Large scale instability of helical flows

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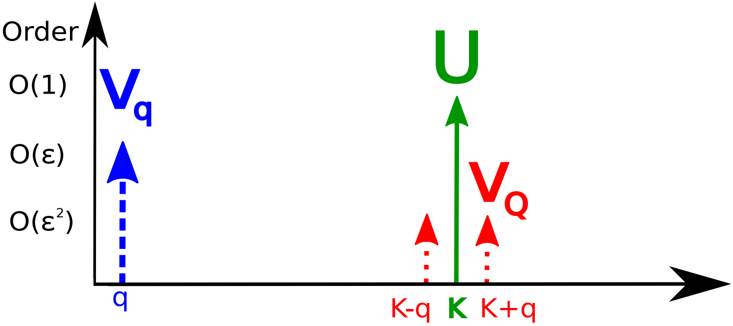
Table of contents

Hand-waving



First order effects: α and *AKA*

Scales

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad , \quad \nabla_x \to \nabla_x + \epsilon^2 \nabla_y .$$

Magnetic

α-effect:

$$(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B) ,$$

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U ,$$

$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle .$$

Kinetic

AKA-effect:

$$(\partial_t - \nu_0 \nabla^2) V = \epsilon \left[-(U \cdot \nabla) V - (V \cdot \nabla) U \right] ,$$

$$(\partial_t - \nu_0 \nabla_x^2) V_1 = -(V_0 \cdot \nabla_x) U ,$$

$$(\partial_T - \nu_0 \nabla_y^2) V_0 = -\langle (U \cdot \nabla_y) V_1 \rangle .$$

Growth rate

$$\boxed{\sigma = \alpha q - \nu q^2} \quad \text{with} \quad \boxed{\alpha = aReU} .$$

Second order effects: β-effect and eddy viscosity

Magnetic

β-effect

Kinetic

Negative eddy viscosity

Hypothesis

i. flow does not have an α- or *AKA*-effects

ii. derive equation to the next order

Growth rate

$$\boxed{\sigma = \beta q^2 - \nu q^2} \quad \text{with} \quad \boxed{\beta = bRe^2 \nu}.$$

Recap

First order

- $\sigma = \alpha q - \nu q^2$
- $\alpha = aReU_0$
- $Re^c = \nu q/(aU_0)$
- $q^c = \alpha/\nu$

Magnetic

- α
- B , η , Rm

Kinetic

- *AKA*
- v , ν , Re

Second order

- $\sigma = \beta q^2 - \nu q^2$
- $\beta = bRe^2\nu$
- $Re^c = b^{-1/2}$
- Switch

Magnetic

- β
- B , η , Rm

Kinetic

- Negative ν
- v , ν , Re

Table of contents

Linearised Navier-Stokes & Floquet Framework

Non-Linear equation

$$\partial_t \boldsymbol{V} = \boldsymbol{V}\times\boldsymbol{\nabla}\times\boldsymbol{V} - \boldsymbol{\nabla}P + \nu\triangle\boldsymbol{V} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla}\cdot\boldsymbol{V}\,.$$

Linearised equation: $\boxed{\boldsymbol{V} = \boldsymbol{U} + \boldsymbol{v}}$ with $||\boldsymbol{v}|| \ll ||\boldsymbol{U}||$

$$\partial_t \boldsymbol{U} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla}P_K + \nu\triangle\boldsymbol{U} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla}\cdot\boldsymbol{U} = 0\,,$$
$$\partial_t \boldsymbol{v} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla}P + \nu\triangle\boldsymbol{v} \quad , \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,.$$

Floquet framework

$$\boldsymbol{v}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{r},t)e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c. \quad , \quad p(\boldsymbol{r},t) = \tilde{p}(\boldsymbol{r},t)e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c.\,,$$
$$\partial_x \boldsymbol{v} = \left[\partial_x\tilde{\boldsymbol{v}}^r - q_x\tilde{\boldsymbol{v}}^i + \imath(q_x\tilde{\boldsymbol{v}}^r + \partial_x\tilde{\boldsymbol{v}}^i)\right] e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c.\,.$$

Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t\tilde{\boldsymbol{v}} = (\boldsymbol{\nabla}\times\boldsymbol{U})\times\tilde{\boldsymbol{v}} + (\imath\boldsymbol{q}\times\tilde{\boldsymbol{v}} + \boldsymbol{\nabla}\times\tilde{\boldsymbol{v}})\times\boldsymbol{U} - (\imath\boldsymbol{q} + \boldsymbol{\nabla})\tilde{p} + \nu(\Delta - \boldsymbol{q}^2)\tilde{\boldsymbol{v}}\,,$$
$$\text{with} \quad \imath\boldsymbol{q}\cdot\tilde{\boldsymbol{v}} + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{v}} = 0\,.$$

Floquet Linear Analysis of Spectral Hydrodynamics (FLASH)

Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{v}} = (\nabla \times \boldsymbol{U}) \times \tilde{\boldsymbol{v}} + (\imath \boldsymbol{q} \times \tilde{\boldsymbol{v}} + \nabla \times \tilde{\boldsymbol{v}}) \times \boldsymbol{U} - (\imath \boldsymbol{q} + \nabla)\tilde{p} + \nu(\Delta - \boldsymbol{q}^2)\tilde{\boldsymbol{v}},$$
$$\text{with} \quad \imath \boldsymbol{q} \cdot \tilde{\boldsymbol{v}} + \nabla \cdot \tilde{\boldsymbol{v}} = 0\,.$$

Numeric method

i. Compute the linear terms in Fourier space.
ii. Compute convective terms in physical space.
iii. Use 4th order explicit RK for the time evolution.

Table of contents

Formalism & Simplification

Mode selection

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_{\boldsymbol{q}}(\boldsymbol{r},t) + \boldsymbol{v}_{\boldsymbol{Q}}(\boldsymbol{r},t) + \boldsymbol{v}_{>}(\boldsymbol{r},t)\,, \tag{1}$$

$$\boldsymbol{v}_{\boldsymbol{q}}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{q},t)\,e^{\imath \boldsymbol{q}\boldsymbol{r}} + c.c.\,, \tag{2}$$

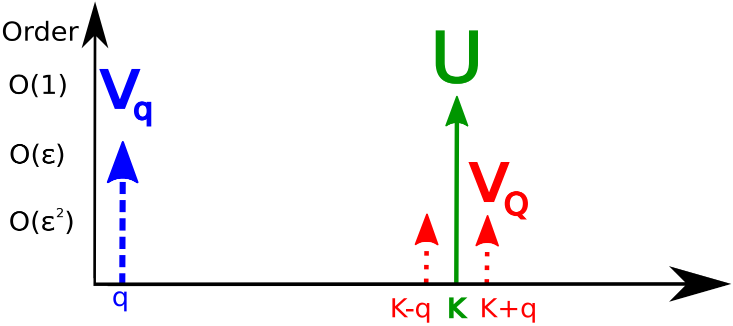
$$\boldsymbol{v}_{\boldsymbol{Q}}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||=1} \tilde{\boldsymbol{v}}(\boldsymbol{q},\boldsymbol{k},t)\,e^{\imath(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.\,, \tag{3}$$

$$\boldsymbol{v}_{>}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||>1} \tilde{\boldsymbol{v}}(\boldsymbol{q},\boldsymbol{k},t)\,e^{\imath(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.\,. \tag{4}$$

Additional hypothesis

- Smallest are greatest: $||\boldsymbol{v}_{>}|| \ll ||\boldsymbol{v}_{\boldsymbol{q}}||\,.$
- Adiabatic hypothesis: $\partial_t \boldsymbol{v}_{\boldsymbol{Q}} \ll \nu \triangle \boldsymbol{v}_{\boldsymbol{Q}}\,.$
- Helical flow: $\boldsymbol{U}_{hel}(\boldsymbol{r}) = K^{-1}\boldsymbol{\nabla}\times\boldsymbol{U}_{hel}(\boldsymbol{r})\,.$

Hand-waving



Equations

Equations before simplification

$$\partial_t \boldsymbol{v_q} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{v_Q} + \boldsymbol{v_Q}\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla} p_{\boldsymbol{q}} + \nu\triangle\boldsymbol{v_q}\,. \tag{5}$$

$$\partial_t \boldsymbol{v_Q} = \boldsymbol{U}\times\boldsymbol{\nabla}\times(\boldsymbol{v_q}+\boldsymbol{v}_>) + (\boldsymbol{v_q}+\boldsymbol{v}_>)\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla} p_{\boldsymbol{Q}} + \nu\triangle\boldsymbol{v_Q}\,. \tag{6}$$

Simplified vorticity equations

$$\nu\triangle\boldsymbol{\omega_Q} = -\boldsymbol{\nabla}\times\left[\boldsymbol{U}_{hel}\times(\boldsymbol{\omega_q} - K\boldsymbol{v_q})\right], \tag{7}$$

$$\partial_t\boldsymbol{\omega_q} = \boldsymbol{\nabla}\times\left[\boldsymbol{U}_{hel}\times(\boldsymbol{\omega_Q} - K\boldsymbol{v_Q})\right] + \nu\triangle\boldsymbol{\omega_q}\,. \tag{8}$$

Prediction for $\lambda-ABC$ flows ($A=1:B=1:C=\lambda$)

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{9}$$

$$\boxed{b = \frac{1-\lambda^2}{4+2\lambda^2}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{10}$$

Modes detailed in 2D

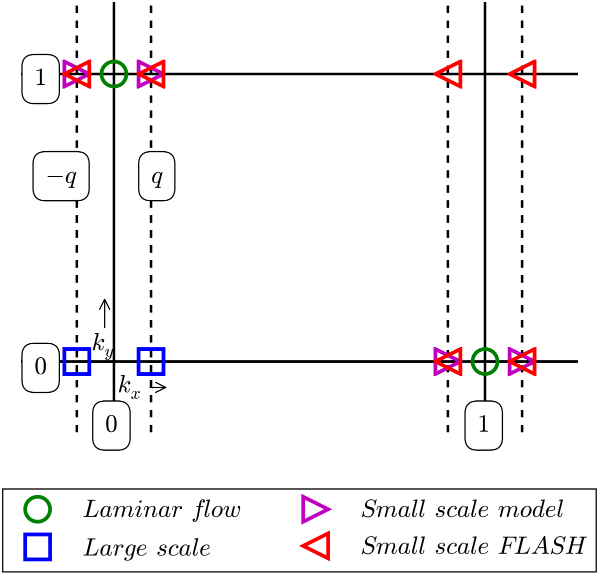


Table of contents

Flow & Theoretical prediction

Flow equation

$$U_x^{Fr87} = U_0 \cos\left(Ky + \nu K^2 t\right), \quad (11)$$

$$U_y^{Fr87} = U_0 \sin\left(Kx - \nu K^2 t\right), \quad (12)$$

$$U_z^{Fr87} = U_x^{Fr87} + U_y^{Fr87}. \quad (13)$$

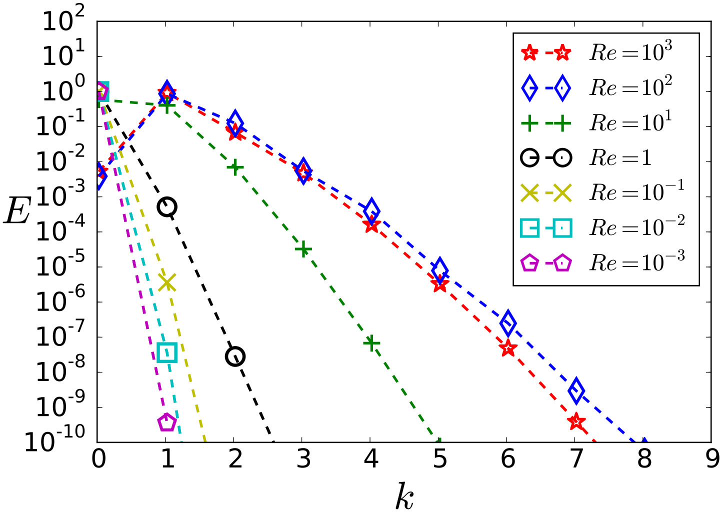
Growth rate of the large scale instability

$$\sigma = \alpha q - \nu q^2 \quad \text{with} \quad \alpha = aReU_0 \quad \text{and} \quad \boxed{a = \frac{1}{2}}. \quad (14)$$

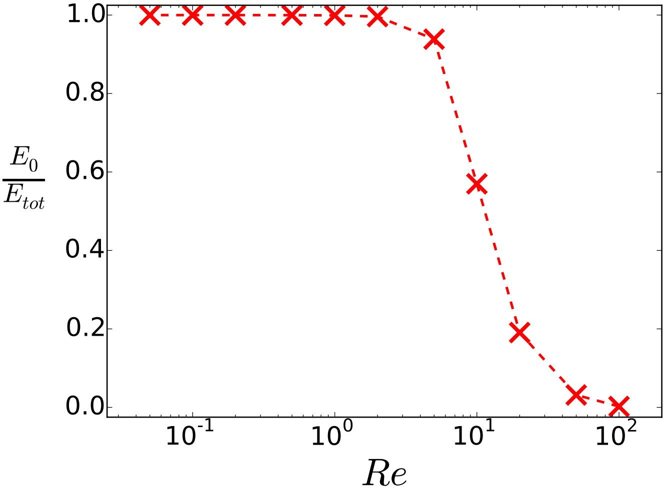
Determining a in the $q \ll 1$ limit

$$\alpha = \left\langle \frac{\sigma}{q} \right\rangle \Longleftrightarrow \frac{\alpha}{U_0} = \frac{1}{U_0} \left\langle \frac{\sigma}{q} \right\rangle \Longleftrightarrow a = \frac{1}{ReU_0} \left\langle \frac{\sigma}{q} \right\rangle = \frac{1}{2}. \quad (15)$$

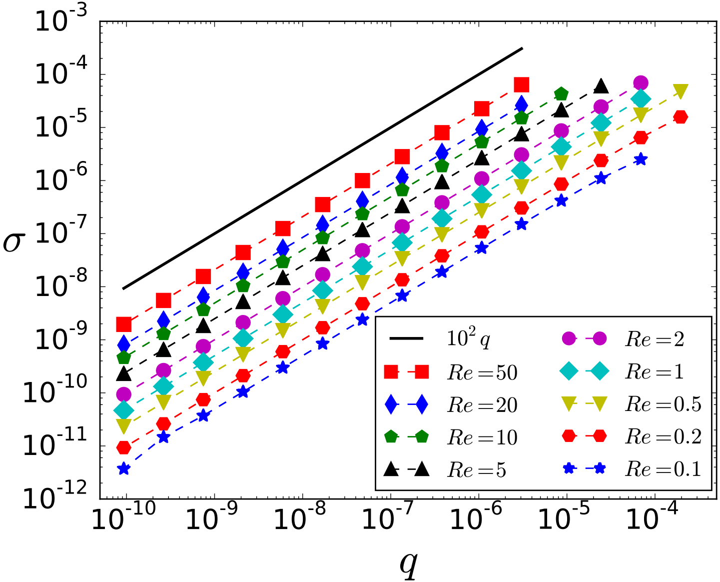
FLASH: Large scale energy ratio



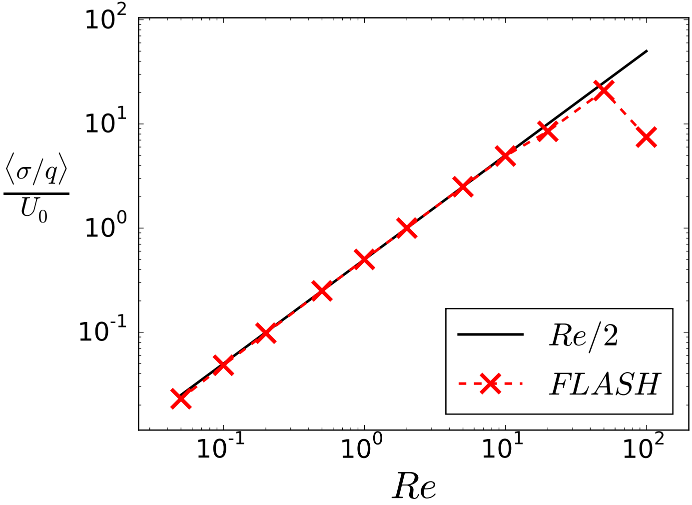
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



FLASH: Power-law pre-factor

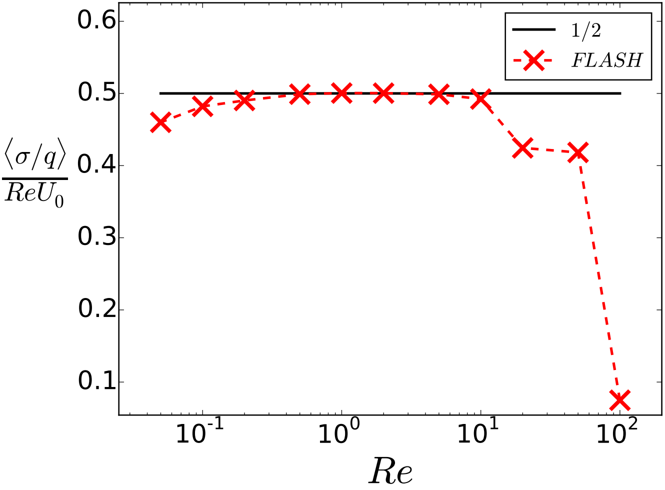


Table of contents

Flow & Theoretical prediction

Flow equation

$$U_x^{Rob} = \cos(Ky)\,, \tag{16}$$

$$U_y^{Rob} = \sin(Kx)\,, \tag{17}$$

$$U_z^{Rob} = \sin(Ky) + \cos(Ky)\,. \tag{18}$$

Growth rate of the large scale instability

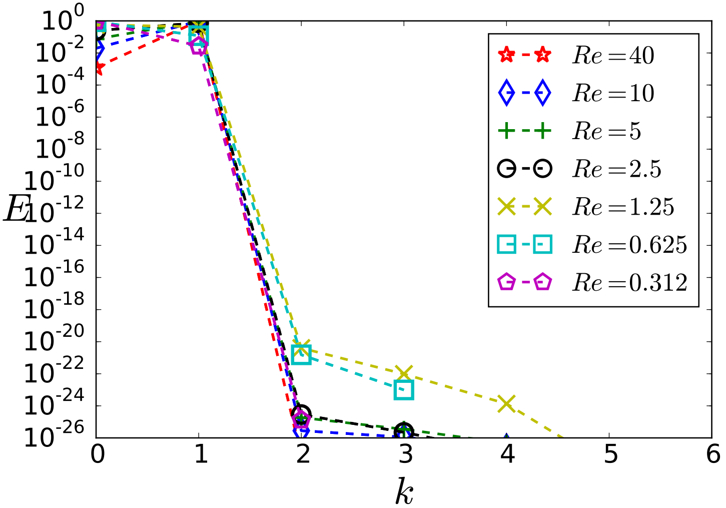
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{19}$$

$$\boxed{b = \frac{1}{4}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{20}$$

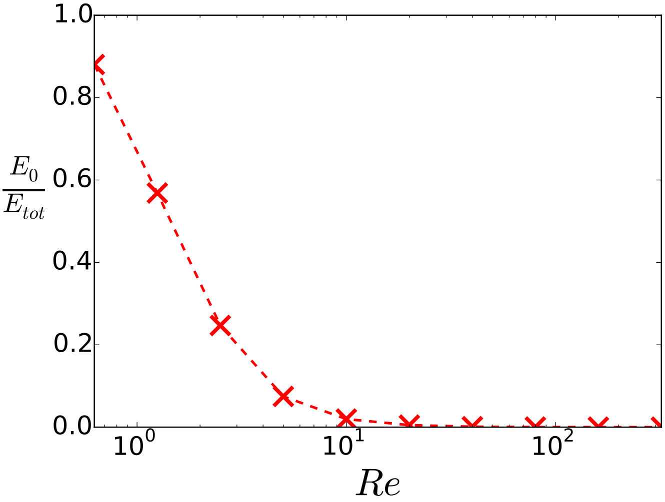
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \Longleftrightarrow \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right) \Longleftrightarrow b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right). \tag{21}$$

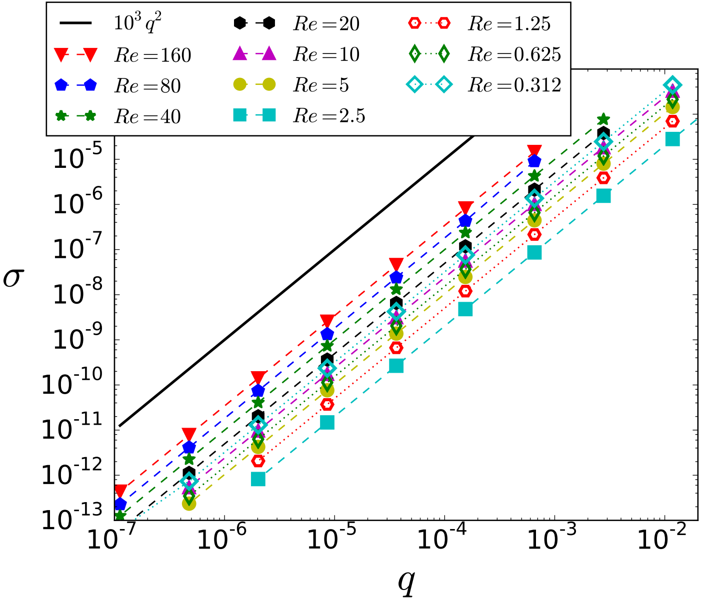
FLASH: Large scale energy ratio



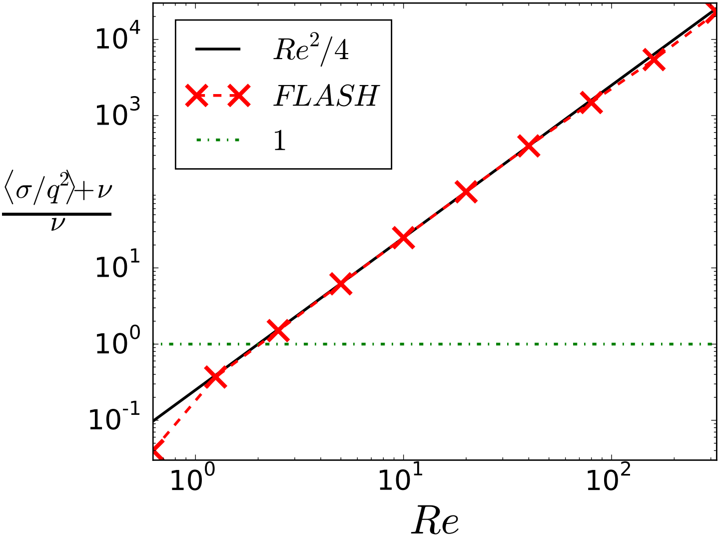
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



FLASH: Power-law pre-factor

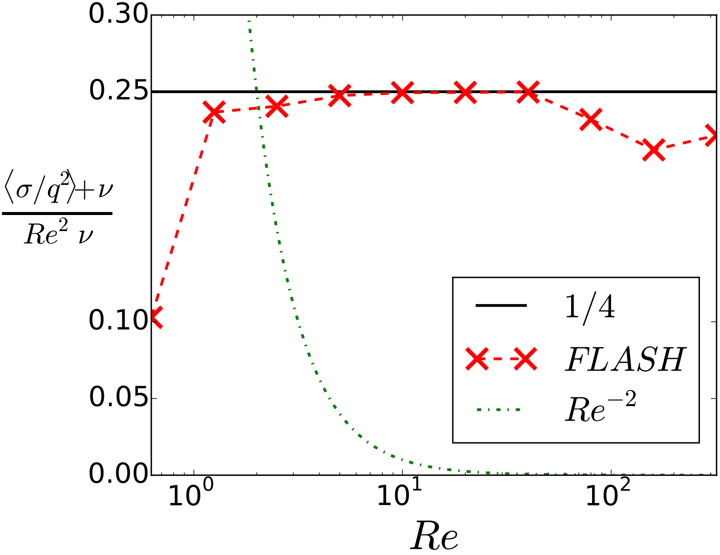


Table of contents

Flow & Theoretical prediction

Flow equation

$$U_x^{equi} = \sin(Kz) + \cos(Ky)\,, \tag{22}$$

$$U_y^{equi} = \sin(Kx) + \cos(Kz)\,, \tag{23}$$

$$U_z^{equi} = \sin(Ky) + \cos(Ky)\,. \tag{24}$$

Growth rate of the large scale instability

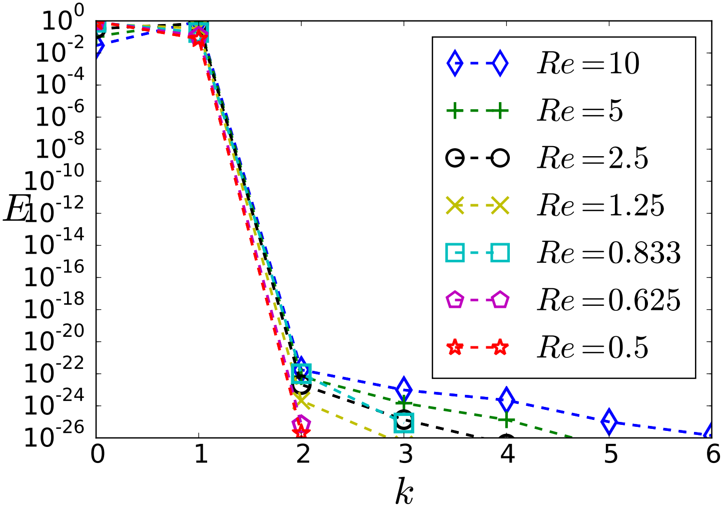
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{25}$$

$$\boxed{b = 0} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{26}$$

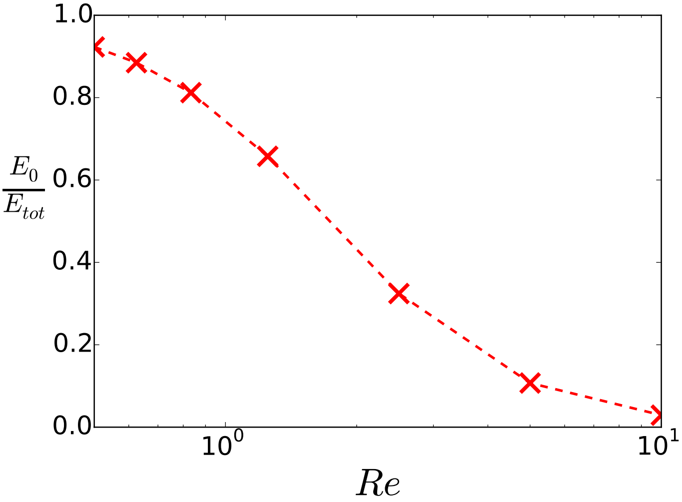
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right) \iff b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right). \tag{27}$$

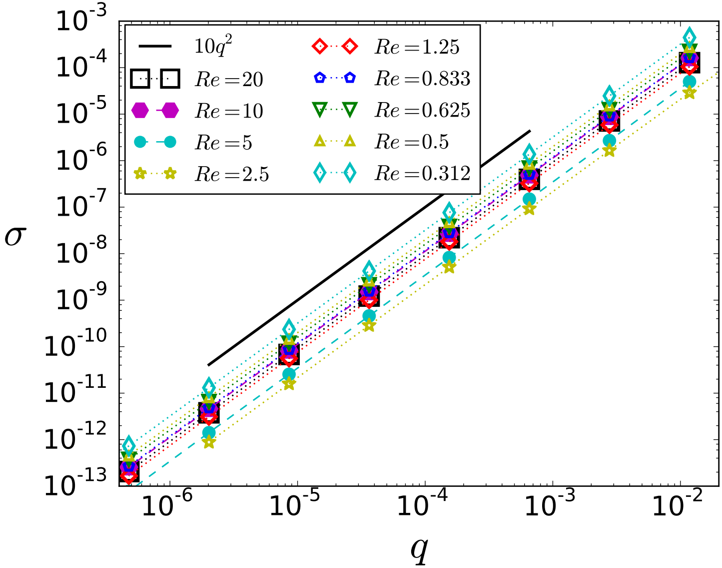
FLASH: Large scale energy ratio



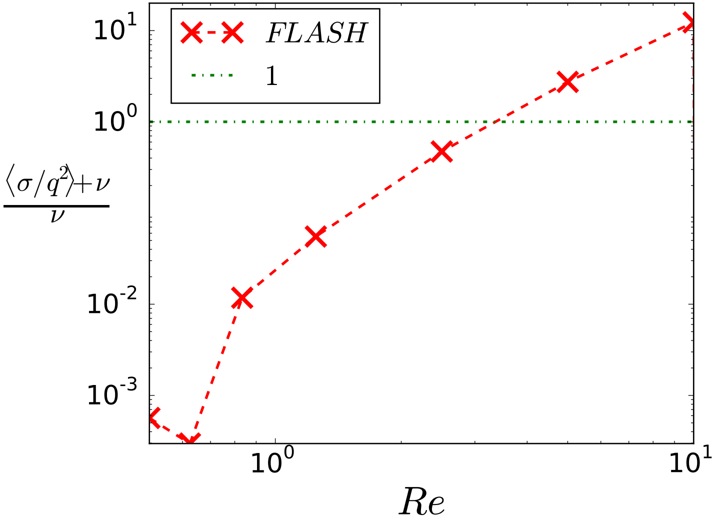
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



FLASH: Power-law pre-factor

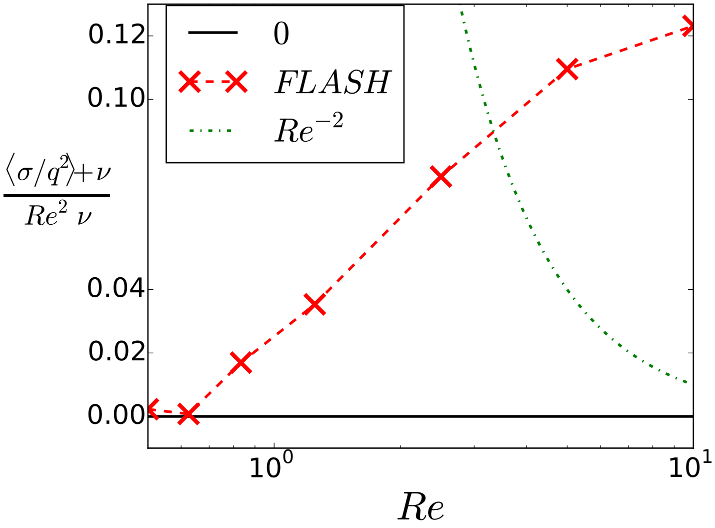


Table of contents

Flow & Theoretical prediction

Flow equation

$$U_x^\lambda = \lambda \sin(Kz) + \cos(Ky)\,, \tag{28}$$

$$U_y^\lambda = \sin(Kx) + \lambda \cos(Kz)\,, \tag{29}$$

$$U_z^\lambda = \sin(Ky) + \cos(Ky)\,. \tag{30}$$

Growth rate of the large scale instability

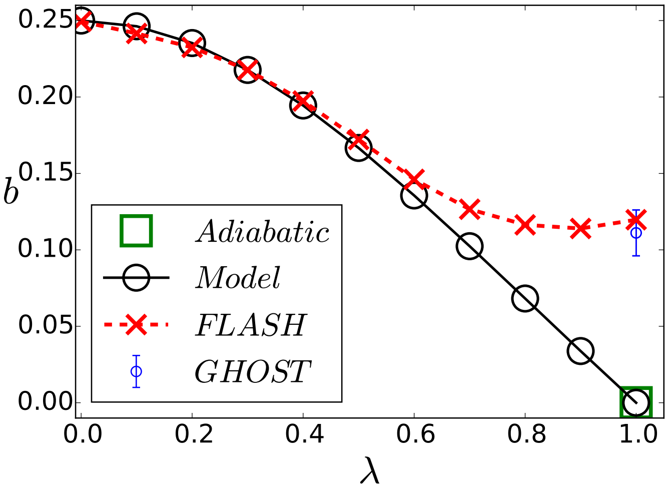
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{31}$$

$$\boxed{b = \frac{1-\lambda^2}{4+2\lambda^2}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{32}$$

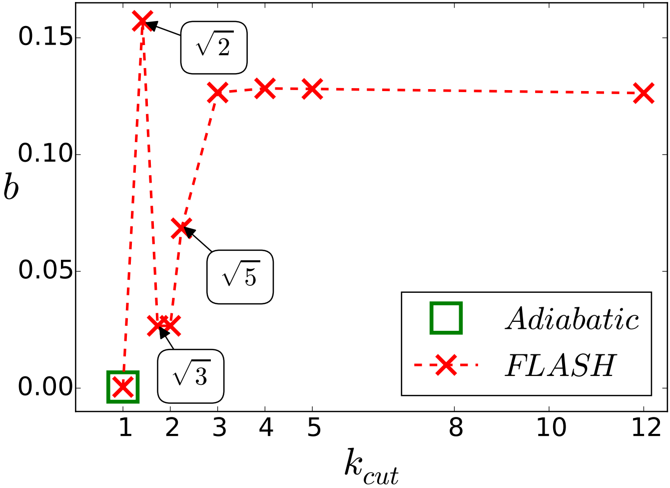
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \Longleftrightarrow \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right) \Longleftrightarrow b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right). \tag{33}$$

FLASH: Power-law pre-factor



FLASH: Fourier truncation



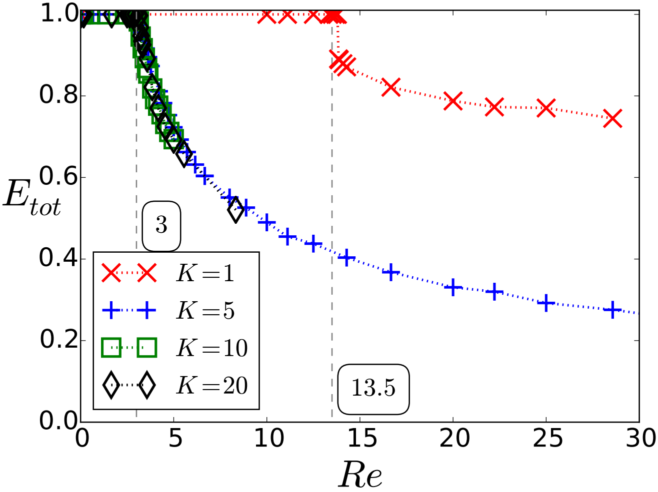
Critical Reynolds number

At the onset of the instability

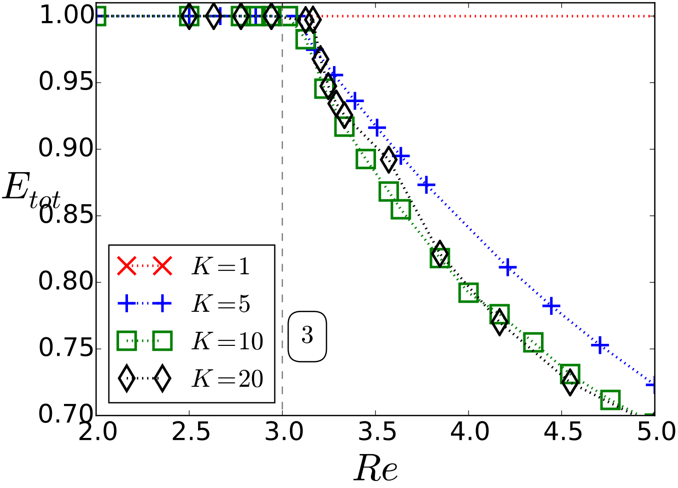
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu \,, \tag{34}$$

$$\sigma = 0 \iff \beta^c = \nu \iff \boxed{b^c = (Re^c)^{-2}} \,. \tag{35}$$

GHOST: Small scale instability



GHOST: Small scale instability zoom



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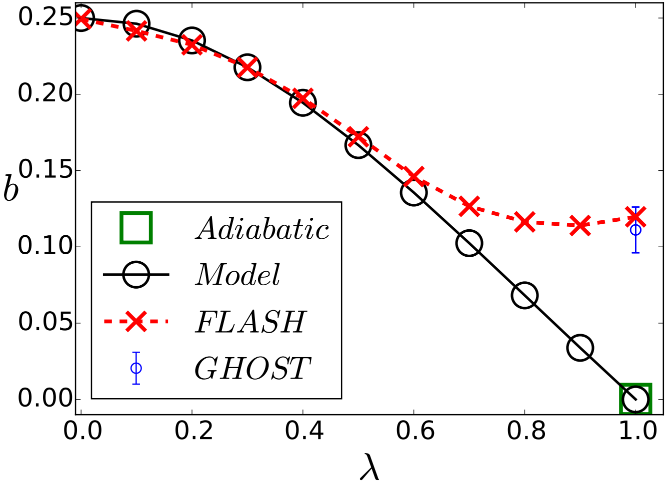


α-effect: overlook

Velocity and magnetic fields are divergence free: $\nabla\cdot U=0 \quad ; \quad \nabla\cdot B=0.$

Induction equation: $\partial_t B=\epsilon\nabla\times(U\times B)+\eta_0\nabla^2 B.$

Scale order: $\partial_t\to\partial_t+\epsilon^{\alpha}\partial_T \quad \text{and} \quad \partial_x\to\partial_x+\epsilon^{\beta}\partial_y.$

$(\partial_t-\eta_0\nabla_x^2)B=\epsilon\nabla_x\times(U\times B)+2\epsilon^{\beta}\eta_0\nabla_y\nabla_x B+\epsilon^{1+\beta}\nabla_y\times(U\times B)+\epsilon^{2\beta}\nabla_y^2 B-\epsilon^{\alpha}\partial_T B,$

$(\epsilon^{\alpha}\partial_T-\epsilon^{2\beta}\nu_0\nabla_y^2)B_0=\epsilon^{1+\beta}\nabla_y\times(U\times\epsilon B_1)=\epsilon^{2+\beta}\nabla_x\times(U\times B_1).$

therefore: $\alpha=2\beta=2+\beta \quad \text{and} \quad \beta=2 \iff \beta=2 \quad \text{and} \quad \alpha=4.$

Scales: $\partial_t\to\partial_t+\epsilon^{4}\partial_T \quad \text{and} \quad \partial_x\to\partial_x+\epsilon^{2}\partial_y.$

Magnetic field expansion: $B=B_0+\epsilon B_1+\sum\epsilon^i B_i,$

$(\partial_t-\eta_0\nabla_x^2)B=\epsilon\nabla_x\times(U\times B)+2\epsilon^2\eta_0\nabla_y\nabla_x B+\epsilon^3\nabla_y\times(U\times B)+\epsilon^4\nabla_y^2 B-\epsilon^4\partial_T B.$

Multi-scales equations:

$(\partial_t-\eta_0\nabla_x^2)B_0=0,$

$$(\partial_t-\eta_0\nabla_x^2)B_1=\nabla_x\times(U\times B_0),$$

$(\partial_t-\eta_0\nabla_x^2)B_2=\nabla_x\times(U\times B_1)+2\eta_0\underline{\nabla_y\nabla_x B_0},$

$(\partial_t-\eta_0\nabla_x^2)B_3=\nabla_x\times(U\times B_2)+2\eta_0\nabla_y\nabla_x B_1+\nabla_y\times(U\times B_0),$

$(\partial_t-\eta_0\nabla_x^2)B_4=\nabla_x\times(U\times B_3)+2\eta_0\nabla_y\nabla_x B_2+\nabla_y\times(U\times B_1)+\nabla_y^2 B_0-\partial_T B_0.$

Solvability equation:

$(\partial_t-\eta_0\nabla_x^2)B_1=\nabla_x\times(U\times B_0)=(B_0\cdot\nabla_x)U,$

$(\partial_T-\eta_0\nabla_y^2)B_0=\langle\nabla_y\times(U\times B_1)\rangle=\nabla_y\times\langle U\times B_1\rangle.$

Cannot simplify too quickly because: $\boldsymbol{\nabla}\cdot B=(\nabla_x\cdot+\epsilon^2\nabla_y\cdot)B=0 \implies \nabla_y\cdot B_1+\nabla_x\cdot B_3=0.$

AKA-effect: overlook

The pressure will not be written, velocities are incompressible: $\nabla\cdot V=0 \quad , \quad \nabla\cdot U=0 .$

Linearised Navier-Stokes: $\partial_t V=\epsilon(-(U\nabla)V-(V\nabla)U)+\nu_0\nabla^2 V .$

Scale order: $\partial_t\to\partial_t+\epsilon^\alpha\partial_T \quad , \quad \partial_x\to\partial_x+\epsilon^\beta\partial_y ,$

$$(\partial_t-\nu_0\nabla_x^2)V=-\epsilon(U\nabla_x)V-\epsilon(V\nabla_x)U+2\epsilon^\beta\nu_0\nabla_y\nabla_x V-\epsilon^{1+\beta}(U\nabla_y)V-\epsilon^{1+\beta}(V\nabla_y)U+\epsilon^{2\beta}\nu_0\nabla_y^2 V-\epsilon^\alpha\partial_T V .$$

AKA expansion $(\epsilon^\alpha\partial_T-\epsilon^{2\beta}\nu_0\nabla_y^2)V_1=-\epsilon^{1+\beta}(U\nabla_y)\epsilon V_2 .$

therefore $\alpha=2\beta \quad \text{and} \quad 2\beta=2+\beta \quad \text{and} \quad \beta=2 \quad \text{and} \quad \alpha=4 .$

Scales: $\partial_t\to\partial_t+\epsilon^4\partial_T \quad , \quad \partial_x\to\partial_x+\epsilon^2\partial_y .$

Velocity expansion: $V=V_1+\epsilon V_2+=\sum\epsilon^{i-1}V_i ,$

$$(\partial_t-\nu_0\nabla_x^2)V=-\epsilon(U\nabla_x)V-\epsilon(V\nabla_x)U+2\epsilon^2\nu_0\nabla_y\nabla_x V-\epsilon^3(U\nabla_y)V-\underline{\epsilon^3(V\nabla_y)U}+\epsilon^4\nu_0\nabla_y^2 V-\epsilon^4\partial_T V .$$

Multi-scale equations:

$(\partial_t-\nu_0\nabla_x^2)V_1=0 \quad\Rightarrow\quad V_1(y,T) ,$

$$(\partial_t-\nu_0\nabla_x^2)V_2=-\underline{(U\nabla_x)V_1}-(V_1\nabla_x)U ,$$

$(\partial_t-\nu_0\nabla_x^2)V_3=-(U\nabla_x)V_2-(V_2\nabla_x)U+\underline{2\nu_0\nabla_y\nabla_x V_1} ,$

$(\partial_t-\nu_0\nabla_x^2)V_4=-(U\nabla_x)V_3-(V_3\nabla_x)U+2\nu_0\nabla_y\nabla_x V_2-(U\nabla_y)V_1 ,$

$(\partial_t-\nu_0\nabla_x^2)V_5=-(U\nabla_x)V_4-(V_4\nabla_x)U+2\nu_0\nabla_y\nabla_x V_4+\Big[-(U\nabla_y)V_2+\nu_0\nabla_y^2 V_1-\partial_T V_1\Big] .$

Solvability equation:

$(\partial_t-\nu_0\nabla_x^2)V_2=-(V_1\nabla_x)U ,$

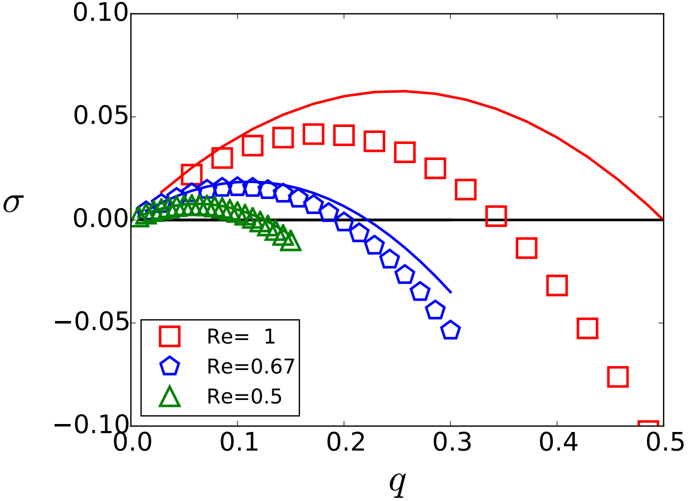
$(\partial_T-\nu_0\nabla_y^2)V_1=-\langle(U\nabla_y)V_2\rangle .$

Fourier space in x,t:

$V_2(k,\omega)=V_1^j\,\frac{-ik_j}{i\omega+\nu_0k^2}\,U(k,\omega) ,$

$$(\partial_T-\nabla_y^2)V_1=\left[\nabla_y V_1^j\right]\sum_{k,\omega}U(-k,\omega)\,\frac{ik_j}{i\omega+\nu_0k^2}\,U(k,\omega) .$$

FLASH: Fr87 growth linear scale



First order effects

α-effect:

$$(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B)\,, \tag{36}$$

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U\,, \tag{37}$$

$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle\,. \tag{38}$$

AKA-effect in vorticity:

$$(\partial_t - \eta_0 \nabla^2)\Omega = \epsilon\, [\nabla \times (U \times \Omega) + \nabla \times (\Omega \times U)]\,, \tag{39}$$

$$(\partial_t - \eta_0 \nabla_x^2)\Omega_1 = (\Omega_0 \cdot \nabla_x) U\,, \tag{40}$$

$$(\partial_T - \eta_0 \nabla_y^2)\Omega_0 = \nabla_y \times \langle U \times \Omega_1 \rangle + \nabla_y \times \langle \Omega_1 \times U \rangle\,. \tag{41}$$

α-effect

Velocity and magnetic fields are divergence free

$$\nabla \cdot U = 0 \quad ; \quad \nabla \cdot B = 0 . \tag{42}$$

Induction equation:

$$\partial_t B = \epsilon \nabla \times (U \times B) + \eta_0 \nabla^2 B . \tag{43}$$

Scale order:

$$\partial_t \rightarrow \partial_t + \epsilon^\alpha \partial_T \quad \text{and} \quad \partial_x \rightarrow \partial_x + \epsilon^\beta \partial_y . \tag{44}$$

$$(\partial_t - \eta_0 \nabla_x^2) B = \epsilon \nabla_x \times (U \times B) + 2\epsilon^\beta \eta_0 \nabla_y \nabla_x B \tag{45}$$

$$+ \epsilon^{1+\beta} \nabla_y \times (U \times B) + \epsilon^{2\beta} \nabla_y^2 B - \epsilon^\alpha \partial_T B . \tag{46}$$

α-effect

α expansion:

$$(\epsilon^{\alpha}\partial_T - \epsilon^{2\beta}\nu_0\nabla_y^2)B_0 = \epsilon^{1+\beta}\nabla_y \times (U \times \epsilon B_1) = \epsilon^{2+\beta}\nabla_x \times (U \times B_1)\,. \quad (47)$$

therefore:

$$\alpha = 2\beta \quad \text{and} \quad \beta = 2 \iff \beta = 2 \quad \text{and} \quad \alpha = 4\,. \quad (48)$$

Scales:

$$\partial_t \to \partial_t + \epsilon^4\partial_T \quad \text{and} \quad \partial_x \to \partial_x + \epsilon^2\partial_y\,. \quad (49)$$

Magnetic field expansion:

$$B = B_0 + \epsilon B_1 + \sum \epsilon^i B_i\,. \quad (50)$$

$$(\partial_t - \eta_0\nabla_x^2)B = \epsilon\nabla_x \times (U \times B) + 2\epsilon^2\eta_0\nabla_y\nabla_x B \quad (51)$$

$$+\epsilon^3\nabla_y \times (U \times B) + \epsilon^4\nabla_y^2 B - \epsilon^4\partial_T B. \quad (52)$$

α-effect

Multi-scales equations:

$$(\partial_t - \eta_0 \nabla_x^2) B_0 = 0\,, \tag{53}$$

$$\boxed{(\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_0)}\,, \tag{54}$$

$$(\partial_t - \eta_0 \nabla_x^2) B_2 = \nabla_x \times (U \times B_1) + 2\eta_0 \underline{\nabla_y \nabla_x B_0}\,, \tag{55}$$

$$(\partial_t - \eta_0 \nabla_x^2) B_3 = \nabla_x \times (U \times B_2) + 2\eta_0 \nabla_y \nabla_x B_1 + \nabla_y \times (U \times B_0)\,, \tag{56}$$

$$(\partial_t - \eta_0 \nabla_x^2) B_4 = \nabla_x \times (U \times B_3) + 2\eta_0 \nabla_y \nabla_x B_2 + \tag{57}$$

$$\boxed{\nabla_y \times (U \times B_1) + \nabla_y^2 B_0 - \partial_T B_0}\,. \tag{58}$$

α-effect

Solvability equation:

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = \nabla_x \times (U \times B_0) = (B_0 \cdot \nabla_x) U \,, \tag{59}$$

$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \langle \nabla_y \times (U \times B_1) \rangle = \nabla_y \times \langle U \times B_1 \rangle \,. \tag{60}$$

Cannot simplify too quickly because:

$$\boldsymbol{\nabla} \cdot B = (\nabla_x \cdot + \epsilon^2 \nabla_y \cdot) B = 0 \tag{61}$$

$$\nabla_y \cdot B_1 + \nabla_x \cdot B_3 = 0 \tag{62}$$

AKA-effect

The pressure will not be written, velocities are incompressible:

$$\nabla \cdot V = 0 \quad , \quad \nabla \cdot U = 0 \,. \tag{63}$$

Linearised Navier-Stokes:

$$\partial_t V = \epsilon(-(U\nabla) V - (V\nabla) U) + \nu_0 \nabla^2 V \,. \tag{64}$$

Scale order:

$$\partial_t \rightarrow \partial_t + \epsilon^\alpha \partial_T \quad , \quad \partial_x \rightarrow \partial_x + \epsilon^\beta \partial_y \,, \tag{65}$$

$$(\partial_t - \nu_0 \nabla_x^2) V = -\epsilon (U\nabla_x) V - \epsilon (V\nabla_x) U + 2\epsilon^\beta \nu_0 \nabla_y \nabla_x V \tag{66}$$

$$-\epsilon^{1+\beta} (U\nabla_y) V - \epsilon^{1+\beta} (V\nabla_y) U + \epsilon^{2\beta} \nu_0 \nabla_y^2 V - \epsilon^\alpha \partial_T V \,. \tag{67}$$

AKA-effect

AKA expansion:

$$(\epsilon^{\alpha}\partial_T - \epsilon^{2\beta}\nu_0\nabla_y^2)V_1 = -\epsilon^{1+\beta}(U\nabla_y)\epsilon V_2\,, \qquad (68)$$

therefore:

$$\alpha = 2\beta \quad \text{and} \quad 2\beta = 2+\beta \quad \text{and} \quad \beta = 2 \quad \text{and} \quad \alpha = 4\,. \qquad (69)$$

Scales:

$$\partial_t \to \partial_t + \epsilon^4\partial_T \quad , \quad \partial_x \to \partial_x + \epsilon^2\partial_y\,. \qquad (70)$$

Velocity expansion:

$$V = V_1 + \epsilon V_2 + = \sum \epsilon^{i-1}V_i\,, \qquad (71)$$

$$(\partial_t - \nu_0\nabla_x^2)V = -\epsilon(U\nabla_x)V - \epsilon(V\nabla_x)U + 2\epsilon^2\nu_0\nabla_y\nabla_x V \qquad (72)$$

$$-\epsilon^3(U\nabla_y)V - \underline{\epsilon^3(V\nabla_y)U} + \epsilon^4\nu_0\nabla_y^2 V - \epsilon^4\partial_T V\,. \qquad (73)$$

AKA-effect

Multi-scale equations:

$$(\partial_t - \nu_0 \nabla_x^2) V_1 = 0 \quad \Rightarrow \quad V_1(y, T)\,, \tag{74}$$

$$(\partial_t - \nu_0 \nabla_x^2) V_2 = -\underline{(U\nabla_x) V_1} - (V_1 \nabla_x) U\,, \tag{75}$$

$$(\partial_t - \nu_0 \nabla_x^2) V_3 = -(U\nabla_x) V_2 - (V_2 \nabla_x) U + \underline{2\nu_0 \nabla_y \nabla_x V_1}\,, \tag{76}$$

$$(\partial_t - \nu_0 \nabla_x^2) V_4 = -(U\nabla_x) V_3 - (V_3 \nabla_x) U + 2\nu_0 \nabla_y \nabla_x V_2 - (U\nabla_y) V_1\,, \tag{77}$$

$$(\partial_t - \nu_0 \nabla_x^2) V_5 = -(U\nabla_x) V_4 - (V_4 \nabla_x) U + 2\nu_0 \nabla_y \nabla_x V_4 \tag{78}$$

$$+ \boxed{\left[-(U\nabla_y) V_2 + \nu_0 \nabla_y^2 V_1 - \partial_T V_1 \right]}. \tag{79}$$

AKA-effect

Solvability equation:

$$(\partial_t - \nu_0 \nabla_x^2)\, V_2 = -(V_1 \nabla_x)\, U\,, \tag{80}$$

$$(\partial_T - \nu_0 \nabla_y^2)\, V_1 = -\langle (U \nabla_y)\, V_2 \rangle\,. \tag{81}$$

Fourier space in x, t :

$$V_2(k,\omega) = V_1^j \frac{-ik_j}{i\omega + \nu_0 k^2}\, U(k,\omega)\,, \tag{82}$$

$$(\partial_T - \nabla_y^2)\, V_1 = \left[\nabla_y V_1^j\right] \sum_{k,\omega} U(-k,\omega) \frac{ik_j}{i\omega + \nu_0 k^2}\, U(k,\omega)\,. \tag{83}$$