Helicity effects on large scale correlation time in turbulence

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Equations

Navier-Stokes equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P + \nu\Delta\boldsymbol{u} + \boldsymbol{F} \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0, \tag{1}$$

Truncated Euler equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0 \quad \text{with} \quad \boldsymbol{u}(k > k_M) = 0, \tag{2}$$

Conserved quantities

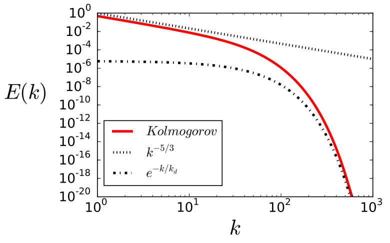
$$E = \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{u} \quad \text{and} \quad H = \int \boldsymbol{u}\cdot\boldsymbol{\omega} \quad \text{with} \quad \boldsymbol{\omega} = \nabla\times\boldsymbol{u}. \tag{3}$$

Kolmogorov theory

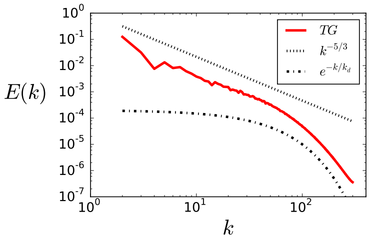
Dimensional analysis ϵ, k

$$E(k) \propto \epsilon^{2/3} k^{-5/3} \tag{4}$$

Theory



Taylor-Green DNS

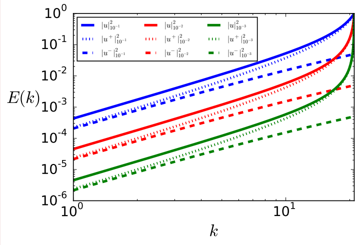


Absolute equilibrium

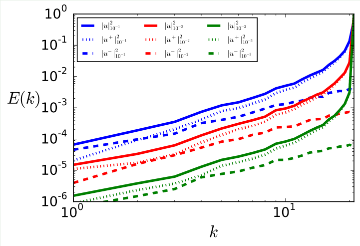
Ensemble Average

$$e_{\boldsymbol{k}} = \langle |\boldsymbol{u}_{\boldsymbol{k}}|^2 \rangle = \frac{2\alpha^{-1}}{1-(\mathcal{H}\, k)^2} \quad \text{and} \quad e^{\pm}_{\boldsymbol{k}} = \left\langle \left| \boldsymbol{u}_{\boldsymbol{k}} \pm \frac{\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{k}}}{k} \right|^2 \right\rangle = \frac{\alpha^{-1}}{1-(\pm\mathcal{H})k} \tag{5}$$

Theory

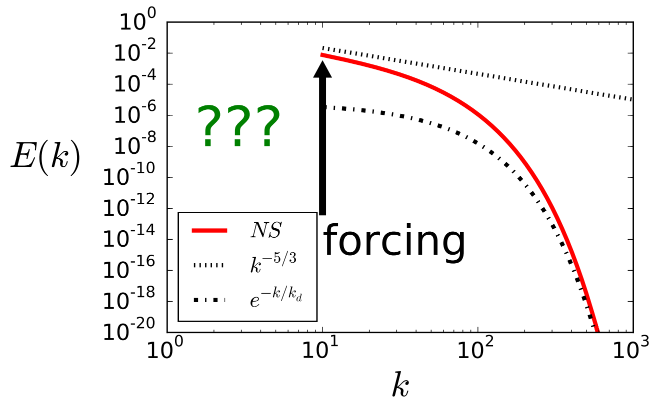


$[0; 2\pi]^3$ periodic DNS



What happens before the forcing wave-number ?

Energy spectrum

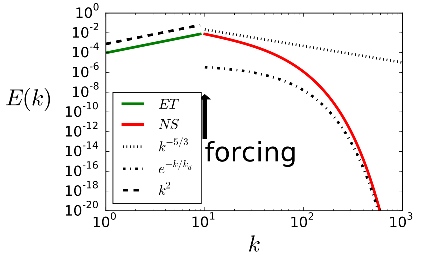


TE behavior?

Turbulence the legacy of A. N. Kolmogorov, U. Frisch p. 209

"Absolute equilibrium solutions seem highly unphysical in view of the approximately $k^{-5/3}$ spectrum of the three-dimensional turbulence. Actually, they are appropriate at the very smallest wave-numbers of turbulent flows maintained by forcing at intermediate wave-numbers (Forster, Nelson and Stephen 1977)."

Energy spectrum



Correlation time : sweeping effect

Definition

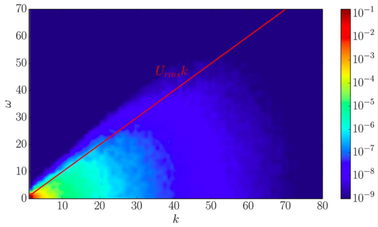
$$\Gamma_k(t) = \overline{(u_k)^*(s)\, u_k(t+s)} \quad \text{with} \quad \overline{G(s)} = \frac{1}{2T}\int_{-T}^{T} G(s)\, ds \tag{6}$$

Dimensional analysis E_{tot}, k

$$\tau \propto \frac{1}{k\sqrt{E_{tot}}} \tag{7}$$

$$\propto \frac{1}{kU_{rms}} \tag{8}$$

DNS



What about the correlation time and correlation function?

Open questions

- What is the correlation function of an absolute equilibrium ?
- What is the correlation time of an absolute equilibrium ?
- What is the correlation time in the large scale for NS ?
- Do they match ?

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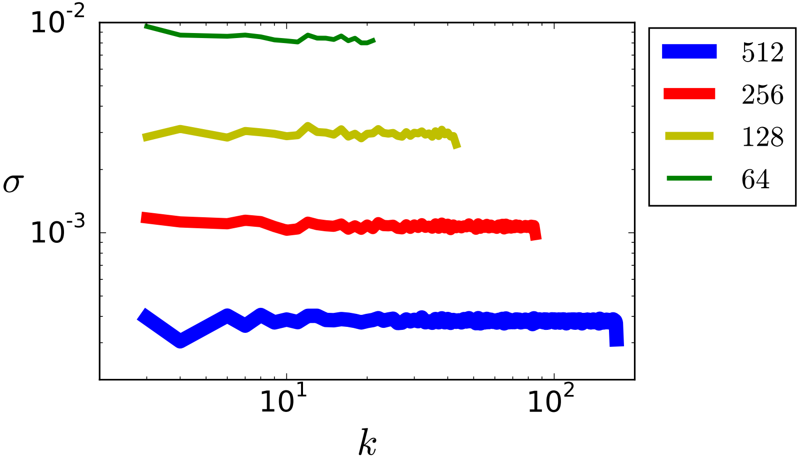
Taylor Green flows

Definition

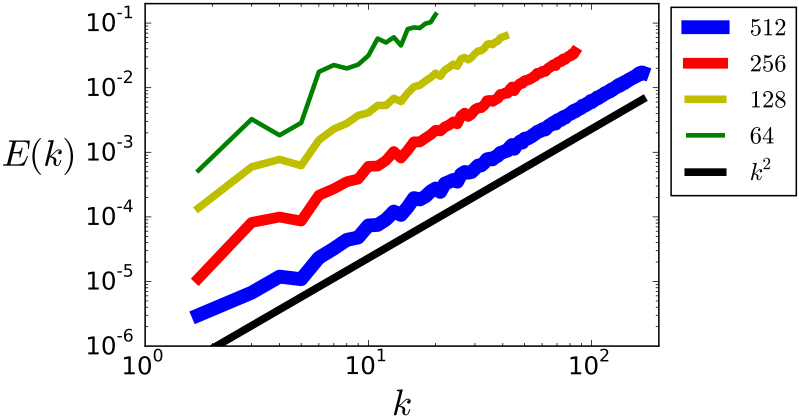
$$\begin{bmatrix} u_{\boldsymbol{r}}^x \\ u_{\boldsymbol{r}}^y \\ u_{\boldsymbol{r}}^z \end{bmatrix} = \sum_{k_x=0}^{\infty} \sum_{k_y=0}^{\infty} \sum_{k_z=0}^{\infty} \begin{bmatrix} u_{\boldsymbol{k}}^x \times \sin k_x x \cos k_y y \cos k_z z \\ u_{\boldsymbol{k}}^y \times \cos k_x x \sin k_y y \cos k_z z \\ u_{\boldsymbol{k}}^z \times \cos k_x x \cos k_y y \sin k_z z \end{bmatrix}, \tag{9}$$

$$\boldsymbol{u}_{\boldsymbol{k}} = \begin{cases} \boldsymbol{U} \in \mathbb{R}^3, \text{ if } k_x, k_y, k_z \text{ are all odd or all even} \\ 0 \text{ otherwise} \end{cases}. \tag{10}$$

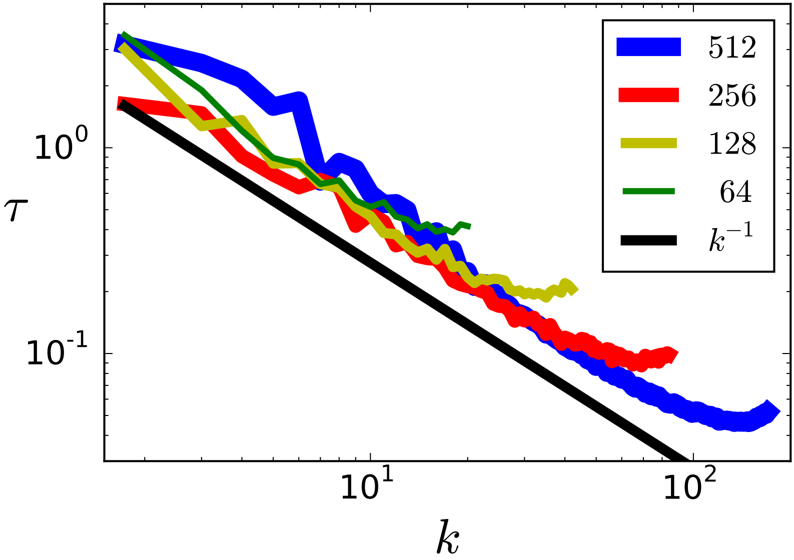
TG Standard deviation



TG Spectrum



TG Correlation time



Correlation time measurement

Procedure

Require: $\boldsymbol{u}(\boldsymbol{k}, n\Delta t)$, n, Δt,

1: Fourier transform: $\boldsymbol{u}(\boldsymbol{k},\omega) = \mathscr{D}\mathscr{F}[\boldsymbol{u}(\boldsymbol{k}, n\Delta t)\text{apodization}(n,\Delta t)](\omega)$

2: Power spectrum: $s(\boldsymbol{k},\omega) = \sum_i |\boldsymbol{u}_i(\boldsymbol{k},\omega)|^2$

3: Shell average: $S(k,\omega) = \sum_{\boldsymbol{k}} \mathbf{1}(k - \frac{1}{2} < |\boldsymbol{k}| \leq k + \frac{1}{2}) s(\boldsymbol{k},\omega)$

4: Correlation function: $\gamma(k,t) = \mathscr{D}\mathscr{F}^{-1}[S(k,\omega)](t)$

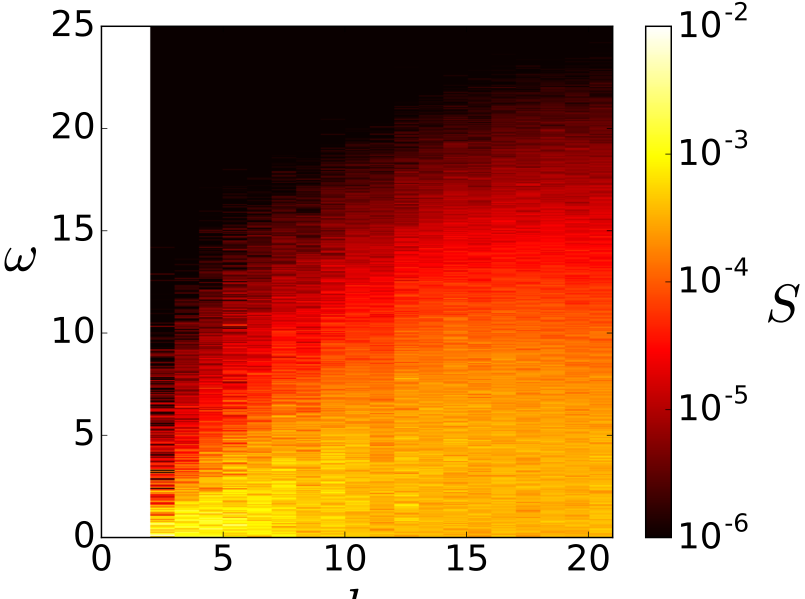
5: Normalized correlation function: $\Gamma(k,t) = \gamma(k,t)/\gamma(k,0)$

6: Correlation time: $\tau(k) = Solve[t, \Gamma(k,t), 1/2]$

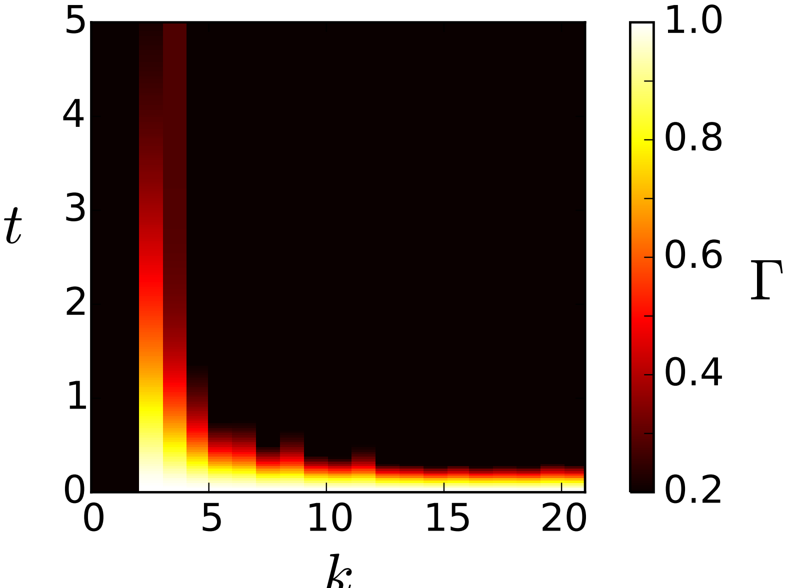
Memory restriction

Modes at $(k_x = 0,\ k_y = 0)$, $(k_y = 0,\ k_z = 0)$ and $(k_x = 0,\ k_z = 0)$ are used.

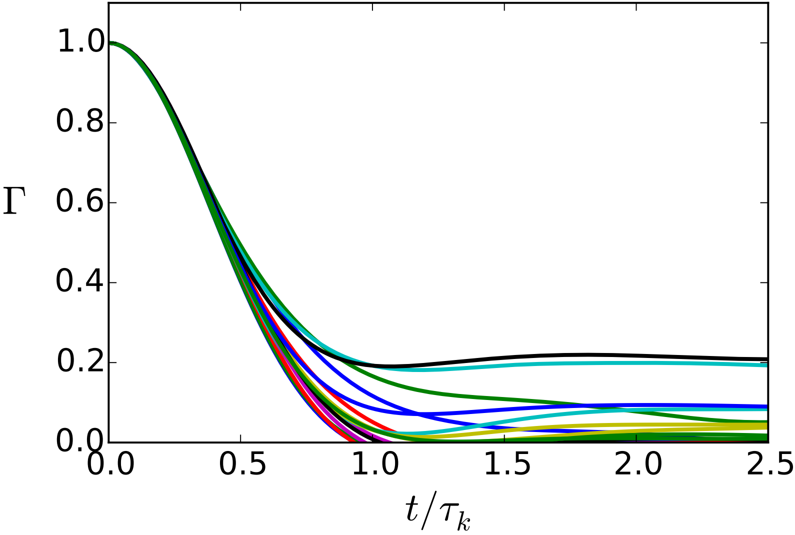
Power Spectrum



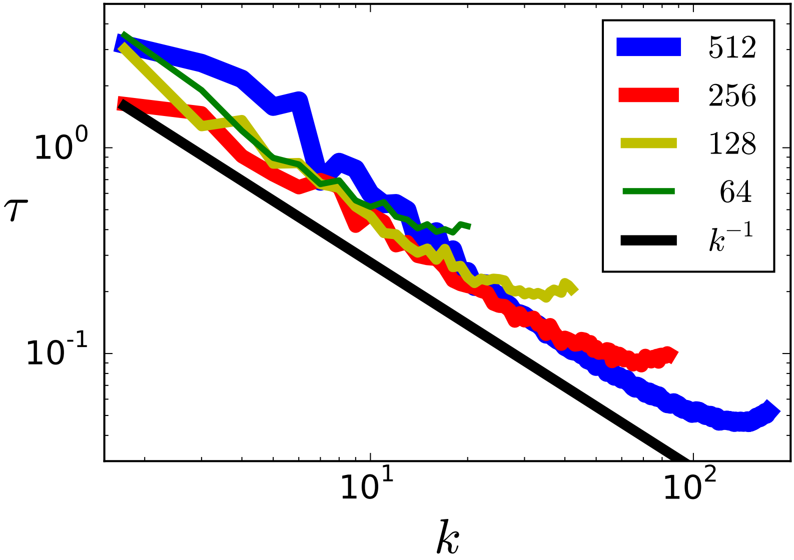
Correlation Spectrum



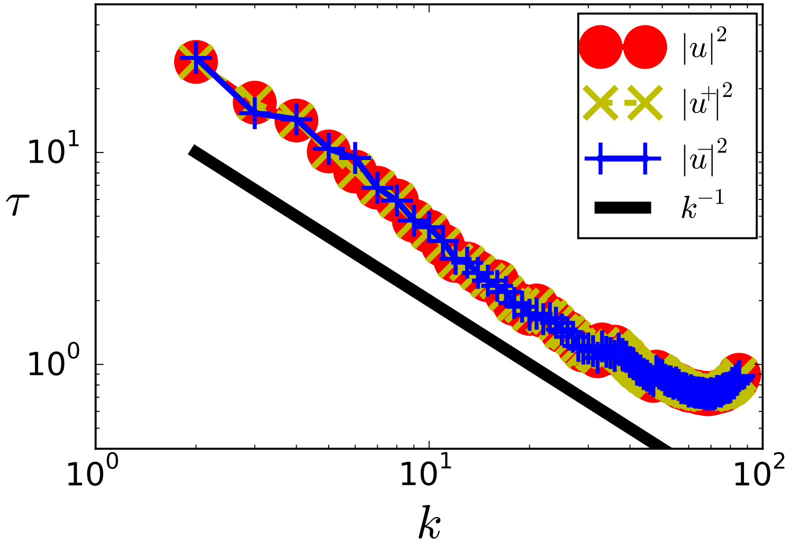
Correlation function



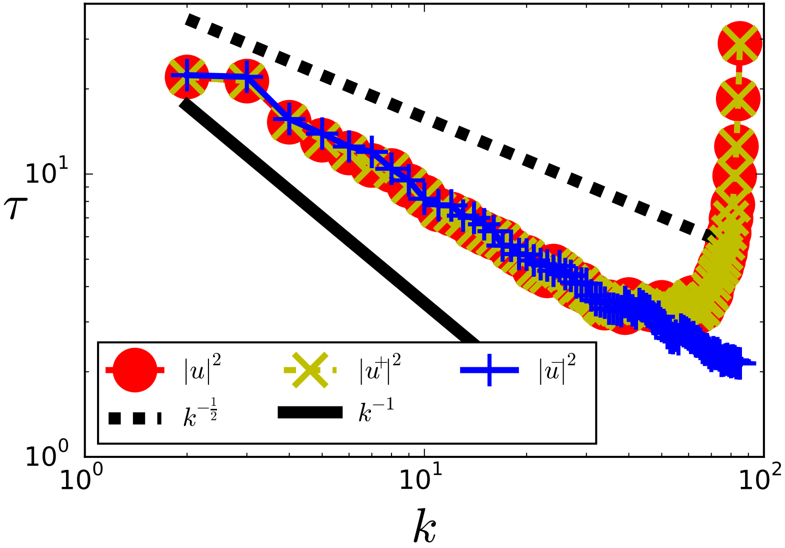
Correlation time



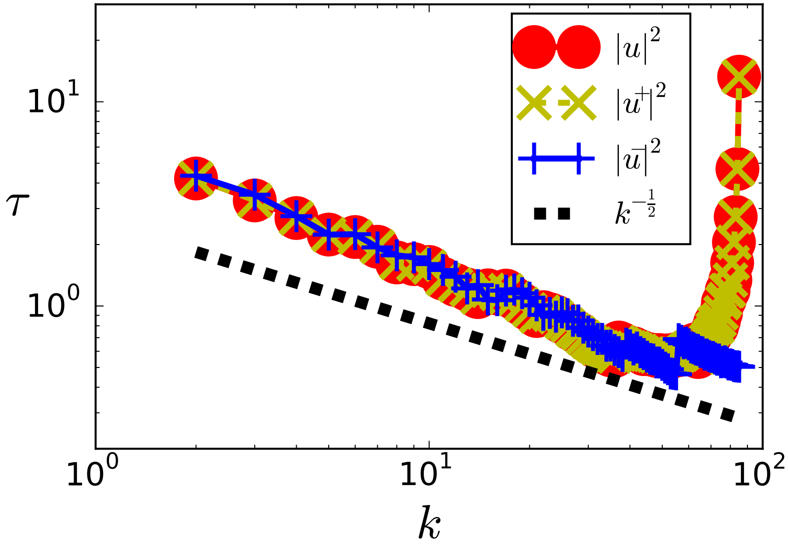
No helicity $\mathcal{H}r = 0$



Some helicity $\mathcal{K}r = 0.8$



Some helicity $\mathcal{K}r = 0.99$



Building time scales

Energy scaling

$$\tau_k^E \propto \frac{1}{kU_{rms}} \propto \frac{1}{\sqrt{k^2 E}} \tag{11}$$

Helicity scaling

$$\tau_k^H \propto \frac{1}{\sqrt{kH}} \tag{12}$$

Parabolic approximation

Correlation function : time average

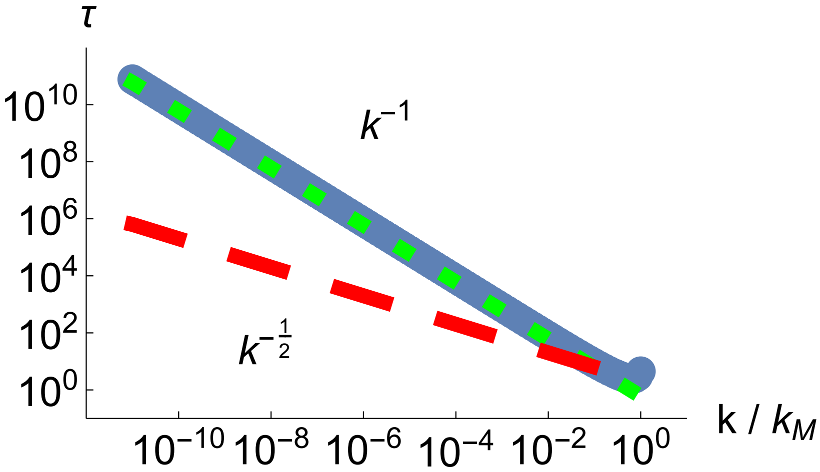
$$\Gamma_{\boldsymbol{k}}(t) = \frac{\overline{\boldsymbol{u}^*_{\boldsymbol{k}}(s)\,\boldsymbol{u}_{\boldsymbol{k}}(s+t)}}{\overline{|\boldsymbol{u}_{\boldsymbol{k}}(s)|^2}} = \frac{\overline{\boldsymbol{u}^*_{\boldsymbol{k}}(s)\,\boldsymbol{u}_{\boldsymbol{k}}(s)}}{\overline{|\boldsymbol{u}_{\boldsymbol{k}}(s)|^2}} + \frac{t^2}{2}\frac{\overline{\boldsymbol{u}^*_{\boldsymbol{k}}(s)\partial_t^2\boldsymbol{u}_{\boldsymbol{k}}(s)}}{\overline{|\boldsymbol{u}_{\boldsymbol{k}}(s)|^2}} + \mathcal{O}(t^4) \qquad (13)$$

Ergodicity : ensemble average

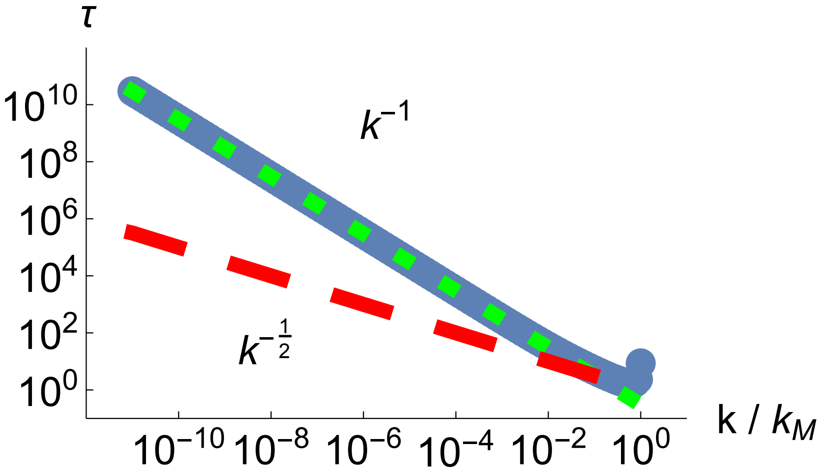
$$\Gamma_{\boldsymbol{k}}(t) = \frac{\langle|\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle}{\langle|\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle} - \frac{t^2}{2}\frac{\langle|\partial_t\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle}{\langle|\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle} + \mathcal{O}(t^4) = 1 - \frac{1}{2}\left(\frac{t}{\tau_{\boldsymbol{k}}}\right)^2 + \mathcal{O}(t^4) \qquad (14)$$

$$\tau_{\boldsymbol{k}} = \sqrt{\frac{\langle|\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle}{\langle|\partial_t\boldsymbol{u}_{\boldsymbol{k}}|^2\rangle}}. \qquad (15)$$

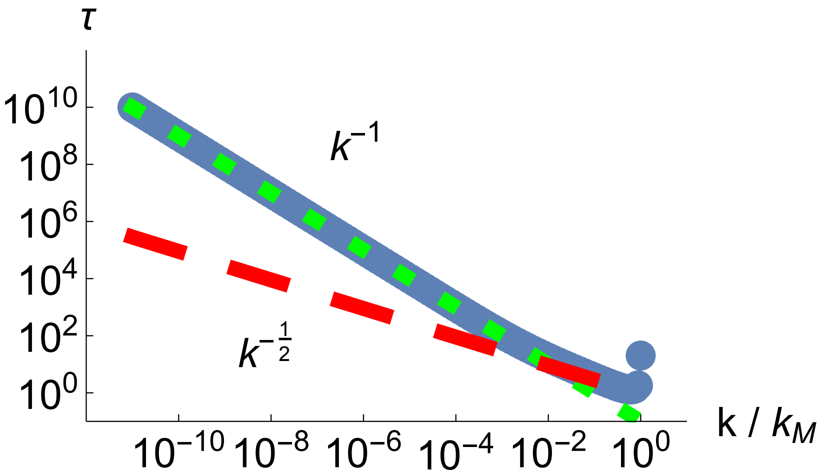
Model prediction $\mathscr{K}r = 1 - 10^{-1}$



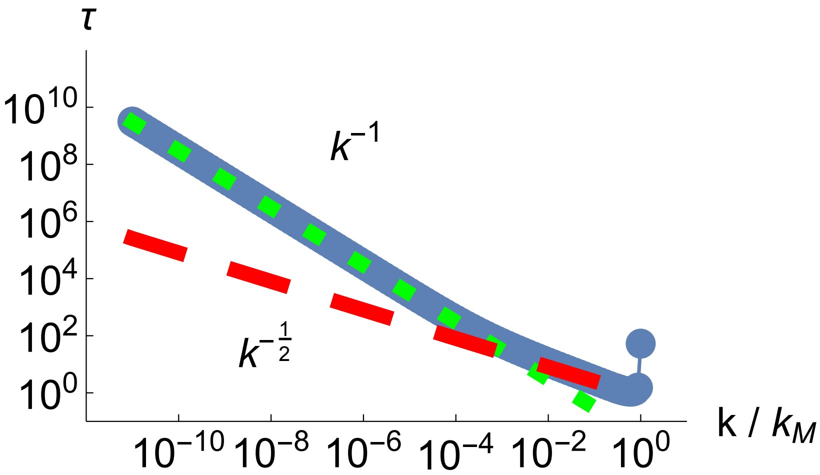
Model prediction $\mathcal{K}r = 1 - 10^{-2}$



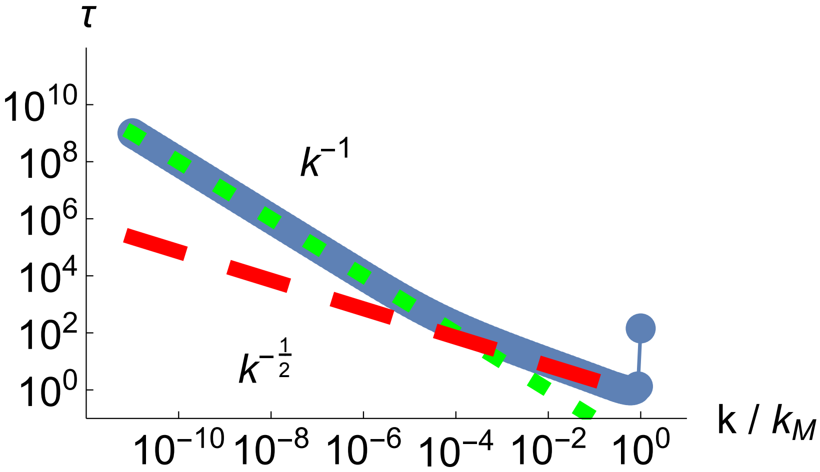
Model prediction $\mathcal{K}r = 1 - 10^{-3}$



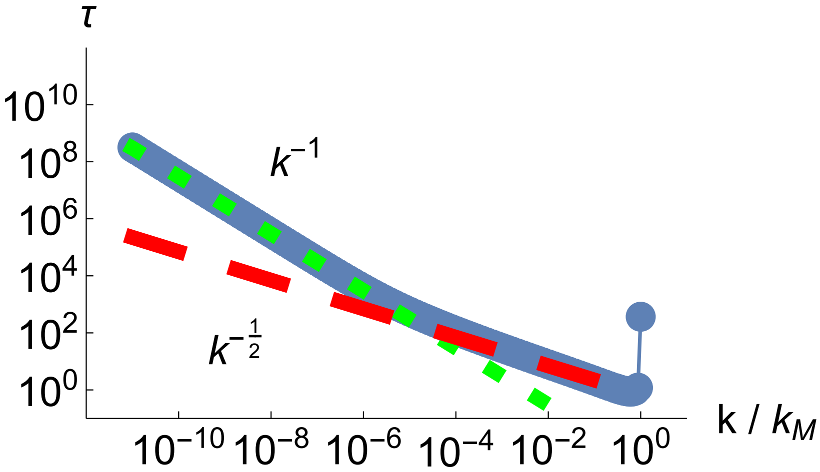
Model prediction $\mathcal{K}r = 1 - 10^{-4}$



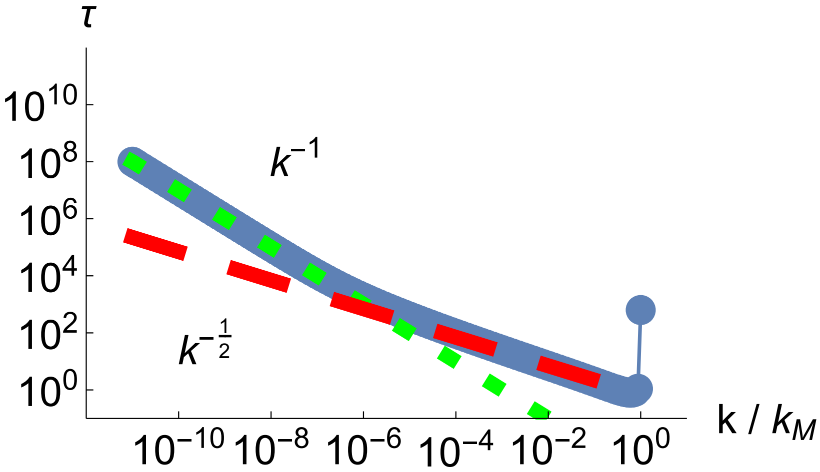
Model prediction $\mathcal{K}r = 1 - 10^{-5}$



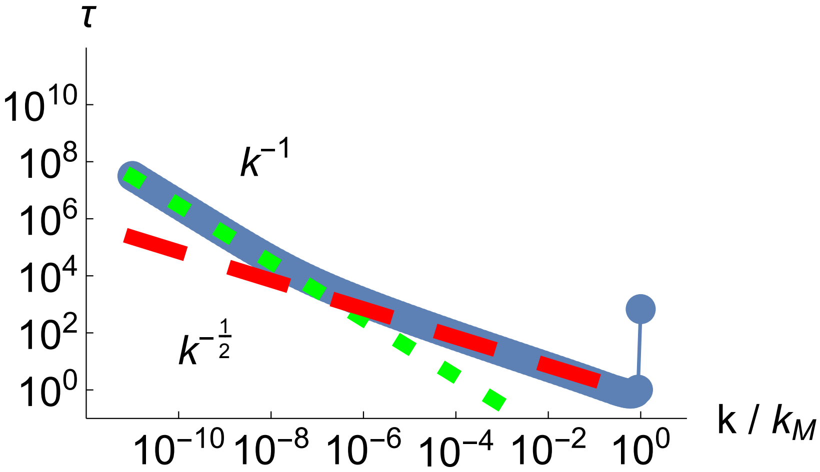
Model prediction $\mathscr{K}r = 1 - 10^{-6}$



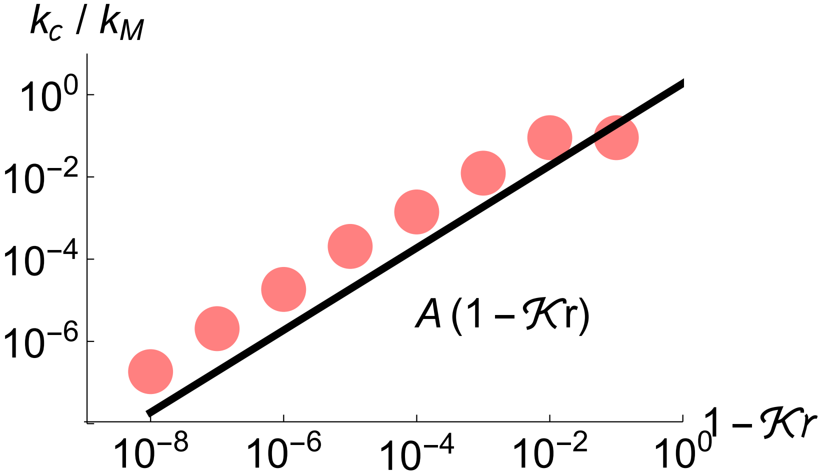
Model prediction $\mathcal{K}r = 1 - 10^{-7}$



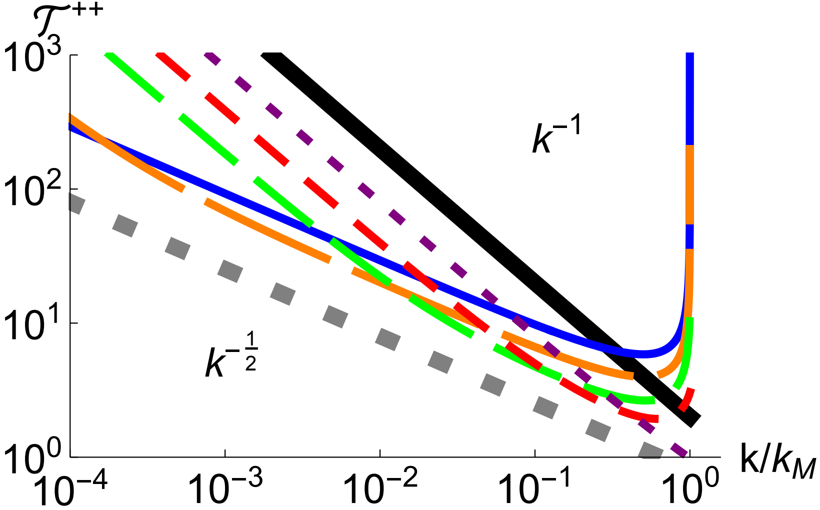
Model prediction $\mathcal{K}r = 1 - 10^{-8}$



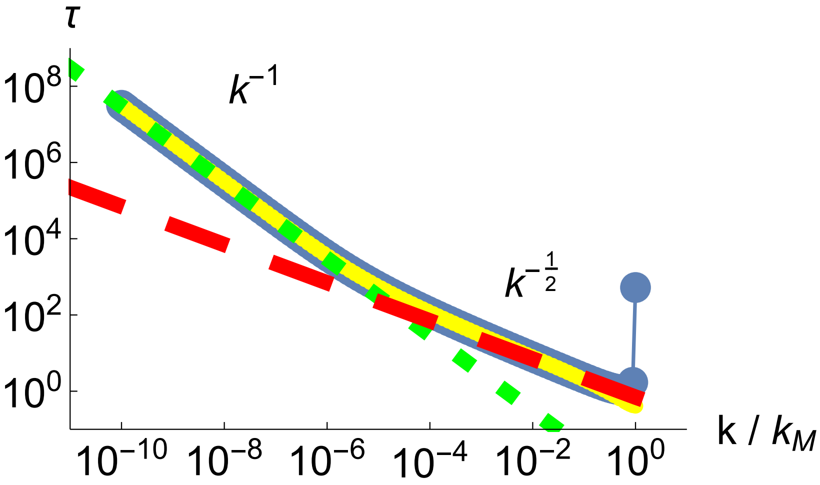
Critical wave length



Correlation time near k_M



Model prediction helical modes



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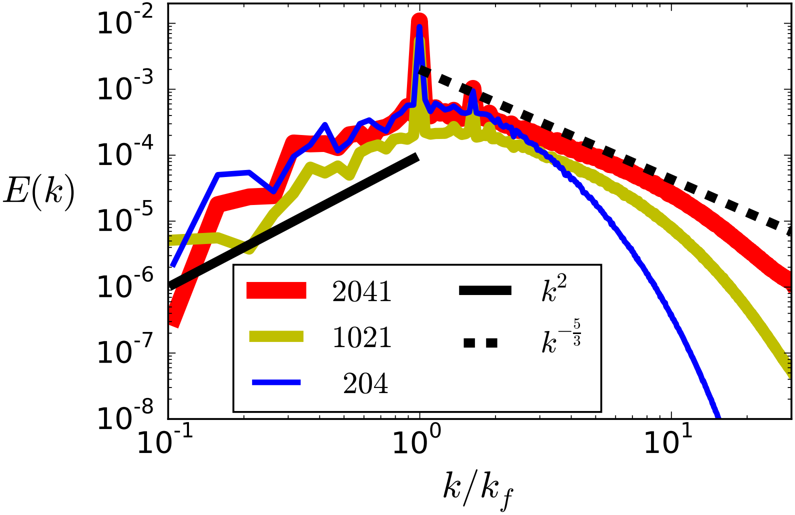
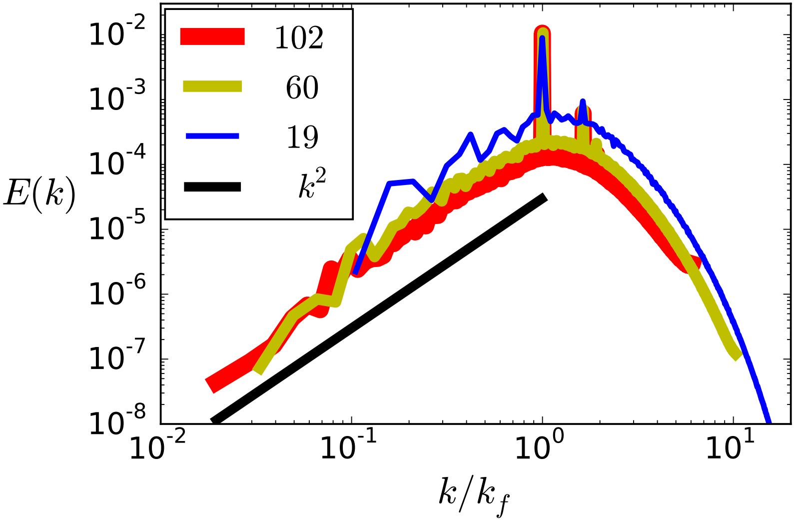
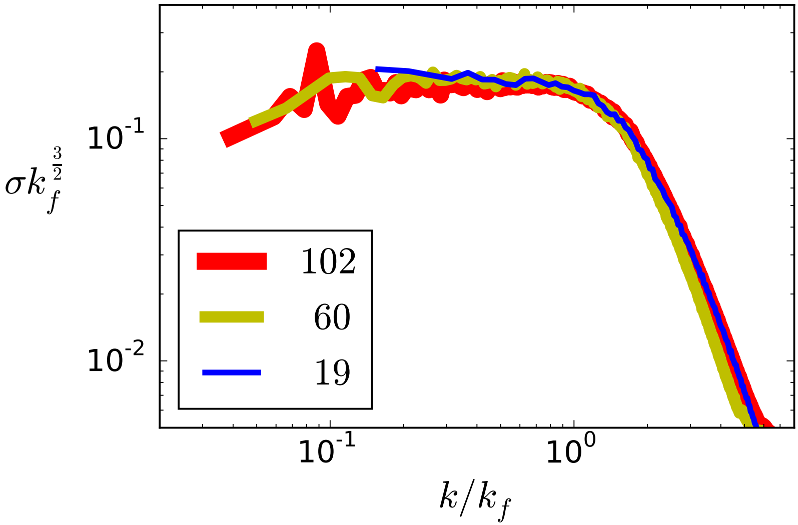
TG Spectrum varying Re, fixed $k_f L$



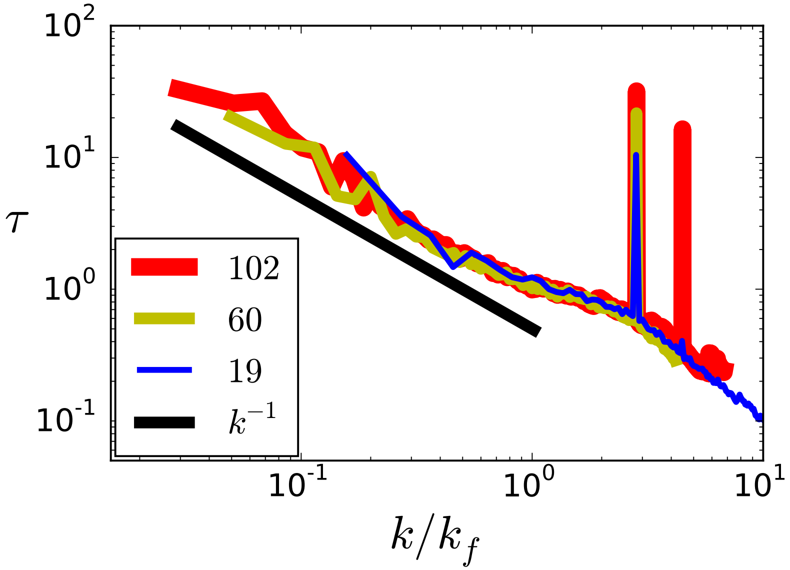
TG Spectrum fixed *Re*, varying $k_f L$



TG standard deviation



TG correlation time



Expression of the flow

ABC flow

$$U_x^{ABC} = (C \sin k_f z + B \cos k_f y) \tag{16}$$

$$U_y^{ABC} = (A \sin k_f x + C \cos k_f z) \tag{17}$$

$$U_z^{ABC} = (B \sin k_f y + A \cos k_f y) \tag{18}$$

$$\boldsymbol{\nabla} \times \boldsymbol{U}^{ABC} = k_f \boldsymbol{U}^{ABC} \tag{19}$$

Non helical ABC

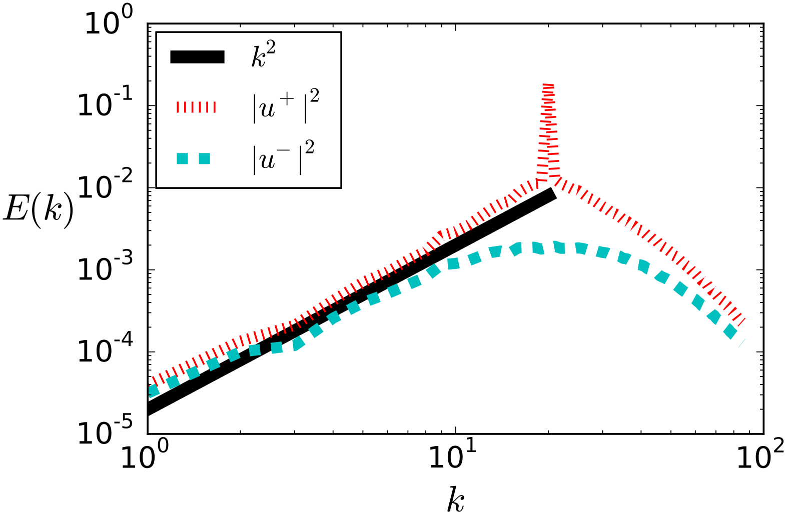
$$U_x^{CBA} = U_0 (C \cos k_f z + B \cos k_f y) \tag{20}$$

$$U_y^{CBA} = U_0 (A \cos k_f x + C \cos k_f z) \tag{21}$$

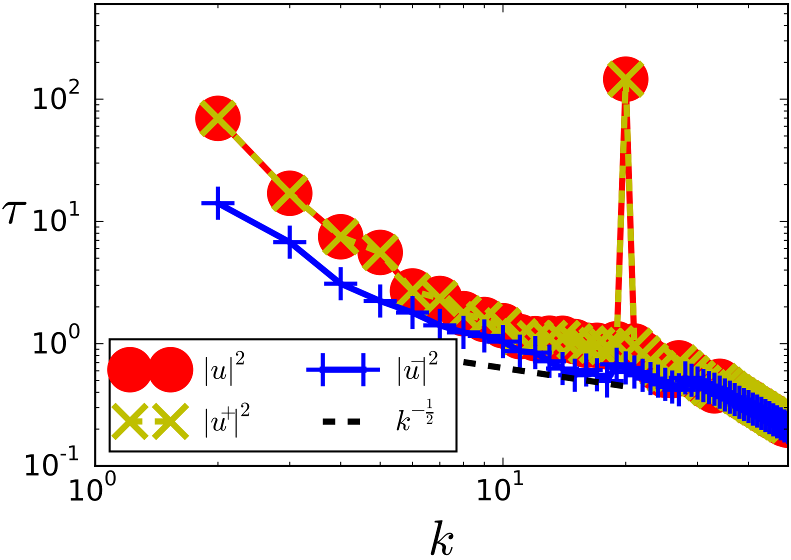
$$U_z^{CBA} = U_0 (B \cos k_f y + A \cos k_f y) \tag{22}$$

$$\int \boldsymbol{\nabla} \times \boldsymbol{U}^{ABC} \cdot \boldsymbol{U}^{ABC} d^3 \boldsymbol{r} = 0 \tag{23}$$

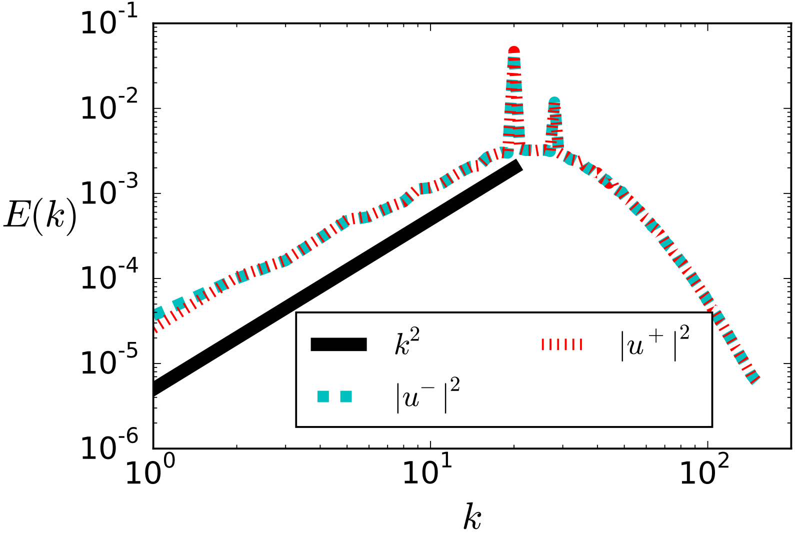
ABC Spectrum



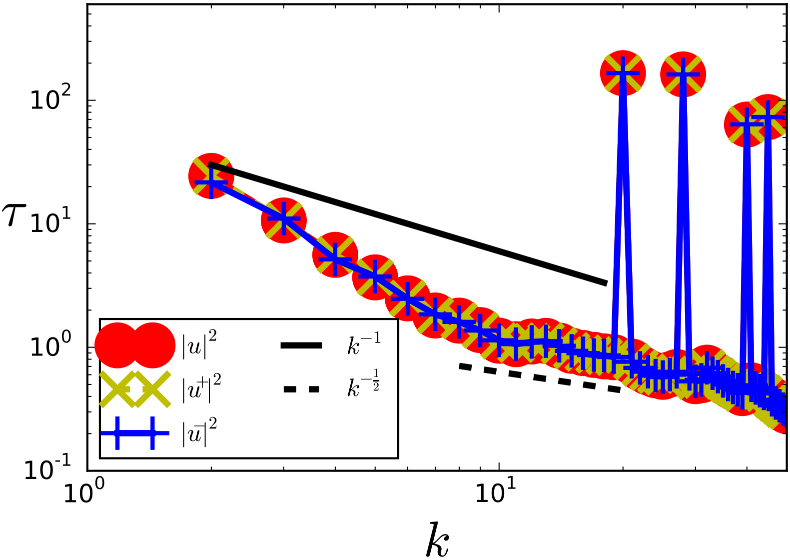
Correlation time ABC flow



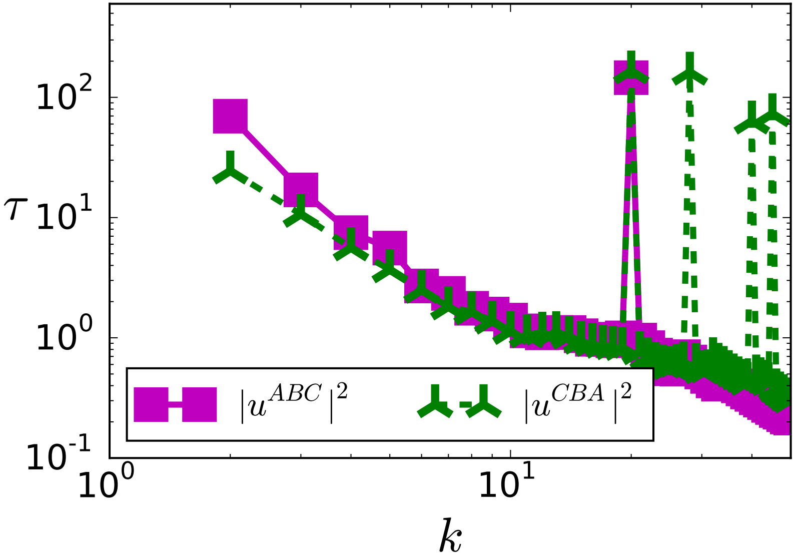
CBA Spectrum



Correlation time CBA flow



Correlation time comparison



END

Thank you for your attention

Craya Herring helical decomposition

Truncated Euler equation in Fourier space

$$\partial_t u_{\boldsymbol{k}}^\alpha = -\frac{\imath}{2}\left(k^\beta P_{\boldsymbol{k}}^{\alpha\gamma} + k^\gamma P_{\boldsymbol{k}}^{\alpha\beta}\right)\sum_{\boldsymbol{p}} u_{\boldsymbol{p}}^\beta u_{\boldsymbol{k}-\boldsymbol{p}}^\gamma \quad \text{and} \quad P_{\boldsymbol{k}}^{\alpha\beta} = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\,. \tag{24}$$

Helical decomposition

$$\boldsymbol{u}_{\boldsymbol{k}}^\pm = \boldsymbol{u}_{\boldsymbol{k}} \pm k^{-1}\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{k}} = u_{\boldsymbol{k}}\boldsymbol{h}_{\boldsymbol{k}}^\pm \quad \text{thus} \quad \boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{k}}^\pm = \pm k\boldsymbol{u}_{\boldsymbol{k}}^\pm\,. \tag{25}$$

Truncated Euler equation with helical decomposition

$$\partial_t (u_{\boldsymbol{k}}^{s_k})^* = \sum_{\substack{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0\\ s_p, s_q}} C_{\boldsymbol{kpq}}^{s_k s_p s_q} u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} \text{ and } C_{\boldsymbol{kpq}}^{s_k s_p s_q} = \frac{-1}{4}(s_p p - s_q q)\left(\boldsymbol{h}_{\boldsymbol{k}}^{s_k}\cdot\boldsymbol{h}_{\boldsymbol{p}}^{s_p}\times\boldsymbol{h}_{\boldsymbol{q}}^{s_q}\right)\,. \tag{26}$$