [1], where it is proved that the permutation group of any set is strongly bounded. Subsequently, intensive research on the subject has been carried on (see, among others [2, 3, 9, 10], and the references in [1]). It is worth noting that a countable group is strongly bounded if and only if it is finite, so that the definition is of interest only for uncountable groups. In Definition A.1, the equivalence between (i) and (ii) is easy and standard; the equivalence between Conditions (i) and (iii) is established in [2] but already apparent in [1].

Fix an integer $n \geq 1$. The purpose of this appendix is to point out the following consequence of the proof of Theorem C in the paper above.

Theorem A.2 *The group* $\mathrm{Homeo}(\mathbf{S}^n)$ *is strongly bounded.*

A weaker version of Theorem A.2 was recently proved in [23, Theorem 1.7]; namely, $\mathrm{Homeo}(\mathbf{S}^n)$ is strongly bounded as a *topological* group (for the uniform convergence); this means that every *continuous* length function is

[1] The following terminologies for the same concept also exist in the literature: G has the Bergman Property; G has the strong Bergman Property; G has uncountable strong cofinality.

bounded. In [23, Theorem 5.4], it was also proved that Homeo($\mathbf{S}^1$) is strongly bounded. In contrast, when $r > 3/2$, the group $\mathrm{Diff}^r(\mathbf{S}^1)$ is not strongly bounded, as it has an unbounded isometric action on a Hilbert space [16].

Proof Clearly, it suffices to show that the subgroup of index two $\mathrm{Homeo}^+(\mathbf{S}^n)$ is strongly bounded. By contradiction, we suppose the existence of an unbounded length function L on G. Let us pick a sequence (h_i) in G satisfying $L(h_i) \geq i^2$ for all i.

Using the notation in the proof of Theorem C, Set $S = \{f_1, \ldots, f_6, T, F, Z_1, Z_2\}$. Then each h_i can be expressed by a word of length $\sim 24i$ in $S^{\pm}$. But, on the subgroup of $\mathrm{Homeo}^+(\mathbf{S}^n)$ generated by the finite set S, the length function L must be dominated by the word length with respect to S. This contradicts the assumption $L(h_i) \geq i^2$ for all i. □

Remark A.3 A similar argument to that of the proof of Theorem A.2 was used in [2] to prove that ω_1-existentially closed groups are strongly bounded. This reasoning is made systematic by Khelif [10]. Let us say that a group is *strongly distorted* (introduced as "Property P" in [10]) if there exists an integer m and an integer-valued sequence (w_n) with the following property: for every sequence (h_n) in G, there exist $g_1, \ldots, g_m \in G$ such that, for every n, one can express h_n as an element of length w_n in the g_i's. Following the proof of Theorem A.2, we get that a strongly distorted group is strongly bounded, and that Homeo($\mathbf{S}^n$) is strongly distorted.

The symmetric group on any set is strongly distorted: a proof can be found in [5], although a weaker result is stated there. In [10], it is claimed that the automorphism group of any 2-transitive chain is strongly distorted; strong boundedness was previously proved in [3]. On the other hand, if F is a non-trivial finite perfect group, then the infinite (unrestricted) direct product $F^{\mathbb{N}}$ is strongly bounded [2]; however it is clearly not strongly distorted since it is infinite and locally finite.

Remark A.4 It follows from Theorem A.2 that Homeo($\mathbf{S}^n$) is Cayley bounded (see Definition A.1); that is, the Cayley graph with respect to any generating subset has bounded diameter. It is natural to ask whether there is a uniform bound on those diameters: the answer is negative. Indeed, endow $\mathbf{S}^n$ with its Euclidean metric, and, for $r > 0$, set $W_r = \{g \in \mathrm{Homeo}(\mathbf{S}^n) \mid \forall x \in \mathbf{S}^n, d(x, g(x)) < r\}$ and $W_r^+ = W_r \cap \mathrm{Homeo}^+(\mathbf{S}^n)$. Then W_r^+ is open in $\mathrm{Homeo}^+(\mathbf{S}^n)$. The group $\mathrm{Homeo}^+(\mathbf{S}^n)$, endowed with the topology of uniform convergence, is connected: this is well-known, and can be deduced, for instance, from Lemma 6.9 above. It follows that W_r^+ generates $\mathrm{Homeo}^+(\mathbf{S}^n)$. Clearly, for every $k \geq 1$, we have $(W_r^+)^k \subset W_{kr}^+$. It follows that if we have chosen $r \leq 2/k$, then $(W_r^+)^k \neq \mathrm{Homeo}^+(\mathbf{S}^n)$. Thus $\mathrm{Homeo}^+(\mathbf{S}^n)$ has Cayley graphs of arbitrary large diameter. A similar argument works for $\mathrm{Homeo}(\mathbf{S}^n)$ as follows: fix a reflection $T \in \mathrm{O}(n+1)$ of $\mathbf{S}^n$, and take $W_r' = W_r \cup TW_r$. Then it is easy to check that $(W_r')^k \subset W_{kr}'$ for all $k \geq 1$, so that $(W_r')^k \neq \mathrm{Homeo}(\mathbf{S}^n)$ if we have chosen $r \leq 2/k$.

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