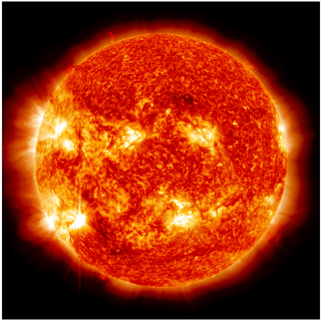
The fate of alpha dynamos at large *Rm*

Alexandre Cameron & Alexandros Alexakis

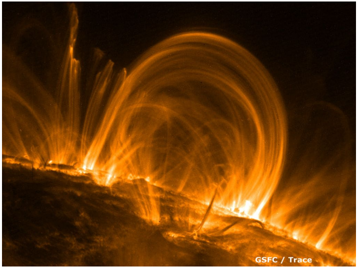
LPS ENS

2nd February 2017

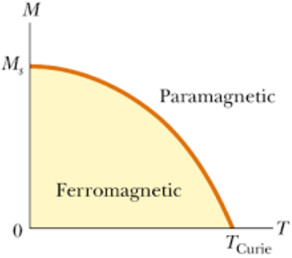


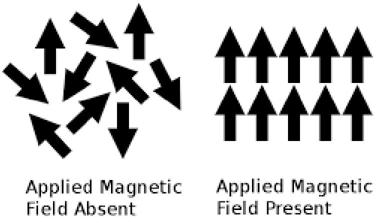
Magnetic activity

Observation

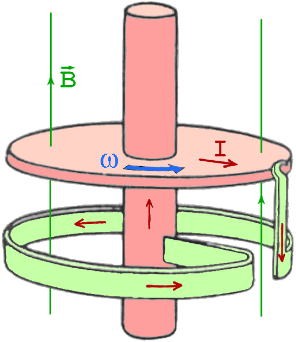


Curie temperature

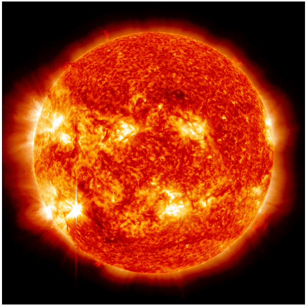


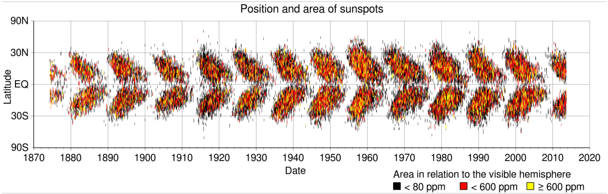


Dynamo mechanism



Where do large magnetic scales come from?





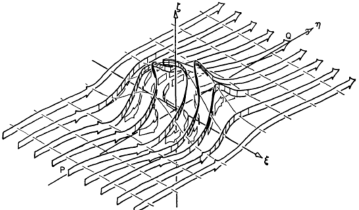
Eugene Parker's 1955 theory

HYDROMAGNETIC DYNAMO MODELS

EUGENE N. PARKER
Department of Physics, University of Utah, Salt Lake City, Utah
Received October 18, 1954; revised May 11, 1955

ABSTRACT

The purpose of this paper is to investigate the steady-state amplification of magnetic fields in a fluid It is shown that a rotating sphere of conducting fluid can regenerate a dipole magnetic field. It is sufficient for the angular velocity of rotation to vary with distance from the axis of rotation and for cyclonic fluid motions to be present The nonuniform rotation generates a toroidal field from the dipole field; the cyclones generate, from the toroidal field, loops of flux in the meridional plane which coalesce to amplify the dipole field The rotating sphere is discussed in relation to the liquid core of the earth and the geomagnetic dipole field. If, instead of a rotating sphere, one has a prismatic volume of fluid, it is possible to construct migratory dynamo waves The dynamo waves are discussed in relation to the solar convective zone; it is shown that such waves can account for many of the principal features of the observed solar magnetic activity



Eugene Parker's 1957 theory

THE SOLAR HYDROMAGNETIC DYNAMO

By E. N. Parker

ENRICO FERMI INSTITUTE FOR NUCLEAR STUDIES AND DEPARTMENT OF PHYSICS,
UNIVERSITY OF CHICAGO

The Babcock magnetograms (Babcock and Babcock, 1955) indicate that the sun possesses a general dipole magnetic field of the order of 1 gauss. The dipole field is observed within about 25° of the poles and, presumably, is responsible for the polar coronal streamers. The field is not observed nearer the equator, and what we know of large-scale solar magnetic fields at lower latitudes is only by inference: it is generally assumed that the bipolar character of the sunspots and of the more diffuse magnetic regions observed by the Babcocks implies the existence of bands of toroidal magnetic field circling the sun beneath the photosphere. Each band is about 30° wide in latitude. Successive bands, and corresponding bands north and south of the equator, have opposite sign, as shown in Figure 4. The bands first make their presence known at about latitude ±40° through the appearance of bipolar magnetic regions at the photosphere. In about 11 years they migrate from latitude ±40° to the equator, where they apparently disappear.

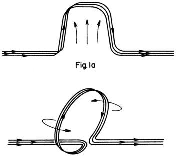


Fig. Ia

Fig. Ib

Fig. 1.—*a*, Schematic diagram of the upward bulging of the lines of force of the toroidal field by a rising convective cell; *b*, twisting of the bulge to form a magnetic loop with nonvanishing projection on the meridional plane.

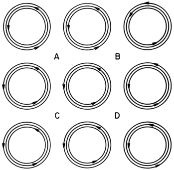


Fig. 2a

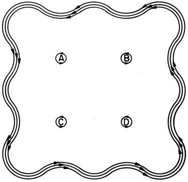
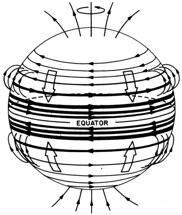


Fig. 2b



M. Steenbeck, F. Krause, K.-H. Rädler (1966)

Berechnung der mittleren Lorentz-Feldstärke $\mathbf{v} \times \mathbf{B}$ für ein elektrisch leitendes Medium in turbulenter, durch Coriolis-Kräfte beeinflusster Bewegung

BAND 21 a ZEITSCHRIFT FÜR NATURFORSCHUNG HEFT 4

Berechnung der mittleren Lorentz-Feldstärke $\overline{\mathfrak{v} \times \mathfrak{B}}$ für ein elektrisch leitendes Medium in turbulenter, durch Coriolis-Kräfte beeinflußter Bewegung

M. Steenbeck, F. Krause und K.-H. Rädler

Institut für Magnetohydrodynamik Jena der Deutschen Akademie der Wissenschaften zu Berlin

(Z. Naturforschg. **21 a**, 369—376 [1966]; eingegangen am 11. November 1965)

A turbulent, electrically conducting fluid containing a magnetic field with non-vanishing mean-value is investigated. The magnetic field strength and the conductivity may be so small that the turbulence is not influenced by the action of the Lorentz force.

The average of the crossproduct of velocity and magnetic field is calculated in a second approximation. It contains the averages of the products of two components of the velocity field, i. e. the components of the correlation tensor.

Here the structure of the correlation tensor is determined for a medium with gradients of density and/or turbulence intensity, furthermore the turbulent motion is influenced by Coriolis forces.

As the main result is shown that in those turbulent velocity fields the average crossproduct of velocity and magnetic field generally has a non-vanishing component *parallel* to the average magnetic field.

Such a turbulence may be present in rotating stars. Consequences concerning the selfexcited build up of steller magnetic fields are discussed in a following paper.

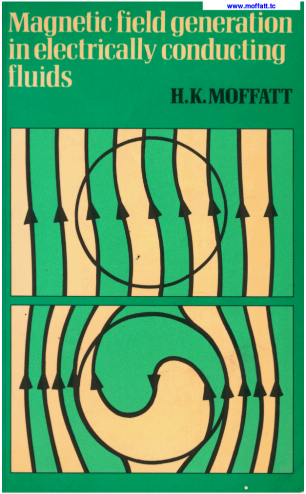
Die Berechnung des Mittelwertes von $\mathfrak{H}$ führen wir in der folgenden Weise durch: Wir setzen

$$\mathfrak{H} = \overline{\mathfrak{H}} + \mathfrak{H}' \qquad (3)$$

und berechnen $\mathfrak{H}'$ im Sinne einer Störungsrechnung für kleine $|\mathfrak{v}|$ aus der Gleichung

$$\frac{\partial \mathfrak{H}'}{\partial t} - \frac{1}{\mu\,\sigma} \Delta \mathfrak{H}' = \operatorname{rot}(\mathfrak{v} \times \overline{\mathfrak{H}}) \,. \qquad (4)$$

Moffatt's Book (1978)



CHAPTER 7

THE MEAN ELECTROMOTIVE FORCE GENERATED BY RANDOM MOTIONS

7.1. Turbulence and random waves

We have so far treated the velocity field $\mathbf{U}(\mathbf{x}, t)$ as a known function of position and time[1]. In this chapter we consider the situation of greater relevance in both solar and terrestrial contexts when $\mathbf{U}(\mathbf{x}, t)$ includes a random ingredient whose statistical (i.e. average) properties are assumed known, but whose detailed (unaveraged) properties are too complicated for either analytical description or observational determination. Such a velocity field generates random perturbations of electric current and magnetic field, and our aim is to determine the evolution of the statistical properties of the magnetic field (and in particular of its local mean value) in terms of the ('given') statistical properties of the **U**-field.

The random velocity field may be a turbulent velocity field as normally understood, or it may consist of a random superposition of interacting wave motions. The distinction can be most easily appreciated for the case of a thermally stratified fluid. If the stratification is unstable (i.e. if the fluid is strongly heated from below) then thermal turbulence will ensue, the net upward transport of heat being then predominantly due to turbulent convection. If the stratification is stable (i.e. if the temperature either increases with height, or decreases at a rate less than the adiabatic rate) then turbulence will not occur, but the medium may support internal gravity waves which will be present to a greater or lesser extent, in proportion to any random influences that may be present, distributed either throughout the fluid or on its boundaries. For example, if the outer core of the Earth is stably stratified (as maintained by Higgins & Kennedy, 1971 – see § 4.4), random inertial waves may be excited either by sedimentation of iron-rich material released from the mantle across the core–mantle interface or by flotation of

[1] From now on, we shall use **U** to represent the total velocity field, reserving **u** for its random ingredient.

Krause & Rädler (1980)

MEAN-FIELD
MAGNETOHYDRODYNAMICS
AND DYNAMO THEORY

by

F. KRAUSE and K.-H. RÄDLER

Zentralinstitut für Astrophysik der
Akademie der Wissenschaften der DDR

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CHAPTER 2

BASIC IDEAS OF MEAN-FIELD ELECTRODYNAMICS

2.1. Basic equations

Our considerations are based on non-relativistic magnetohydrodynamics. Let $\boldsymbol{B}$ be the magnetic flux density, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{E}$ the electric field, $\boldsymbol{j}$ the electric current density, μ the permeability, σ the electrical conductivity, and $\boldsymbol{u}$ the fluid velocity; μ and σ shall be supposed to be constant.

We shall adopt the magnetohydrodynamic approximation, i.e. suppose that these fields are related by the equations

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \mathrm{curl}\,\boldsymbol{H} = \boldsymbol{j}, \quad \mathrm{div}\,\boldsymbol{B} = 0, \tag{2.1}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}). \tag{2.2}$$

These equations determine $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{j}$ from an assigned $\boldsymbol{u}$. They show that

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{curl}\,(\boldsymbol{u}\times\boldsymbol{B}) + \frac{1}{\mu\sigma}\Delta\boldsymbol{B}, \quad \mathrm{div}\,\boldsymbol{B} = 0. \tag{2.3}$$

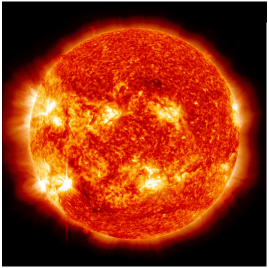
Once $\boldsymbol{B}$ has been determined from this equation, $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{j}$ may be obtained from (2.1) and (2.2).

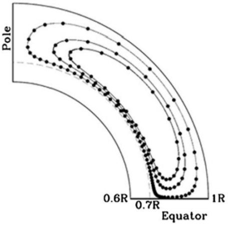
Here, and in most of the following chapters, we shall consider $\boldsymbol{u}$ to be given. In this way we shall avoid the problems raised by the reaction of the magnetic field on the motion, but occasionally discuss this effect qualitatively. Finally, in chapter 10, we shall give a survey of the results obtained in mean-field magnetohydrodynamics up to now.

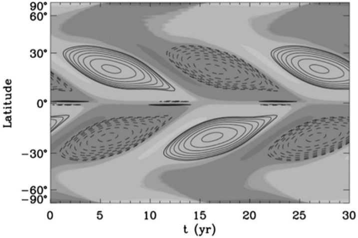
2.2. Averaging operations

In a turbulent medium, all the fields so far introduced vary irregularly in space and time. Let F be such a fluctuating field, considered as a random

Alpha modeling is at the heart of todays Solar models!







$$\frac{\partial B}{\partial t} = -\frac{1}{r}\left[\frac{\partial}{\partial r}(r u_r B) + \frac{\partial}{\partial \theta}(u_\theta B)\right] + r\sin\theta(\mathbf{B}_p.\nabla)\Omega \qquad - \hat{\mathbf{e}}_\phi.[\nabla\eta \times \nabla \times B\hat{\mathbf{e}}_\phi] + \eta\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)B,$$

$$\frac{\partial A}{\partial t} = -\frac{1}{r\sin\theta}(\mathbf{u}.\nabla)(r\sin\theta A) + \eta\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)A \quad + \frac{S(r,\theta)B}{1+(B/B_0)^2} + \frac{\alpha B}{1+(B/B_0)^2}$$

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M. Dikpati and P. A. Gilman, Astrophys. J. 649, 498 (2006).
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Formulation

$$\partial_t \mathbf{B} = \nabla \times \mathbf{u} \times \mathbf{B} + Rm^{-1}\nabla^2\mathbf{B},$$

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- $\mathbf{B}$ Magnetic field evolving in a domain of size L.

$$\ell \ll L$$

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$$\text{Averaging procedure for f}(\mathbf{x}) : \qquad \langle f \rangle = \frac{1}{V_\ell} \int_{V_\ell} f(\mathbf{X} + \mathbf{x}) d\mathbf{x}^3$$

where V_ℓ a volume centered at $\mathbf{X}$ with $\ell^3 \ll V_\ell \ll L^3$

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where V_ℓ a volume centered at $\mathbf{X}$ with $\ell^3 \ll V_\ell \ll L^3$
Let $\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}$ then

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \langle \mathbf{u} \times \mathbf{b} \rangle + Rm^{-1}\nabla^2 \langle \mathbf{B} \rangle,$$

and

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle) + \nabla \times (\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle) + Rm^{-1}\nabla^2 \mathbf{b}$$

Formulation

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \mathcal{E} + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle,$$
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where

$$\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle \qquad \text{and} \qquad \mathcal{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle$$

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Gradient expansion ...

$$\mathcal{E}^i = \alpha^{ij} \langle \mathbf{B} \rangle^j + \beta^{ijk} \nabla^j \langle \mathbf{B} \rangle^k + \dots$$

then for isotropic flows

$$\partial_t \langle \mathbf{B} \rangle = \alpha \nabla \times \langle \mathbf{B} \rangle + (Rm^{-1} + \beta) \nabla^2 \langle \mathbf{B} \rangle + \dots,$$

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$$\partial_t \langle \mathbf{B} \rangle = \alpha \nabla \times \langle \mathbf{B} \rangle + (Rm^{-1} + \beta) \nabla^2 \langle \mathbf{B} \rangle + \dots,$$

Always unstable for $\alpha \neq 0$ & $L \gg \ell$ with growth-rate γ

$$\gamma = \alpha k - (Rm^{-1} + \beta) k^2 + \dots$$

Small Rm limit: $Rm = u\ell/\eta \ll 1$ and $\ell/L = \mathcal{O}(Rm^2)$

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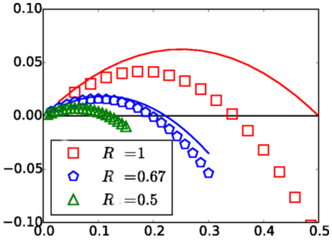
$$\mathbf{b} = Rm \nabla^{-2} \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle)$$

$$\mathcal{E} = \alpha \langle \mathbf{B} \rangle = \langle \mathbf{u} \times \mathbf{b} \rangle$$



$$\gamma \simeq \alpha k - Rm^{-1} k^2$$

Multi-scale expansion (Childress 1969)

Expanding $\mathbf{B} = \mathbf{B_0} + \mathbf{R}\,\mathbf{B_1}\ldots$

$$\partial_t = R^{-1}\partial_\tau + R^3\partial_T, \quad \nabla = \nabla_{\boldsymbol{x}} + R^2\nabla_{\boldsymbol{X}}, \quad \boldsymbol{B} = \boldsymbol{B}_0 + R\boldsymbol{B}_1 + \cdots$$

From the induction equation at the first few orders we have

$$\begin{aligned}
(\partial_\tau - \nabla^2_{\boldsymbol{x}})\boldsymbol{B}_0 &= 0,\\
(\partial_\tau - \nabla^2_{\boldsymbol{x}})\boldsymbol{B}_1 &= \nabla_{\boldsymbol{x}} \times (\boldsymbol{u} \times \boldsymbol{B}_0),\\
(\partial_\tau - \nabla^2_{\boldsymbol{x}})\boldsymbol{B}_2 &= \nabla_{\boldsymbol{x}} \times (\boldsymbol{u} \times \boldsymbol{B}_1) + 2\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{X}}\boldsymbol{B}_0,\\
(\partial_\tau - \nabla^2_{\boldsymbol{x}})\boldsymbol{B}_3 &= \nabla_{\boldsymbol{x}} \times (\boldsymbol{u} \times \boldsymbol{B}_2) + \nabla_{\boldsymbol{X}} \times (\boldsymbol{u} \times \boldsymbol{B}_0) + 2\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{X}}\boldsymbol{B}_1,\\
(\partial_\tau - \nabla^2_{\boldsymbol{x}})\boldsymbol{B}_4 &= \nabla_{\boldsymbol{x}} \times (\boldsymbol{u} \times \boldsymbol{B}_3) + \nabla_{\boldsymbol{X}} \times (\boldsymbol{u} \times \boldsymbol{B}_1) + 2\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{X}}\boldsymbol{B}_2\\
&\qquad - (\partial_T - \nabla^2_{\boldsymbol{X}})\boldsymbol{B}_0,
\end{aligned}$$

and from $\nabla \cdot \boldsymbol{B} = 0$,

$$\begin{aligned}
\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}_0 &= 0,\\
\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}_1 &= 0,\\
\nabla_{\boldsymbol{X}} \cdot \boldsymbol{B}_0 + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}_2 &= 0.
\end{aligned}$$

Small τ approximation: correlation time $\tau \ll \ell/\boldsymbol{u}$

Small τ approximation: correlation time $\tau \ll \ell/u$

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \mathcal{E} + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle,$$

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle) + \cancel{\nabla \times \mathcal{G}} + \cancel{Rm^{-1} \nabla^2 \mathbf{b}}$$

Small τ approximation: correlation time $\tau \ll \ell/u$

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \mathcal{E} + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle,$$

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle) + \cancel{\nabla \times \mathcal{G}} + \cancel{Rm^{-1} \nabla^2} \mathbf{b}$$

$$\mathbf{b} \simeq \tau \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle)$$

$$\mathcal{E} = \alpha \langle \mathbf{B} \rangle = \langle \mathbf{u} \times \mathbf{b} \rangle$$

Small τ approximation: correlation time $\tau \ll \ell/u$

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \mathcal{E} + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle,$$
$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle) + \cancel{\nabla \times \mathcal{G}} + \cancel{Rm^{-1} \nabla^2 \mathbf{b}}$$

$$\mathbf{b} \simeq \tau \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle)$$
$$\mathcal{E} = \alpha \langle \mathbf{B} \rangle = \langle \mathbf{u} \times \mathbf{b} \rangle$$

Assuming isotropy

$$\alpha = -\frac{\tau}{3} \langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle$$

α is proportional to the helicity of the flow!

The stances in the community

Pro-Alpha groups

Measuring α by expansions and Numerical simulations

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Some quotes

"… We argue that the interpretation of these quantities (α, β) in terms of the evolution of the large-scale field may be **fundamentally flawed**."
F. Cattaneo and D. W. Hughes, Mon. Not. R. Astron. Soc. 395, L51 (2009).

"It is stressed that the connection of the mean electromotive force with the mean magnetic field and its first spatial derivatives is in general neither local nor instantaneous and that **quite a few claims concerning pretended failures of the mean-field concept result from ignoring this aspect**."
K.-H. Rädler Astron. Nachr. 335, 459 (2014)

When can we neglect $\mathcal{G}$?

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$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \mathcal{E} + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle,$$
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where

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If $\mathcal{G} \neq 0$ the linear relation $\mathcal{E} = \alpha \langle \mathbf{B} \rangle + \dots$ does not hold!

A simple example

$$q \ll k$$

$$\begin{aligned} \dot{b}_q &= -\eta q^2 \quad b_q \quad +\alpha q \quad b_k \, , \\ \dot{b}_k &= \alpha k \quad b_q \quad +\gamma_{SSD} \quad b_k \end{aligned}$$

where $\gamma_{SSD} = u_k k - \eta k^2$ is the growth rate of the small scale dynamo obtained by setting $\alpha = 0$.

$$\gamma = \frac{1}{2} \left[\gamma_{SSD} - \eta q^2 \pm \sqrt{\gamma_{SSD}^2 + 4\alpha^2 kq + 2\gamma_{SSD}\eta q^2 + \eta^2 q^4} \right]$$

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$$\gamma_{SSD} < 0$$

$$\gamma \simeq \alpha^2 kq/|\gamma_{SSD}| = \mathcal{O}(q)$$

$$b_q/b_k \simeq (|\gamma_{SSD}|/\alpha k) = \mathcal{O}(1)$$

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$$\gamma \simeq \gamma_{SSD} = \mathcal{O}(1)$$

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The Floquet (Bloch) Formalism for periodic flows

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Let $\mathbf{u}(\mathbf{x}, t)$ be a spatially periodic flow of a given spatial period $\ell = 2\pi/k$.

The Floquet theory states that the magnetic field can be decomposed as $\mathbf{B}(\mathbf{x}, t) = e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{\mathbf{b}}(\mathbf{x}, t) + c.c.$ where $\tilde{\mathbf{b}}(\mathbf{x}, t)$ is a complex vector field that satisfies the same spatial periodicity as the velocity field $\mathbf{u}$, and $\mathbf{q}$ is an arbitrary wave number.

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$$\boxed{\partial_t \tilde{\mathbf{b}} = i\mathbf{q} \times (\mathbf{u} \times \tilde{\mathbf{b}}) + \nabla \times (\mathbf{u} \times \tilde{\mathbf{b}}) + \eta(\nabla + i\mathbf{q})^2 \tilde{\mathbf{b}}}$$

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Fields with $q = 0$ and $\langle \tilde{\mathbf{b}} \rangle = 0$ correspond to purely small scale fields. For $0 < q/k < 1$ the dynamo mode has in general a finite projection to the large scales measured by $\langle \tilde{\mathbf{b}} \rangle$.

A twofold gain

- it provides with a clear way to disentangle dynamos that involve only small scales (for which $\mathbf{q}/k \in \mathbb{Z}^3$) from dynamos that involve large scales ($0 < q/k \ll 1$)

- it allows to investigate numerically arbitrary large scale separations $q \ll k$ with no additional numerical cost.

The examined Flows

$$\mathbf{u} = U \begin{bmatrix} \sin(ky+\phi_2) & + & \cos(kz+\psi_3), \\ \sin(kz+\phi_3) & + & \cos(kx+\psi_1), \\ \sin(kx+\phi_1) & + & \cos(ky+\psi_2) \end{bmatrix}.$$

- (A) $\phi_i = \psi_i = 0$ ie the helical ABC flow
- (B) $\phi_i = \psi_i - \pi/2 = 0$ a non-helical flow

$Rm = U/k\eta$

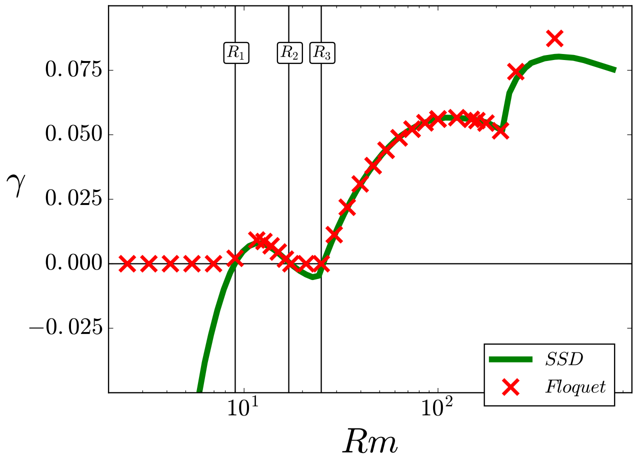
and γ is measured in units of Uk

- (C) $\phi_i = \psi_i$ change randomly every time τ

$Rm = (U/k\eta) \times (\tau Uk) = U^2\tau/\eta$

and γ is measured in units of $U^2k^2\tau$

Results: (A) ABC flow, $q = 10^{-3}$



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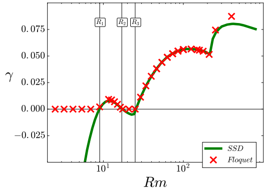
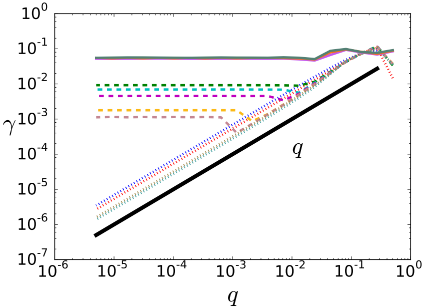
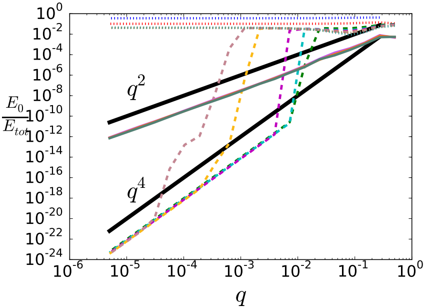


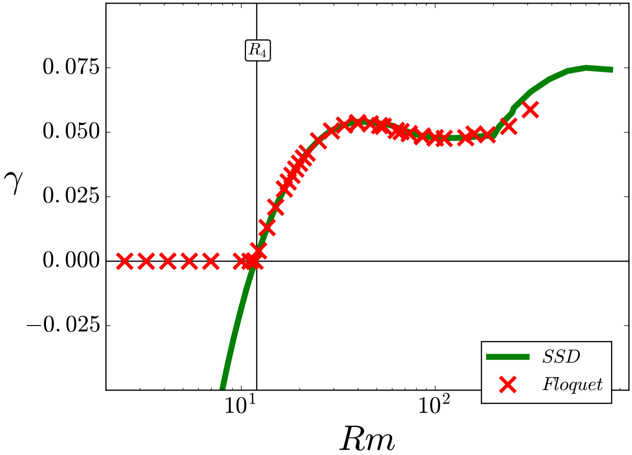
γ

$E_0/E = \langle\tilde{\mathbf{b}}\rangle^2/\langle\tilde{\mathbf{b}}^2\rangle$





Results: (B) Non-helical flow, $q = 10^{-3}$



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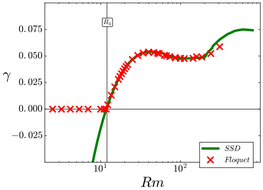
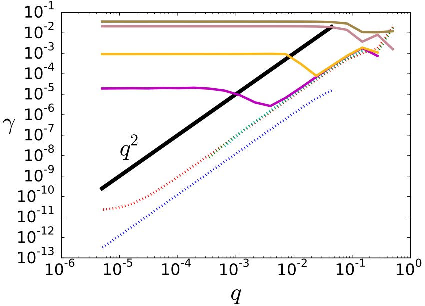


γ



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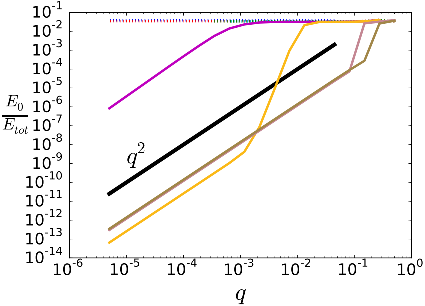
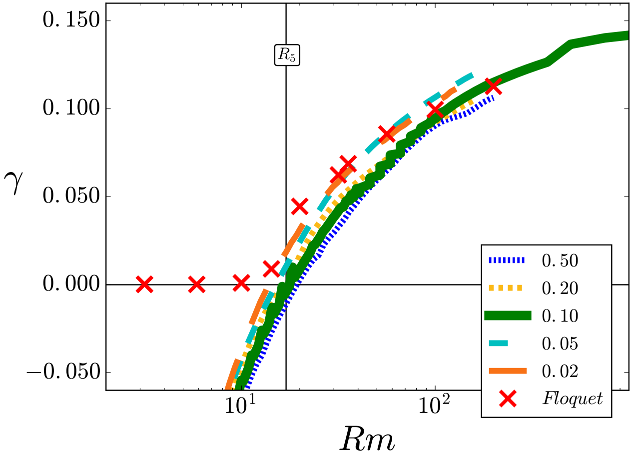
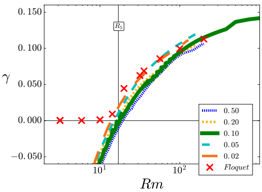
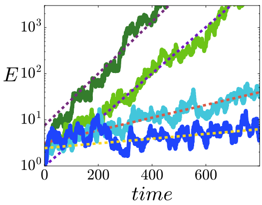


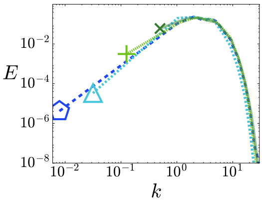
Results: (R) Random helical flow, $q = 10^{-3}$



Results: (C) Random helical flow







γ

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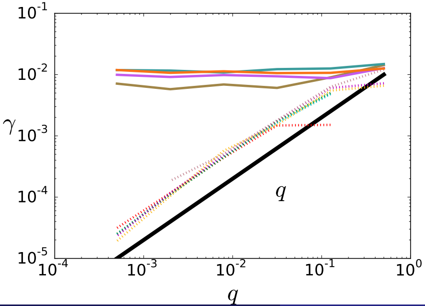

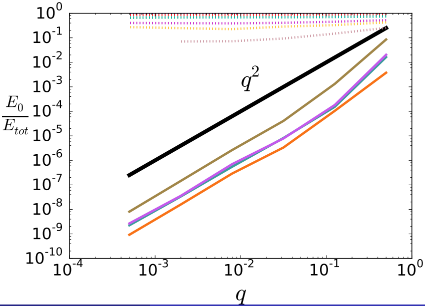

Large scale projection

$$q \to \epsilon q \ll 1$$

$$\gamma \tilde{\mathbf{b}} = i\epsilon \mathbf{q} \times (\mathbf{u} \times \tilde{\mathbf{b}}) + \nabla \times (\mathbf{u} \times \tilde{\mathbf{b}}) + \eta(\nabla + i\epsilon \mathbf{q})^2 \tilde{\mathbf{b}}$$

Let $\gamma = \gamma_0 + \epsilon\gamma_1 + \dots$ and $\tilde{\mathbf{b}} = \tilde{\mathbf{b}}_0 + \epsilon\tilde{\mathbf{b}}_1 \dots$

- 0th order

$$\gamma_0 = \gamma_{SSD} \quad \text{and} \quad \langle \tilde{\mathbf{b}}_0 \rangle = 0$$

- 1st order

$$\gamma_0 \langle \tilde{\mathbf{b}}_1 \rangle = i\mathbf{q} \times \langle \mathbf{u} \times \tilde{\mathbf{b}}_0 \rangle \qquad \textit{and} \qquad \gamma_1 = i\langle \tilde{\mathbf{b}}_0^{\dagger} \cdot (\mathbf{q} \times \mathbf{u} \times \tilde{\mathbf{b}}_0) \rangle$$

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$$E_0/E = \langle \tilde{\mathbf{b}} \rangle^2 / \langle \tilde{\mathbf{b}}^2 \rangle \propto q^2 \qquad \textit{if} \qquad \langle \mathbf{u} \times \tilde{\mathbf{b}}_0 \rangle \neq 0$$

The Large scale behavior

Back to the gradient expansion ...

$$\mathcal{E}^i = \mathcal{E}_0^i + \alpha^{ij}\langle \mathbf{B}\rangle^j + \beta^{ijk}\nabla^j\langle \mathbf{B}\rangle^k + \dots$$

then for isotropic flows

$$\partial_t\langle \mathbf{B}\rangle = \mathcal{E}_0 + \alpha\nabla\times\langle \mathbf{B}\rangle + (Rm^{-1}+\beta)\nabla^2\langle \mathbf{B}\rangle + \dots,$$

where

$$\mathcal{E}_0 \propto e^{(\gamma_{SSD}\, t)}$$

and could be modeled as noise for a turbulent flow.

The end

Thank you for your attention