Large scale instability of helical flows

A. Cameron ; A. Alexakis ; M.-É. Brachet

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Département de Physique, ENS Paris

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Incompressible Navier-Stokes

Equations

Incompressibility condition: $\nabla \cdot \mathbf{v} = 0\,.$

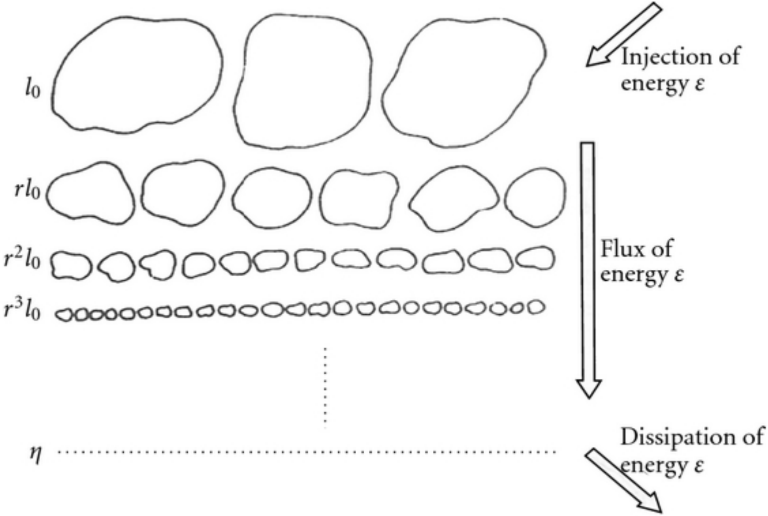
Equation of evolution, Navier-Stokes:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + Re^{-1}\Delta\mathbf{v} + \mathbf{F} \quad , \quad Re = \frac{VL}{\nu}\,. \tag{1}$$

Variables

- $\mathbf{v}$ is the dimensionless velocity flow.
- $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is the transport term.
- ∇P is the pressure term.
- $Re^{-1}\Delta\mathbf{v}$ is the viscous term.
- $\mathbf{F}$ is the dimensionless forcing.
- Re is the Reynolds number.

Richardson Cascade



Energy spectrum

Local Reynolds number and energy

$$Re(k) = \frac{V}{\nu}\ell = \frac{V}{\nu}k^{-1} \quad , \quad E(k) = \sum_{|\mathbf{k}|=k} |\mathbf{v}(\mathbf{k})|^2 \,. \tag{2}$$

Viscous regime: $k \gg \frac{\nu}{V}$

Viscous limit: $Re(k) \ll 0$. Energy distribution

$$E(k) \propto \, exp(-\delta k)\,. \tag{3}$$

Kolmogorov regime: $k \ll \frac{\nu}{V}$

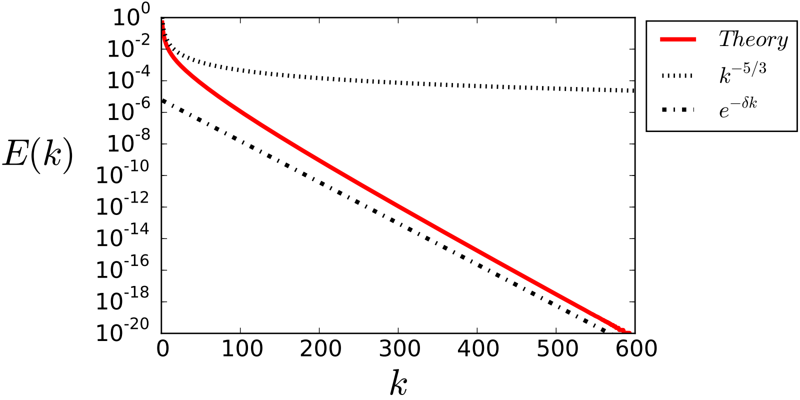
Kolmogorov limit: $Re(k) \gg 1$. Energy distribution

$$E(k) \propto \, k^{-5/3} \tag{4}$$

Viscous energy spectrum

Energy spectrum

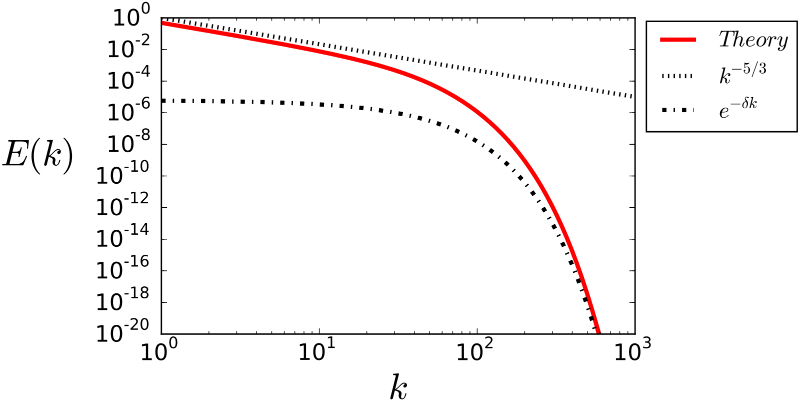
$$E(k) \propto \ exp(-\delta k) \, .$$



Kolmogorv energy spectrum

Energy spectrum

$$E(k) \propto k^{-5/3}.$$



Taylor-Green flow

Velocity decomposition

$$v_x(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} u_x(m,n,p,t) \times \sin mx \cos ny \cos pz\,, \tag{5}$$

$$v_y(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} u_y(m,n,p,t) \times \cos mx \sin ny \cos pz\,, \tag{6}$$

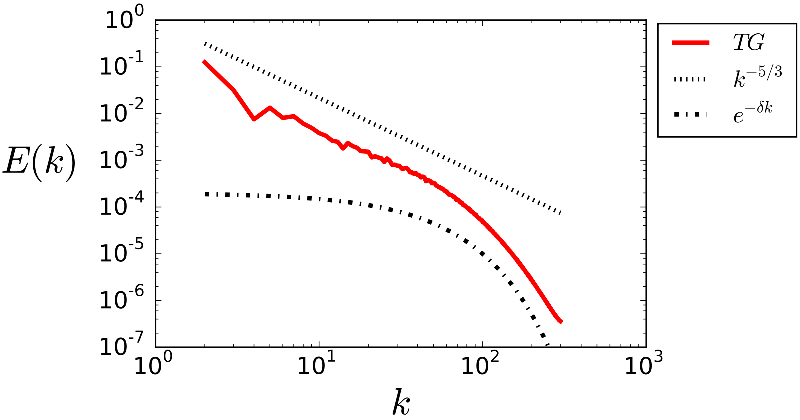
$$v_z(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} u_z(m,n,p,t) \times \cos mx \cos ny \sin pz\,. \tag{7}$$

Energy injection

Mode (1,1,1) set at

$$u_x(1,1,1,t) = -u_y(1,1,1,t) = u_0 \quad \text{and} \quad u_z(1,1,1,t) = 0\,. \tag{8}$$

Taylor-Green at $Re = 1020$



Truncated Euler equation

Equations

Incompressibility condition: $\nabla \cdot \mathbf{v} = 0\,.$

Equation of evolution, Euler equation:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P\,. \tag{9}$$

Truncation of the Fourier series: $\mathbf{v}(|\mathbf{k}| > K_M) = 0\,.$

Variables

- $\mathbf{v}$ is the dimensionless velocity flow.
- $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is the transport term.
- ∇P is the pressure term.
- K_M truncation wave-number.
- No forcing no viscous term.

Equi-partition of energy

Conservation of energy

Euler equation conserves energy: $E_{tot} = \frac{1}{2}\int \mathbf{v}(\mathbf{x})^2 d^3\mathbf{x}$.

$$\frac{1}{2}\partial_t \mathbf{v}^2 = \mathbf{v}\partial_t \mathbf{v} = \mathbf{v}\cdot\left[\mathbf{v}\times\nabla\times\mathbf{v} - \nabla\left(P + \frac{1}{2}\mathbf{v}^2\right)\right]. \quad (10)$$

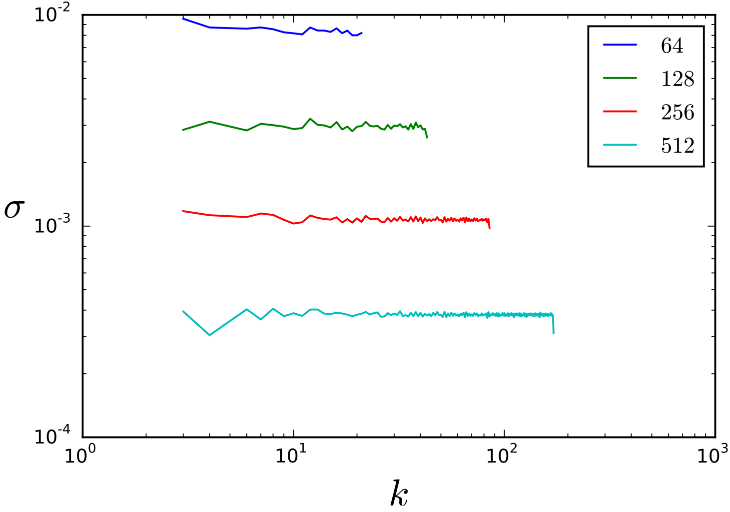
Equivalence with thermodynamics

Energy in Fourier space: $E_{tot} = \frac{1}{2}\sum_{\mathbf{k}} |\mathbf{v}(\mathbf{k})|^2$.
Equi-partition: $\sigma^2 = \langle|\mathbf{v}(\mathbf{k})|^2\rangle - |\langle\mathbf{v}(\mathbf{k})\rangle|^2$.

Energy spectrum

$$E(k) = \sum_{|\mathbf{k}|=k} |\mathbf{v}(\mathbf{k})|^2 \propto 4\pi k^2 \sigma^2 .$$

Taylor-Green standard deviation



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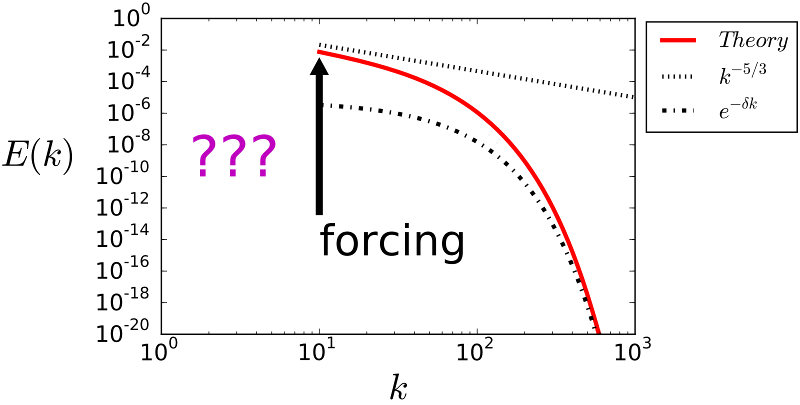
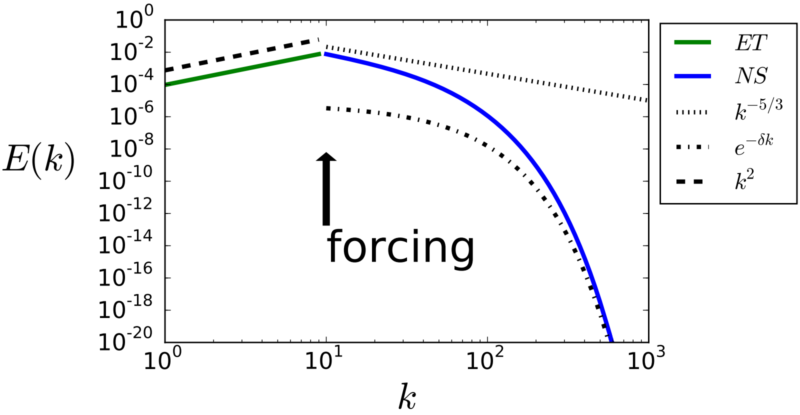
Small scale forcing

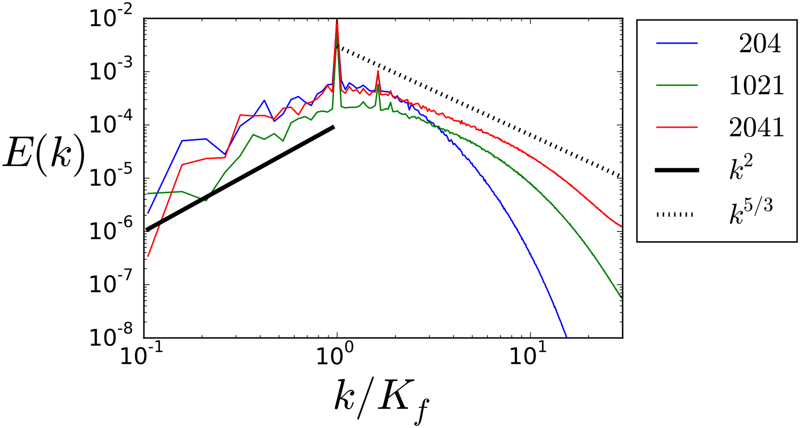


Eddy noise inject energy at large scale modes.

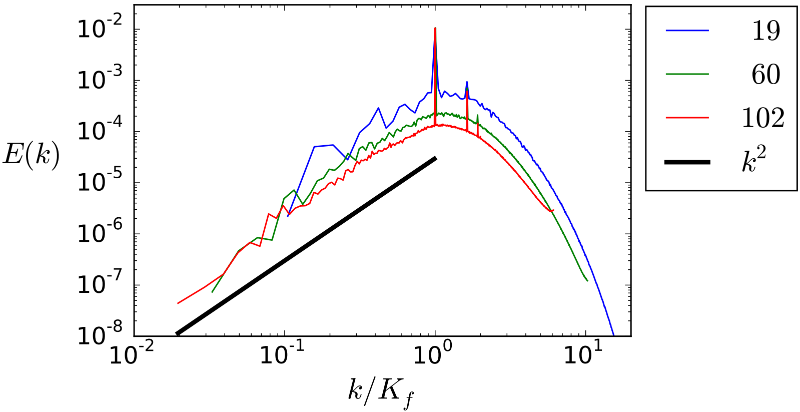
Prediction


$E(k)$
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{0}
10^{1}
10^{2}
10^{3}
k
forcing
ET
NS
$k^{-5/3}$
$e^{-\delta k}$
k^2

Taylor-Green flow, $K_f = 19$



Taylor-Green flow, $Re = 204$



Another invariant helicity

Conservation of helicity

Euler equation conserves helicity: $H_{tot} = \int \mathbf{v} \cdot \nabla \times \mathbf{v}(\mathbf{x})\, d^3\mathbf{x}$.

$$\partial_t \nabla \times \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \nabla \times (\mathbf{v} \times \nabla \times \mathbf{v}) + \nabla \times \mathbf{v} \cdot \left[\mathbf{v} \times \nabla \times \mathbf{v} - \nabla \left(P + \frac{1}{2} \mathbf{v}^2 \right) \right] . \quad (11)$$

Kraichnan absolute equilibrium

The invariant is: $\alpha \sum_{\mathbf{k}} |\mathbf{v}(\mathbf{k})|^2 + \beta \sum_{\mathbf{k}} \nabla \times \mathbf{v} \cdot \mathbf{v}(\mathbf{k})$.

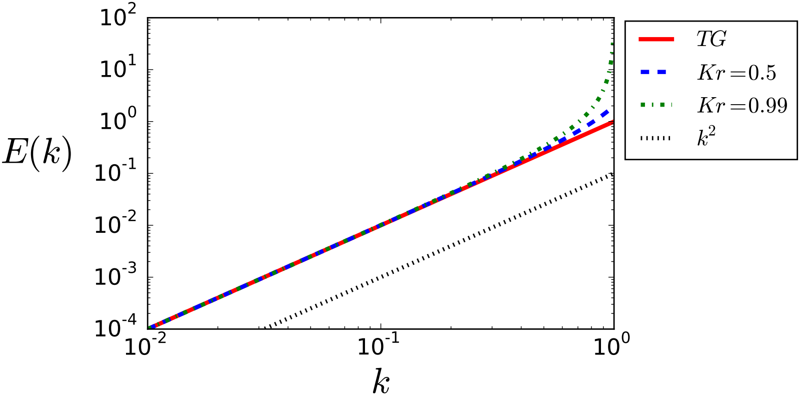
The energy spectrum follows:

$$E(k) \propto \frac{k^2}{\alpha^2 - \beta^2 k^2} \quad \text{and} \quad Kr = \frac{\beta K_M}{\alpha} . \quad (12)$$

Energy spectrum power law

Energy spectrum

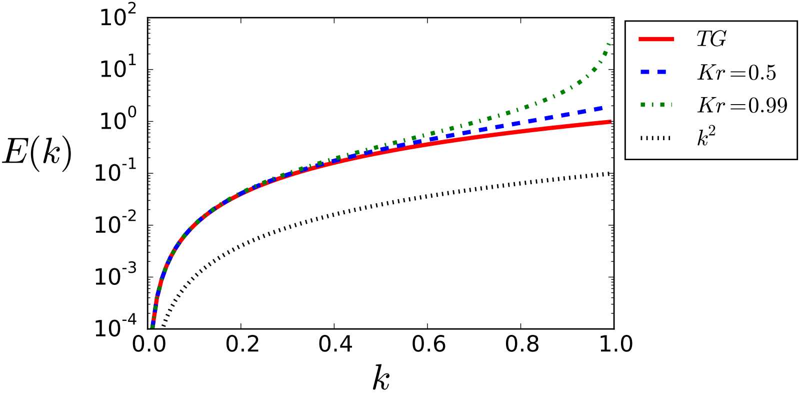
$$E(k) \propto \frac{k^2}{\alpha^2 - \beta^2 k^2} \quad \text{and} \quad Kr = \frac{\beta K_M}{\alpha}$$



Energy spectrum deviation

Energy spectrum

$$E(k) \propto \frac{k^2}{\alpha^2 - \beta^2 k^2} \quad \text{and} \quad Kr = \frac{\beta K_M}{\alpha}$$



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Expression of the ABC flow

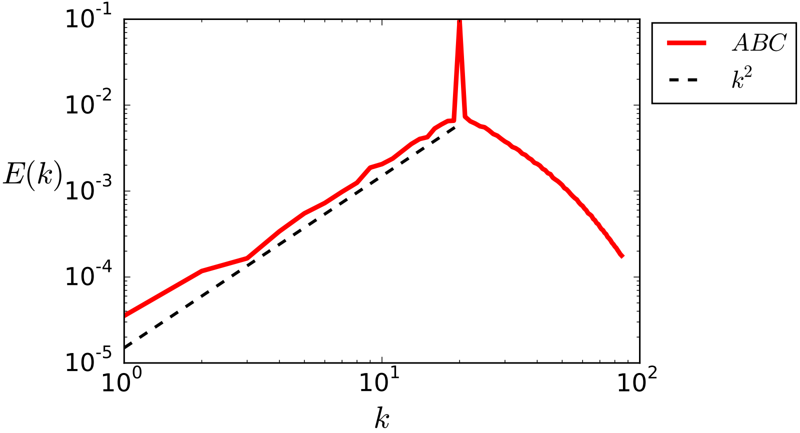
Definition

$$U_x^{ABC} = C\sin(Kz) + B\cos(Ky)\,, \tag{13}$$
$$U_y^{ABC} = A\sin(Kx) + C\cos(Kz)\,, \tag{14}$$
$$U_z^{ABC} = B\sin(Ky) + A\cos(Ky)\,. \tag{15}$$

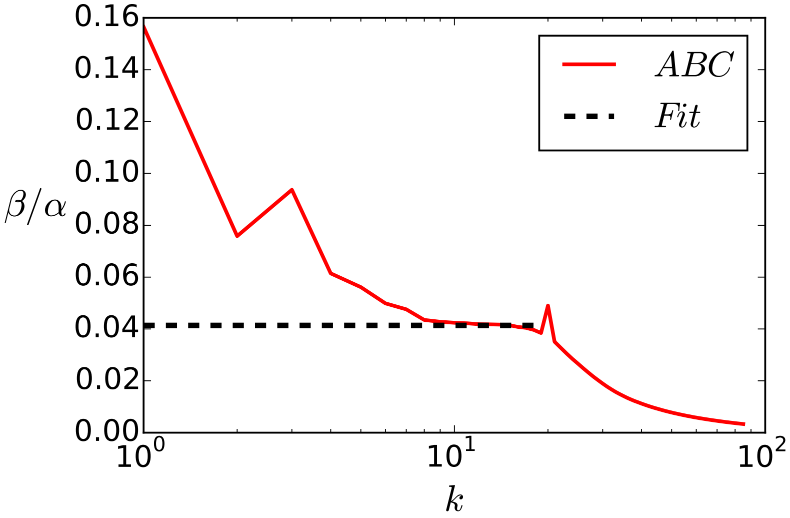
Vorticity and helicity

$$\nabla \times U_{ABC} = KU_{ABC} \quad \text{and} \quad H = K\langle |U_{ABC}|^2\rangle\,. \tag{16}$$

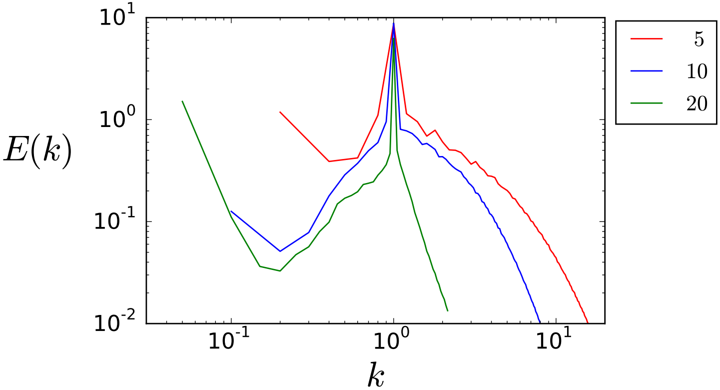
Energy spectrum



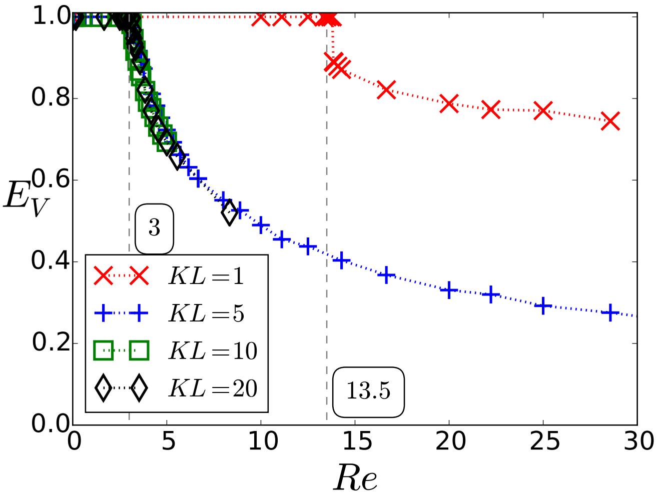
α/β ratio



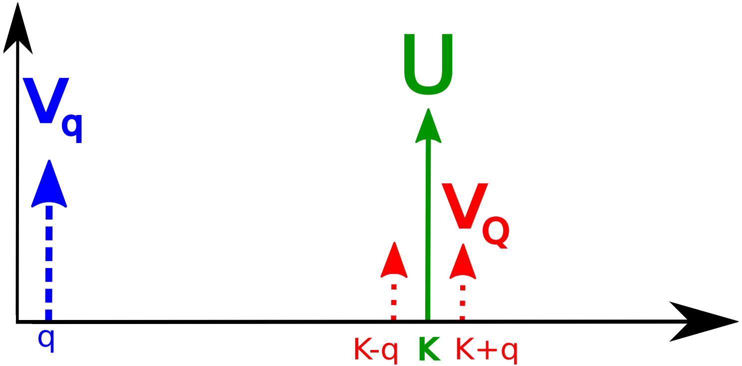
Low *Re*, large scale instability



Low *Re*, large scale instability bifurcation



Three-mode model



Floquet decomposition

Floquet framework

$$V = U + v, \tag{17}$$

$$v(r,t) = \tilde{v}(r,t)e^{\imath q \cdot r} + c.c.. \tag{18}$$

Linearised Navier-Stokes

$$\partial_t \tilde{v} = (\nabla \times U) \times \tilde{v} + (\imath q \times \tilde{v} + \nabla \times \tilde{v}) \times U - (\imath q + \nabla)\tilde{p} + \nu(\Delta - q^2)\tilde{v}, \tag{19}$$

$$0 = \imath q \cdot \tilde{v} + \nabla \cdot \tilde{v}. \tag{20}$$

Flow & Theoretical prediction

Flow equation

$$U_x^\lambda = \lambda \sin(Kz) + \cos(Ky)\,, \tag{21}$$

$$U_y^\lambda = \sin(Kx) + \lambda \cos(Kz)\,, \tag{22}$$

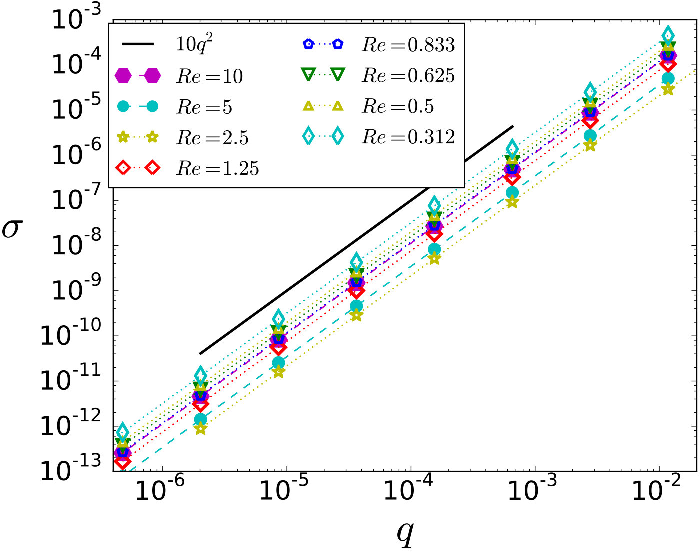
$$U_z^\lambda = \sin(Ky) + \cos(Ky)\,. \tag{23}$$

Growth rate of the large scale instability

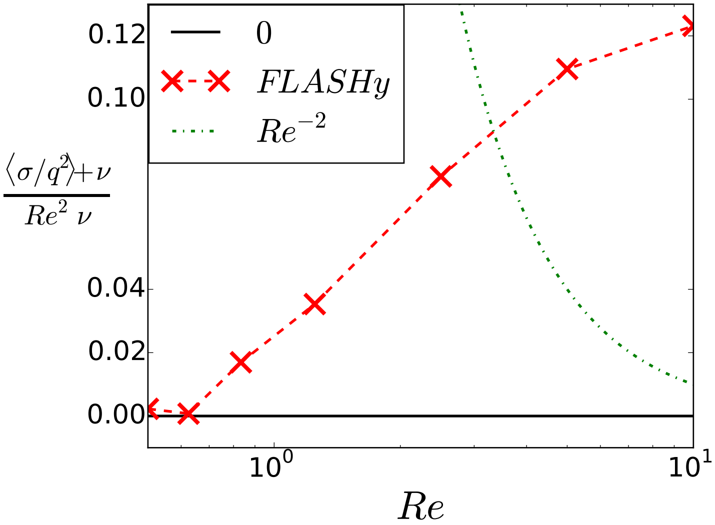
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu \quad \Longrightarrow b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right), \tag{24}$$

$$\boxed{b = \frac{1-\lambda^2}{4+2\lambda^2}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{25}$$

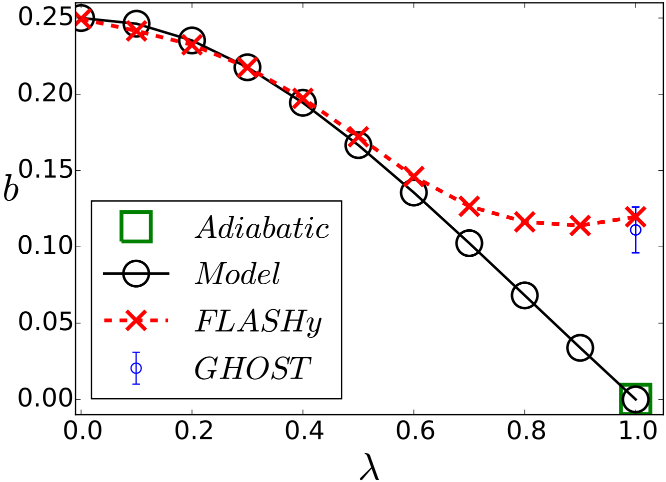
Growth rate


$10q^2$
$Re = 10$
$Re = 5$
$Re = 2.5$
$Re = 1.25$
$Re = 0.833$
$Re = 0.625$
$Re = 0.5$
$Re = 0.312$
σ
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
q

Power-law pre-factor



Instability threshold



Thank you for your attention