# Large scale instability of 3D helical flows

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Magnetic and kinetic similarities	FLASH 000000	Fr87 benchmark	Helical flows
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# Magnetic and kinetic similarities

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- Roberts flow
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  - Linear problem
  - Full non-linear problem

FLASH 00000C Fr87 benchmark

Helical flows

## Large scale magnetic fields



Magnetic	and	kinetic	simi	larities
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Fr87 benchmark

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## The induction and vorticity equations

Magnetic	Kinetic
$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}$	$\nabla \cdot \omega = 0$
$\partial_t B = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$	$\partial_t \omega = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \boldsymbol{v} \Delta \boldsymbol{\omega}$
	$\omega = \boldsymbol{\nabla} \times \boldsymbol{u}$

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Hand-waving			



Fr87 benchmark

Helical flows

#### **First order effects:** $\alpha$ **and** *AKA*

**Scales** 

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad , \quad \nabla_x \to \nabla_x + \epsilon^2 \nabla_y.$$

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 $\alpha$ -effect:  $\eta = \epsilon \eta_0$ 

 $(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B),$ 

 $(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U,$  $(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle.$  **Kinetic:**  $u = U + \epsilon V$ ;  $V = \epsilon^i V_i$ 

AKA-effect:  $v = \epsilon v_0$ 

$$\begin{aligned} (\partial_t - v_0 \nabla^2) V &= \\ \epsilon \left[ -(U \cdot \nabla) V - (V \cdot \nabla) U \right], \\ (\partial_t - v_0 \nabla_x^2) V_1 &= -(V_0 \cdot \nabla_x) U, \\ (\partial_T - v_0 \nabla_y^2) V_0 &= -\langle (U \cdot \nabla_y) V_1 \rangle. \end{aligned}$$

[Frisch et al. Phys. D 87]

### **Growth rate**

$$\sigma = \alpha q - \nu q^2$$
 with  $\alpha = a Re U$ 

Magnetic	and	kinetic	similarities	

Fr87 benchmark

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#### Second order effects: $\beta$ -effect and eddy viscosity

### Magnetic

 $\beta$ -effect

#### **Kinetic**

Negative eddy viscosity [Dubrulle & Frisch PRA91]

### **Growth rate**

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = b R e^2 \nu \,.$$

Fr87 benchmark

Helical flows

### Recap

First order		Second order	
• $\sigma = \alpha q - v q$	$\tau^2$	• $\sigma = \beta q^2 - v$	$q^2$
• $\alpha = aReU_0$		• $\beta = bRe^2 v$	
• $Re^c = vq/(a)$	$uU_0$ )	• $Re^c = b^{-1/2}$	
• $q^c = \alpha / v$		• Switch	
Magnetic	Kinetic	Magnetic	Kinetic
• α	• AKA	• β	• v < 0
• <i>B</i> , η, <i>Rm</i>	• v, v, Re	• <i>B</i> , η, <i>Rm</i>	• v, v, Re

Magnetic	and	kin	etic	ilar	ities	
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Fr87 benchmark

Helical flows

#### Derivation

#### **Linearised Navier-Stokes & Floquet Framework**

### **Non-Linear equation**

$$\partial_t \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{u} - \boldsymbol{\nabla} P + \boldsymbol{v} \Delta \boldsymbol{u} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{u}.$$

**Linearised equation:** 

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v}$$
 with  $||\boldsymbol{v}|| \ll ||\boldsymbol{U}||$ 

$$\partial_t \boldsymbol{U} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} P_K + v \Delta \boldsymbol{U} + \boldsymbol{F} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0 ,$$
  
$$\partial_t \boldsymbol{v} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{\nabla} \times \boldsymbol{U} - \boldsymbol{\nabla} P + v \Delta \boldsymbol{v} \quad , \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

#### **Floquet framework**

$$\boldsymbol{\nu}(\boldsymbol{r},t) = \tilde{\boldsymbol{\nu}}(\boldsymbol{r},t) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c. , \quad \boldsymbol{\rho}(\boldsymbol{r},t) = \tilde{\boldsymbol{\rho}}(\boldsymbol{r},t) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c.$$
  
$$\partial_{\boldsymbol{\chi}}\boldsymbol{\nu} = \left[\partial_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{r} - q_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{i} + \iota(q_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{r} + \partial_{\boldsymbol{\chi}}\tilde{\boldsymbol{\nu}}^{i})\right] e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + c.c. .$$

Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{\nu}} = (\boldsymbol{\nabla} \times \boldsymbol{U}) \times \tilde{\boldsymbol{\nu}} + (\iota \boldsymbol{q} \times \tilde{\boldsymbol{\nu}} + \boldsymbol{\nabla} \times \tilde{\boldsymbol{\nu}}) \times \boldsymbol{U} - (\iota \boldsymbol{q} + \boldsymbol{\nabla}) \tilde{\boldsymbol{p}} + \nu (\Delta - \boldsymbol{q}^2) \tilde{\boldsymbol{\nu}},$$
  
with  $\iota \boldsymbol{q} \cdot \tilde{\boldsymbol{\nu}} + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\nu}} = 0.$ 

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Derivation			

Floquet Linear Analysis of Spectral Hydrodynamics (FLASH)

#### Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{v}} = (\boldsymbol{\nabla} \times \boldsymbol{U}) \times \tilde{\boldsymbol{v}} + (\iota \boldsymbol{q} \times \tilde{\boldsymbol{v}} + \boldsymbol{\nabla} \times \tilde{\boldsymbol{v}}) \times \boldsymbol{U} - (\iota \boldsymbol{q} + \boldsymbol{\nabla}) \tilde{\boldsymbol{p}} + \nu(\Delta - \boldsymbol{q}^2) \tilde{\boldsymbol{v}},$$
  
with  $\iota \boldsymbol{q} \cdot \tilde{\boldsymbol{v}} + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{v}} = 0.$ 

#### Numeric method

- i. Compute the linear terms in Fourier space.
- **ii.** Compute convective terms in physical space.
- iii. Use 4th order explicit RK for the time evolution.

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3M model			
Hand-waving			



Magnetic and kinetic similarities	FLASH ○○○●○○	Fr87 benchmark	Helical flows
3M model			

#### **Formalism & Simplification**

### **Mode selection**

$$v(r, t) = v_q(r, t) + v_Q(r, t) + v_>(r, t), \qquad (1)$$

$$\boldsymbol{v}_{\boldsymbol{q}}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{q},t)e^{l\boldsymbol{q}\boldsymbol{r}} + c.c., \qquad (2)$$

$$\boldsymbol{\nu}_{\boldsymbol{Q}}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||=1} \tilde{\boldsymbol{\nu}}(\boldsymbol{q},\boldsymbol{k},t) e^{i(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c., \qquad (3)$$

$$\boldsymbol{\nu}_{>}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||>1} \tilde{\boldsymbol{\nu}}(\boldsymbol{q},\boldsymbol{k},t) e^{i(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.$$
(4)

#### **Additional hypothesis**

- Smallest are greatest:
- Adiabatic hypothesis:
- Helical flow:

$$\begin{split} ||\boldsymbol{v}_{>}|| \ll ||\boldsymbol{v}_{\boldsymbol{q}}|| \, . \\ \partial_{t} \boldsymbol{v}_{\boldsymbol{Q}} \ll \boldsymbol{v} \Delta \boldsymbol{v}_{\boldsymbol{Q}} \, . \\ \boldsymbol{U}_{hel}(\boldsymbol{r}) = K^{-1} \boldsymbol{\nabla} \times \boldsymbol{U}_{hel}(\boldsymbol{r}) \, . \end{split}$$

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3M model			
Equations			

### **Equations before simplification**

$$\partial_t \boldsymbol{v}_{\boldsymbol{q}} = \boldsymbol{U} \times \boldsymbol{\nabla} \times \boldsymbol{v}_{\boldsymbol{Q}} + \boldsymbol{v}_{\boldsymbol{Q}} \times \boldsymbol{\nabla} \times \boldsymbol{\mathcal{U}}^{\mathsf{K} \boldsymbol{U}_{hel}} - \boldsymbol{\nabla} p_{\boldsymbol{q}} + \boldsymbol{v} \Delta \boldsymbol{v}_{\boldsymbol{q}}.$$

$$\partial_t \boldsymbol{v}_{\boldsymbol{Q}} = \boldsymbol{U} \times \boldsymbol{\nabla} \times (\boldsymbol{v}_{\boldsymbol{q}} + \boldsymbol{v}_{\boldsymbol{S}}) + (\boldsymbol{v}_{\boldsymbol{q}} + \boldsymbol{v}_{\boldsymbol{S}}) \times \boldsymbol{\nabla} \times \boldsymbol{\mathcal{U}}^{\mathsf{K} \boldsymbol{U}_{hel}} - \boldsymbol{\nabla} p_{\boldsymbol{Q}} + \boldsymbol{v} \Delta \boldsymbol{v}_{\boldsymbol{Q}}.$$

### Simplified vorticity equations

$$v\Delta\omega_{\boldsymbol{Q}} = -\boldsymbol{\nabla} \times \left[ \boldsymbol{U}_{hel} \times (\boldsymbol{\omega}_{\boldsymbol{q}} - K\boldsymbol{v}_{\boldsymbol{q}}) \right], \tag{5}$$

$$\partial_t \boldsymbol{\omega}_{\boldsymbol{q}} = \boldsymbol{\nabla} \times \left[ \boldsymbol{U}_{hel} \times (\boldsymbol{\omega}_{\boldsymbol{Q}} - K \boldsymbol{\nu}_{\boldsymbol{Q}}) \right] + \boldsymbol{\nu} \Delta \boldsymbol{\omega}_{\boldsymbol{q}}.$$
(6)

### **Prediction for** $\lambda$ -*ABC* **flows (**A=1:B=1:C= $\lambda$ **)**

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu, \tag{7}$$
$$b = \frac{1 - \lambda^2}{4 + 2\lambda^2} \quad \text{and} \quad Re = \frac{U}{K\nu}. \tag{8}$$

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Fr87 benchmark

Helical flows

#### **Flow & Theoretical prediction**

### **Flow equation**

$$U_x^{Fr87} = U_0 \cos(Ky + \nu K^2 t),$$
 (9)

$$U_{y}^{Fr87} = U_{0} \sin\left(Kx - vK^{2}t\right),$$
 (10)

$$U_z^{Fr87} = U_x^{Fr87} + U_y^{Fr87}.$$
 (11)

### Growth rate of the large scale instability

$$\sigma = \alpha q - \nu q^2$$
 with  $\alpha = a Re U_0$  and  $a = \frac{1}{2}$ . (12)

#### **Determining** *a* in the $q \ll 1$ limit

$$\alpha = \left\langle \frac{\sigma}{q} \right\rangle \iff \frac{\alpha}{U_0} = \frac{1}{U_0} \left\langle \frac{\sigma}{q} \right\rangle \iff a = \frac{1}{ReU_0} \left\langle \frac{\sigma}{q} \right\rangle = \frac{1}{2}.$$
 (13)

Fr87 benchmark

Helical flows

#### FLASH: Large scale energy ratio



Magnetic and kinetic similarities	FLASH	Fr87 benchmark	Helical flows

## FLASH: Large scale energy ratio



Helical flows

#### **FLASH: Growth rate**



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#### **FLASH: Power-law**



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#### Roberts flow

### Flow & Theoretical prediction

## **Flow equation**

$$U_x^{Rob} = \cos(Ky), \qquad (14)$$

$$U_{y}^{Rob} = \sin(Kx), \qquad (15)$$

$$U_z^{Rob} = \sin(Kx) + \cos(Ky).$$
(16)

### Growth rate of the large scale instability

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v, \tag{17}$$
$$b = \frac{1}{4} \quad \text{and} \quad R e = \frac{U}{K v}. \tag{18}$$

#### **Determining** *b*

$$\beta - v = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{v} = \frac{1}{v} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + v \right) \iff b = \frac{1}{Re^2 v} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + v \right).$$
(19)

Fr87 benchmark

Helical flows ○●○○○○○○○○○○○○○○○○

#### Roberts flow

FLASH: Large scale energy ratio



Fr87 benchmark

Helical flows

#### Roberts flow

### FLASH: Large scale energy ratio



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Fr87 benchmark

Helical flows

#### Roberts flow

#### **FLASH: Growth rate**



Fr87 benchmark

Helical flows

#### Roberts flow

### **FLASH: Power-law**



#### Equilateral ABC flow

### Flow & Theoretical prediction

## **Flow equation**

$$U_x^{equi} = \sin(Kz) + \cos(Ky), \qquad (20)$$

$$U_{y}^{equi} = \sin(Kx) + \cos(Kz), \qquad (21)$$

$$U_z^{equi} = \sin(Ky) + \cos(Ky).$$
<sup>(22)</sup>

## Growth rate of the large scale instability

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v, \tag{23}$$
$$\boxed{b=0} \quad \text{and} \quad R e = \frac{U}{K v}. \tag{24}$$

#### **Determining** *b*

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{\nu} = \frac{1}{\nu} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right) \iff b = \frac{1}{Re^2\nu} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + \nu \right).$$
(25)

Fr87 benchmark

Helical flows

#### Equilateral ABC flow

FLASH: Large scale energy ratio



Fr87 benchmark

Helical flows

#### Equilateral ABC flow

### FLASH: Large scale energy ratio



Helical flows

#### Equilateral ABC flow

#### **FLASH: Growth rate**



Magnetic and kinetic similarities	FLASH 000000	Fr87 benchmark	<b>Helical flows</b>
Equilateral ABC flow			
FLASH: Power-law			



#### $\lambda - ABC$ flows

### Flow & Theoretical prediction

## **Flow equation**

$U_x^{\lambda} = \lambda \sin(Kz) + \cos(Ky)$ ,	(26)
$U_x^{\lambda} = \lambda \sin(Kz) + \cos(Ky),$	(26

$$J_{y}^{\lambda} = \sin(Kx) + \lambda \cos(Kz), \qquad (27)$$

$$U_z^{\lambda} = \sin(Ky) + \cos(Kx).$$
<sup>(28)</sup>

### Growth rate of the large scale instability

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2 \nu, \tag{29}$$
$$b = \frac{1 - \lambda^2}{4 + 2\lambda^2} \quad \text{and} \quad Re = \frac{U}{K\nu}. \tag{30}$$

### **Determining** *b*

$$\beta - v = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{v} = \frac{1}{v} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + v \right) \iff b = \frac{1}{Re^2 v} \left( \left\langle \frac{\sigma}{q^2} \right\rangle + v \right).$$
(31)

Fr87 benchmark

Helical flows

#### $\lambda$ –ABC flows

#### **FLASH: Power-law pre-factor**



Magnetic and kinetic similarities	FLASH 000000	Fr87 benchmark	Helical flows
$\lambda - ABC$ flows			





Fr87 benchmark

Helical flows

 $\lambda$ –ABC flows

#### **GHOST : Instability spectrum** *KL* = 20



Fr87 benchmark

Helical flows

#### $\lambda$ –ABC flows

#### **GHOST : Instability spectra**



Fr87 benchmark

Helical flows

#### $\lambda$ –ABC flows

### **GHOST: Small scale instability**



[Dombre et al. 86; Podvigina & Pouquet Phys. D 94]

Fr87 benchmark

Helical flows

#### $\lambda$ –ABC flows

#### GHOST: Small scale instability zoom



Magnetic	and	kine	tic	sim	ilarit	

Fr87 benchmark

Helical flows

 $\lambda$ –ABC flows

**Critical Reynolds number** 

## At the onset of the instability

$$\sigma = \beta q^2 - v q^2 \quad \text{with} \quad \beta = b R e^2 v, \quad (32)$$
$$\sigma = 0 \iff \beta^c = v \iff b^c = (R e^c)^{-2}. \quad (33)$$

Magnetic and kinetic similarities	FLASH 000000	Fr87 benchmark	Helical flows ○○○○○○○○○○○○○○○○○○●
$\lambda - ABC$ flows			
$\lambda - ABC$ flows graph			



On arXiv, submitted to PR Fluids

#### Thank you for your attention

## • Also investigated in HD:

- Turbulent ABC flow
- Study of  $E_0/E_{tot}$

## • Also investigated in Induction:

- Study of the *ABC*-dynamo
- Study of magnetic response to non-helical flow
- Study of magnetic response to  $\delta\text{-correlated}$  flows

On arXiv, submitted to PRL

# Thank you for your attention