Large scale instability of 3D helical flows

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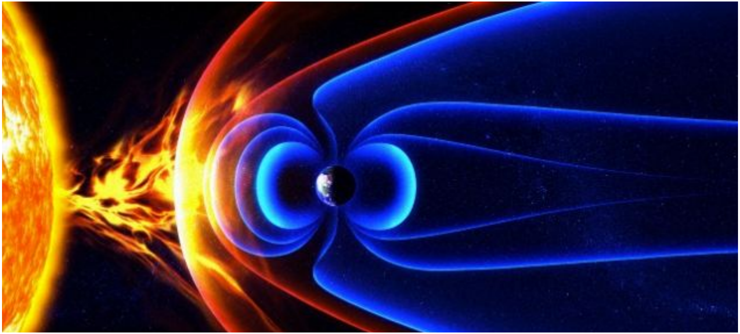
Large scale magnetic fields



The induction and vorticity equations

Magnetic

$$\nabla \cdot B = 0$$

$$\partial_t B = \nabla \times (u \times B) + \eta \Delta B$$

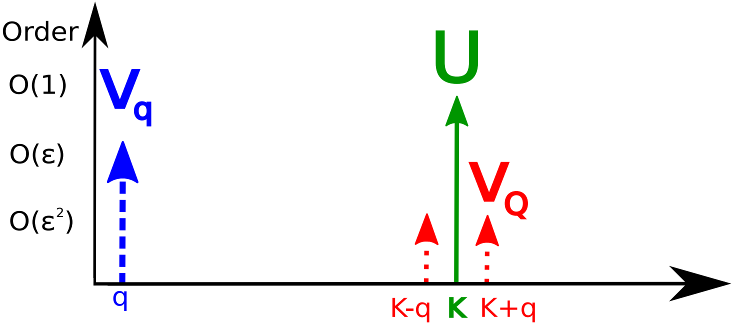
Kinetic

$$\nabla \cdot \omega = 0$$

$$\partial_t \omega = \nabla \times (u \times \omega) + \nu \Delta \omega$$

$$\omega = \nabla \times u$$

Hand-waving



Scales

- K : forcing scale
- q : large scale
- k : Fourier mode

First order effects: α and *AKA*

Scales

$$\partial_t \to \partial_t + \epsilon^4 \partial_T \quad , \quad \nabla_x \to \nabla_x + \epsilon^2 \nabla_y .$$

Magnetic $B = \epsilon^i B_i$

α-effect: $\eta = \epsilon \eta_0$

$$(\partial_t - \eta_0 \nabla^2) B = \epsilon \nabla \times (U \times B) ,$$

$$(\partial_t - \eta_0 \nabla_x^2) B_1 = (B_0 \cdot \nabla_x) U ,$$
$$(\partial_T - \eta_0 \nabla_y^2) B_0 = \nabla_y \times \langle U \times B_1 \rangle .$$

Kinetic: $u = U + \epsilon V$; $V = \epsilon^i V_i$

AKA-effect: $\nu = \epsilon \nu_0$

$$(\partial_t - \nu_0 \nabla^2) V = \epsilon \left[-(U \cdot \nabla) V - (V \cdot \nabla) U \right] ,$$

$$(\partial_t - \nu_0 \nabla_x^2) V_1 = -(V_0 \cdot \nabla_x) U ,$$
$$(\partial_T - \nu_0 \nabla_y^2) V_0 = -\langle (U \cdot \nabla_y) V_1 \rangle .$$

[Frisch *et al.* Phys. D 87]

Growth rate

$$\boxed{\sigma = \alpha q - \nu q^2} \quad \text{with} \quad \boxed{\alpha = aReU} .$$

Second order effects: β-effect and eddy viscosity

Magnetic

β-effect

Kinetic

Negative eddy viscosity
[Dubrulle & Frisch PRA91]

Growth rate

$$\boxed{\sigma = \beta q^2 - \nu q^2} \quad \text{with} \quad \boxed{\beta = bRe^2\nu}.$$

Recap

First order

- $\sigma = \alpha q - \nu q^2$
- $\alpha = aReU_0$
- $Re^c = \nu q / (aU_0)$
- $q^c = \alpha / \nu$

Magnetic

- α
- B, η, Rm

Kinetic

- AKA
- v, ν, Re

Second order

- $\sigma = \beta q^2 - \nu q^2$
- $\beta = bRe^2 \nu$
- $Re^c = b^{-1/2}$
- Switch

Magnetic

- β
- B, η, Rm

Kinetic

- $\nu < 0$
- v, ν, Re

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Linearised Navier-Stokes & Floquet Framework

Non-Linear equation

$$\partial_t \boldsymbol{u} = \boldsymbol{u}\times\nabla\times\boldsymbol{u} - \nabla P + \nu\Delta\boldsymbol{u} + \boldsymbol{F} \quad , \quad \nabla\cdot\boldsymbol{u}.$$

Linearised equation: $\boxed{\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v}}$ with $||\boldsymbol{v}|| \ll ||\boldsymbol{U}||$

$$\partial_t \boldsymbol{U} = \boldsymbol{U}\times\nabla\times\boldsymbol{U} - \nabla P_K + \nu\Delta\boldsymbol{U} + \boldsymbol{F} \quad , \quad \nabla\cdot\boldsymbol{U} = 0,$$
$$\partial_t \boldsymbol{v} = \boldsymbol{U}\times\nabla\times\boldsymbol{v} + \boldsymbol{v}\times\nabla\times\boldsymbol{U} - \nabla P + \nu\Delta\boldsymbol{v} \quad , \quad \nabla\cdot\boldsymbol{v} = 0.$$

Floquet framework

$$\boldsymbol{v}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{r},t)e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c. \quad , \quad p(\boldsymbol{r},t) = \tilde{p}(\boldsymbol{r},t)e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c.,$$
$$\partial_x \boldsymbol{v} = \left[\partial_x\tilde{\boldsymbol{v}}^r - q_x\tilde{\boldsymbol{v}}^i + \imath(q_x\tilde{\boldsymbol{v}}^r + \partial_x\tilde{\boldsymbol{v}}^i)\right] e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c..$$

Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{v}} = (\nabla\times\boldsymbol{U})\times\tilde{\boldsymbol{v}} + (\imath\boldsymbol{q}\times\tilde{\boldsymbol{v}} + \nabla\times\tilde{\boldsymbol{v}})\times\boldsymbol{U} - (\imath\boldsymbol{q} + \nabla)\tilde{p} + \nu(\Delta - \boldsymbol{q}^2)\tilde{\boldsymbol{v}},$$
$$\text{with} \quad \imath\boldsymbol{q}\cdot\tilde{\boldsymbol{v}} + \nabla\cdot\tilde{\boldsymbol{v}} = 0.$$

Floquet Linear Analysis of Spectral Hydrodynamics (FLASH)

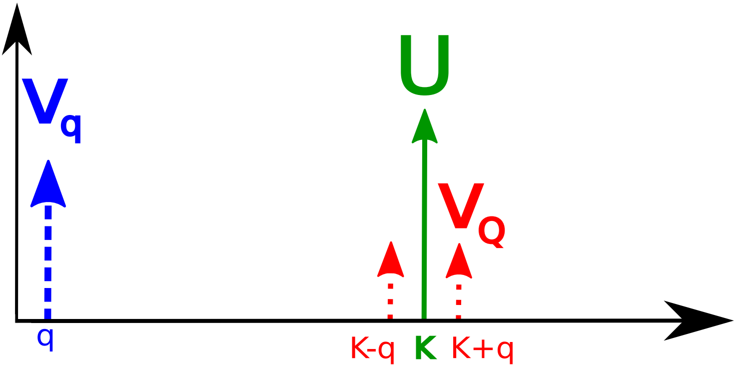
Linearised Navier-Stokes equations with the Floquet framework

$$\partial_t \tilde{\boldsymbol{v}} = (\nabla \times \boldsymbol{U}) \times \tilde{\boldsymbol{v}} + (\imath \boldsymbol{q} \times \tilde{\boldsymbol{v}} + \nabla \times \tilde{\boldsymbol{v}}) \times \boldsymbol{U} - (\imath \boldsymbol{q} + \nabla) \tilde{p} + \nu (\Delta - \boldsymbol{q}^2) \tilde{\boldsymbol{v}},$$
$$\text{with} \quad \imath \boldsymbol{q} \cdot \tilde{\boldsymbol{v}} + \nabla \cdot \tilde{\boldsymbol{v}} = 0\,.$$

Numeric method

i. Compute the linear terms in Fourier space.
ii. Compute convective terms in physical space.
iii. Use 4th order explicit RK for the time evolution.

Hand-waving



Formalism & Simplification

Mode selection

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v_q}(\boldsymbol{r},t) + \boldsymbol{v_Q}(\boldsymbol{r},t) + \boldsymbol{v}_>(\boldsymbol{r},t)\,, \tag{1}$$

$$\boldsymbol{v_q}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{q},t)\,e^{\imath \boldsymbol{q}\boldsymbol{r}} + c.c.\,, \tag{2}$$

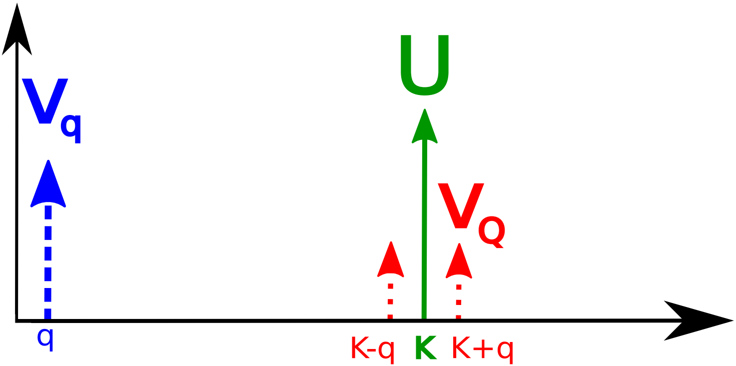
$$\boldsymbol{v_Q}(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||=1} \tilde{\boldsymbol{v}}(\boldsymbol{q},\boldsymbol{k},t)\,e^{\imath(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.\,, \tag{3}$$

$$\boldsymbol{v}_>(\boldsymbol{r},t) = \sum_{||\boldsymbol{k}||>1} \tilde{\boldsymbol{v}}(\boldsymbol{q},\boldsymbol{k},t)\,e^{\imath(\boldsymbol{q}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r})} + c.c.\,. \tag{4}$$

Additional hypothesis

- Smallest are greatest: $||\boldsymbol{v}_>|| \ll ||\boldsymbol{v_q}||\,.$
- Adiabatic hypothesis: $\partial_t \boldsymbol{v_Q} \ll \nu \Delta \boldsymbol{v_Q}\,.$
- Helical flow: $\boldsymbol{U}_{hel}(\boldsymbol{r}) = K^{-1}\boldsymbol{\nabla}\times\boldsymbol{U}_{hel}(\boldsymbol{r})\,.$

Hand-waving



Equations

Equations before simplification

$$\partial_t \boldsymbol{v_q} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{v_Q} + \boldsymbol{v_Q}\times\underbrace{\boldsymbol{\nabla}\times\boldsymbol{U}}_{\to K\boldsymbol{U}_{hel}} - \boldsymbol{\nabla} p_{\boldsymbol{q}} + \nu\Delta\boldsymbol{v_q}\,.$$

$$\cancel{\partial_t \boldsymbol{v_Q}} = \boldsymbol{U}\times\boldsymbol{\nabla}\times(\boldsymbol{v_q} + \cancel{\boldsymbol{v_Q}}) + (\boldsymbol{v_q} + \cancel{\boldsymbol{v_Q}})\times\underbrace{\boldsymbol{\nabla}\times\boldsymbol{U}}_{\to K\boldsymbol{U}_{hel}} - \boldsymbol{\nabla} p_{\boldsymbol{Q}} + \nu\Delta\boldsymbol{v_Q}\,.$$

Simplified vorticity equations

$$\nu\Delta\boldsymbol{\omega_Q} = -\boldsymbol{\nabla}\times\left[\boldsymbol{U}_{hel}\times(\boldsymbol{\omega_q} - K\boldsymbol{v_q})\right], \tag{5}$$

$$\partial_t\boldsymbol{\omega_q} = \boldsymbol{\nabla}\times\left[\boldsymbol{U}_{hel}\times(\boldsymbol{\omega_Q} - K\boldsymbol{v_Q})\right] + \nu\Delta\boldsymbol{\omega_q}\,. \tag{6}$$

Prediction for $\lambda - ABC$ flows ($A=1:B=1:C=\lambda$)

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{7}$$

$$\boxed{b = \frac{1-\lambda^2}{4+2\lambda^2}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{8}$$

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Flow & Theoretical prediction

Flow equation

$$U_x^{Fr87} = U_0 \cos\left(Ky + \nu K^2 t\right), \quad (9)$$
$$U_y^{Fr87} = U_0 \sin\left(Kx - \nu K^2 t\right), \quad (10)$$
$$U_z^{Fr87} = U_x^{Fr87} + U_y^{Fr87}. \quad (11)$$

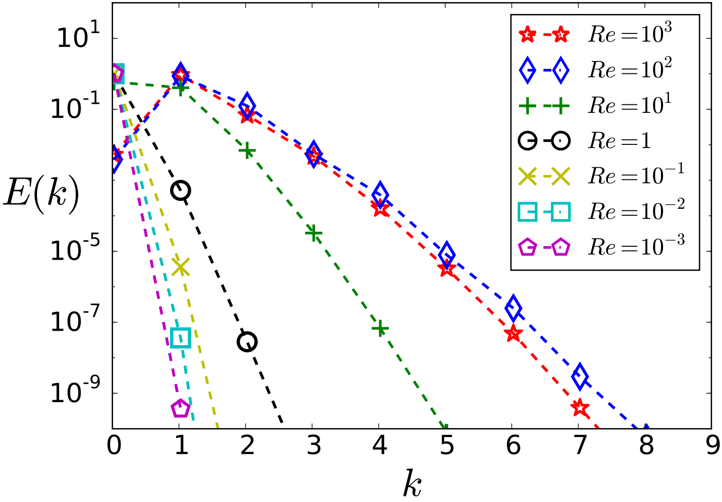
Growth rate of the large scale instability

$$\sigma = \alpha q - \nu q^2 \quad \text{with} \quad \alpha = aReU_0 \quad \text{and} \quad \boxed{a = \frac{1}{2}}. \quad (12)$$

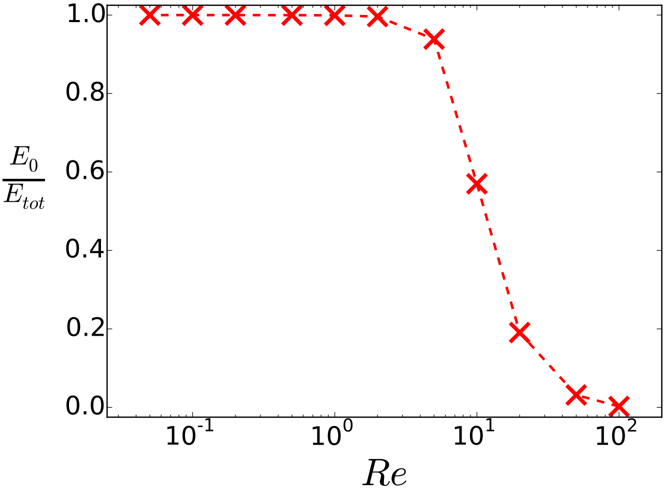
Determining a in the $q \ll 1$ limit

$$\alpha = \left\langle \frac{\sigma}{q} \right\rangle \iff \frac{\alpha}{U_0} = \frac{1}{U_0}\left\langle \frac{\sigma}{q} \right\rangle \iff a = \frac{1}{ReU_0}\left\langle \frac{\sigma}{q} \right\rangle = \frac{1}{2}. \quad (13)$$

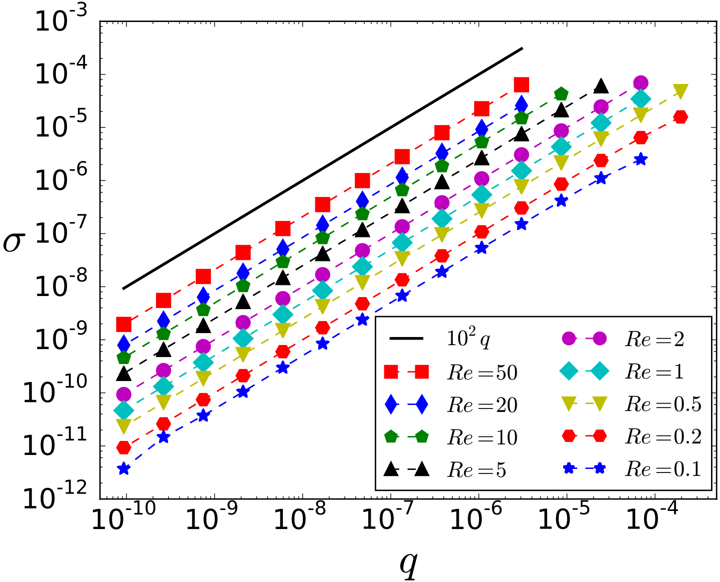
FLASH: Large scale energy ratio



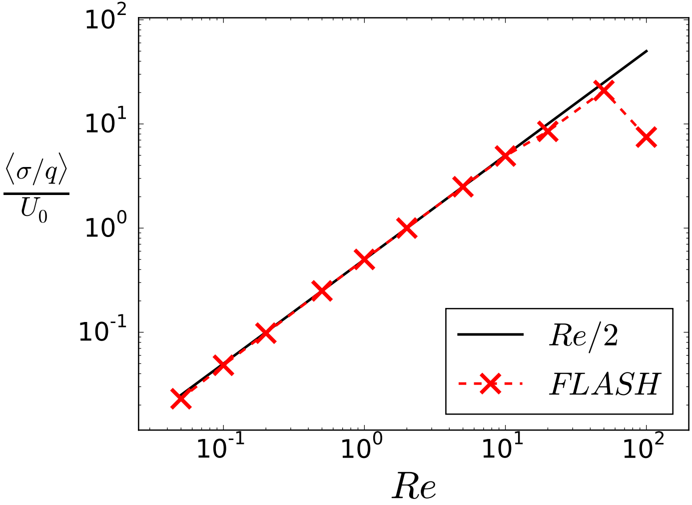
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



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Flow & Theoretical prediction

Flow equation

$$U_x^{Rob} = \cos(Ky)\,, \qquad (14)$$

$$U_y^{Rob} = \sin(Kx)\,, \qquad (15)$$

$$U_z^{Rob} = \sin(Kx) + \cos(Ky)\,. \qquad (16)$$

Growth rate of the large scale instability

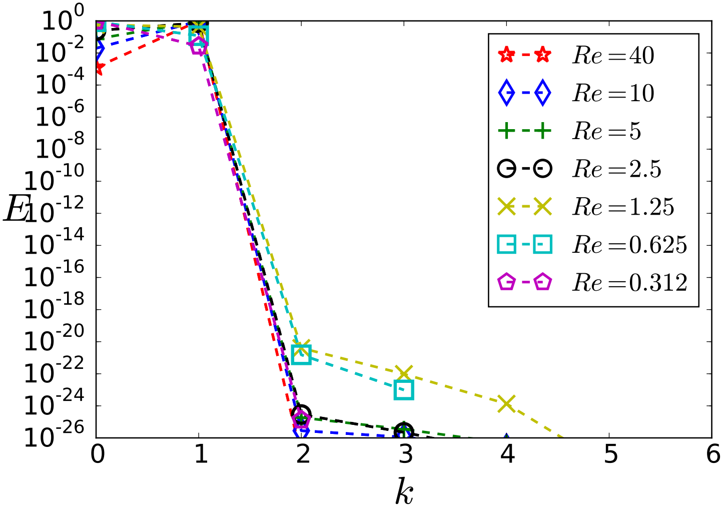
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \qquad (17)$$

$$\boxed{b = \frac{1}{4}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \qquad (18)$$

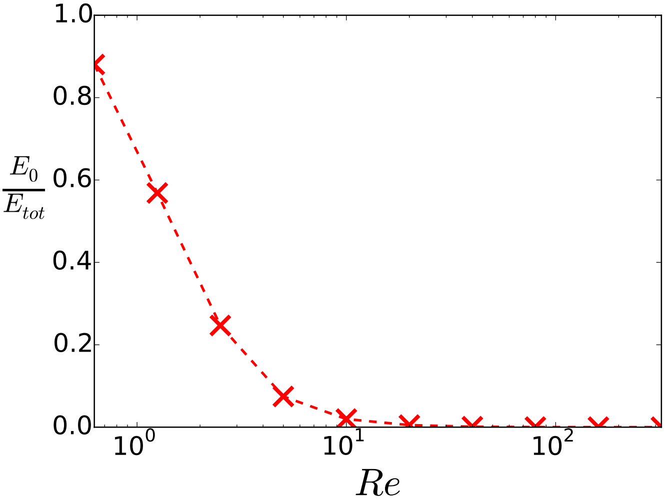
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \iff \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right) \iff b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right). \qquad (19)$$

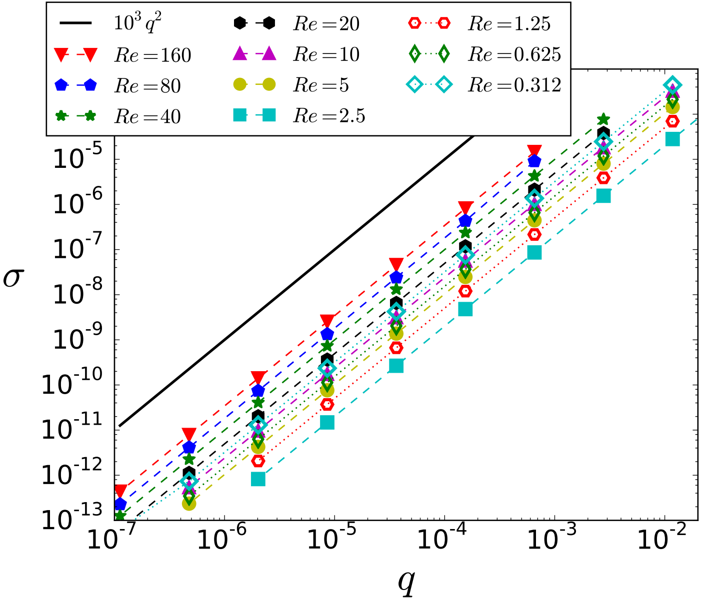
FLASH: Large scale energy ratio



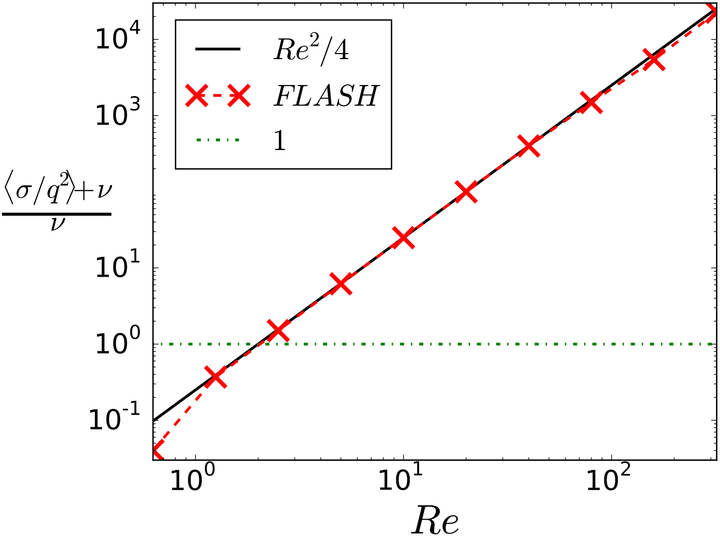
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



Flow & Theoretical prediction

Flow equation

$$U_x^{equi} = \sin(Kz) + \cos(Ky)\,, \tag{20}$$

$$U_y^{equi} = \sin(Kx) + \cos(Kz)\,, \tag{21}$$

$$U_z^{equi} = \sin(Ky) + \cos(Ky)\,. \tag{22}$$

Growth rate of the large scale instability

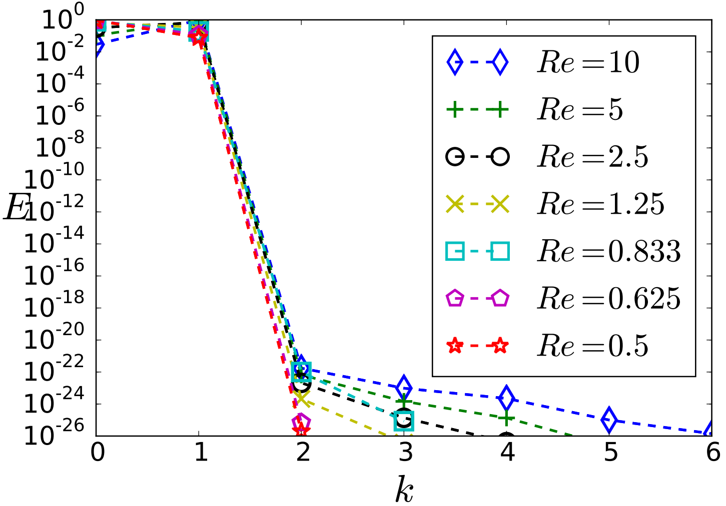
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{23}$$

$$\boxed{b = 0} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{24}$$

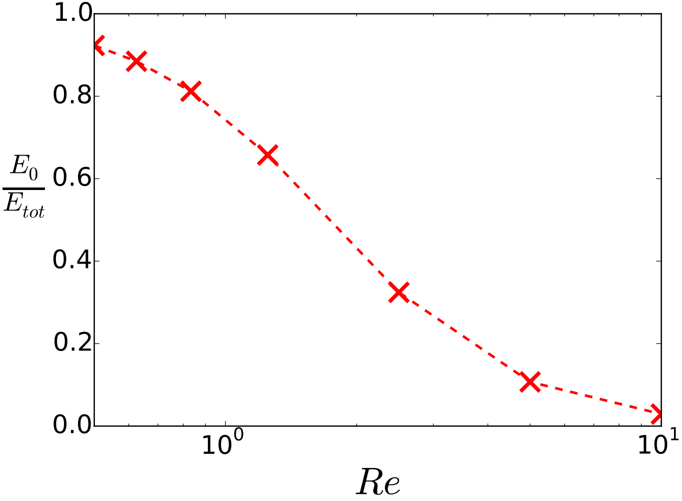
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \Longleftrightarrow \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right) \Longleftrightarrow b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right). \tag{25}$$

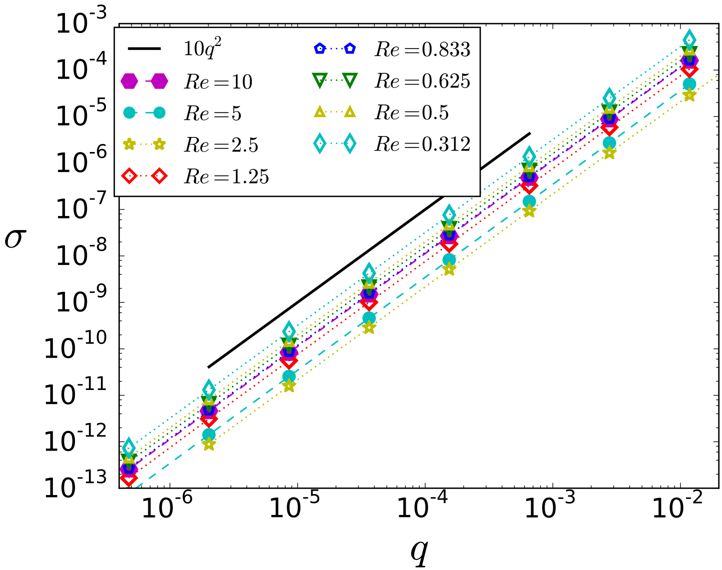
FLASH: Large scale energy ratio



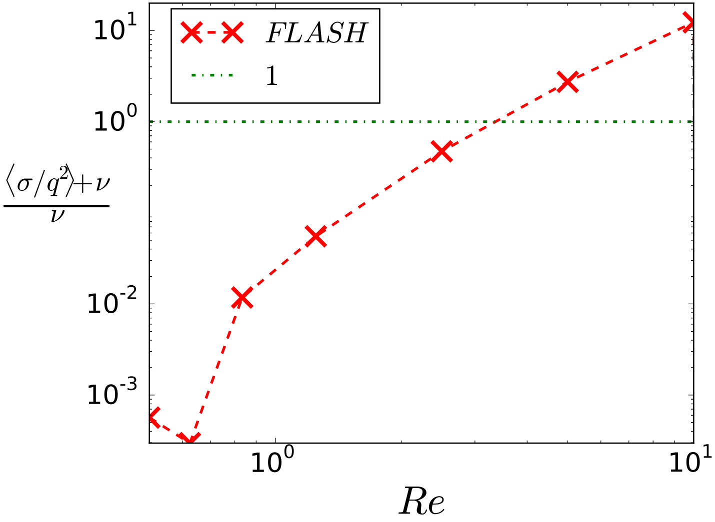
FLASH: Large scale energy ratio



FLASH: Growth rate



FLASH: Power-law



Flow & Theoretical prediction

Flow equation

$$U_x^\lambda = \lambda \sin(Kz) + \cos(Ky)\,, \tag{26}$$

$$U_y^\lambda = \sin(Kx) + \lambda \cos(Kz)\,, \tag{27}$$

$$U_z^\lambda = \sin(Ky) + \cos(Kx)\,. \tag{28}$$

Growth rate of the large scale instability

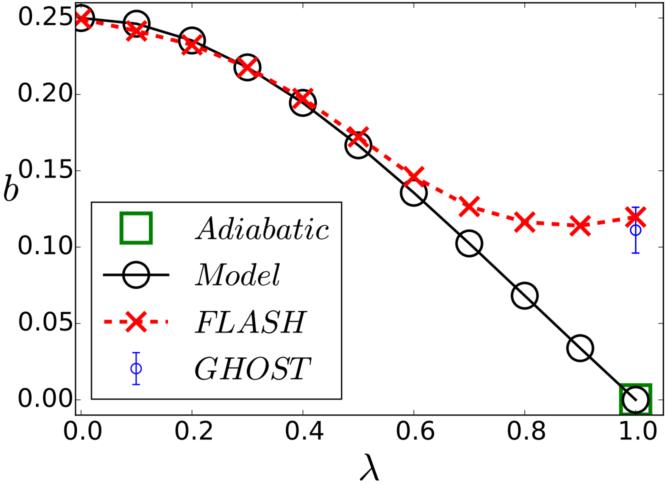
$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{29}$$

$$\boxed{b = \frac{1-\lambda^2}{4+2\lambda^2}} \quad \text{and} \quad Re = \frac{U}{K\nu}\,. \tag{30}$$

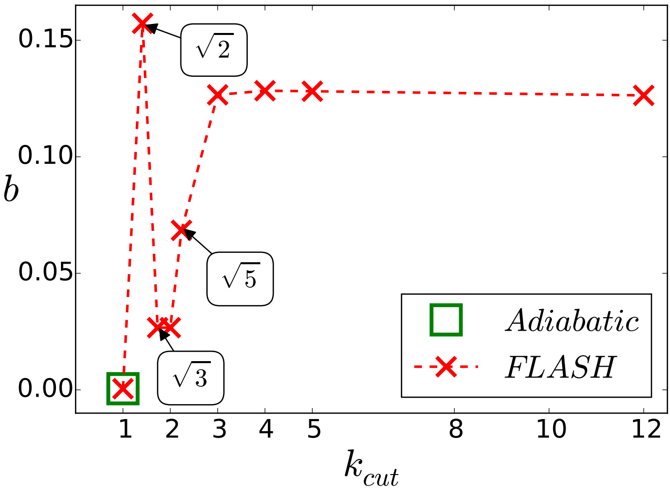
Determining b

$$\beta - \nu = \left\langle \frac{\sigma}{q^2} \right\rangle \Longleftrightarrow \frac{\beta}{\nu} = \frac{1}{\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right) \Longleftrightarrow b = \frac{1}{Re^2\nu}\left(\left\langle \frac{\sigma}{q^2} \right\rangle + \nu\right). \tag{31}$$

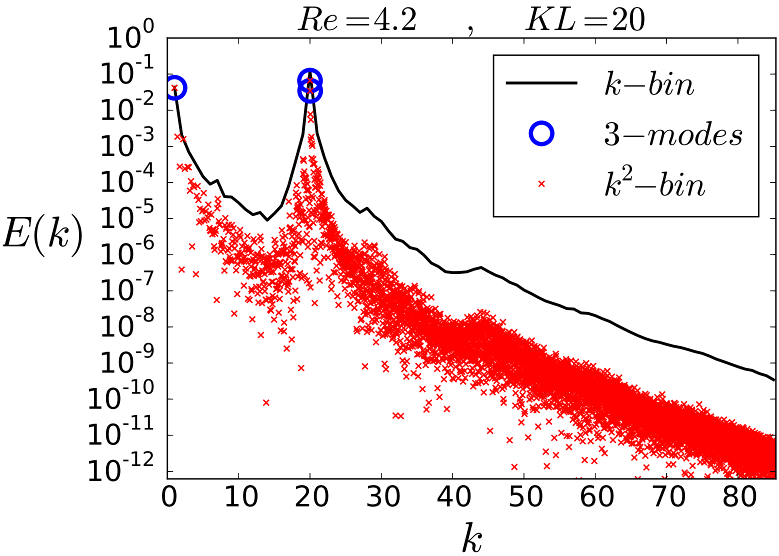
FLASH: Power-law pre-factor



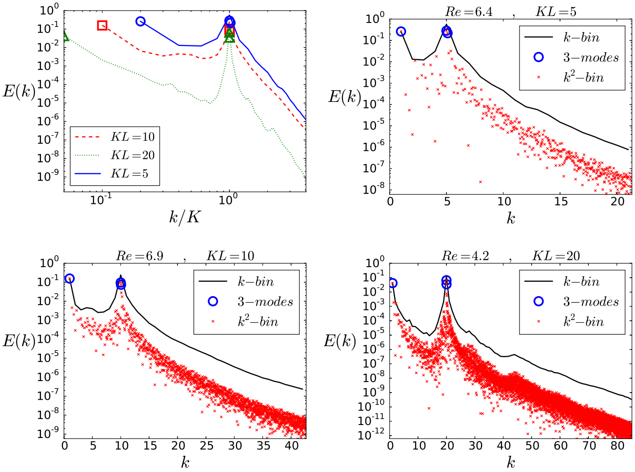
FLASH: Fourier truncation



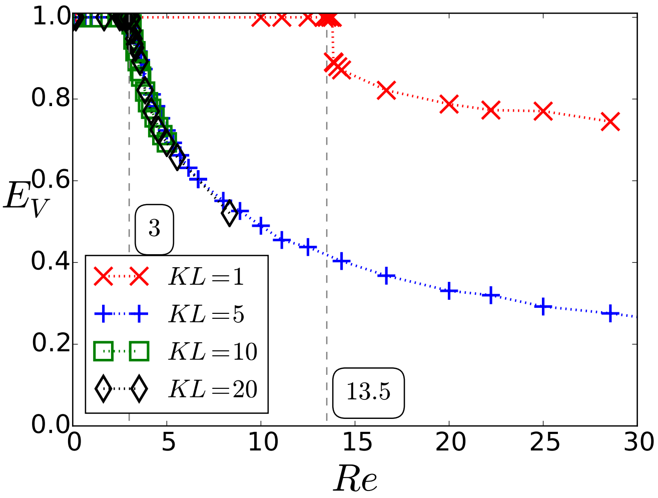
GHOST : Instability spectrum $KL = 20$



GHOST : Instability spectra

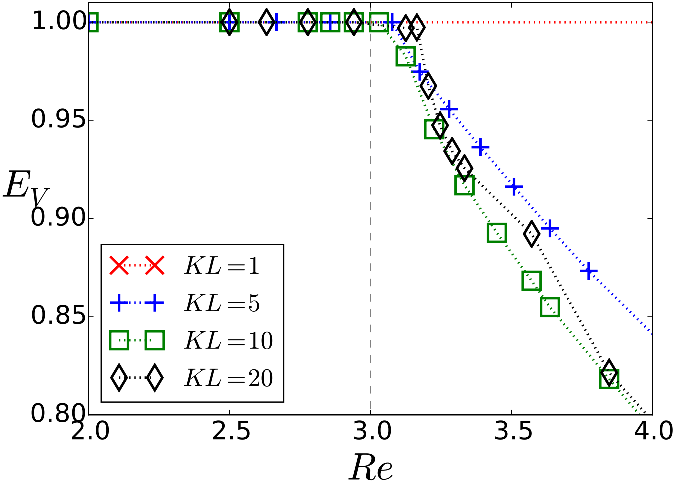


GHOST: Small scale instability



[Dombre *et al.* 86; Podvigina & Pouquet Phys. D 94]

GHOST: Small scale instability zoom



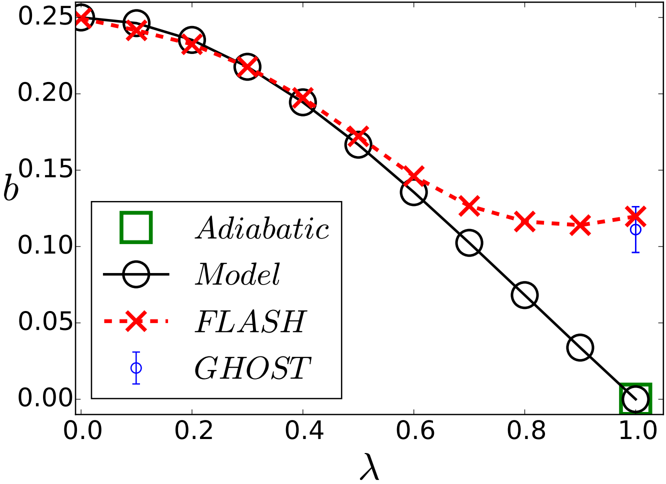
Critical Reynolds number

At the onset of the instability

$$\sigma = \beta q^2 - \nu q^2 \quad \text{with} \quad \beta = bRe^2\nu\,, \tag{32}$$

$$\sigma = 0 \iff \beta^c = \nu \iff \boxed{b^c = (Re^c)^{-2}}\,. \tag{33}$$

$\lambda-ABC$ flows graph



On arXiv, submitted to PR Fluids

Thank you for your attention

- Also investigated in HD:
  - Turbulent *ABC* flow
  - Study of E_0 / E_{tot}
- Also investigated in Induction:
  - Study of the *ABC*-dynamo
  - Study of magnetic response to non-helical flow
  - Study of magnetic response to δ-correlated flows

On arXiv, submitted to PRL

Thank you for your attention