# ON BLOW-UP AND DYNAMICS NEAR GROUND STATES FOR SOME SEMILINEAR EQUATIONS

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ABSTRACT. This is the companion document to the talk the author gave at IHÉS for the Laurent Schwartz seminar on May 3, 2016. The main issue is the study of qualitative properties for two canonical semilinear equations:

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad \partial_{tt} u = \Delta u + |u|^{p-1}u.$$

Its aim is threefold: introduce nonspecialists to the blow-up issue and dynamics near stationary states sometimes leading to their concentration, give an up to date bibliography on this subject for the two equations and on the existence and properties of stationary states and backward self-similar solutions, and a presentation of the author's work on this issue with a sketch of proof for a recent result obtained in collaboration with F. Merle and P. Raphaël.

## 1. Introduction

We consider two model equations, the semilinear focusing heat equation

$$(NLH) \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + |u(t,x)|^{p-1}u(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^d \text{ or } \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \dagger \quad (1.1)$$

on $\mathbb{R}^d$ (we will at some point consider a smooth bounded domain $\Omega \subset \mathbb{R}^d$ in which case we add the Dirichlet boundary condition $u(t,x) = 0$ if $x \in \partial\Omega$), and the semilinear focusing wave equation

$$(NLW) \begin{cases} \partial_{tt} u(t,x) = \Delta u(t,x) + |u(t,x)|^{p-1}u(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

on $\mathbb{R}^d$, where $p > 1$ and $\Delta := \sum_1^d \partial_{x_i x_i}$. The two underlying linear equations, the heat equation introduced by Fourier in 1811, and the wave equation which is the first partial differential equation formulated by d'Alembert in 1747, have been the subject of a huge amount of work. In mathematical physics however, many models involve nonlinear variants of these two equations, and (1.1) and (1.2) appear as canonical nonlinear extensions in which the space is still homogeneous and the nonlinearity is a monomial. The study of qualitative properties of solutions of such equations gained an increasing interest since the middle of the twentieth century and we refer to the monographs [44, 50] for a general approach to these equations.

For regular and well-localized initial data $u_0$, $(NLH)$ and $(NLW)$ possess a unique solution $u(t)$, defined on a maximal time interval $[0,T)$. The issue of the optimality of the functional framework in which such a statement holds is now well-understood, and we refer to [4, 18, 29, 49, 52] for more on the Cauchy problem. If $T < +\infty$, this means that some singularity happens that prevents the solution to be extended further. In that case the solution is said to blow-up at time $T$, and if $T = +\infty$ it is said to be global. The main issue at stake here is the asymptotic behavior as $t \to T$ of the solutions.

In the two equations, the linear term makes the solution decrease, by dissipation for $(NLH)$ and dispersion for $(NLW)$, and is in competition with a nonlinear term that makes the solution increase (the equations are then called focusing). Therefore, it is interesting to know wether or not these two forces can interact and create special behaviors, and if a general description of arbitrary solutions is available.

The paper is organized as follows. First, we recall some basic features of the equations in Section 2. Then we prove formally (and state some rigorous results) a convergence to stationary or backward self-similar profiles during blow-up in Sections 3 and 4. Results concerning the scale instability of stationary states are given in Section 5 and eventually in Section 6 the proof of the classification of the dynamics in a special case obtained in [7].

The notation $a \lesssim b$ means that there exists an independent constant $C>0$ such that $a \leq Cb$, and $a \ll b$ means that $\frac{a}{b} = o(1)$.

## 2. **Structure, symmetries and criticality**

We first recall some special features of the equations, among which some will be crucial in the sequel.

2.1. **Energy.** The two equations admit an energy. $(NLH)$ admits the following gradient flow structure:

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}, \ \frac{d}{dt}E(u) = -\int u_t^2 \leq 0, \tag{2.1}$$

and for $(NLW)$ a similar energy is conserved:

$$E(u) = \frac{1}{2}\int |\nabla u|^2 + |\partial_t u|^2 - \frac{1}{p+1}\int |u|^{p+1}, \ E(u) = E(u(0)).$$

2.2. **Invariances.** The two equations are invariant by time and space translations. They are also invariant by scaling: if $u$ is a solution, then so is

$$\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) =: (u_\lambda(\lambda^2 t, \cdot))(x),$$

for $(NLH)$, and

$$\lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) = (u_\lambda(\lambda t, \cdot))(x),$$

for $(NLW)$. The infinitesimal generator of this semi group is the operator:

$$\Lambda u := \frac{2}{p-1}u + x.\nabla u.$$

The transformation appearing, $u \mapsto u_\lambda$, is an isometry on the following homogeneous Sobolev space:

$$\| u_\lambda \|_{\dot{H}^{s_c}} = \| u \|_{\dot{H}^{s_c}}, \quad s_c := \frac{d}{2} - \frac{2}{p-1}, \tag{2.2}$$

where for $0 \leq s < \frac{d}{2}$, denoting by $\hat{u}$ the Fourier transform of $u$:

$$\dot{H}^s = \dot{H}^s(\mathbb{R}^d) := \{u, \ \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}|^2||\}, \quad \| u \|_{\dot{H}^s} := \| |\xi|^s \hat{u} \|_{L^2} .$$

Note that $(NLW)$ possesses other important symmetries such as the Lorentz transform, which we avoid here.

2.3. **Criticality.** From the above scaling invariance, one distinguishes between three problems, according to the relative position of the energy (2.1) and the critical Sobolev space (2.2):

(i) If $1 < p < \frac{d+2}{d-2} := p_c$ ($p_c = +\infty$ for $d = 1, 2$), then $s_c < 1$ and the problem is said to be energy subcritical.
(ii) If $p = p_c$ then $s_c = 1$ and the problem is energy critical.
(ii) If $p > p_c$, $s_c > 1$ and the problem is energy supercritical.

## 3. The renormalized flow and asymptotic elliptic equations

The symmetries of the equations play a crucial role in the description of their solutions. We describe here formally a renormalization procedure for singularity formation for $(NLH)$, but a similar approach can be made for $(NLW)$ and for large time behaviors $t \to +\infty$ of global solutions as well. As in renormalization group theory in physics, the aim is to use the symmetries of the equation to reduce the number of degrees of freedom of the system in a particular regime; here we will kill time dependence during singularity formation.

3.1. **The renormalized flow for $(NLH)$.** For a scale $\lambda(t) > 0$, we define the renormalized time $s$ as a solution of the differential equation:

$$s(0) = s_0, \ \frac{ds}{dt} = \frac{1}{\lambda^2}. \tag{3.1}$$

Then if $u$ is a solution of (1.1),

$$v(s, \cdot) = u_\lambda(t, \cdot) = \lambda^{\frac{2}{p-1}} u(t, \lambda \cdot) \tag{3.2}$$

is a solution of the renormalized heat equation

$$v_s - \frac{\lambda_s}{\lambda} \Lambda v = \Delta v + |v|^{p-1} v, \quad \Lambda v = \frac{2}{p-1} v + x.\nabla v. \tag{3.3}$$

3.2. **Asymptotic equations for blow-up.** What follows is a formal reasoning. Assume that $u$ is radially symmetric and decreasing and blows up at $(0, T)$: $|u(t, 0)| \to +\infty$ as $t \to T$. We take $\lambda(t) = \parallel u(t) \parallel_{L^\infty}^{-\frac{p-1}{2}}$ and assume $\lambda_t < 0$, so that $v$ given by (3.2) is such that

$$\forall s, \quad |v(0, s)| = 1, \quad \parallel v \parallel_{L^\infty} = 1. \tag{3.4}$$

Now assume that $v$ is not only bounded but also convergent (think of compactness coming from parabolic regularization):

$$v \to w, \quad \partial_s v \to 0. \tag{3.5}$$

$\lambda(t)$ is then the right scale to zoom on what is happening near 0 as $t \to T$. The diffusion speed yields the bound $|\lambda(t)| \lesssim \sqrt{T-t}$ (rigorously this bound is implied by the lower bound $\parallel u(t) \parallel_{L^\infty} \geq \frac{C}{(T-t)^{\frac{1}{p-1}}}$ coming from the Cauchy theory in $L^\infty$). In renormalized time this means

$$-\frac{\lambda_s}{\lambda} \lesssim 1.$$

It also implies from its definition (3.1) that $\lim_{t \to T} s(t) = +\infty$. Therefore there are two subcases:

$$\text{Case 1} \quad \text{lim-sup}_{s \to T} \ -\frac{\lambda_s}{\lambda} = c > 0, \quad (\text{then } \lambda \sim \sqrt{T-t}) \tag{3.6}$$

$$\text{Case 2} \quad \text{lim-sup}_{s \to T} \ -\frac{\lambda_s}{\lambda} = 0 \quad (\text{then } \lambda \ll \sqrt{T-t}). \tag{3.7}$$

$$\tag{3.8}$$

3.2.1. *Case 1: asymptotically self-similar blow-up.* . From (3.3), (3.5) and (3.6) $w$ must solve:

$$\frac{1}{2}\Lambda w = \Delta w + |w|^{p-1}w \tag{3.9}$$

(one can indeed change $c$ to $\frac{1}{2}$ by a scale change). This equation is called the backward self-similar equation. If $w$ solves (3.9) then one obtains an exact solution of (1.1) under the form

$$u(t,x) = \frac{1}{(T-t)^{\frac{2}{p-1}}} w\left(\frac{x}{\sqrt{T-t}}\right)$$

which is a solution blowing up at 0. Self-similar solutions are then exact solutions for which the scale shrinks at the universal diffusion speed.

3.2.2. *Case 2: asymptotically stationary blow-up.* . From (3.3), (3.5) and (3.7) $w$ must solve:

$$0 = \Delta w + |w|^{p-1}w. \tag{3.10}$$

If $w$ solves (3.9) then $w$ is a stationary solution of (1.1). The meaning of this situation is the following: for a blow-up that happens slower (3.7) than the natural blow-up speed (3.6), at main order diffusion must cancel nonlinear effects (3.10).

3.3. **On self-similar solutions for** $(NLH)$**.** For any $p > 1$ there exist the following solutions to (3.9)

$$0, \quad \kappa := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \quad -\kappa,$$

corresponding to the constant in space ODE blow-up $u(t) = \frac{\kappa}{(T-t)^{\frac{1}{p-1}}}$. They are the only bounded solutions for $1 < p \leq p_c$ in the non-radial class [17]. Then two particular numbers arise [25, 27]:

$$p_{JL} := 1 + \frac{4}{d - 4 - 2\sqrt{d-1}} > p_c, \ p_L := 1 + \frac{6}{d-10} > p_{JL} \tag{3.11}$$

($p_L = p_{JL} = +\infty$ if $d \geq 10$). If $p_c < p < p_{JL}$ then there exists an countable family of new radial solutions to (3.9) [2, 3, 51]. If $p_{JL} < p < p_L$ then there exists a finite number of radial solutions [27] and if $p > p_L$ no radial positive solutions other than $\kappa$ exist [40]. Note that the unicity of these solutions in the radial class is still open, as well as the existence of nonradial solutions.

3.4. **On stationary solutions.** For $1 < p < p_c$ the only nonnegative or radial solution is 0 [16]. For $p = p_c$ the only positive radially decaying solution with $w(0) = 1$,

$$Q(x) := \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{2}{p-1}}}, \quad \| Q \|_{\dot{H}^{s_c}} < +\infty, \tag{3.12}$$

is the Talenti-Aubin profile [1, 48]. For $p > p_c$ there exists also an only positive radially decaying solution with $w(0) = 1$ [9, 19, 28] and its asymptotic behavior depends on $p$. For $p > p_{JL}$:

$$Q(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}} + \frac{a_1}{|x|^\gamma} + o(|x|^{-\gamma}) \quad \text{as } |x| \to +\infty,\ a_1 \neq 0, \tag{3.13}$$

with

$$c_\infty := \left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}},$$

$$\gamma := \frac{1}{2}(d-2-\sqrt{\triangle}), \quad \triangle := (d-2)^2 - 4pc_\infty^{p-1} \quad (\triangle > 0 \text{ iff } p > p_{JL}),$$

whereas for $p_c < p < p_{JL}$, defining $\omega := \sqrt{-\triangle}$:

$$Q(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}} + \frac{a_2 \sin(\omega \log(r) + c)}{|x|^{\frac{d-2}{2}}} + o\left(\frac{1}{r^{\frac{d-2}{2}}}\right) \quad \text{as } |x| \to +\infty, \quad a_2 \neq 0,\ c \in \mathbb{R}. \tag{3.14}$$

In all cases, the action of the symmetry groups gives a $d+1$ manifold of stationary states:

$$\left\{\frac{1}{\lambda^{\frac{2}{p-1}}} Q\left(\frac{x-y}{\lambda}\right),\ \lambda > 0,\ y \in \mathbb{R}^d\right\}.$$

Note that in the case $p = p_c$ non-radial solutions do exist [8, 10].

## 4. **A priori description of blow-up profiles**

We now state some rigorous results in the radial case concerning the convergence of (1.1) and (1.2) to the asymptotic elliptic equations (3.9) and (3.10) done formally in the previous section.

**Theorem 4.1** ([17, 37, 32])**.** *Assume $u$ is a radial bounded solution of $(NLH)$ that blows up at the origin at $T > 0$. Then there exists $t_n \to T$ such that:*

$$\lambda_n := \| u(t_n) \|_{L^\infty}^{-\frac{p-1}{2}} \to 0, \quad \lambda_n \lesssim \sqrt{T-t}, \quad u_{\lambda_n}(t_n) \to w$$

*where $w \neq 0$ solves either the stationnary equation or the self-similar equation (up to scale change).*

The study of self-similar blow-up for the wave equation has attracted a great amount of work. However we will only give a result on the blow-up profile in the nonself-similar setting as it is more related to the results in the sequel. We refer to [12] for an up to date version of this result.

**Theorem 4.2** ([11])**.** *Let $p = p_c$, and $u$ be a radial solution of (1.2) blowing-up at the origin at $T > 0$ such that $\limsup_{t \to T} \| u(t), \partial_t u(t) \|_{\dot H^{s_c} \times \dot H^{s_c - 1}} < +\infty$. Then there exist $(\lambda_n, t_n)$ such that*[1]*:*

$$t_n \to T, \quad \lambda_n \to 0, \quad \lambda_n \ll T-t, \quad u_{\lambda_n}(t_n) \underset{\dot H^{s_c}}{\rightharpoonup} Q, \quad \Delta Q + Q^p = 0.$$

[1] Here the diffusion speed $\sqrt{T-t}$ has to be replaced with the sound speed $T-t$ associated to the wave propagation.

The interpretation of these two results is clear: the universal mechanisms for singularity formation are either self-similarity or the concentration of a stationary state by scale instability. If self-similar solutions naturally shrink and lead to blow-up, stationary states themselves are harmless and their possible scale instability has to be investigated. It is thus unclear if the concentration of a stationary state can happen, and the above classification results must be completed by existence or nonexistence results. This is the purpose of the next subsection.

## 5. On blow-up with a stationary state as blow-up profile

We recall that the smallest value of $p$ for which radial stationary states exist is $p_c$. In this energy critical case, in low dimensions, the existence of solutions blowing-up by concentration of the radial stationary state (3.12) has been shown by various authors in low dimensions.

**Theorem 5.1** ([15, 47])**.** *For $(NLH)$, $p = p_c$, $d = 3, 4, 5, 6$, there exists a radial solution blowing-up at time $T > 0$ such that:*

$$u(t) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \varepsilon, \quad \varepsilon(t) \underset{\dot{H}^{s_c}}{\longrightarrow} u^*, \quad \lambda(t) \ll \sqrt{T-t}.$$

In fact, there exists a countable family of scale speeds $\lambda_n(t)$ in [15] and in [46] only the fundamental one is rigorously constructed. For the wave equation, finite speed of propagation and singularity propagation allow for a continuum of blow-up speeds [26]. Classification of the possible blow-up speeds under a suitable regularity assumption is a very interesting question.

**Theorem 5.2** ([22, 23, 26])**.** *For $(NLW)$, $p = p_c$, $d = 3, 4, 5$ there exists a radial solution blowing-up at time $T > 0$ such that:*

$$u(t) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \varepsilon, \quad (\varepsilon(t), \partial_t u(t)) \underset{\dot{H}^{s_c}\times \dot{H}^{s_c-1}}{\longrightarrow} (u^*, v^*),$$
$$\lambda(t) \ll T-t.$$

Surprisingly, due to the difference of the asymptotic behaviors of the radial stationary states for $p = p_c$, $p_c < p < p_{JL}$ and $p > p_{JL}$ ((3.12), (3.13) and (3.14)), blow-up by concentration of a stationary state ceases to exist for $p_c < p < p_{JL}$ for $(NLH)$ [31, 41]. It is a very interesting open problem to know what happens in the case of $(NLW)$. Eventually, When $p > p_{JL}$, concentration of the radial stationary state is again possible.

**Theorem 5.3** ([21, 30, 38, 39, 42])**.** *For (NLH), $p > p_{JL}$, $\Omega = \mathbb{R}^d$, there exist a sequence $(u_\ell)_{\ell > \frac{\gamma}{2} - \frac{1}{p-1}}$ of smooth radial solutions blowing-up with a stationary state as blow-up profile, such that the blow-up rates are quantized:*

$$\| u_\ell(t) \|_{L^\infty} \sim \frac{c_\ell}{(T-t)^{\frac{2\ell}{\left(\gamma - \frac{2}{p-1}\right)(p-1)}}}.$$

*Moreover they are the only possible rates for radial bounded solutions.*

The existence of such blow-up solutions was formally predicted in [21] and was a breakthrough formal computation using matched asymptotics. The authors gave a rigorous proof in an unpublished paper, and later the rigorous proof was done in [38]. These works however strongly used parabolic techniques that were only available for radial parabolic problems, and which cannot be applied to the nonradial case or to other equations. Using techniques developed in the context of energy critical dispersive equations [45], the author was able to give a more detailed construction in the non-radial case and to study the geometrical structure of these particular solutions, highlighting the role played by the stationary state.

**Theorem 5.4** ([6]). *Let $p > p_{JL}$ and $\Omega \subset \mathbb{R}^d$ be a smooth open bounded domain. For $x_0 \in \Omega$ let $\chi(x_0)$ be a smooth cut-off function around $x_0$ with support in $\Omega$. For all $\ell \in \mathbb{N}$ satisfying $2\ell > \gamma - \frac{2}{p-1}$ there exists a solution $u$ of* (1.1) *blowing up in finite time $T > 0$ at a point $x_0' \in \Omega$ with $|x_0' - x_0| \ll 1$:*

$$u(t,x) = \chi_{x_0}(x)\frac{1}{\lambda(t)^{\frac{2}{p-1}}}Q\left(\frac{x-x_0'}{\lambda(t)}\right) + v, \quad \lambda(t) = c(u_0)(1+o_{t\to T}(1))(T-t)^{\frac{\ell}{\gamma-\frac{2}{p-1}}}.$$

*The Sobolev norms below scaling remain bounded*

$$\limsup_{t\uparrow T} \parallel u(t) \parallel_{H^s(\Omega)} < +\infty \quad \text{for all} \quad 1 \leq s < s_c$$

*and the convergence to $Q$ in rescaled variables is ensured by*

$$\lim_{t\to T}\left\|\lambda(t)^{\frac{2}{p-1}} v\left(t, x_0 + \lambda(t)x\right)\right\|_{H^s(\lambda(t)^{-1}(\Omega - \{x_0\}))} = 0 \quad \text{for all} \quad s_c < s \leq s_+, \quad s_+ \gg 1.$$

**Theorem 5.5** ([5]). *Let $p > p_{JL}$ and $\ell \in \mathbb{N}$ with $\ell > \gamma - \frac{2}{p-1}$. Then there exists a Lipschitz manifold of codimension $\ell - 1$ ($\geq 2$) in a suitable space of regular radially symmetric initial data $(u_0, u_1) \in H^{s_+} \times H^{s_+-1}(\mathbb{R}^d)$ such that the corresponding solution to $(NLW)$ blows up in finite time $T > 0$ with*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(Q+\varepsilon)\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) = c(u_0)(1+o_{t\uparrow T}(1))(T-t)^{\frac{\ell}{\gamma-\frac{2}{p-1}}}$$

*with the boundedness of the solution below scaling:*

$$\limsup_{t\uparrow T} \parallel u(t), \partial_t u(t) \parallel_{\dot{H}^s \times \dot{H}^{s-1}} < +\infty \quad \text{for all} \quad 1 \leq s < s_c$$

*and the convergence to the stationary state in rescaled variables*

$$\lim_{t\uparrow T} \|\varepsilon(t,\cdot), \lambda(\partial_t u)_\lambda(t,\cdot)\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0 \quad \text{for all} \quad s_c < s \leq s_+, \quad s_+ \gg 1.$$

Note that there are some technical conditions that were omitted in the statement of the theorems for the sake of simplicity.

## 6. Dynamics near the radial stationary state in large dimension for the energy critical $(NLH)$

The previous section was filled with exemples of solutions concentrating the radial stationary state. However, no result existed for the energy critical case in large dimension. Recently, the author, in a joint work with F. Merle and P. Raphaël, classified the behavior of all solutions of $(NLH)$ starting close to $Q$ in an optimal topology, ruling out the existence of scale instability in a neighborhood of $Q$. Nonetheless, it is still an open problem wether or not slow blow-up can happen in the radial case for large perturbations of $Q$. Also, all the results mentioned here deal with only one nonlinear object, $Q$, but some blow-up phenomena could involve towers of radial stationary states concentrating at different speeds. This is an amazing open problem and we refer to [24] for a related result.

**Theorem 6.1** ([7]). *Take $p = p_c$, $d \geq 7$ and $\Omega = \mathbb{R}^d$. For any $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\parallel u_0 - Q \parallel_{\dot{H}^1} \ll 1$, the solution has one the of the following behavior:*

(i) Stability*: the solution is global and*

$$\exists(\lambda_\infty, z_\infty), \quad u \underset{\dot{H}^1}{\longrightarrow} \frac{1}{\lambda_\infty^{\frac{2}{p-1}}} Q\left(\frac{x-z_\infty}{\lambda_\infty}\right) \quad \text{as } t \to +\infty.$$

(ii) Dissipation*: the solution is global and*

$$u \underset{\dot{H}^1}{\longrightarrow} 0 \quad \text{as } t \to +\infty.$$

(ii) Self-similar ODE blow up*: the solution blows up with the self-similar profile* $\kappa$*:*

$$\| u(t) \|_{L^\infty} \sim \kappa (T-t)^{-\frac{1}{p-1}} \quad \text{as } t \to T.$$

Note that this theorem is optimal in view of Theorem 5.1. It is an interesting question to know if this result holds true for $(NLW)$. We now give a sketch of the proof and fix for the rest of this section $d \geq 11$, the analysis becoming more degenerate as $d$ is close to 7. The first thing to do is to study the linearized dynamics, which was already known [46] (except for the coercivity). Since the potential is radial, the results below can be shown using ODE techniques, Sturm Liouville arguments, and calculus of variation for the coercivity.

**Proposition 6.2.** *$H := -\Delta - pQ^{p-1}$ has the following spectral structure .*

(i) *One negative eigenvalue $-e_0$ associated to a strictly positive and well-localized eigenfunction $\mathcal{Y}$.*
(ii) *$Ker(H) = Span(\Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q)$ (the natural invariances).*
(iii) *Coercivity: if $\varepsilon \in Span(\mathcal{Y}, \Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q)^\perp$ then*

$$\int \varepsilon H^i \varepsilon \gtrsim \| \varepsilon \|_{\dot{H}^i}^2, \quad i = 1,2,3. \tag{6.1}$$

Therefore, there is one direction associated to a well-localized linear instability, and the orthogonal to the manifold of stationary states and to this instability, $H$ dissipates like the standard Laplacian (6.1). The second step is to decompose any solution close to $Q$ according to the above spectral structure. The following lemma is a consequence of the implicit function theorem.

**Lemma 6.3.** *Any $u \in \dot{H}^1$ satisfying*

$$\inf_{\overline{z}\in\mathbb{R}^d,\ \overline{\lambda}>0} \left\| u - \frac{1}{\overline{\lambda}^{\frac{2}{p-1}}} Q\left(\frac{x-\overline{z}}{\overline{\lambda}}\right) \right\|_{\dot{H}^1} \ll 1,$$

*can be written in a unique way:*

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + a\mathcal{Y} + \varepsilon)\left(\frac{x-z}{\lambda}\right), \quad a \in \mathbb{R},$$

$$\varepsilon \in \dot{H}^1, \quad \varepsilon \in Span(\mathcal{Y}, \Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q)^\perp. \tag{6.2}$$

We now have a suitable geometrical decomposition of any solution close to $Q$. If $u$ is a solution of (1.1), then under this decomposition it must solve the equation

$$\partial_s \varepsilon + a_s \mathcal{Y} - \frac{x_s}{\lambda}.\nabla(Q + a\mathcal{Y} + \varepsilon) - \frac{\lambda_s}{\lambda}\Lambda(Q + a\mathcal{Y} + \varepsilon) = -H\varepsilon + e_0 a\mathcal{Y} + NL.$$

Performing computations on this equation with the help of the orthogonality condition (6.2), one can quantify how each piece of the decomposition interact with the others. For the parameters such equations are called modulation equations, and for the part of the solution on the infinite dimensional subspace, we use energy methods that are adapted at the linear level (6.1).

**Lemma 6.4.** *There hold the differential bounds*

$$|a_s - e_0 a| + \left|\frac{z_s}{\lambda}\right| + \left|\frac{\lambda_s}{\lambda}\right| \lesssim |a|^2 + \| \varepsilon \|^2_{\dot H^2},$$

$$\frac{d}{ds}\left(\| \varepsilon \|^2_{\dot H^1}\right) \approx \frac{d}{ds}\left(\int \varepsilon H \varepsilon\right) \lesssim - \| \varepsilon \|^2_{\dot H^2} + O(|a|^4),$$

$$\frac{d}{ds}\left(\| \varepsilon \|^2_{\dot H^2}\right) \approx \frac{d}{ds}\left(\int \varepsilon H^2 \varepsilon\right) \lesssim - \| \varepsilon \|^2_{\dot H^3} + O(|a|^4 + \| \varepsilon \|^4_{\dot H^2}).$$

The interpretation of the above estimates is clear: the instable part evolves according to a linear unstable dynamics plus nonlinear terms, the stable part dissipates at the linear level and undergo nonlinear effects, and the scale and the central points are only affected by nonlinear effects. A striking additional estimate is given by the dissipation of the energy (2.1)

$$|E(u) - E(Q)| \lesssim \inf_{\overline{z}\in\mathbb{R}^d,\ \overline{\lambda}>0} \left\| u - \frac{1}{\overline{\lambda}^{\frac{2}{p-1}}} Q\left(\frac{x-\overline{z}}{\overline{\lambda}}\right)\right\|^2_{\dot H^1}, \quad \frac{d}{ds}\left(E(u)\right) \lesssim -a^2 - \| \varepsilon \|^2_{\dot H^2},$$

which implies the a priori averaged control of the nonlinear term appearing in the above identities:

$$\int_{s_0}^{s_1} a^2 + \| \varepsilon \|^2_{\dot H^2} \lesssim \sup_{s_0\leq s\leq s_1} \inf_{\overline{z}\in\mathbb{R}^d,\ \overline{\lambda}>0} \left\| u - \frac{1}{\overline{\lambda}^{\frac{2}{p-1}}} Q\left(\frac{x-\overline{z}}{\overline{\lambda}}\right)\right\|^2_{\dot H^1}.$$

We can now enter in the dynamical system approach. First, as there is only one direction of linear instability, we construct and characterize the instable manifold that contains all elements staying close to the manifold of stationary states as $t \to -\infty$. The existence and unicity of solutions having such a behavior as $t \to -\infty$ follows from a fixed point arguments involving the estimates of the previous lemma. Their behavior forward in time is studied using comparison principles, parabolic regularizing effects and convexity for the blow-up. We also use the fact that any positive blow-up solution is of type I taken from [31].

**Theorem 6.5.** *There exist two strictly positive radial solutions $Q^+$ and $Q^-$ defined on $(-\infty, t_0] \times \mathbb{R}^d$ such that:*

$$Q^{\pm} = Q \pm e^{e_0 t}\mathcal{Y} + O(e^{2e_0 t}) \quad on\ (-\infty, t_0].$$

*$Q^+$ blows up with self-similar blow-up with profile $\kappa$ forward in time. $Q^-$ is global and dissipates toward $0$. Moreover if $u$ is a solution of (1.1) on $(-\infty, 0]$ such that:*

$$sup_{t\leq 0}\ inf_{\lambda>0,\ z\in\mathbb{R}^d} \| u(t) - Q_{z,\lambda} \|_{\dot H^1} \ll 1$$

*then $u = Q^{\pm}$ or $u = Q$ up to the symmetries of the flow.*

The behaviors associated to $Q^+$ and $Q^-$ are moreover stable. The stability of dissipation is rather easy to show but the stability of the self-similar blow-up with profile $\kappa$ is more involved and adapts to the energy critical setting an argument from [14]. To end the proof of Theorem 6.1, we now show that for any solution starting close to $Q$, either the linear instability dominates and make the solution exit a universal neighborhood of $Q$ close to $Q^+$ or $Q^-$, either it never takes control, meaning that the solution is located on the stable infinite dimensional subspace (6.2) and undergoes dissipation toward $Q$.

**Lemma 6.6.** *If for all times* $t \in [0, T)$,

$$|a(t)| \lesssim \| \varepsilon(t) \|_{\dot{H}^2}^2$$

*then* $u$ *is in a dissipative regime, is global* $T = +\infty$ *and converges toward a renormalized stationary state* $\frac{1}{\lambda_\infty^{\frac{2}{p-1}}} Q\left(\frac{x-z_\infty}{\lambda}\right)$. *If for some time* $T_{ins}$,

$$|a(T_{ins})| \approx \| \varepsilon(T_{ins}) \|_{\dot{H}^2}^2$$

*then* $u$ *enters an instable regime, and there exists* $T_{exit} > T_{ins}$ *such that:*

$$\text{either} \quad \| u(T_{exit}) - Q^+ \|_{\dot{H}^1} \ll 1 \quad \text{or} \ \| u(T_{exit}) - Q^- \|_{\dot{H}^1} \ll 1.$$

If a solution enters the instable regime, it will then have the same behavior as $Q^+$ or $Q^-$ since they are stable for the $\dot{H}^1$ topology. The proof of the above Lemma is more technical and we refer to [7] for more details.

Let us end this document by mentioning some fundamental works related to the above result. First, a similar case where strong scale instability cannot happen was studied in [20]. There exist some other classification results, [33, 34, 35, 43], for the generalized KdV equation and for the nonlinear klein gordon equations, but the scenarios are less precise. Finally, the study of the instable manifold was done in [13] for the wave equation, relying on the study of minimal elements started in [36] .

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