# Introduction to singularity formation for the Burgers, Prandtl and Keller-Segel equations

July 28, 2023

**Abstract**

These lecture notes are for graduate students, and introduce to the topic of singularity formation for some model fluid and reaction-diffusion equations. We describe with details the simplest example of shocks for a compressible fluid: the Burgers equation, and then the simplest example of a singularity for an incompressible fluid: the homogeneous and inviscid Prandtl system. For these two models singularity formation can be understood via representation formulae. Then, we turn to the parabolic-elliptic Keller-Segel system describing bacteria motion. We present an explicit singular solution that is backward self-similar in the mass supercritical case, as well as another singular solution whose self-similarity is of the second kind in the critical case. We develop some tools that are commonly used to address the stability of singular solutions in the absence of explicit representation formulae: renormalisation techniques and spectral analysis.

These are the lecture notes for a mini-course that C. Collot gave at the Chinese Academy of Sciences in 2023, at the Academy of Mathematical and System Sciences, Beijing. He thanks warmly the institution and in particular H. Nguyen for the invitation. The lecture notes were taken by Ruilin Hu and Shumao Wang, whom C. Collot would like to thank warmly for their work. They are based on the joint works [3–6] of C. Collot with Ghoul, Masmoudi and Nguyen, and on the work [9] of Glogić and Schorkhüber.

## Contents

# 1 Singularity formation for the Burgers equation

This section is taken from [3] where the reader may find additional information. We describe completely how singularities are formed for the simplest compressible fluid which is Burgers' equation

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

## 1.1 The Cauchy problem for classical solutions

In this subsection we explain how to find solutions by the methods of characteristics. This allows us to show the existence of $C^1$ solutions locally in time, and to find a sharp formula for their maximal time of existence.

**Definition 1.1** *Given $X \in \mathbb{R}$ and $u_0(X)$ the characteristics are defined as the solutions to the ODE*

$$\begin{cases} \frac{d}{dt}x = u, \\ \frac{d}{dt}u = 0, \\ (x(t=0), u(t=0)) = (X, u_0(X)). \end{cases} \tag{1.2}$$

We remark that the solution to (1.2) is

$$\begin{cases} x(t,X) = X + tu_0(X), \\ u(t,X) = u_0(X) \end{cases}$$

Knowing the characteristics is equivalent to knowing the solution. From the above explitic formula one can obtain an explicit formula for the solution to the Burgers equation.

**Theorem 1.2** *Let $u_0 \in C^1(\mathbb{R})$ with $\partial_x u_0 \in L^\infty$, and define*

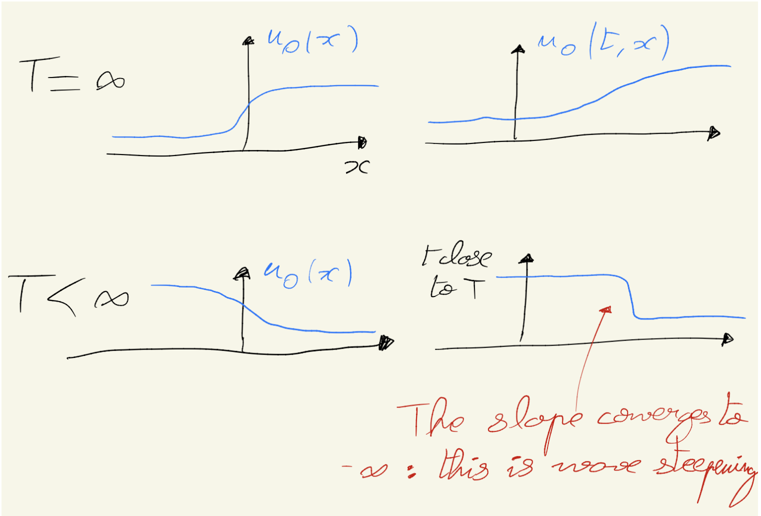
$$T = \begin{cases} \infty, & \text{if} \quad u_0 \geq 0 \\ \frac{1}{-\inf_{\mathbb{R}} \partial_x u_0}, & \text{else.} \end{cases}$$

*Then,*

*(i) There exists a unique solution $u \in C^1([0,T) \times \mathbb{R})$ to Burgers' equation* (1.1).

*(ii) If $T < \infty$, then the solution will be singular at $T$, in the sense that* $\lim_{t\to T} \|\partial_x u(t)\|_{L^\infty} = \infty$.

**Example 1.3** *Below are two examples, one of a global solution and the other of a solution that becomes singular in finite time.*




**Proof. Step 1**. *Existence.* The map $x(t,\cdot):\mathbb{R}\to\mathbb{R}$, $X\to x(t,X)$ given by characteristics is bijective, for $0\le t<T$. Define $X(t,\cdot)=x(t,\cdot)^{-1}$, its inverse map. Define $u$ by

$$u(t,x)=u_0(t,u_0(t,X)).$$

Then it is a solution to (1.1), because

$$u(t,x(t,X))=u_0(X)$$

which implies

$$\frac{d}{dt}u(t,x(t,X))+\frac{\partial}{\partial t}x\frac{\partial}{\partial x}u(t,x(t,X))=0$$

which using $\frac{\partial}{\partial t}x=u(t,x(t,X))$ gives

$$\partial_t u+u\partial_x u=0.$$

**Step 2**. *Proofs of uniqueness and ii)*. They are left as exercise. ■

## 1.2 Symmetries and backward self-similar solutions

In this subsection we present certain transformations of the set of functions that leave the set of solutions to the Burgers equation invariant. We then present backward self-similar solutions. These are solutions that are invariant by scaling transformations. They become singular in finite time by concentrating to smaller and smaller scales while keeping the same shape. We classify these solutions for the Burgers equation, and give an explicit formula for them.

**Lemma 1.4** *If $u$ is solution of the Burgers equation* (1.1)*, then* $\forall(\lambda,\mu,C,x_0,t_0)\in(0,\infty)^2\times\mathbb{R}^3$,

$$\frac{\mu}{\lambda}u(\frac{t}{\lambda},\frac{x-x_0-ct}{\mu})+c$$

*is also a solution.*

**Proof.** Exercise. ■

**Definition 1.5** *A backward self-similar solution of the Burgers equation* (1.1) *is a solution of the form*

$$u(t,x) = (-t)^{\alpha-1}\Psi(\frac{x}{(-t)^{\alpha}})$$

*defined for* $t \in (-\infty, 0)$, $\psi$ *is called the profile, and* $\alpha \in \mathbb{R}$ *is called the scaling exponent.*

**Remark 1.6** *i) For any* $\mu \in (0,\infty)$ *we have* $u(t,x) = \mu^{1-\frac{1}{\alpha}}u(\frac{t}{\mu^{\frac{1}{\alpha}}}, \frac{x}{\mu})$ *for all* $(t,x) \in (-\infty,0)\times \mathbb{R}$, *so that* $u$ *is invariant under the action of 1-D group of symmetries* ($(0,\infty)$ *with multiplication law). This is the general definition of self-similarity from physicists' point of view.*
*ii) For any* $\mu, t, x^*, c$,

$$u(t,x) = (T-t)^{\alpha-1}\mu\Psi(\frac{x - x^* + c(T-t)}{\mu(T-t)^{\alpha}})$$

*is also a solution.*

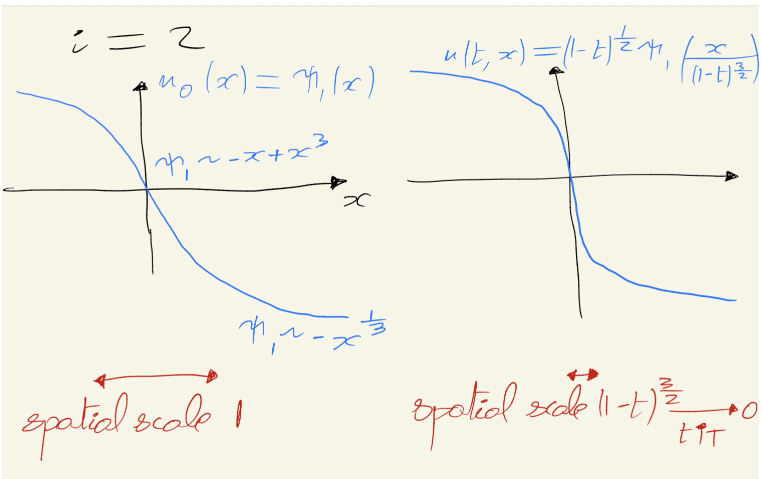
The following theorem gives explicit backward self-similar solutions, and classifies them. To our knowledge, these self-similar solutions were only found in [8], and then studied in depth in [3].

**Theorem 1.7** *i) For* $i \in \mathbb{N}$, *the mapping* $\Phi_i : \mathcal{Y} \to -\mathcal{Y} - \mathcal{Y}^{2i+1}$ *is bijective on* $\mathbb{R}$, *let* $\psi$ *be its inverse, then*

$$u(t,x) = (-t)^{\frac{1}{2i}}\Psi_i(\frac{x}{(-t)^{1+\frac{1}{2i}}}) \tag{1.3}$$

*is a backward self-similar solution of the Burgers equation* (1.1).
*ii) Conversely, assume* $u$ *is a backward self-similar solutions, with a profile that satisfies* $\lim_{x\to\infty}\Psi(x) = 0$, *then* $\exists i \in \mathbb{N}$, $\mu > 0$, *such that* $u$ *is given by* (1.3).

**Example 1.8** *Below is the example of the first profile* $\Psi_1$ *and of the corresponding backward self-similar solution.*




**Proof.** Substitute (1.3) in Burger equation, $\Psi$ must solve

$$(1-\alpha)\Psi + (\alpha\mathcal{X} + \Psi)\Psi' = 0. \tag{1.4}$$

This equation is called the stationary equation in self-similar variables.

**Step 1**. *A change of variables.* Let $I$ be a maximal interval on which $\Psi' \neq 0$, $J = \Psi(I) = (a,b)$, if $a \neq -\infty$. Let $\Phi : (a,b) \to I$ denote the inverse of $\Psi$. Then note that $a \neq -\infty$ if and only if

$$\lim_{Y\to a}(|\Phi| + |\Phi'|)(Y) = \infty. \tag{1.5}$$

A similar result holds at b.

**Step 2**. *A formula in the new variable.* We claim that if $J$ has a negative element, then $(-\infty, 0) \subset J$, and there

$$\Phi(y) = K_-|y|^{\frac{\alpha}{\alpha-1}} - y, \quad K_- \in \mathbb{R}. \tag{1.6}$$

Indeed, injecting $\Psi'(x) = \frac{1}{\Phi'(\psi(x))}$ in (1.4) gives

$$(1-\alpha)\Psi\Phi'(\Psi(x)) + \alpha x + \Psi(x) = 0.$$

Define $y = \Psi(x)$, then

$$(1-\alpha)y\Phi' + \alpha\Phi = -y.$$

Solving this linear equation shows (1.6).

**Step 3**: End of the proof (left as exercise). ■

### 1.3 Resolution into self-similar solutions for singular solutions

In this subsection we show that if a solution becomes singular in finite time, it concentrates near the singularities a backward self-similar solution. Thus, backward solutions describe to leading order any singularity. They are the attractors of singular solutions locally near the singularities.

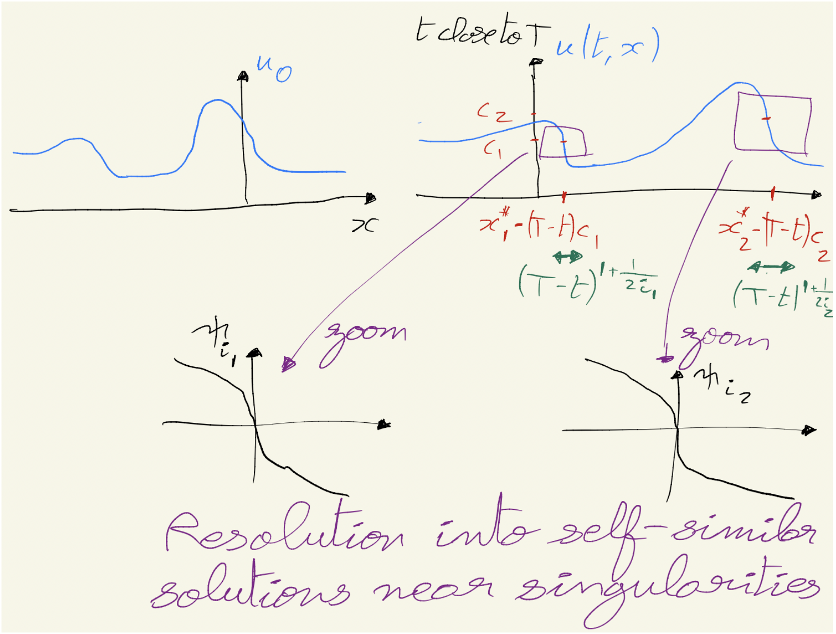
**Theorem 1.9** *Let $u$ be a singular solution of Burgers' equation* (1.1)*, assume $u_0$ is analytic, and $\lim_{|x|\to\infty} \partial_x u_0 = 0$. Then there exists $J \in \mathbb{N}$, $(x_j, c_j, \mu_j, i_j)|_{1\leq j\leq J}$, such that*
*i) Outside any neighborhood of $\{(T, x_1), ..., (T, x_J)\}$ in $[0, T] \times \mathbb{R}$, $u$ and its derivatives are locally uniformly bounded.*
*ii) For each $1 \leq j \leq J$ near $(T, x_j)$, one has*

$$u(t,x) = c_j + \mu_j(T-t)^{\frac{1}{2i_j}}\Psi_{i_j}\left(\frac{x - x_j + c_j(T-t)}{\mu_j(T-t)^{1+\frac{1}{2i_j}}}\right) + v_j(t,x)$$

*with $v_j$ lower order term, in the sense that*

$$\lim_{(t,x)\to(T,x_j)} \frac{v_j(t,x)}{(T-t)^{\frac{1}{2i_j}}\psi_{i_j}\left(\frac{x-x_j+c_j(T-t)}{\mu_j(T-t)^{1+\frac{1}{2i_j}}}\right)} = 0$$

**Example 1.10** *Below is the example of a solution that becomes singular at two points, and of the corresponding resolution into self-similar solutions.*




To prove Theorem 1.9 we will use the following Lemma.

**Lemma 1.11** *Assume* $\frac{d}{dx}u_0$ *has a negative minimum at* $X_0$ *at which*

$$\frac{d^j}{dx^j}u_0(X_0) = 0 \quad \textit{for} \quad j = 2, 3, ..., 2i \quad \textit{and} \quad \frac{d^{2i+1}}{dx^{2i+1}}u_0(X_0) > 0.$$

*Then* $u$ *becomes singular at time* $T = -\frac{1}{\partial_x u_0(X_0)}$ *and there exists* $(\mu, c, x^*) \in (0, \infty) \times \mathbb{R}^2$ *such that*

$$u = c + \mu(T - t)\Psi_i\left(\frac{x - x^* + c(T - t)}{\mu(T - t)^{1+\frac{1}{2i}}}\right)(1 + \varepsilon(t, x)), \quad \lim_{(t,x)\to(T,x^*)} \varepsilon(t, x) = 0 \tag{1.7}$$

**Proof for Theorem 1.9..** As $u_0$ is analytic, and $\lim_{x\to\infty} \partial_x u_0 = 0$, the set E where $\partial_x u_0$ attains its minimum is made of finitely many points $(X_1, ..., X_J)$. Moreover, at each point $X_j$ of $E$, (1.7) is always satisfied for some $i$ by analyticity. Then the theorem is a result of Lemma 1.11.

■

**Proof for Lemma 1.11.** **Step 1**. *Simplification using symmetries.* We first remark that it suffices first to prove the result assuming $x_0 = 0$, $u_0(x_0) = 0$, $\frac{d}{dx}u_0(x_0) = -1$ and $\frac{d^{2i+1}}{dx^{2i+1}}u_0(x_0) = (2i + 1)!$. The general case follows from applying Lemma 1.4.

**Step 2**. *Proof assuming the hypotheses of Step 1.*. By (1.7) and Taylor expansion, we get

$$u_0(X) = -X + X^{2i+1} + ... \tag{1.8}$$

where ... denotes higher order terms. Then characteristics are

$$\begin{aligned} x(t, X) &= X + tu_0(X) \\ &= (1 - t)X + X^{2i+1} + ... \end{aligned}$$

We remark that

$$(1 - t)X + X^{2i+1} = -\phi_{(1-t)^{1+\frac{1}{2i}}} \circ \Phi_i \circ \phi_{\frac{1}{(1-t)^{\frac{1}{2i}}}}$$

where $\Phi_i$ is given by Theorem 1.9, and $\phi_\nu(g) = g\nu$. Hence the inverse of the mapping $X \mapsto (1-t)X + X^{2i+1}$ is

$$
\begin{aligned}
-\phi^{-1}_{\frac{1}{(1-t)^{\frac{1}{2i}}}} \circ \Phi_i^{-1} \circ \phi^{-1}_{(1-t)^{1+\frac{1}{2i}}} &= -\phi^{-1}_{\frac{1}{(1-t)^{\frac{1}{2i}}}} \circ \Psi_i \circ \phi^{-1}_{(1-t)^{1+\frac{1}{2i}}} \\
&= -(1-t)^{\frac{1}{2i}} \Psi_i\left(\frac{x}{(1-t)^{1+\frac{1}{2i}}}\right)
\end{aligned}
$$

The inverse of the full characteristics of the solutions are then close to

$$
X(t,x) = -(1-t)^{\frac{1}{2i}} \Psi_i\left(\frac{x}{(1-t)^{1+\frac{1}{2i}}}\right) + \dots \tag{1.9}
$$

The solution found by characteristics is

$$
\begin{aligned}
u(t,x) &\overset{(1.8)}{=} u_0(X(t,x)) = -X(t,x) + X^{2i+1} + \dots \\
&\overset{(1.9)}{=} (1-t)^{\frac{1}{2i}} \Psi_i\left(\frac{x}{(1-t)^{1+\frac{1}{2i}}}\right) + \dots
\end{aligned}
$$

This proves the Theorem, upon proving suitable estimates for the ... remainders above, which is left as exercise.

∎

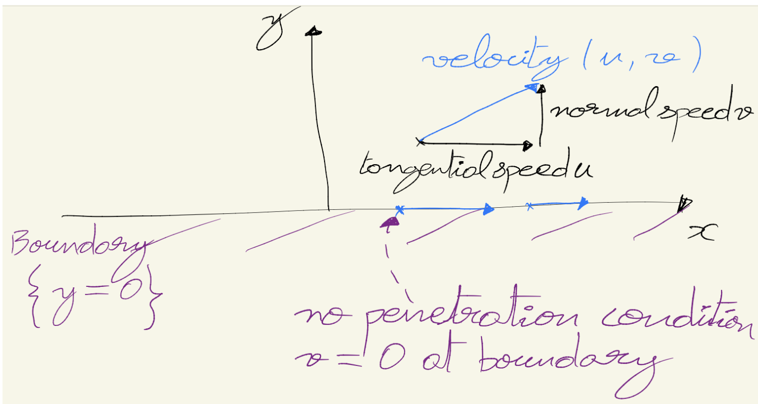
# 2 Singularity formation for the homogeneous inviscid Prandtl system

This section is based on [4] and references therein. Based on our understanding of singularity formation for the simplest compressible fluid in the previous section, we now turn to the simplest incompressible fluid that admits singularity formation. It is the following homogeneous and inviscid Prandtl system on the upper half-plane

$$
(HI-Prandtl) \qquad \begin{cases} u_t + uu_x + vu_y = 0 \\ u_x + v_y = 0 \\ v|_{y=0} = 0, \ \lim_{y\to\infty} u = 0 \end{cases}
$$

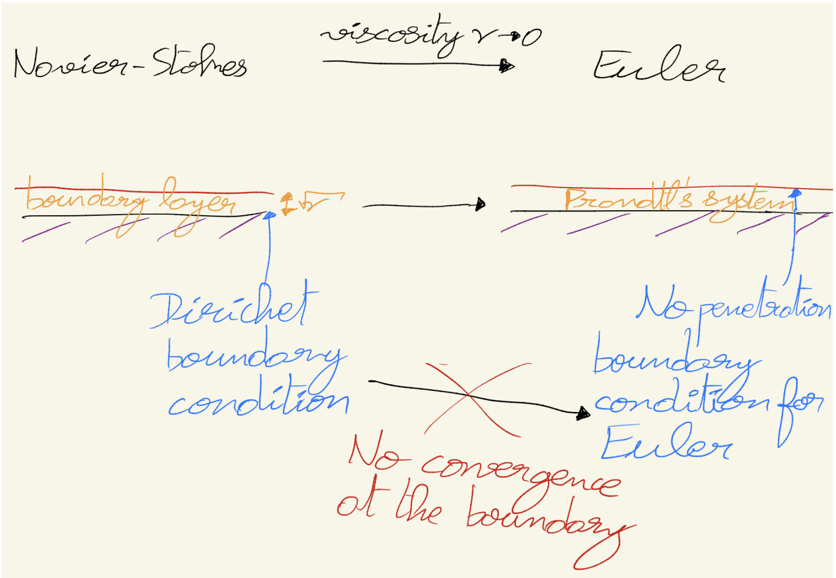
The first equation is the convection equation for the horizontal speed $u$ of the fluid. The second equation is the incompressibility equation for the fluid velocity $(u, v)$. The third equation is the no-penetration condition at the boundary $\{y = 0\}$ and the zero-flow matching condition as $y \to \infty$.

Below is the physical representation of the equation:

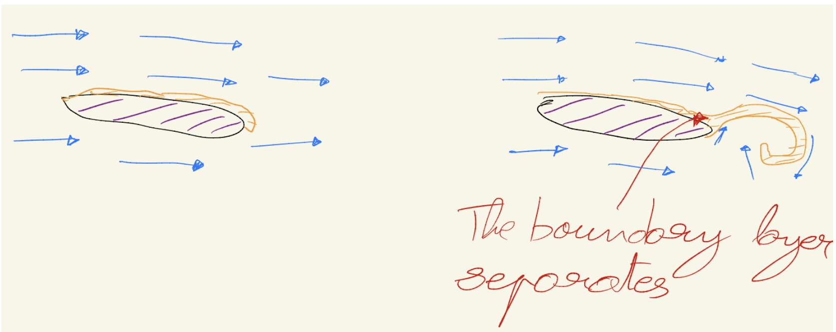


## 2.1 Origin of the equation

(HI-Prandtl) equation is a simplication of the usual Prandtl equation. The usual Prandtl system comes from the approximation of Navier-Stokes system by the Euler system in the vanishing viscosity $\nu \to 0$ limit. The boundary conditions cannot converge, and hence this approximation is not valid near the boundary. Hence the introduction of a boundary layer where the flow transitions from the Dirichlet boundary condition to the outer Euler flow.




An interesting phenomenon to describe for this boundary layer is the boundary layer separation. It occurs when a layer of fluid detaches off the boundary and is ejecting away. The following drawing represents the occurence of boundary layer separation for the flow past an obstacle:




From the works of Goldstein in the steady case and of Sears and Telionis in the unsteady case (among others), the following observation was made by physicists: boundary layer separation should occur if and only if the solution of the the boundary layer becomes singular.

## 2.2 Cauchy problem for classical solutions

In this subsection we obtain the existence of classical solutions by the methods of characteristics again. However, the characteristics are more subtle for this fluid and we have to rely on its incompressibility.

The characteristics are for an initial position $(X, Y)$ of the particles and an initial horizontal speed $u_0(X, Y)$ the solutions to

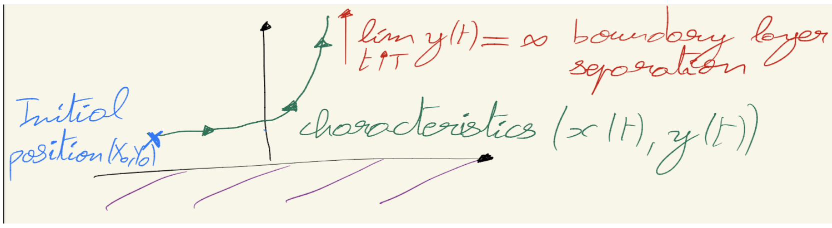
$$\begin{cases} \frac{d}{dt}x = u(t, x(t), y(t)) \\ \frac{d}{dt}y = v(t, x(t), y(t)) \\ \frac{d}{dt}u = 0 \\ (x, y, u)|_{t=0} = (X, Y, x_0(X, Y)) \end{cases} \tag{2.1}$$

It has a solution as

$$\begin{cases} u(t, x(t), y(t)) = u_0(X, Y) \\ x = X + tu_0(X, Y) \\ \frac{d}{dt}y = v(t, x(t), y(t)) \end{cases} \tag{2.2}$$

Using the characteristics, the simplest definition of boundary layer separation is the following one.

**Definition 2.1** *Boundary layer separation occurs at time $T$ if there exists $(X, Y) \in [0, \infty)$ such that the associated characteristics $(x(t), y(t))$, such that $\lim_{t\to T} y(t) = \infty$.*




One notices that in (2.2) the horizontal position of the particles $x$ can be easily determined. The next lemma shows that the vertical position $y$ can be retrieved from $x$ using incompressibility.

**Lemma 2.2** *Let $x \in C^2(\mathbb{R} \times (0, \infty))$ and $y \in C^1$ be two functions, then the following are equivlent,*
*i) The map $\mathbb{R} \times (0, \infty) \to \mathbb{R} \times (0, \infty)$, $(X, Y) \to (x(X, Y), y(X, Y))$ is a volume presevering diffeomorphism.*
*ii) $x$ satisfies $\nabla_{X,Y} x \neq 0$ where $\nabla_{X,Y} x = (\partial_X x, \partial_Y x)$ denotes the gradient in $(X, Y)$ variables, its level sets are diffeomorphic to $[0, \infty)$, and $y$ is given by*

$$y(X, Y) = \int_{\Gamma_{X,Y}} \frac{1}{|\nabla_{X,Y} x(\gamma(s))|} ds \tag{2.3}$$

*where $\Gamma_{X,Y}$ is the curve that connects $(X, Y)$ to the boundary $\mathbb{R} \times \{0\}$ on the level set $\{x(\tilde{X}, \tilde{Y}) = x(X, Y)\}$ with parametrisation $\gamma$ by arc length parametrisation $s$.*

Lemma 2.2 tells us that in order to find the characteristics, it is sufficient to know the horizontal displacement $x$, and then the nornmal displacement $y$ can be recovered from $x$. It is illustrated below:

**Proof.**
The proof can be done via standard Calculus techniques, and left to the reader.

■

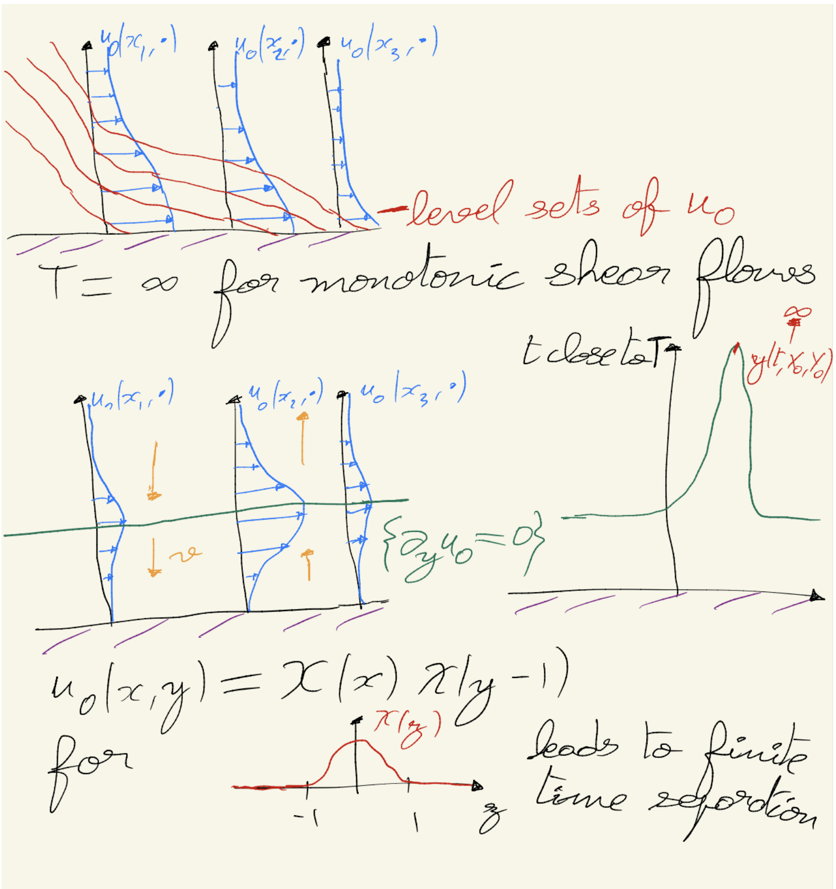
**Theorem 2.3** *Assume $u_0 \in C^2(\mathbb{R} \times [0, \infty)$ satisfies $\lim_{|x|,|y|\to\infty} \partial_x u_0(x, y) = 0$, and $\partial_y u_0 \in L^\infty$, and let*

$$T = \begin{cases} \infty, & \text{if } \partial_x u_0 \geq 0 \text{ on the set } \{\partial_y u = 0\} \cup \{y = 0\} \\ -\frac{1}{\inf_{\{\partial_y u=0\}\cup\{y=0\}} \partial_x u_0}, & \text{else} \end{cases}$$

*Then*
*i) $\exists!$ $C^1$ solution $u$ to (HI-Prandtl) equation.*
*ii) The following are equivalent.*

- $T < \infty$
- $\Leftrightarrow \lim_{t\to T} \|\partial_x u(t)\|_{L^\infty} = \infty$
- $\Leftrightarrow$ *Either a shock happen at boundary $\{y = 0\}$, or there is boundary layer separation at time $T$.*

**Example 2.4** *The drawing below gives two examples. One of a global solution, happening for a monotonic shear flow $u_0(x, y) = \phi(y - c(x))$ where $\phi' \neq 0$ and $c' \neq 0$, and the other one of a solution that becomes singular in finite time triggering boundary layer separation.*




**Proof. Step 1** *Existence and uniqueness* Exercise, use characteristics like Burgers equation and Lemma 1.4.

**Step 2**. *Characterisation of singularity formation.* We prove that $T < \infty \Rightarrow$ separation or shock at $\{y = 0\}$

Case 1: Assume that on the set $\{\partial_y u_0 = 0\} \cup \{y = 0\}$, the function $\partial_x u_0$ attains its minimum $-\frac{1}{T}$ at a point $(X_0, Y_0)$ such that $Y_0 > 0$. Then we have $\partial_y u_0(X_0, Y_0) = 0$. Then the characteristics give

$$x(t, X, Y) = X + tu_0(X, Y),$$

so that

$$\partial_X x(t, X_0, Y_0) = 1 - \frac{t}{T} \quad \text{and} \quad \partial_Y x(t, X_0, Y_0) = 0.$$

Then $\lim_{t\to T} \nabla_{X,Y} x(t) = 0$. Using Lemma 2.2 we obtain that

$$\begin{aligned} y(t, X_0, Y_0) &= \int_{\Gamma_{X_0,Y_0}} \frac{ds}{|\nabla_{X,Y} x(\gamma(s))|} \\ &\to \infty \end{aligned}$$

where the second line is because the integrand converges to $\infty$ (a rigorous proof is left as exercise).

Case 2: Assume that on the set $\{\partial_y u_0 = 0\} \cup \{y = 0\}$, the function $\partial_x u_0$ attains its minimum $-\frac{1}{T}$ at a point $(X_0, Y_0)$ such that $Y_0 = 0$. Then at the boundary $\{y = 0\}$ the (HI-Prandtl) system gives

$$\partial_t u|_{y=0} + u|_{y=0} \partial_x u|_{y=0} = 0$$

which is the Burgers equation (1.1)! So by Theorem 1.2, $u|_{y=0}$ becomes singular at time $T$. Other proofs of implications are left as exercise.

■

## 2.3 Backward self-similar solutions

In this subsection we study the symmetries of the (HI-Prandtl) system and address the existence and description of backward self-similar solutions.

If u is a solution, then so is

$$c + \frac{\mu}{\lambda} u(\frac{t - t_0}{\lambda}, \frac{x - x_0 - ct}{\mu}, \frac{y}{\nu}) \tag{2.4}$$

for any $(\lambda, \mu, \nu, c, x_0, t_0) \in (0, \infty)^3 \times \mathbb{R}^3$. A backward self-similar solution is then a solution of the form

$$u(t, x, y) = (-t)^{\alpha-1} \Theta(\frac{x}{(-t)^\alpha}, \frac{y}{(-t)^\beta})$$

defined for $t < 0$, for two exponents $\alpha, \beta \in \mathbb{R}^2$. The profile $\Theta$ solves the stationary equation in self-similar variables

$$\begin{cases} (1 - \alpha)\Theta + (\alpha\mathcal{X} + \Theta)\partial_{\mathcal{X}}\Theta + (\beta\mathcal{Y} + \Upsilon)\partial_{\mathcal{Y}}\Theta = 0 \\ \partial_{\mathcal{X}}\Theta + \partial_{\mathcal{Y}}\Upsilon = 0 \\ \Upsilon|_{\mathcal{Y}=0} = 0 \end{cases} \tag{2.5}$$

**Theorem 2.5** *Consider the mapping* $\Phi(a, b) = (a + b^2 + a^3, \int_{-\infty}^{b} \frac{1}{1+3\Psi_1^2(a+a^3+b^2-\tilde{b}^2)} d\tilde{b})$*, where* $\Psi_1$ *is defined in Theorem 1.7, then*

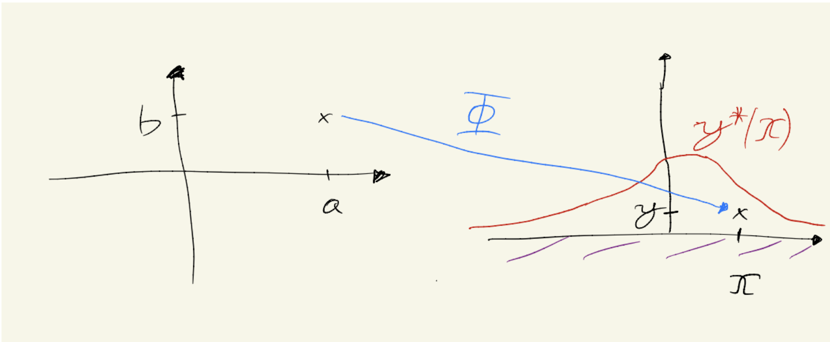
*i)* $\Phi$ *is a volume presevering diffeomorphism between* $\mathbb{R}^2$ *and the following set:* $\{(\mathcal{X}, \mathcal{Y}), 0 < \mathcal{Y} < \mathcal{Y}^*(\mathcal{X})\}$*, where* $\mathcal{Y}^*(\mathcal{X}) = \int_{-\infty}^{\infty} \frac{1}{1+3\Psi_1^2(\mathcal{X}-\tilde{b}^2)} d\tilde{b}$.

*ii) Define its inverse* $\Phi^{-1} = (\Phi_1^{-1}, \Phi_2^{-1})$ *and* $\Theta(\S, \dagger) = -\Phi_1^{-1}(\S, \dagger)$. *Then* $\Theta$ *is a self-similar profile in the sense that*

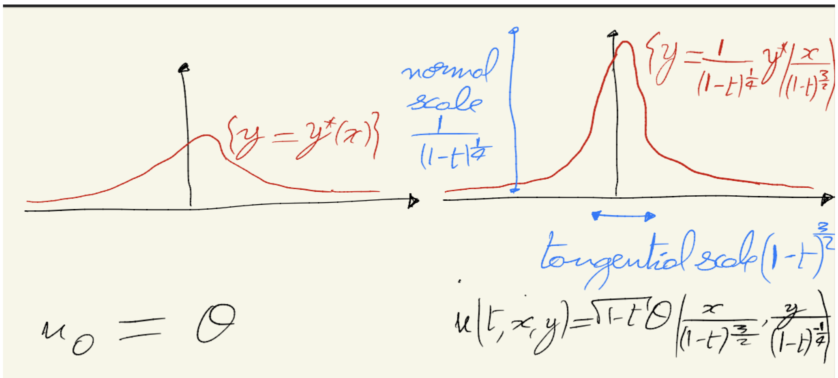
$$u(t,x,y) = (-t)^{\frac{1}{2}}\Theta(\frac{x}{(-t)^{\frac{3}{2}}}, \frac{y}{(-t)^{-\frac{1}{4}}})$$

*solves (HI) Prandtl on the set* $\{0 < t, 0 < y < (T-t)^{-\frac{1}{4}}\mathcal{Y}^*(\frac{x}{(T-t)^{\frac{3}{2}}})\}$

The first illustration below represents the mapping $\Phi$ and the upper boundary $\{\mathcal{Y} = \mathcal{Y}^*(\mathcal{X})\}$ of the support of the profile $\Theta$




The second represents the evolution of the support of the associated backward self-similar solution




**Proof.** $\Theta$ solves (2.5), which is

$$-\frac{1}{2}\Theta + (\frac{3}{2}\mathcal{X} + \Theta)\partial_{\mathcal{X}}\Theta + (-\frac{1}{4}\mathcal{Y} + \Upsilon)\partial_{\mathcal{Y}}\Theta = 0. \tag{2.6}$$

We write this equation in the form

$$v(\Theta) = \frac{1}{2}\Theta$$

where $v$ is the vector field defined by its action on functions:

$$v(f) = (\frac{3}{2}\mathcal{X} + \Theta)\partial_{\mathcal{X}} f + (-\frac{1}{4}\mathcal{Y} + \Upsilon)\partial_{\mathcal{Y}} f.$$

Since by definition we have $\Theta = -a$ then

$$v(a) = \frac{1}{2}a. \tag{2.7}$$

We have also by definition of $v$

$$v(\mathcal{X}) = \frac{3}{2}\mathcal{X} + \Theta \tag{2.8}$$

We now change variables and instead of considering the $(\mathcal{X}, \mathcal{Y})$ variables we consider the $(a, b)$ variables. This is a modified Crocco transform. The general expression of $v$ in variables $(a, b) = (-\Theta, b)$ is

$$v(f) = v(a)\partial_a f + v(b)\partial_b f.$$

By (2.7),

$$v = \frac{1}{2}a\partial_a + v(b)\partial_b.$$

To find $v(b)$, we have by volume preseveration of the change of variable$(a, b) \to (x, y)$, that

$$\mathrm{div}_{x,y}\, v = \mathrm{div}_{a,b}\, v.$$

Therefore

$$\begin{aligned}
&\partial_x(\frac{3}{2}x + \Theta) + \partial_y(-\frac{1}{4}y + \Upsilon) = \partial_a(\frac{1}{2}a) + \partial_b(v(b)) \\
\iff &\frac{3}{2} - \frac{1}{4} + \partial_x\Theta + \partial_y\Upsilon = \frac{1}{2} + \frac{1}{2}\partial_b(v(b)), \\
\iff &\partial_b v = \frac{3}{4}
\end{aligned}$$

where we used the incompressibility $\partial_x\Theta + \partial_y\Upsilon = 0$. So, using in addition the boundary condition, $v(b) = \frac{3}{4}b$. Hence

$$v = \frac{1}{2}a\partial_a + \frac{3}{4}b\partial_b.$$

The equation (2.8) becomes in $(a, b)$ variables

$$\frac{1}{2}a\partial_a\mathcal{X} + \frac{3}{4}b\partial_b\mathcal{X} = \frac{3}{2}\mathcal{X} + a$$

One can check that $\mathcal{X} = a + a^3 + b^2$ is a solution of the above equation. One can check that this equation is equivalent to the original equation (2.6) (we skip the details which are left as exercise), and hence $\Theta$ is a self-similar profile.

∎

## 2.4 Generic singularity associated to separation

We now prove that for singular solutions, generically the self-similar profile (2.5) describes to leading order the singularity. By "generic", we mean here that singular solutions which are described by $\Theta$ are stable (the corresponding set of iniital data is open), and occur frequently (the set is dense). Other singularities exist, but they are less "frequent".

**Theorem 2.6** *Among the set of initial data of Theorem 2.3 that are $C^4(\mathbb{R} \times [0, \infty))$ and whose solutions become singular in finite time with boundary layer separation, there exists a subset that is dense and open (generic) such that the following hold for the corresponding solutions. There exist $\mu, \nu > 0$, $\iota \in \{\pm 1\}$, $x^*, c \in \mathbb{R}$, such that*

$$u(t, x, y) = c + \mu\iota(T - t)^{\frac{1}{2}}\Theta\left(\iota\frac{x - x^* + c(T - t)}{\mu(T - t)^{\frac{3}{2}}}, \frac{y}{\nu(T - t)^{-\frac{1}{4}}}\right) + v(t, x, y),$$

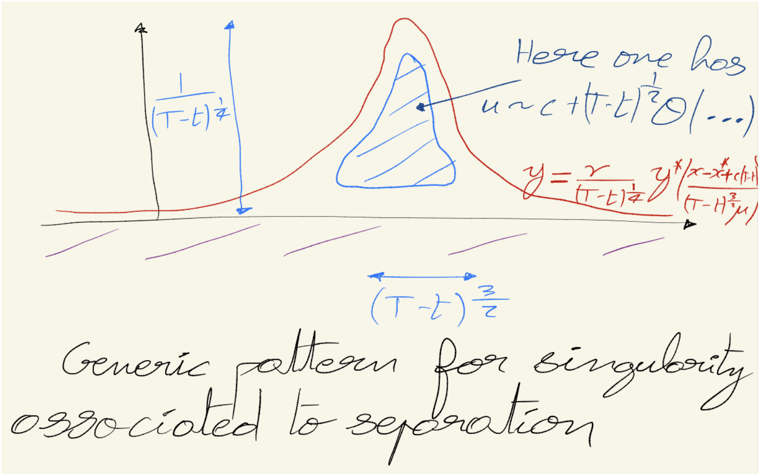
*where for any $\delta > 0$,*

$$\lim_{t\to T} \frac{|v(t, x, y)|}{|\Theta(\iota\frac{x - x^* + C(T-t)}{\mu(T-t)^{\frac{3}{4}}}, \frac{y}{\nu(T-t)^{-\frac{1}{4}}})|} = 0$$

*uniformly for $(x, y) \in \mathbb{R} \times [0, \infty)$ with*

$$|x - x^* + c(T - t)| \leq \frac{1}{\delta(T - t)^{\frac{3}{2}}} \quad \textit{and} \quad \frac{\delta}{(T - t)^{\frac{1}{4}}} \leq y \leq \nu\frac{\mathcal{Y}^*(\frac{x - x^* + C(T-t)}{\mu(T-t)^{\frac{3}{2}}}) - \delta}{(T - t)^{\frac{1}{4}}}$$

Below is an illustration of Theorem 2.6




**Proof.**
We will give a sketch for the proof.

**Step 1**: *Generic expansion for the characteristics.* Let $(X_0, Y_0)$ be defined as in Case 1 of the proof of Theorem 2.3. One can show that, for an open and dense set of initial data leading to separation, $(X_0, Y_0)$ is unique and one has (up to renomalizing the solution by the symmetries of the (HI-Prandtl) system (2.4)) that $X_0 = 0$ and near $(0, Y_0)$,

$$u_0(\tilde{X}, Y_0 + \tilde{Y}) = -\tilde{X} + \tilde{X}^3 + (\tilde{X} - \tilde{Y})^2 + ... \tag{2.9}$$

where "..." denotes higher order terms.

**Step 2**: *Computation of the characteristics.* . The characteristics given by (2.2) are

$$\begin{aligned} x(t, \tilde{X}, y_0 + \tilde{Y}) &= \tilde{X} + tu_0(\tilde{X}, \tilde{Y}) \\ &= \tilde{X}(1-t) + \tilde{X}^3 + (\tilde{X} - \tilde{Y})^2 + ... \end{aligned}$$

We can change variables, and write

$$(\tilde{X}, \tilde{Y}) = (\sqrt{1-t}a - (1-t)^{\frac{3}{4}}b, \sqrt{1-t}a + (1-t)^{\frac{3}{4}}b).$$

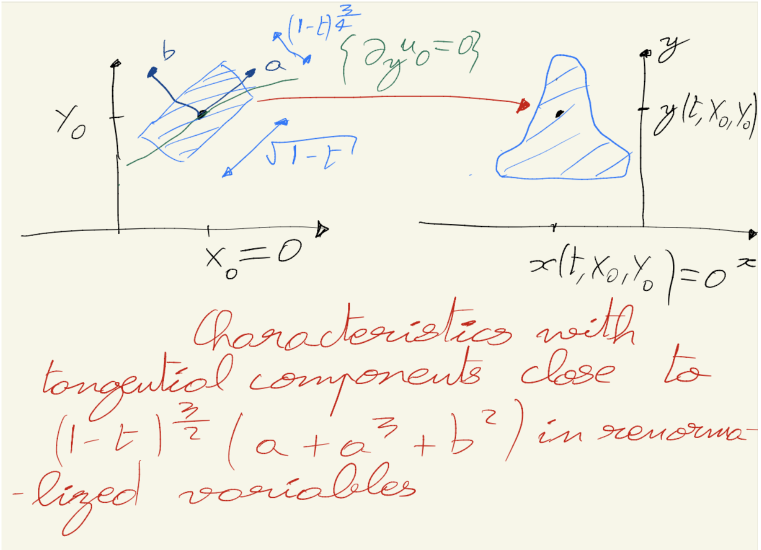
Then the above becomes

$$\begin{aligned} x &= (1-t)^{\frac{3}{2}}(a + a^3 + b^2)... \\ &= (1-t)^{\frac{3}{2}}\Phi_1(a, b) + ... \end{aligned} \tag{2.10}$$

where we used the definition of $\Phi$ from Theorem 2.5. Using Lemma 2.2 and the fact that $\Phi = (\Phi_1, \Phi_2)$ is an incompressible change of variables from Theorem 2.5, we get

$$y = \frac{1}{(T-t)^{\frac{1}{4}}}\Phi_2(a, b) + ...$$

where the remainder "..." can be estimated by tedious computations that we skip here. Below is an illustration of the above reasoning to compute the characteristics




**Step 3**: *Computation of the solutions.* Computing the solution using the characteristics, then injecting (2.9) and changin variables $(\tilde{X},\tilde{Y}) \mapsto (a,b)$ we get:

$$
\begin{aligned}
u(t,x,y) &= u_0(X,Y) \\
&= -\tilde{X} + \ldots \\
&= -\sqrt{1-t}a + \ldots
\end{aligned}
$$

Inverting the characteristics map (2.10), then using $\Theta = -\Phi_1^{-1}$ one finds

$$
a = \Phi_1^{-1}\left(\frac{x}{(1-t)^{\frac{3}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}}\right) + \ldots = -\Theta\left(\frac{x}{(1-t)^{\frac{3}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}}\right) + \ldots
$$

Combining we obtain

$$
u(t,x,y) = \sqrt{1-t}\Theta\left(\frac{x}{(T-t)^{\frac{3}{2}}}, \frac{y}{(T-t)^{-\frac{1}{4}}}\right) + \ldots
$$

which is the desired result. The precise estimates for the "..." remainders can be tedious and can be found in [4]

■

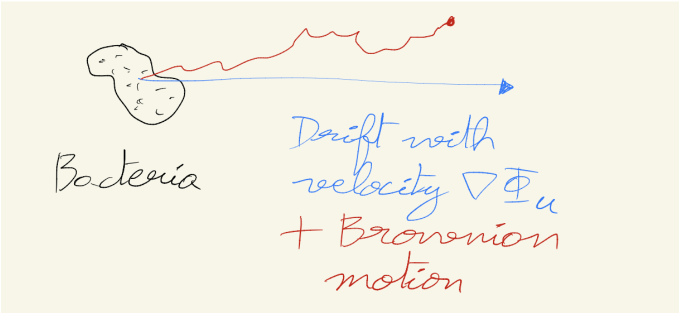
# 3 Preliminaries on the Keller-Segel system

We will now study singularity formation for the parabolic-elliptic Keller-Segel system

$$
(K-S) \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \Phi_u) \\ -\Delta \Phi_u = u. \end{cases} \tag{3.1}
$$

It is a model for the propagation of bacteria:

- $u$ is the density of bacteria.
- $\Phi_u$ is the concentration of the chemo-attractant. It is related to $u$ by Fick's law.
- Below is an illustration of the equation as a mean-field equation arising from particles in interactions




In this section we study the Cauchy problem for (3.1) and the parabolic regularisation effects.

## 3.1 The Cauchy problem in supercritical Lebesgue spaces

### 3.1.1 Solving the Poisson equation

We first solve the Poisson equation $-\Phi_u = u$ in (3.1). This gives $\Phi_u = k * u$ where $k$ is the Green function of the Laplacian. Thus we have

$$\begin{aligned}\nabla\Phi_u &= \nabla k * u\\ &= -\frac{1}{|\mathbb{S}^d|}\frac{x}{|x|^d} * u\end{aligned}$$

where $|\mathbb{S}^d|$ is the surface of the sphere.

**Lemma 3.1** *For any* $1 < p < n$ *if* $u \in L^p$ *then* $\nabla\Phi_u \in L^{\frac{np}{n-p}}$ *with*

$$\|\nabla\Phi_u\|_{L^{\frac{np}{n-p}}} \lesssim \|u\|_{L^p}.$$

**Proof.**
On the Fourier side the Laplace equation is $|\xi|^2\widehat{\Phi_u}(\xi) = \hat{u}(\xi)$. Hence after differentiation:

$$\widehat{\frac{\partial^2}{\partial x_j \partial x_k}\Phi_u}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}\hat{u}(\xi).$$

The Fourier multiplier $m(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}$ satisfies $|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ for any multi-index $\alpha \in \mathbb{N}$, where $|\alpha| = \alpha_1 + ... + \alpha_n$. We can thus apply the Hörmander-Mihlin multiplier Theorem and get that for any $u \in L^p$ we have $\frac{\partial^2}{\partial x_j\partial x_k}\Phi_u \in L^p$ with

$$\|\frac{\partial^2}{\partial x_j \partial x_k}\Phi_u\|_{L^p} \lesssim \|u\|_{L^p}.$$

We conclude that $\|\frac{\partial}{\partial x_k}\Phi_u\|_{L^{\frac{np}{n-p}}} \lesssim \|u\|_{L^p}$ by the homogeneous Sobolev embedding.

∎

### 3.1.2 Solving the heat equation

We next solve the heat equation. We denote the solution for heat equations $f_t = \Delta f$ as

$$S(t)f_0 = K(t) * f_0$$

where $K(t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$.

We recall the standard heat kernel estimates:

**Lemma 3.2** *For any $1 \leq p \leq q \leq \infty$, if $u \in L^p$ then we have $\nabla S(t)u \in L^q$ with*

$$\|\nabla S(t)u\|_{L^q} \lesssim \frac{1}{t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}}$$

**Proof.**
As $\nabla S(t)u = \nabla K_t * u$ we have by the Young inequality for convolution

$$\|\nabla S(t)u\|_{L^q} \lesssim \|\nabla K_t\|_{L^{(1+\frac{1}{q}-\frac{1}{p})^{-1}}} \|u\|_{L^p}.$$

An explicit computation gives $\|\nabla K_t\|_{L^r} = \frac{1}{t^{\frac{n}{2}-\frac{n}{2r}+\frac{1}{2}}}$ what concludes the proof.

■

### 3.1.3 Solving the Keller-Segel system

We now show that the Keller-Segel system has solution in a rather weak sense. The standard method we present is to consider the solution for short times as a perturbation of the heat equation, and to use dissipative estimates to control the effects of the nonlinearity.

**Definition 3.3** *We say $u$ is an integral solution of the Keller-Segel system* (3.1) *in $L^p$ on $[0, T_0]$ if for every $t \in [0, T_0]$ we have $K(t-\cdot)\nabla \cdot (u\nabla\Phi_u)(\cdot) \in L^1([0,t], L^p)$ and*

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\nabla \cdot (u\nabla\Phi_u)(s)ds.$$

We remark that this is the Duhamel formula for (3.1). We first prove a bilinear estimate.

**Lemma 3.4** *If $u \in L^p$, $\frac{n}{2} < p < n$, and $u \in L^q$ for $q > \frac{n}{2}$, then for any $r \geq (\frac{1}{p} + \frac{1}{q} - \frac{1}{n})^{-1}$ we have*

$$\|S(t)\nabla \cdot (u\nabla\Phi_u)\|_{L^r} \lesssim \frac{\|u\|_{L^p}\|u\|_{L^q}}{t^{\frac{n}{2}(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}-\frac{1}{n})+\frac{1}{2}}}$$

**Proof.** By Lemma 3.1, $\|\nabla\Phi_u\|_{L^a} \lesssim \|u\|_{L^p}$ for $\frac{1}{a} = \frac{1}{p} - \frac{1}{n}$. By Hölder, $\|u\nabla\Phi_u\|_{L^b} \lesssim \|u\|_{L^p}\|u\|_{L^q}$, with $\frac{1}{b} = \frac{1}{a} + \frac{1}{q} = \frac{1}{p} + \frac{1}{q} - \frac{1}{n}$. Then conclude by Lemma 3.1.2. ■

**Theorem 3.5** *For any $u_0 \in L^p$, $\frac{n}{2} < p < n$, there exists $T_0(\|u_0\|_{L^p}) > 0$ and a unique integral solution $u$ in $L^p$ on $[0, T_0]$ of the (K-S) system* (3.1)*.*

**Remark 3.6** *The Cauchy problem is also locally well posed in the case $p = \frac{n}{2}$ in large dimensions $n \geq 3$. For the semilinear heat equation $u_t = \Delta u + u^2$, the Cauchy problem is ill posed if $p = \frac{n}{2}$ in dimension $n = 2$. We refer to [2, 14]. For the well-posedness in other critical spaces, see [12].*

**Proof.** We prove by fixed point theorem. We define

$$\varphi(v) = S(t)u_0 + \int_0^t S(t-s)\nabla \cdot (v\Phi_v)(s)ds$$

We claim that $\varphi$ is a contraction on the ball $B(0, 2\|u_0\|_{L^p})$ of $X = C([0, T_0], L^p)$, for $T_0$ small enough. Its fixed point is the desired solution. Indeed,

$$\|S(t)u_0\|_{L^p} \leq \|K_t\|_{L^1}\|u_0\|_{L^p} \leq \|u_0\|_{L^p}$$

by the Young inequality and the fact that $\|K_t\|_{L^1} = 1$ which is an explicit computation. We have

$$\|S(t-s)\nabla \cdot (v\nabla\Phi_v)\|_{L^p} \lesssim \frac{\|v\|^2_{L^p}}{(t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{n})+\frac{1}{2}}}$$

by Lemma 3.4. So if $T_0$ small enough, we have

$$\| \int_0^t S(t-s)\nabla\cdot(v\nabla\Phi_v)(s)ds\|_{L^p} \lesssim C\|u_0\|_{L^p}^2 \int_0^t (t-s)^{-\alpha}ds \lesssim \|u_0\|_{L^p}^2 T_0^{1-\alpha} \leq \frac{\|u_0\|_{L^p}^2}{2},$$

where $\alpha = \frac{1}{2}(\frac{1}{p}-\frac{1}{n})+\frac{1}{2} < 1$. We leave as an exercise to prove that $\varphi(v) \in C([0,T],L^p)$ by similar estimates. Therefore, $\varphi$ maps the ball $B(0,2\|u_0\|_{L^p})$ of $X = C([0,T_0],L^p)$ onto itself. We also leave as an exercise to prove that it is a contraction by similar estimates again.

■

## 3.2 Parabolic regularisation

In this subsection we prove that the integral solutions provided by Theorem 3.5 are instantaneously regularised: they are immediately bounded after $t=0$, and are infinitely differentiable with bounded derivatives as well. Thus, they in fact are classical solutions right after the initial time. As a consequence, we obtain a singularity formation criterion: the solution becomes singular in finite time $T$ if it is unbounded as $t \to T$. We introduce the standard method of parabolic bootstrap in order to prove this regularisation effect.

**Proposition 3.7** *Assume $u$ is an integral solution in $L^p$ for some $\frac{n}{2} < p < n$ with $u \in C([0,T_0],L^q)$ for some $q > \frac{n}{2}$.*

- *If $\frac{1}{q}+\frac{1}{p}-\frac{2}{n} > 0$. then for any $q \leq \lambda < (\frac{1}{q}+\frac{1}{p}-\frac{2}{n})^{-1}$ we have $u \in C((0,T_0],L^\lambda)$.*
- *If $\frac{1}{q}+\frac{1}{p}-\frac{2}{n} \leq 0$. then for any $q \leq \lambda \leq \infty$ we have $u \in C((0,T_0],L^\lambda)$.*

**Proof.**
The proof can be proved as Theorem 3.5, relying on Lemma 3.4. We leave this as an exercise. ■

**Corollary 3.8** *Such integral solutions are in fact in $C((0,T_0],L^\infty)$*

**Proof.** Apply the first point of Proposition 3.7 for the first time to obtain $u \in C((0,T_0],L^{q_1})$. Then apply again with $q = q_1$, and obtain $u \in C((0,T_0],L^{q_2})$ with $q_2 > q_1$. Then repeat the procedure until $u \in C((0,T_0],L^{q_N})$, with $\frac{1}{q_N}+\frac{1}{p}-\frac{n}{2} \leq 0$. Then apply the second point of proposition 3.7 to get the result. ■

**Proposition 3.9** *If $u$ is an integral solution in $L^p$ for $\frac{n}{2} < p < n$ with $u \in C([0,T_0],W^{k,\infty}\cap W^{k,p})$ for some $k \geq 0$, then $u \in C((0,T_0],W^{k+1,\infty}\cap W^{k+1,p})$.*

**Proof.**
The proof relies on the following useful observation. The proof of Theorem 3.5 is based on a fixed point argument which shows that the solution enjoys the same properties as the solution to the linear heat equation with the same initial data. We could thus incorporate additional properties of solutions to the linear heat equation in the space in which we perform our fixed point.

Let $Y$ be the Banach space with the following norm

$$\|v\|_Y = \|v\|_{L^\infty([0,T_0],W^{k,\infty}\cap W^{k,p})} + \sup_{t\in(0,T_0]} \sqrt{t}\|\nabla^{k+1}v(t)\|_{L^\infty\cap L^p}.$$

We note that the second quantity encodes a regularisation effect $\|\nabla^{k+1}v(t)\|_{L^p\cap L^\infty} \leq \frac{\|v\|_Y}{\sqrt{t}}$ with the same rate as the regularisation of the heat semigroup from $W^{k,\infty}\cap W^{k,p}$ to $W^{k+1,\infty}\cap W^{k+1,p}$, because of Lemma 3.1.2.

We leave as an exercise to do a fixed point argument in the space Y like before in the proof of Theorem 3.5. ■

**Corollary 3.10** *Such integral solutions are in fact in $C([0,T_0], W^{j,\infty} \cap W^{j,p})$ for $\forall j \geq 0$, and in fact $u \in C^\infty((0,T_0] \times \mathbb{R})$, which means that $u$ is a classical solution.*

**Proof.** The proof for corollary 3.10 is just the use of bootstrap, and is left as exercise.
■

We can define the maximal time of existence $T$ of a solution in a standard way. As a consequence of the above Propositions above, there holds the following singularity formation criterion, the proof of which is omitted:

$$T < \infty \qquad \Leftrightarrow \qquad \lim_{t \to T} \|u(t)\|_{L^\infty} = \infty.$$

# 4 Stable backward self-similar singularity in three dimensions and higher

In this section we show the existence of an explicit backward self-similar solution. Moreover, we will see that this explicit solution is stable, which was showed in a recent article by Glocic and Schorkhuber [9]. This is the reason why we chose the Keller-Segel system for this mini-course: it is the only parabolic equation that is rather easy to introduce, for which there is an explicit stable backward self-similar solution (this is not the case for the apparently simpler model $u_t = \Delta u + u^2$). The proof of stability is not easy, but we will give some tools that have been developped over the last three decades to address this problem.

## 4.1 Criticality

For (K-S) equation, we notice the following scaling invariance and conserved quantity.

Scaling invariance: If u is a solution, then

$$\frac{1}{\lambda^2} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{4.1}$$

is also a solution. The scaled Lebesgue norms are

$$\|\frac{1}{\lambda^2} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})\|_{L^p} = \lambda^{-2+\frac{n}{p}} \|u_0\|_{L^p} \underset{\lambda \to \infty}{\longrightarrow} \begin{cases} 0, & p > \frac{n}{2} \\ \|u_0\|_{L^p}, & p = \frac{n}{2} \\ \infty, & p < \frac{p}{2} \end{cases} \tag{4.2}$$

Mass invariance: If $u$ is a solution, then

$$\int u(t) dx = \int u_0 dx.$$

This is because

$$\frac{d}{dt} \int u(t) dx = \int \nabla \cdot (\nabla u \cdot u \nabla \Phi_u) dx = 0.$$

We remark that if $u$ is positive, then its $L^1$ norm remains bounded $\|u\|_{L^1} = \int u dx = \int u_0 dx$.

**Definition 4.1** • *The Lebesgue space $L^{\frac{n}{2}}$ is called the* critical space*. This is because its norm is invariant by the rescaling* (4.2)*. The space $L^p$ is* subcritical *if $p < \frac{n}{2}$, and supercritical if $p > \frac{n}{2}$. Informally, a supercritical norm detects a concentration according to the scaling of the equation in which case it diverges according to* (4.2)*, while a subcritical norm cannot. This is a heuristic for general evolution PDEs explaining why subcritical norms are not expected to be suitable for local well-posedness, while supercritical may be.*

- *The (K-S) system is said to be* mass critical *if* $\frac{n}{2} = 1$*,* mass supercritical *if* $\frac{n}{2} > 1$*, and* mass subcritical *if* $\frac{n}{2} < 1$*. This compares the Lebesgue exponent of the critical space,* $\frac{n}{2}$*, with that of the conservation law,* $1$*. In the subcritical case the solutions are global, because the* $L^1$ *norm controls the evolution on uniform time intervals. Formally, the critical case is the boundary case at which this fails, and usually different dynamics occur as we will see.*

### 4.2 Radial solutions and partial masses

In this subsection we describe how the (K-S) equations simplify into a local equation in the case of radial solutions.

**Definition 4.2** *For* $u$ *a radially symmetric solution, its partial mass is* $m(r) = \frac{1}{r^n|\mathbb{S}^n|}\int_{|x|\leq r} u(x)dx$

**Lemma 4.3** *If* $u$ *is radial, then it is a solution of (K-S) iff its partial mass solves*

$$(KSm) \qquad \partial_t m = \partial_r^2 m + \frac{n+1}{r}\partial_r m + nm^2 + rm\partial_r m.$$

**Proof.**
Using the second equation in (3.1), then applying the divergence Theorem, we have

$$m = \frac{1}{r^n|\mathbb{S}^n|}\int_{|x|\leq r} u dx = \frac{1}{r^n|\mathbb{S}^n|}\int_{|x|=r} \nabla\Phi_u \cdot \frac{x}{|x|}dx = \frac{1}{r}\partial_r\Phi_u.$$

The proof is then left as exercise, using similar integration by parts. ∎

### 4.3 An explicit stable backward self-similar solution

Most evolution PDEs do not have backward self-similar solutions in closed form. Their existence is in almost all cases showed by solving the corresponding stationary equation in self-similar variables. Fortunately, the (K-S) system admits an explicit one! It was found in [1, 11]

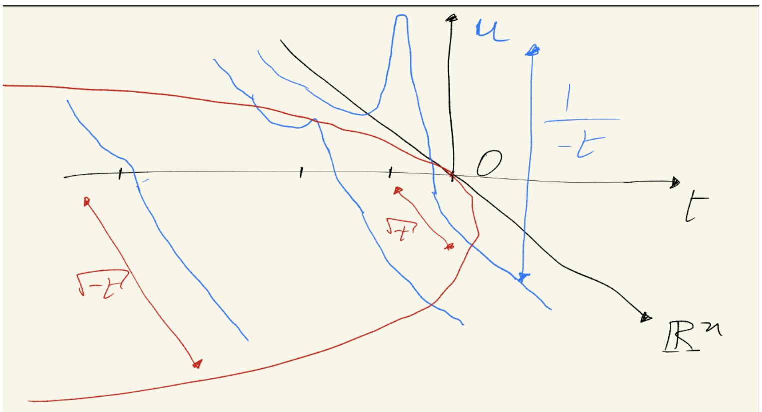
**Lemma 4.4** *For* $n \geq 3$*, there exists a backward self-similar solution of the Keller-Segel system* (3.1)

$$u(t,x) = \frac{1}{-t}\Psi\left(\frac{x}{\sqrt{-t}}\right)$$

*defined for* $t < 0$*, with profile*

$$\Psi(y) = \frac{4(n-2)(2n+|y|^2)}{(2(2n-2)+|y|^2)^2}.$$

The drawing below illustrates the backward self-similar solution:



**Proof.** By Lemma 4.3, u solves (KS) iff m solves $(KS_m)$. Denote $m(t,x) = \frac{1}{-t}\phi(\frac{r}{\sqrt{-t}})$, $\phi(\rho) = \frac{1}{\rho^n|\mathbb{S}^n|}\int_0^\rho \Psi(\tilde{\rho})\tilde{\rho}^{n-1}d\tilde{\rho}$. By a direct computation,

$$\phi(\rho) = \frac{4}{2(n-2)+\rho^2}. \tag{4.3}$$

We have that $\frac{1}{-t}\phi(\frac{r}{\sqrt{-t}})$ solves $KS_m$ iff $\phi$ solves the stationary equation for partial masses in self-similar variables

$$-\partial_{\rho\rho}\phi - \frac{n+1}{\rho}\partial_\rho\phi + n\phi^2 + \rho\phi\partial_\rho\phi + \frac{1}{2}\Lambda\phi = 0 \tag{4.4}$$

where $\Lambda$ is the infinitesimal generator of the scaling group

$$\Lambda f = \rho\partial_\rho f + 2f.$$

By a direct computation, $\phi$ given by (4.3) solves (4.4), which proves the Lemma.

■

The backward self-similar profile $\Psi$ is associated to a stable singularity, as proved by Glogic and Schorkhuber [9].

**Theorem 4.5** *There exists $\varepsilon > 0$, such that if $u_0 = \psi + \tilde{u}_0$, for $\tilde{u}_0 \in \mathcal{S}$ with $\|\tilde{u}_0\|_{H^3} < \varepsilon$, then $u$ becomes singular at some time $T$ with*

$$u(t,x) = \frac{1}{T-t}\psi(\frac{x}{(T-t)^{\frac{1}{2}}}) + \tilde{u}(t,x),$$

*where*

$$\lim_{t\to T}(T-t)\|\tilde{u}\|_{L^\infty}(t) = 0.$$

We will now develop key techniques for studying solutions that are close to a backward self-similar solution, which are often used in the proof of such stability result.

### 4.4 Renormalization techniques

In this subsection we introduced renormalisation techniques. They allow to zoom on the solution in a time-dependent way, at the relevant scale where the solution is close to the backward self-similar profile. They lead to the linearization around the self-similar profile of the evolution equation, in renormalized variables, which is the key equation driving the evolution for the remainder.

We first introduce trapped functions. They are close to the full set of rescaled self-similar profiles $\{\frac{1}{\lambda^2}\Psi(\frac{x}{\lambda})_{\lambda>0}$. This closeness is measured with respect to the nearest rescaled profile. Let $\delta > 0$ small.

**Definition 4.6** *We say that $m$ is trapped if $\exists\tilde{\lambda} > 0$, such that $m(r) = \frac{1}{\tilde{\lambda}^2}\phi(\frac{r}{\tilde{\lambda}}) + \tilde{m}$, with $\|\tilde{m}\|_{L^\infty} \le \frac{\delta}{\tilde{\lambda}^2}$.*

It is always possible to find a scale $\tilde{\lambda}$ for the rescaled self-similar profile such that $m - \frac{1}{\tilde{\lambda}^2}\phi(\frac{x}{\tilde{\lambda}})$ is orthogonal to the tangent space of the full family of rescaled self-similar profiles at $\frac{1}{\tilde{\lambda}^2}\phi(\frac{x}{\tilde{\lambda}})$. We introduce the scalar product

$$\langle f,g\rangle = \int fg\sigma d\rho, \qquad \sigma(\rho) = \frac{\rho^{n+2}}{\phi^2(\rho)}e^{-\frac{\rho^2}{4}}.$$

**Lemma 4.7** *If $m$ is trapped, there exists a unique $\tilde{\lambda} > 0$ and $q$, with*

$$\|q\|_{L^\infty} \lesssim \frac{\delta}{\tilde{\lambda}^2} \tag{4.5}$$

*such that*

$$m(r) = \frac{1}{\tilde{\lambda}^2}(\phi + q)(\frac{r}{\tilde{\lambda}}),$$

*with*

$$\langle q, \Lambda\phi \rangle = 0. \tag{4.6}$$

**Proof. Step 1** *Case* $\tilde{\lambda} = 1$. Consider the function

$$\begin{array}{llll} \theta: & L^\infty \times (0,\infty) & \to & \mathbb{R} \\ & (u,\mu) & \to & \langle \mu^2 m(\mu\rho) - \phi, \Lambda\phi \rangle \end{array}$$

We have $\theta(\phi, 1) = 0$ and $\frac{\partial\theta}{\partial\mu}(\phi, 1) = \langle \Lambda\phi, \Lambda\phi \rangle > 0$. Therefore, the result follows from implicit function theorem.

**Step 2**. *Case* $\tilde{\lambda} \neq 1$*:* We use the scaling transformation (4.1) to map this case back to the case $\tilde{\lambda} = 1$ treated previously. We leave the details in exercise.

■

**Definition 4.8** *We say a solution is trapped on* $[0, T_0]$ *if it is trapped at each* $t \in [0, T_0]$*.*

For $\lambda \in C^1([0,T),(0,\infty))$, we define the *renormalized variables*

$$\rho = \frac{r}{\lambda(t)}, \qquad s(t) = \int_0^t \frac{ds}{\lambda^2(s)} \tag{4.7}$$

and the renormalized unknown

$$m(t,r) = \frac{1}{\lambda^2}(\phi(\rho) + q(s,\rho)). \tag{4.8}$$

**Proposition 4.9** *If the solution of* (3.1) *is trapped on* $[0,T]$*, there exists a unique* $\lambda \in C^\infty((0,T])$ *such that the renormalised unknown satisfies the orthogonality* (4.6) *and the estimate* (4.5)*. Moreover, the remainder $q$ solves the* linearized evolution equation in renormalized variables

$$\partial_s q + \mathcal{L}q = (\frac{\lambda_s}{\lambda} + \frac{1}{2})\Lambda(\phi + q) + nq^2 + \rho q \partial_\rho q \tag{4.9}$$

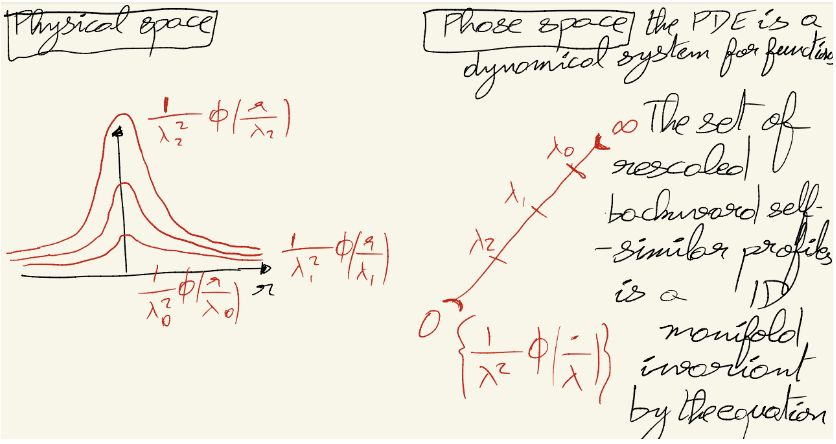
*where*

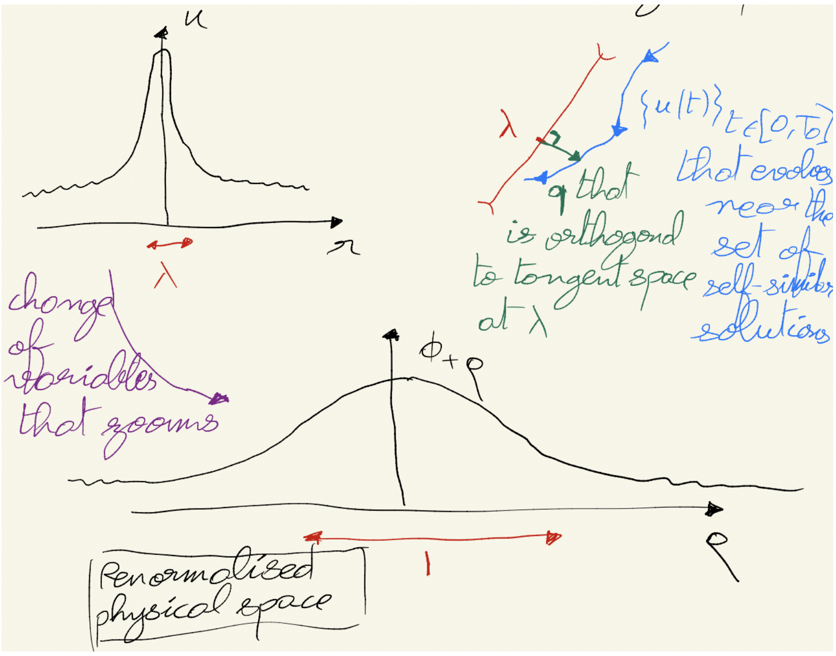
$$\mathcal{L}q = -\partial_{\rho\rho}q - \frac{n+1}{\rho}\partial_\rho q + \frac{1}{2}\Lambda q - \rho\phi\partial_\rho q - (\rho\partial_\rho\phi + 2n\phi)q$$

*is the linearized operators in renormalized variables.*

The term $q$ is called the *radiation remainder*. In (4.9), the term $(\frac{\lambda_s}{\lambda} + \frac{1}{2})\Lambda(\phi + q)$ is called the *modulation term* and is due to the difference between the renormalisation rate $\frac{\lambda_s}{\lambda}$ of the equation and the exact self-similar renormalisation rate $-\frac{1}{2}$ of the self-similar solution. The term $nq^2 + \rho q\partial_\rho q$ is called the nonlinear term.

The drawings below illustrate the geometrical interpretation of Lemma 4.7 and Proposition 4.9




**Proof.** Apply Lemma 4.7 to define $\lambda$. By parabolic regularization of Section 3 before, everything is $C^\infty$. The equation (4.9) then follows from a direct computation. ■

### 4.5 Spectral analysis

In this subsection we introduce some tools from spectral analysis to study the linearized operator $\mathcal{L}$ in self-similar variables.

**Proposition 4.10** *$\mathcal{L}$ is self-adjoint on $L^2(\sigma d\rho)$, $\sigma(\rho) = \frac{\rho^{n+2}}{\phi^2(\rho)} e^{-\frac{\rho^2}{4}}$ with compact resolvant. It is diagonalisable in a Hilbert basis $(f_i)|_{i\geq 1}$ of eigenfunctions with eigenvalues*

$$\lambda_1 = -1 < 0 < \lambda_2 < ... < \lambda_n \overset{n\to\infty}{\to} \infty.$$

*The first eigenfunction is $f_1 = \Lambda\phi$.*

**Proof. Step 1** *Self-adjointness.* For $q, q' \in \mathcal{S}$,

$$\langle \mathcal{L}q, q' \rangle = \int (\partial_\rho q \partial_\rho q' - (\rho\partial_\rho\phi + 2n\phi) q q')\sigma(\rho) d\rho$$

by IBP, so $\mathcal{L}$ is symmetric.
By the Hardy and Poincare inequalities,

$$\int |\partial_\rho q|^2 \sigma d\rho \gtrsim \int (\frac{1}{\rho^2} + \rho^2) q^2 \sigma d\rho \tag{4.10}$$

(the proof is left as exercise). This leads to the self-adjointness of $\mathcal{L}$, other details are left as exercise.

**Step 2**. *Compactness of the resolvant.* We decompose $\mathcal{L} = \mathcal{L}_0 + V$, where

$$\mathcal{L}_0 = -\partial_{\rho\rho} - \frac{n+1}{\rho}\partial_\rho + \frac{1}{2}\Lambda + \rho\phi\partial_\rho.$$

Then $\mathcal{L}_0$ has a compact resolvant because of (4.10) and of the compactness of Sobolev embedding $H^1_{loc} \to L^2_{loc}$. The operator $q \mapsto Vq$ is a compact perturbation of $\mathcal{L}_0$ (Kato theory). Hence $\mathcal{L}$ has compact resolvant.

**Step 3**. *Diagonalization.* This is a consequence of the spectral theorem.

**Step 4**. *Proof that $\Lambda\phi$ is an eigenfunction.* This is a very general fact that for any self-similar solution, any other symmetry of the equation produces an eigenfunction. We give a very general proof that is applicable to any other context. Let $M(t,x) = \frac{1}{-t}\phi(\frac{r}{\sqrt{-t}})$. Then $\lambda^2 M(\lambda^2 t, \lambda r) = M(t,r)\ \forall\lambda$. Apply $\frac{\partial}{\partial\lambda}$, obtain

$$\tilde{\Lambda} M = 0, \quad \tilde{\Lambda} = 2t\partial_t + \Lambda. \tag{4.11}$$

Compute commutator with other invariance generator $\partial_t$

$$[\partial_t, \tilde{\Lambda}] = \partial_t \tilde{\Lambda} - \tilde{\Lambda}\partial_t = 2\partial_t \tag{4.12}$$

Combining (4.11) and (4.12) we get

$$\tilde{\Lambda}\partial_t M = -2\partial_t M \tag{4.13}$$

We now apply the additional time translation symmetry and define $M_{t_0}(t,r) = M(t+t_0, r)$, it solves

$$\partial_t M_{t_0} = \partial_{rr} M_{t_0} + \frac{n+1}{r}\partial_r M_{t_0} + nM_{t_0}^2 + rM_{t_0}\partial_r M_{t_0}$$

Apply $\frac{\partial}{\partial_{t_0}}$, use $\frac{\partial}{\partial t_0} M_{t_0} = \partial_t M(t+t_0)$, and take $t_0 = 0$, and by (4.13), use $\tilde{\Lambda} = 2t\partial_t + \Lambda$, by (4.13),

$$\tilde{\Lambda}\partial_t M = [2t(\partial_{rr} + \frac{n+1}{r}\partial_r + 2nM + r\partial_r M + rM\partial_r) + \Lambda]\partial_t M = -2\partial_t M$$

Take $t = -1$ at which $\partial_t M = \Lambda\phi$ and $M = \phi$ to conclude.

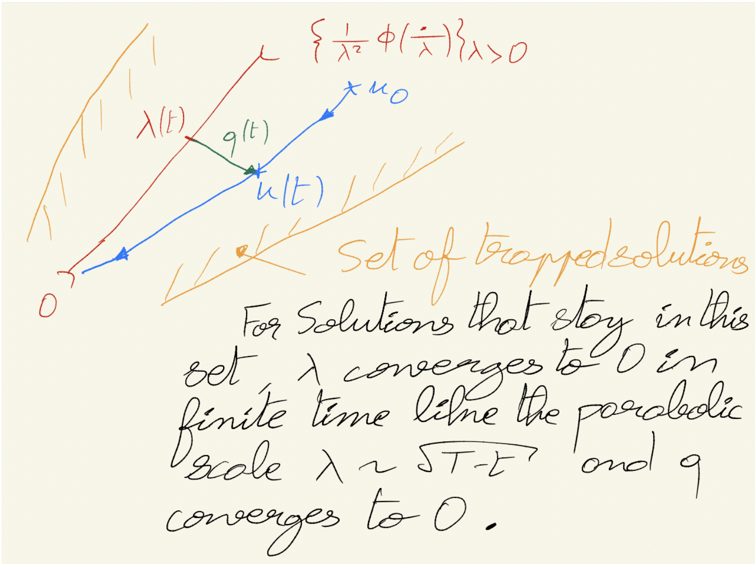
**Step 5**. *Positivity of $\lambda_2$.* We refer to [9] where a counting argument from [7] is used. ■

## 4.6 Asymptotic stability for trapped solutions

We will not give a full proof of Theorem 4.5. Rather, we will prove the following intermediate result: if a solution remains close to the set of all rescaled profiles, then it has to become singular in finite time $T$, concentrating at scale $\sqrt{T-t}$, and converges asymptotically to the self-similar profile in renormalised variables.

**Proposition 4.11** *Assume that a solution is trapped on its whole maximal interval of existence* $[0,T)$*, then*
*i) it becomes singular in finite time* $T<\infty$*.*
*ii)* $m(t,r)=\frac{1}{T-t}\phi(\frac{r}{\sqrt{T-t}})+\frac{1}{T-t}\tilde m$*, where* $\lim_{t\to T}\|\tilde m\|_{L^2(\sigma)}=0$*.*

Below is the geometrical interpretation of Proposition 4.11




In order to prove Proposition 4.11, we will rely on several Lemmas. The first one establishes the *modulation equation* which describes the evolution of the *modulation parameter* $\lambda$.

**Lemma 4.12** *Under the assumptions of Proposition 4.11,*

$$\frac{1}{\lambda}\frac{d}{ds}\lambda=-\frac{1}{2}+N_1(q) \tag{4.14}$$

*where* $|N_1(q)|\lesssim\min\{\|q\|^2_{L^2(\sigma)},\|q\|^2_{L^\infty}\}$

**Proof.** Differentiating the orthogonality condition $\langle q,\Lambda\phi\rangle=0$ gives

$$\langle q_s,\Lambda\Phi\rangle=0.$$

Injecting the evolution equation (4.9) for $q$,

$$\langle-\mathcal{L}q,\Lambda\phi\rangle+(\frac{\lambda_s}{\lambda}+\frac{1}{2})\langle\Lambda\phi,\Lambda\phi+\Lambda q\rangle+\langle nq^2+\rho q\partial_\rho q,\Lambda\phi\rangle=0.$$

Using that $\mathcal{L}$ is self-adjoint, that $\Lambda\phi$ is an eigenfunction and that $q$ is orthogonal to $\Lambda\phi$:

$$\langle\mathcal{L}q,\Lambda\phi\rangle=\langle q,\mathcal{L}\Lambda\phi\rangle=-\langle q,\Lambda\phi\rangle=0.$$

Hence (4.14) with $N_1=\frac{\langle nq^2+\rho q\partial_\rho q,\Lambda\phi\rangle}{\langle\Lambda\phi,\Lambda\phi\rangle+\langle q,\Lambda\phi\rangle}$. We estimate that $|N_1(q)|\lesssim\min(\|q\|^2_{L^\infty},\|q\|^2_{L^2_\sigma})$ after IBP. ∎

The second lemma performs an *energy estimate in renormalized variables.*

**Lemma 4.13** *Under the assumptions of Proposition 4.11,*

$$\frac{d}{ds}\|q\|^2_{L^2(\sigma)}=-2\langle\mathcal{L}q,q\rangle+N_2(q) \tag{4.15}$$

*with* $|N_2(q)|\lesssim\|\partial_\rho q\|^2_{L^2(\sigma)}\|q\|_{L^\infty}$

**Proof.** The evolution equation (4.9) of $q$ gives the energy identity

$$\frac{d}{ds}\|q\|^2_{L^2(\sigma)} = -2\langle \mathcal{L}q, q\rangle + 2(\frac{\lambda_s}{\lambda} + \frac{1}{2})(\langle \Lambda\phi, q\rangle + \langle \Lambda q, q\rangle) + 2\langle nq^2 + \rho q\partial_\rho q, q\rangle.$$

Using the orthogonality $\langle q, \Lambda\phi\rangle = 0$, and the modulation equation (4.14) we get the identity (4.15) with

$$N_2(q) = 2N_1(q)\langle \Lambda q, q\rangle + 2\langle nq^2 + \rho q\partial_\rho q, q\rangle.$$

Using Lemma 4.12, Cauchy-Schwarz and the estimate (4.10) we have

$$|N_1(q)\langle \Lambda q, q\rangle| \lesssim \|q\|^2_{L^\infty}\|\Lambda q\|_{L^2(\sigma)}\|q\|_{L^2_\sigma} \lesssim \|q\|^2_{L^\infty}\|\partial_\rho q\|^2_{L^2(\sigma)}.$$

Using the Hölder inequality and (4.10) we have

$$\left|2\langle nq^2 + \rho q\partial_\rho q, q\rangle\right| \lesssim \|q\|_{L^\infty}(\|q\|^2_{L^2(\sigma)} + \|\rho q\|_{L^2(\sigma)}\|\partial_\rho q\|_{L^2(\sigma)}) \lesssim \|q\|_{L^\infty}\|\partial_\rho q\|^2_{L^2(\sigma)}.$$

Combining the two above inequalities, using that $\|q\|_{L^\infty} \lesssim \delta \ll 1$ from (4.5) yields the desired estimate for $N_2(q)$.

■

**Proof for Proposition 4.11.** By Lemma 4.12 and 4.13, we have

$$\begin{cases} \frac{1}{\lambda}\frac{d}{ds}\lambda = -\frac{1}{2} + N_1(q) \\ \frac{d}{ds}\|q\|^2_{L^2(\sigma)} = -2\langle \mathcal{L}q, q\rangle + O(\|\partial_\rho q\|^2_{L^2(\sigma)}\|q\|_{L^\infty}) \end{cases} \tag{4.16}$$

**Step 1:** *Decay for the remainder $q$.* By Prop 4.10, since $\langle q, \Lambda\phi\rangle = \langle q, f_1\rangle = 0$, we have on the one hand

$$-\langle \mathcal{L}q, q\rangle = -\sum_{i=1}^{\infty}\lambda_i|\langle q, f_i\rangle|^2 = -\sum_{i=2}^{\infty}\lambda_i|\langle q, f_i\rangle|^2 \le -\lambda_2\|q\|^2_{L^2(\sigma)}$$

which is called a *spectral gap estimate.* On the other hand, an explicit computation gives

$$-\langle \mathcal{L}q, q\rangle = -\int |\partial_\rho q|^2\sigma + \int (2 - 2\phi - \rho\partial_\rho\phi)q^2\sigma.$$

Combining, this implies for any $\kappa > 0$ that

$$\begin{aligned} -\langle \mathcal{L}q, q\rangle &= -(1-\kappa)\langle \mathcal{L}q, q\rangle - \kappa\langle \mathcal{L}q, q\rangle \\ &\le -\lambda_2(1-\kappa)\|q\|^2_{L^2(\sigma)} - \kappa\int |\partial_\rho q|^2\sigma + \kappa\int (2 - 2\phi - \rho\partial_\rho\phi)q^2\sigma \\ &\le -(\lambda_2 - c\kappa)\|q\|^2_{L^2(\sigma)} - \kappa\int |\partial_\rho q|^2\sigma. \end{aligned}$$

for some fixed constant $c > 0$, where we used that $\phi$ and $\rho\partial_\rho\phi$ are bounded for the last inequality. Therefore, the second differential inequality in (4.16) gives

$$\frac{d}{ds}\|q\|^2_{L^2(\sigma)} \le -2(\lambda_2 - c\kappa)\|q\|^2_{L^2(\sigma)} - 2\kappa\int |\partial_\rho q|^2\sigma + C\|\partial_\rho q\|^2_{L^2(\sigma)}\|q\|_{L^\infty}.$$

Recalling that $\|q\|_{L^\infty} \lesssim \delta$, choosing $\kappa$ small enough and then $\delta$ small enough leads to

$$\frac{d}{ds}\|q\|^2_{L^2(\sigma)} \le -\lambda_2\|q\|^2_{L^2(\sigma)}.$$

Integrate with time to obtain

$$\|q(s)\|_{L^2(\sigma)} \le \|q_0\|_{L^2(\sigma)} e^{-\frac{\lambda_2}{2}s}. \tag{4.17}$$

**Step 2.** *Asymptotic for the scale* $\lambda$. Inject (4.17) back to (4.16),

$$\frac{d}{ds}\ln\lambda = -\frac{1}{2} + N_1, \qquad |N_1| \lesssim \min(\delta^2, e^{-\lambda_2 s}).$$

Integrating with time one finds $\lambda(s) = \lambda_0 e^{-\frac{s}{2}s} e^{\int_0^s N_1(\tau)d\tau}$ so

$$\lambda(s) = \mu e^{-\frac{s}{2}} + O(e^{(-\frac{1}{2}+\lambda_2)s})$$

for $\mu = \exp(\int_0^\infty N_1)$. As $\frac{ds}{dt} = \frac{1}{\lambda^2}$, we have $\frac{dt}{ds} = \lambda^2$, so $\frac{dt}{ds} = \mu^2 e^{-s} + O(e^{-(1+\lambda_2)s})$, so

$$t(s) = \mu^2 \int_0^t e^{-s'}ds' + \int_0^s O(e^{-(1+\lambda_2)s'})ds' = T - \mu^2 e^{-s} + O(e^{-(1+\lambda_2)s})$$

for $T = \mu^2 \int_0^\infty O(e^{-(1+\lambda_2)s})ds$, which implies

$$e^{-s} = \frac{T-t}{\mu^2} + O((T-t)^{1+\lambda_2})$$

So $\lambda = \sqrt{T-t} + O((T-t)^{\frac{1}{2}+\lambda_2}) \sim \sqrt{T-t}$.

■

# 5 Stable singularity that is self-similar of the second kind in two dimensions

In this section we describe a stable singularity formation in two dimensions. It is not equivalent to a backward self-similar solution and differs from the one seen in Section 4. We fix the dimension $n = 2$.

**Lemma 5.1** $U(x) = \frac{8}{(1+|x|^2)^2}$ *is a stationary solution of* (3.1).

**Proof.** This can be proved by a direct computation using the partial mass and Lemma 4.4. ■

**Remark 5.2** *Stationary states are "self-similar solutions" because invariant under 1D symmetry group of time translations.*

The following theorem was obtained in [5, 6] with previous seminal results [10, 13]

**Theorem 5.3** *There exists an open set of data for norm*

$$\|u\|^2 = \sum_{k=0}^{2} \int \langle x\rangle^{\frac{3}{2}+k} |\nabla^k u|^2 dx$$

*such that*
*(i) The solution becomes singular at time* $T$
*(ii) It can be decomposed as*

$$u(t,x) = \frac{1}{\lambda^2(t)} U\left(\frac{x - x^*(t)}{\lambda(t)}\right) - \tilde{u}(t,x)$$

*where* $x^* \to x^*(T)$,

$$\lambda(t) \sim 2e^{\frac{2+r}{2}} \sqrt{T-t}\, e^{-\sqrt{\frac{|\ln(T-t)|}{2}}}$$

*and* $\lambda^2(t)\|\tilde{u}(t)\|_{L^\infty} \underset{t\to T}{\to} 0$.

We now compare the various type of singularity formation patterns studied in these lecture notes. Wether for the Burgers equation in Theorem 1.9, the homogeneous inviscid Prandtl system in Theorem 2.6, the mass-supercritical Keller-Segel system in Theorem 4.5, or the mass critical Keller-Segel system in Theorem 5.3, to leading order the solution keeps the same shape, and is localised at a scale that converges to 0 in finite time. The shape is given by a backward self-similar profile or a stationary state. The concentration rate of the scale is given by a family of possible scaling exponents, or the parabolic scale, or a degenerate perturbation of the parabolic scale.

**Definition 5.4** *For mathematicians, for a singular solution of the Keller-Segel system, if* $\|u(t)\|_{L^\infty} \approx \frac{1}{T-t}$*, the singularity is called of type I, and if* $\|u(t)\|_{L^\infty} \gg \frac{1}{T-t}$ *of type II.*
*For physicists following Barrenblatt and Zeldovich, there is self-similarity of the first kind and of the second kind. If the scaling group of invariance is 1D, backward self-similar solutions have only one possible scaling exponent, and the corresponding self-similar solutions are called self-similar of the 1st kind. If the scaling group is dD,* $d \geq 2$*, then the scaling exponents are not known a priori, and this is called self-similarity of 2nd kind. If the scaling group is 1D but the singular solution does not approach a backward self-similar solution, then the scaling parameter is not known a priori and this is also called self-similarity of 2nd kind.*

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