

Large scale instability of helical flows

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Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000
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Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ -ABC flows



Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000

First order effects: α **and** *AKA*

Scales

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad , \quad \nabla_x \to \nabla_x + \epsilon^2 \nabla_y.$$

Magnetic	Kinetic
α -effect:	AKA-effect:
$(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B),$	$(\partial_t - v_0 \nabla^2) V = \epsilon \left[-(U \cdot \nabla) V - (V \cdot \nabla) U \right],$
$(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U,$	$(\partial_t - v_0 \nabla_x^2) V_1 = -(V_0 \cdot \nabla_x) U,$
$(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle.$	$(\partial_T - v_0 \nabla_y^2) V_0 = -\langle (U \cdot \nabla_y) V_1 \rangle.$

Growth rate

$$\sigma = \alpha q - v q^2$$
 with $\alpha = a ReU$.

Kinetic

Negative eddy viscosity

Second order effects: β -effect and eddy viscosity

Magnetic

 β -effect

Hypothesis

- i. flow does not have an α or *AKA*-effects
- ii. derive equation to the next order

Growth rate

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v \,.$$

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows
Recap						

First order		Second order				
• $\sigma = \alpha q - v q$	q^2	• $\sigma = \beta q^2 - v$	q^2			
• $\alpha = aReU_0$		• $\beta = bRe^2 v$				
• $Re^c = vq/(a)$	aU_0)	• $Re^c = b^{-1/2}$				
• $q^c = \alpha / v$		• Switch				
Magnetic	Kinetic	Magnetic	Kinetic			
• α	• AKA	• β	 Negative 			
• B, η, Rm	• v, v, Re	• B, η, Rm V				
			• v, v, Re			

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Linearised Navier-Stokes & Floquet Framework

Non-Linear equation

$$\partial_t \boldsymbol{V} = \boldsymbol{V} \times \boldsymbol{\nabla} \times \boldsymbol{V} - \boldsymbol{\nabla} \boldsymbol{P} + \boldsymbol{v} \bigtriangleup \boldsymbol{V} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{V}.$$

Linearised equation:

$$V = \boldsymbol{U} + \boldsymbol{v} \mid \text{with } ||\boldsymbol{v}|| \ll ||\boldsymbol{U}||$$

$$\partial_t \boldsymbol{U} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} P_K + \boldsymbol{v} \bigtriangleup \boldsymbol{U} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0, \\ \partial_t \boldsymbol{v} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} P + \boldsymbol{v} \bigtriangleup \boldsymbol{v} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

Floquet framework

$$\boldsymbol{\nu}(\boldsymbol{r},t) = \tilde{\boldsymbol{\nu}}(\boldsymbol{r},t)e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c. , \quad \boldsymbol{\rho}(\boldsymbol{r},t) = \tilde{\boldsymbol{\rho}}(\boldsymbol{r},t)e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c., \\ \partial_{\boldsymbol{\chi}}\boldsymbol{\nu} = \left[\partial_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{r} - q_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{i} + \iota(q_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{r} + \partial_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{i})\right]e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c..$$

Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{\nu}} = (\boldsymbol{\nabla} \times \boldsymbol{U}) \times \tilde{\boldsymbol{\nu}} + (\iota \boldsymbol{q} \times \tilde{\boldsymbol{\nu}} + \boldsymbol{\nabla} \times \tilde{\boldsymbol{\nu}}) \times \boldsymbol{U} - (\iota \boldsymbol{q} + \boldsymbol{\nabla}) \tilde{\boldsymbol{p}} + \nu (\Delta - \boldsymbol{q}^2) \tilde{\boldsymbol{\nu}},$$

with $\iota \boldsymbol{q} \cdot \tilde{\boldsymbol{\nu}} + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\nu}} = 0.$

Floquet Linear Analysis of Spectral Hydrodynamics (FLASH)

Linearised Navier-Stokes equations with the Floquet framework

$$\begin{aligned} \partial_t \tilde{\boldsymbol{v}} &= (\boldsymbol{\nabla} \times \boldsymbol{U}) \times \tilde{\boldsymbol{v}} + (\iota \boldsymbol{q} \times \tilde{\boldsymbol{v}} + \boldsymbol{\nabla} \times \tilde{\boldsymbol{v}}) \times \boldsymbol{U} - (\iota \boldsymbol{q} + \boldsymbol{\nabla}) \tilde{\boldsymbol{p}} + \nu (\Delta - \boldsymbol{q}^2) \tilde{\boldsymbol{v}}, \\ \text{with} \quad \iota \boldsymbol{q} \cdot \tilde{\boldsymbol{v}} + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{v}} = 0. \end{aligned}$$

Numeric method

- i. Compute the linear terms in Fourier space.
- ii. Compute convective terms in physical space.
- iii. Use 4th order explicit RK for the time evolution.

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Formalis	n & Sim	plificatio	n			

Mode selection

$$v(r, t) = v_q(r, t) + v_Q(r, t) + v_>(r, t), \qquad (1)$$

$$\boldsymbol{v}_{\boldsymbol{q}}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{q},t)e^{l\boldsymbol{q}\boldsymbol{r}} + c.c., \qquad (2)$$

$$\boldsymbol{\nu}_{\boldsymbol{Q}}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||=1} \tilde{\boldsymbol{\nu}}(\boldsymbol{q},\boldsymbol{k},t) e^{i(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c., \qquad (3)$$

$$\boldsymbol{\nu}_{>}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||>1} \tilde{\boldsymbol{\nu}}(\boldsymbol{q},\boldsymbol{k},t) e^{i(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.$$
(4)

Additional hypothesis

- Smallest are greatest:
- Adiabatic hypothesis:
- Helical flow:

$$\begin{split} ||\boldsymbol{v}_{>}|| \ll ||\boldsymbol{v}_{\boldsymbol{q}}|| \, . \\ \partial_{t} \boldsymbol{v}_{\boldsymbol{Q}} \ll \boldsymbol{v} \, \Delta \boldsymbol{v}_{\boldsymbol{Q}} \, . \\ \boldsymbol{U}_{hel}(\boldsymbol{r}) = K^{-1} \boldsymbol{\nabla} \times \boldsymbol{U}_{hel}(\boldsymbol{r}) \, . \end{split}$$

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Equations

Equations before simplification

$$\partial_t \boldsymbol{v}_{\boldsymbol{q}} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{v}_{\boldsymbol{Q}} + \boldsymbol{v}_{\boldsymbol{Q}} \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} p_{\boldsymbol{q}} + \boldsymbol{v} \bigtriangleup \boldsymbol{v}_{\boldsymbol{q}}.$$
(5)

$$\partial_t \boldsymbol{v}_{\boldsymbol{Q}} = \boldsymbol{U} \times \boldsymbol{\nabla} \times (\boldsymbol{v}_{\boldsymbol{q}} + \boldsymbol{v}_{\boldsymbol{>}}) + (\boldsymbol{v}_{\boldsymbol{q}} + \boldsymbol{v}_{\boldsymbol{>}}) \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} p_{\boldsymbol{Q}} + \boldsymbol{v} \bigtriangleup \boldsymbol{v}_{\boldsymbol{Q}}.$$
 (6)

Simplified vorticity equations

$$\boldsymbol{v} \bigtriangleup \boldsymbol{\omega}_{\boldsymbol{Q}} = -\boldsymbol{\nabla} \times \left[\boldsymbol{U}_{hel} \times (\boldsymbol{\omega}_{\boldsymbol{q}} - K\boldsymbol{v}_{\boldsymbol{q}}) \right], \tag{7}$$

$$\partial_t \boldsymbol{\omega}_{\boldsymbol{q}} = \boldsymbol{\nabla} \times \left[\boldsymbol{U}_{hel} \times (\boldsymbol{\omega}_{\boldsymbol{Q}} - K \boldsymbol{v}_{\boldsymbol{Q}}) \right] + \boldsymbol{v} \bigtriangleup \boldsymbol{\omega}_{\boldsymbol{q}}.$$
(8)

Prediction for λ -*ABC* **flows (**A=1:B=1:C= λ **)**

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v, \tag{9}$$
$$b = \frac{1 - \lambda^2}{4 + 2\lambda^2} \quad \text{and} \quad R e = \frac{U}{K v}. \tag{10}$$



Modes detailed in 2D



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Flow & Theoretical prediction

Flow equation

$$U_x^{Fr87} = U_0 \cos(Ky + \nu K^2 t), \qquad (11)$$

$$U_{y}^{Fr87} = U_{0} \sin\left(Kx - vK^{2}t\right),$$
 (12)

$$U_z^{Fr87} = U_x^{Fr87} + U_y^{Fr87}.$$
 (13)

Growth rate of the large scale instability

$$\sigma = \alpha q - \nu q^2$$
 with $\alpha = a Re U_0$ and $a = \frac{1}{2}$. (14)

Determining *a* in the $q \ll 1$ limit

$$\alpha = \left\langle \frac{\sigma}{q} \right\rangle \iff \frac{\alpha}{U_0} = \frac{1}{U_0} \left\langle \frac{\sigma}{q} \right\rangle \iff a = \frac{1}{ReU_0} \left\langle \frac{\sigma}{q} \right\rangle = \frac{1}{2}.$$
 (15)



FLASH: Large scale energy ratio





FLASH: Large scale energy ratio



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FLASH: Growth rate



Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ -ABC flows
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FLASH: Power-law pre-factor



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Flow & Theoretical prediction

Flow equation

$$U_x^{Rob} = \cos(Ky), \qquad (16)$$

$$U_{y}^{Rob} = \sin(Kx), \qquad (17)$$

$$U_z^{Rob} = \sin(Ky) + \cos(Ky).$$
(18)

Growth rate of the large scale instability

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v, \tag{19}$$
$$b = \frac{1}{4} \quad \text{and} \quad R e = \frac{U}{K v}. \tag{20}$$

Determining *b*

$$\beta - v = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{v} = \frac{1}{v} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + v \right) \iff b = \frac{1}{Re^2 v} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + v \right).$$
(21)



FLASH: Large scale energy ratio





FLASH: Large scale energy ratio







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FLASH: Power-law



Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ – <i>ABC</i> flows 000000

FLASH: Power-law pre-factor



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Flow & Theoretical prediction

Flow equation

$$U_x^{equi} = \sin(Kz) + \cos(Ky), \qquad (22)$$

$$U_{y}^{equi} = \sin(Kx) + \cos(Kz), \qquad (23)$$

$$U_z^{equi} = \sin(Ky) + \cos(Ky).$$
⁽²⁴⁾

Growth rate of the large scale instability

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu, \tag{25}$$
$$\boxed{b=0} \quad \text{and} \quad Re = \frac{U}{K\nu}. \tag{26}$$

Determining *b*

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{\nu} = \frac{1}{\nu} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right) \iff b = \frac{1}{Re^2\nu} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right).$$
(27)



FLASH: Large scale energy ratio





FLASH: Large scale energy ratio





FLASH: Growth rate









FLASH: Power-law pre-factor



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Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows
Linear problem						

Flow & Theoretical prediction

Flow equation

$$U_x^{\lambda} = \lambda \sin(Kz) + \cos(Ky), \qquad (28)$$

$$U_{y}^{\lambda} = \sin(Kx) + \lambda \cos(Kz), \qquad (29)$$

$$U_z^{\lambda} = \sin(Ky) + \cos(Ky). \tag{30}$$

Growth rate of the large scale instability

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu, \tag{31}$$
$$b = \frac{1 - \lambda^2}{4 + 2\lambda^2} \quad \text{and} \quad Re = \frac{U}{K\nu}. \tag{32}$$

Determining *b*

$$\beta - v = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{v} = \frac{1}{v} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + v \right) \iff b = \frac{1}{Re^2 v} \left(\left\langle \frac{\sigma}{q^2} \right\rangle + v \right).$$
(33)











Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000
Full non-linear pr	oblem					
Critical R	eynolds	number				

At the onset of the instability

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu, \quad (34)$$
$$\sigma = 0 \iff \beta^c = \nu \iff b^c = (Re^c)^{-2}. \quad (35)$$

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000					
Full non-linear pr	oblem										
GHOST: S	GHOST: Small scale instability										





GHOST: Small scale instability zoom



Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ -ABC flows

Thank you for your attention



$$\begin{array}{lll} \mbox{Velocity and magnetic fields are divergence free:} & \nabla \cdot U = 0 \ ; \ \nabla \cdot B = 0. \\ \mbox{Induction equation:} & \partial_t B = \epsilon \nabla \times (U \times B) + \eta_0 \nabla^2 B. \\ \mbox{Scale order:} & \partial_t - \partial_t + e^a \partial_T \ \mbox{and} \ \partial_x - \partial_x + e^\beta \partial_y. \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B = \epsilon \nabla_x \times (U \times B) + 2e^\beta \eta_0 \nabla_y \nabla_x B + e^{1+\beta} \nabla_y \times (U \times B) + e^{2\beta} \nabla_y^2 B - e^a \partial_T B, \\ \mbox{(}e^a \partial_T - e^2\beta v_0 \nabla_y^2) B_0 = e^{1+\beta} \nabla_y \times (U \times EB_1) = e^{2+\beta} \nabla_x \times (U \times B_1). \\ \mbox{therefore:} & a = 2\beta = 2+\beta \ \mbox{and} \ \beta = 2 \iff \beta = 2 \ \mbox{and} \ a = 4. \\ \mbox{Scales:} & \partial_t - \partial_t + e^4 \partial_T \ \mbox{and} \ \partial_x - \partial_x + e^2 \partial_y. \\ \mbox{Magnetic field expansion:} & B = B_0 + EB_1 + \sum e^i B_i, \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_0 = e^{\nabla_x} \times (U \times B) + 2e^2 \eta_0 \nabla_y \nabla_x B + e^3 \nabla_y \times (U \times B) + e^4 \nabla_y^2 B - e^4 \partial_T B. \\ \mbox{Multi-scales equations:} \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_0 = 0, \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_1) + 2\eta_0 \nabla_y \nabla_x B_0, \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_2 = \nabla_x \times (U \times B_1) + 2\eta_0 \nabla_y \nabla_x B_1 + \nabla_y \times (U \times B_0), \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_2) + 2\eta_0 \nabla_y \nabla_x B_2 + \nabla_y \times (U \times B_1) + \nabla_y^2 B_0 - \partial_T B_0. \\ \mbox{Solvability equation:} \\ \mbox{(}\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_0) = (B_0 \cdot \nabla_x) U, \\ \mbox{(}\partial_t - \eta_0 \nabla_y^2) B_0 = (\nabla_y \cdot (U \times B_1)) = \nabla_y \times (U \times B_1). \\ \mbox{Cannot simplify too quickly because:} \\ \end{tabular} \end{tabular} \quad \nabla \cdot B = (\nabla_x \cdot e^2 \nabla_y) B = 0 \implies \nabla_y \cdot B_1 + \nabla_x \cdot B_3 = 0. \\ \end{array}$$

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000			
AKA-effect: overlook									

$$\begin{split} \text{The pressure will not be written, velocities are incompressible:} & \nabla \cdot V = 0 \ , \ \nabla \cdot U = 0 \ . \\ \text{Linearised Navier-Stokes:} & \partial_t V = e(-(U\nabla)V - (V\nabla)U) + v_0 \nabla^2 V \ . \\ \text{Scale order:} & \partial_t \to \partial_t + e^{i\partial}\partial_T \ , \ \partial_x \to \partial_x + e^{i\partial}\partial_y, \\ (\partial_t - v_0 \nabla^2_x)V = -e(U\nabla_x)V - e(V\nabla_x)U + 2e^{i\partial}v_0 \nabla_y \nabla_x V - e^{1+\beta}(U\nabla_y)V - e^{1+\beta}(V\nabla_y)U + 2e^{2i}v_0 \nabla^2_y V - e^{i\partial}\partial_T V \\ AKA expansion & (e^{i\partial}\partial_T - e^{2i}\beta_{i0} \nabla^2_y)V_1 = -e^{1+\beta}(U\nabla_y)eV_2 \ . \\ \text{therefore} & a = 2\beta \ \text{and} \ 2\beta = 2+\beta \ \text{and} \ \beta = 2 \ \text{and} \ a = 4 \ . \\ \text{Scales:} & \partial_t \to \partial_t + e^{i\partial}T \ , \ \partial_x \to \partial_x + e^{i\partial}y \ . \\ \text{Velocity expansion:} & V = V_1 + eV_2 + \sum e^{i-1}V_i \ . \\ (\partial_t - v_0 \nabla^2_x)V = -e(U\nabla_x)V - e(V\nabla_x)U + 2e^2v_0\nabla_y \nabla_x V - e^{3}(U\nabla_y)V - e^{3}(V\nabla_y)U + e^{4}v_0 \nabla^2_y V - e^{4}\partial_T V \ . \\ \text{Multi-scale equations:} & (\partial_t - v_0 \nabla^2_x)V_1 = 0 \ \Rightarrow \ V_1(y,T), & (\partial_t - v_0 \nabla^2_x)V_2 = -(U\nabla_x)V_1 - (V_1\nabla_x)U, \\ (\partial_t - v_0 \nabla^2_x)V_3 = -(U\nabla_x)V_2 - (V_2\nabla_x)U + 2v_0\nabla_y \nabla_x V_1 \ . \\ (\partial_t - v_0 \nabla^2_x)V_3 = -(U\nabla_x)V_2 - (V_2\nabla_x)U + 2v_0\nabla_y \nabla_x V_2 \ . \\ (\partial_t - v_0 \nabla^2_x)V_3 = -(U\nabla_x)V_2 - (V_2\nabla_x)U + 2v_0\nabla_y \nabla_x V_2 \ . \\ (\partial_t - v_0 \nabla^2_x)V_3 = -(U\nabla_x)V_4 - (V_4\nabla_x)U + 2v_0\nabla_y \nabla_x V_4 \ . \\ (\partial_t - v_0 \nabla^2_x)V_5 = -(U\nabla_x)V_4 - (V_4\nabla_x)U + 2v_0\nabla_y \nabla_x V_4 \ . \\ (\partial_t - v_0 \nabla^2_y)V_2 = -(U\nabla_y)V_1 \ . \\ (\partial_t - v_0 \nabla^2_y)V_2 = -(U\nabla_y)V_1 \ . \\ (\partial_t - v_0 \nabla^2_y)V_5 = -(U\nabla_y)V_2 \ . \\ \text{Fourier space in } x, t \ . \\ V_2(k, \omega) = V_1^1 \frac{-ik_j}{i\omega + v_0k^2}U(k, \omega), \\ (\partial_t - \nabla^2_y)V_1 = \left[\nabla_y V_1^i \right] \Sigma_{k,\omega} U(-k,\omega) \frac{ik_j}{i\omega + v_0k^2}U(k,\omega) \ . \end{aligned}$$



FLASH: Fr87 growth linear scale



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TT 1						
First orde	er effects	3				

α -effect:

$$(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B), \qquad (36)$$

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U, \qquad (37)$$

$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle.$$
(38)

AKA-effect in vorticity:

$$(\partial_t - \eta_0 \nabla^2)\Omega = \epsilon \left[\nabla \times (U \times \Omega) + \nabla \times (\Omega \times U)\right], \tag{39}$$

$$(\partial_t - \eta_0 \nabla_x^2) \Omega_1 = (\Omega_0 \cdot \nabla_x) U, \qquad (40)$$

$$(\partial_T - \eta_0 \nabla_y^2) \Omega_0 = \nabla_y \times \langle U \times \Omega_1 \rangle + \nabla_y \times \langle \Omega_1 \times U \rangle.$$
(41)



Velocity and magnetic fields are divergence free

$$\nabla \cdot U = 0 \quad ; \quad \nabla \cdot B = 0. \tag{42}$$

Induction equation:

$$\partial_t B = \epsilon \nabla \times (U \times B) + \eta_0 \nabla^2 B.$$
(43)

Scale order:

$$\partial_t \to \partial_t + \epsilon^{\alpha} \partial_T \quad \text{and} \quad \partial_x \to \partial_x + \epsilon^{\beta} \partial_y.$$
 (44)

$$(\partial_t - \eta_0 \nabla_x^2) B = \epsilon \nabla_x \times (U \times B) + 2\epsilon^\beta \eta_0 \nabla_y \nabla_x B$$
(45)

$$+\epsilon^{1+\beta}\nabla_{y}\times(U\times B)+\epsilon^{2\beta}\nabla_{y}^{2}B-\epsilon^{\alpha}\partial_{T}B.$$
 (46)

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α_effect						

 α expansion:

$$(\epsilon^{\alpha}\partial_{T} - \epsilon^{2\beta}\nu_{0}\nabla_{y}^{2})B_{0} = \epsilon^{1+\beta}\nabla_{y} \times (U \times \epsilon B_{1}) = \epsilon^{2+\beta}\nabla_{x} \times (U \times B_{1}).$$
(47)

therefore:

$$\alpha = 2\beta$$
 and $\beta = 2 \iff \beta = 2$ and $\alpha = 4$. (48)

Scales:

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad \text{and} \quad \partial_x \to \partial_x + \epsilon^2 \partial_y.$$
 (49)

Magnetic field expansion:

$$B = B_0 + \epsilon B_1 + \sum \epsilon^i B_i.$$
⁽⁵⁰⁾

$$(\partial_t - \eta_0 \nabla_x^2) B = \epsilon \nabla_x \times (U \times B) + 2\epsilon^2 \eta_0 \nabla_y \nabla_x B$$
(51)

$$+\epsilon^{3}\nabla_{y}\times(U\times B)+\epsilon^{4}\nabla_{y}^{2}B-\epsilon^{4}\partial_{T}B.$$
(52)

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ -ABC flows
α -effect						

Multi-scales equations:

$$(\partial_t - \eta_0 \nabla_x^2) B_0 = 0, \qquad (53)$$

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_0)$$
(54)

$$(\partial_t - \eta_0 \nabla_x^2) B_2 = \nabla_x \times (U \times B_1) + 2\eta_0 \underline{\nabla_y \nabla_x B_0},$$
(55)

$$(\partial_t - \eta_0 \nabla_x^2) B_3 = \nabla_x \times (U \times B_2) + 2\eta_0 \nabla_y \nabla_x B_1 + \nabla_y \times (U \times B_0), \quad (56)$$

$$(\partial_t - \eta_0 \nabla_x^2) B_4 = \nabla_x \times (U \times B_3) + 2\eta_0 \nabla_y \nabla_x B_2 +$$
(57)

$$\nabla_{y} \times (U \times B_{1}) + \nabla_{y}^{2} B_{0} - \partial_{T} B_{0}$$
(58)

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<u>.</u>			

Solvability equation:

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_0) = (B_0 \cdot \nabla_x) U, \tag{59}$$

$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \langle \nabla_y \times (U \times B_1) \rangle = \nabla_y \times \langle U \times B_1 \rangle.$$
(60)

Cannot simplify too quickly because:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = (\nabla_x \cdot + \epsilon^2 \nabla_y \cdot) \boldsymbol{B} = \boldsymbol{0}$$
(61)

$$\nabla_y \cdot B_1 + \nabla_x \cdot B_3 = 0 \tag{62}$$



The pressure will not be written, velocities are incompressible:

$$\nabla \cdot V = 0 \quad , \quad \nabla \cdot U = 0 \,. \tag{63}$$

Linearised Navier-Stokes:

$$\partial_t V = \epsilon(-(U\nabla)V - (V\nabla)U) + \nu_0 \nabla^2 V.$$
(64)

Scale order:

$$\partial_t \to \partial_t + \epsilon^{\alpha} \partial_T \quad , \quad \partial_x \to \partial_x + \epsilon^{\beta} \partial_y,$$
 (65)

$$(\partial_t - \nu_0 \nabla_x^2) V = -\epsilon (U \nabla_x) V - \epsilon (V \nabla_x) U + 2\epsilon^\beta \nu_0 \nabla_y \nabla_x V$$
(66)

$$-\epsilon^{1+\beta}(U\nabla_{y})V - \epsilon^{1+\beta}(V\nabla_{y})U + \epsilon^{2\beta}v_{0}\nabla_{y}^{2}V - \epsilon^{\alpha}\partial_{T}V.$$
(67)

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows 000000
AKA-effec	t					

AKA expansion:

$$(\epsilon^{\alpha}\partial_T - \epsilon^{2\beta}\nu_0\nabla_y^2)V_1 = -\epsilon^{1+\beta}(U\nabla_y)\epsilon V_2, \qquad (68)$$

therefore:

$$\alpha = 2\beta$$
 and $2\beta = 2 + \beta$ and $\beta = 2$ and $\alpha = 4$. (69)

Scales:

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad , \quad \partial_x \to \partial_x + \epsilon^2 \partial_y.$$
 (70)

Velocity expansion:

$$V = V_1 + \epsilon V_2 + \sum \epsilon^{i-1} V_i, \qquad (71)$$

$$(\partial_t - \nu_0 \nabla_x^2) V = -\epsilon (U \nabla_x) V - \epsilon (V \nabla_x) U + 2\epsilon^2 \nu_0 \nabla_y \nabla_x V$$
(72)

$$-\epsilon^{3}(U\nabla_{y})V - \underline{\epsilon^{3}(V\nabla_{y})U} + \epsilon^{4}\nu_{0}\nabla_{y}^{2}V - \epsilon^{4}\partial_{T}V.$$
(73)

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	$\lambda - ABC$ flows
AKA_offer	4					

Multi-scale equations:

$$(\partial_t - v_0 \nabla_x^2) V_1 = 0 \quad \Rightarrow \quad V_1(y, T), \tag{74}$$

$$(\partial_t - \nu_0 \nabla_x^2) V_2 = -\underline{(U\nabla_x) V_1} - (V_1 \nabla_x) U, \qquad (75)$$

$$(\partial_t - \nu_0 \nabla_x^2) V_3 = -(U \nabla_x) V_2 - (V_2 \nabla_x) U + \underline{2\nu_0 \nabla_y \nabla_x V_1}, \qquad (76)$$

$$(\partial_t - \nu_0 \nabla_x^2) V_4 = -(U\nabla_x) V_3 - (V_3 \nabla_x) U + 2\nu_0 \nabla_y \nabla_x V_2 - (U\nabla_y) V_1, \quad (77)$$

$$(\partial_t - \nu_0 \nabla_x^2) V_5 = -(U\nabla_x) V_4 - (V_4 \nabla_x) U + 2\nu_0 \nabla_y \nabla_x V_4$$
(78)

$$+ \left[- (U\nabla_y)V_2 + v_0\nabla_y^2 V_1 - \partial_T V_1 \right] .$$
 (79)

Previous work	FLASH	3M model	Fr87 benchmark	Roberts flow	Equilateral ABC flow	λ -ABC flows
AKA-effec	t					

Solvability equation:

$$(\partial_t - v_0 \nabla_x^2) V_2 = -(V_1 \nabla_x) U,$$
(80)

$$\left(\partial_T - \nu_0 \nabla_y^2\right) V_1 = -\langle (U \nabla_y) V_2 \rangle \,. \tag{81}$$

Fourier space in *x*, *t* :

$$V_{2}(k,\omega) = V_{1}^{j} \frac{-ik_{j}}{i\omega + v_{0}k^{2}} U(k,\omega), \qquad (82)$$

$$(\partial_T - \nabla_y^2) V_1 = \left[\nabla_y V_1^j\right] \sum_{k,\omega} U(-k,\omega) \frac{ik_j}{i\omega + v_0 k^2} U(k,\omega).$$
(83)