Large scale effects in Turbulence

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Topics addressed in the thesis

Results presented: Large scale effects in 3D Hydrodynamics

- Low Reynolds number: Large-scale instabilities of helical flows,
  A. Cameron, A. Alexakis and M.-É. Brachet, Phys. Rev. Fluids 1, 063601
- High Reynolds number: The effect of helicity on the correlation time of large scale turbulent flows,
  A. Cameron, A. Alexakis and M.-É. Brachet, arXiv:1705.05281

Other results

- Magnetohydrodynamics: Fate of Alpha Dynamos at Large *Rm*,
  A. Cameron and A. Alexakis, Phys. Rev. Lett. 117, 205101
- Semi-Lagrangian schemes: Multi-stage high order semi-Lagrangian schemes for incompressible flows in Cartesian geometries,
  A. Cameron, R. Raynaud and E. Dormy, IJNMF

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"Is it possible to define turbulence?"

***Turbulence in Fluids*, Marcel LESIEUR, Springer 2008 ***

Turbulence is a dangerous topic [. . .] it represents extremely different points of view, all of which have in common their complexity, as well as an inability to solve the problem. **It is even difficult to agree on what exactly is the problem to be solved.**

* N.B.: The title of the first section of the book is:

"Is it possible to define turbulence?"

L. da Vinci (1452-1519)

Big eddies break into small eddies

L. Euler (1707-1783)

Idea

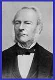
Describe the properties of fluids as a function of space and time.

Euler's equation for incompressible fluids

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = 0 \quad \text{with} \quad \boxed{P = \frac{p}{\rho}}. \qquad (1)$$

- $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$: transport/convective term.
- $\nabla\cdot\boldsymbol{u}$: incompressibility condition.
- $-\nabla P$: pressure gradient.

C. NAVIER (1785-1836) and G. STOKES (1819-1903)



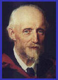
Idea

Use the mechanic properties of the continuous medium derived by Navier on a fluid in motion.

Navier-Stokes equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\nabla P + \boxed{\nu \Delta \boldsymbol{u}} + \boxed{\boldsymbol{F}} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \,. \tag{2}$$

- $\nu \Delta \boldsymbol{u}$: viscous term.
- $\boldsymbol{F}$: forcing field.



O. Reynolds (1842-1912)

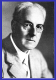
Idea

The Navier-Stokes equation can be expressed with one parameter which defines the regime of the flow.

Reynolds number

$$Re = \frac{[[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]]}{[[\nu\Delta\boldsymbol{u}]]} = \frac{UL}{\nu}. \quad (3)$$

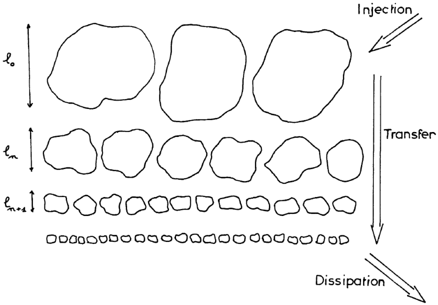
- $Re \ll 1$: laminar regime.
- $1 \ll Re$: turbulent regime.



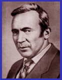
L. Richardson (1881-1953)

Idea

At fixed U and ν, diffusive effect appears for small length scale.



Only for 3D flows. In 2D, dual energy-enstrophy cascade.



A. Kolmogorov (1903-1987)

Idea

The energy cascade can be described using:

- ϵ: energy dissipated per units of mass and time.
- k: the wavenumber $k \sim 1/\ell$.

Energy spectrum

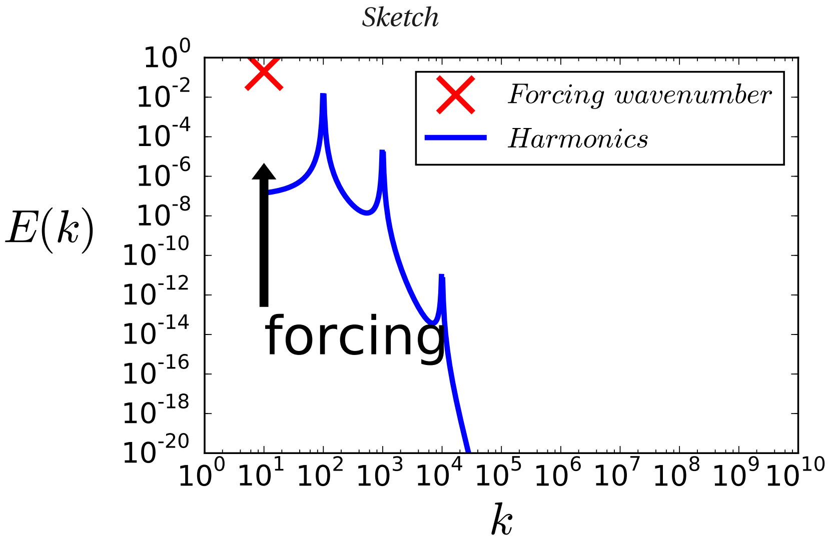
$$E(k) \propto \epsilon^{2/3} k^{-5/3} \quad \text{with} \quad E = \int E(k)\, dk. \tag{4}$$

Validity

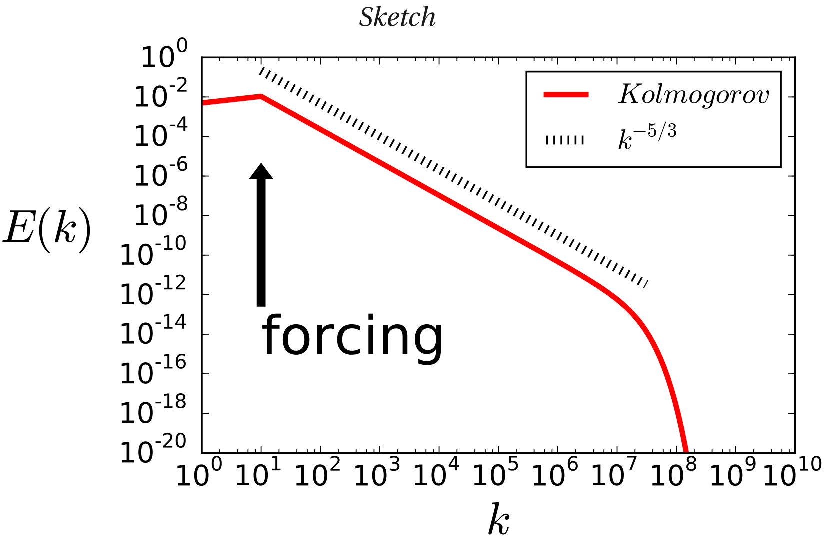
The range of validity of the energy cascade can be found using:

- k_f: the forcing wavenumber.
- ν: viscosity ($k_\nu = k_f Re^{3/4}$).

Laminar energy spectrum



Kolmogorov's prediction for the energy spectrum



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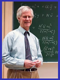


K. Moffatt

Conserved quantities

$$E = \frac{1}{L^3}\int (|\boldsymbol{u}|^2)\, d^3\boldsymbol{r} \quad \text{and} \quad H = \frac{1}{L^3}\int (\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{u})\, d^3\boldsymbol{r} \tag{5}$$

$$\text{with} \quad L^3 = \int d^3\boldsymbol{r}. \tag{6}$$

The Roberts flow (helical)

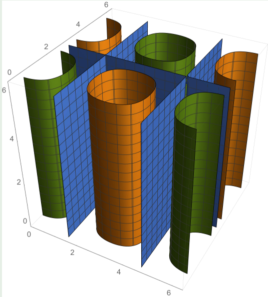
Expression

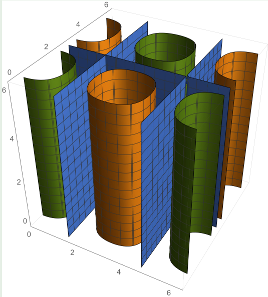
$$u^x = \cos(ky) + 0 \tag{7}$$

$$u^y = 0 \quad + \sin(kx) \tag{8}$$

$$u^z = \cos(kx) + \sin(ky) \tag{9}$$

Isosurface: u^2 and $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$





An Taylor-Green symmetric flow (non-helical)

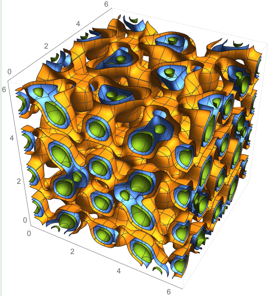
An example of flow with Taylor-Green symmetries

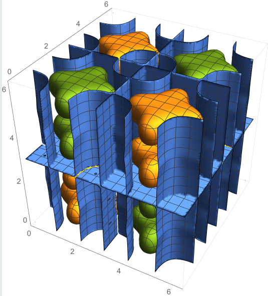
$$u^x = \sin(kx)\cos(ky)\cos(kz) + \sin(2kx)\cos(2ky)\cos(2kz) \quad (10)$$

$$u^y = \cos(2kx)\sin(2ky)\cos(2kz) - \cos(kx)\sin(ky)\cos(kz) \quad (11)$$

$$u^z = -2\cos(2kx)\cos(2ky)\sin(2kz) \quad (12)$$

Isosurface: u^2 and $\boldsymbol{u}\cdot\nabla\times\boldsymbol{u}$





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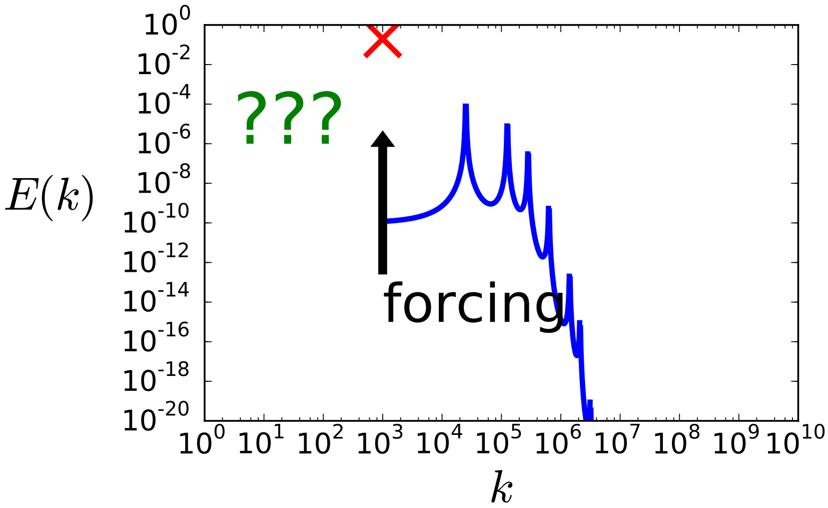
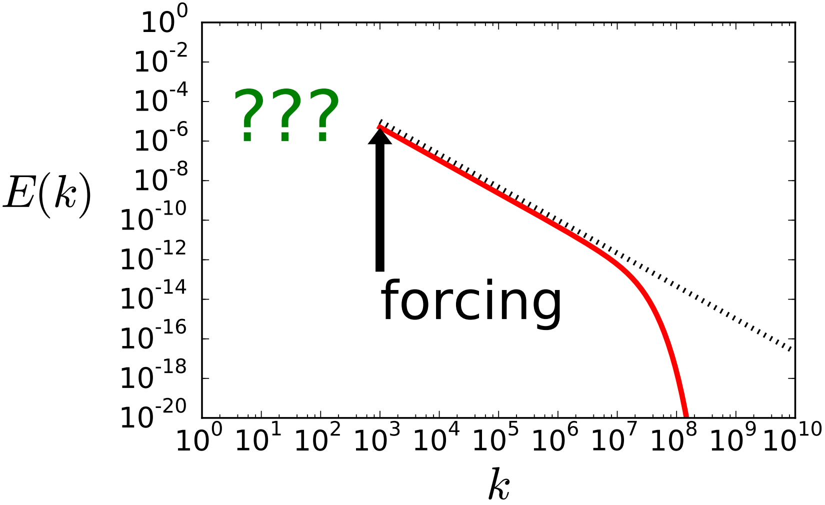
What happens before the forcing scale at low *Re*?



What happens before the forcing scale at high *Re*?



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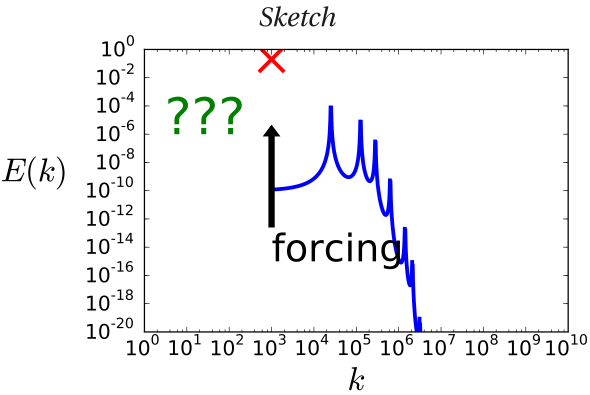
Behavior at small *Re*



Characteristics

- Kolmogorov's scaling: not present.
- Viscous solution: present (red cross)
- Large scale instabilities: possible.

An example of large scale instability: the AKA-instability

Important features

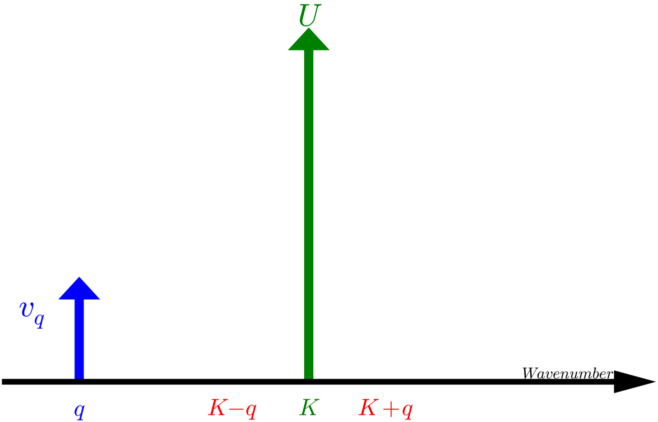
- Large scale instability.
- Unstable in the large scales even when $Re \to 0$.
- Multi-scale expansion at low Reynolds number.

Reference

Large-scale flow driven by the Anisotropic Kinetic Alpha-effect, U. FRISCH, Z.S. SHE and P.L. SULEM, Physica D 1987

Three-mode constructive interaction

Sketch: How to generate a large scale instability ?

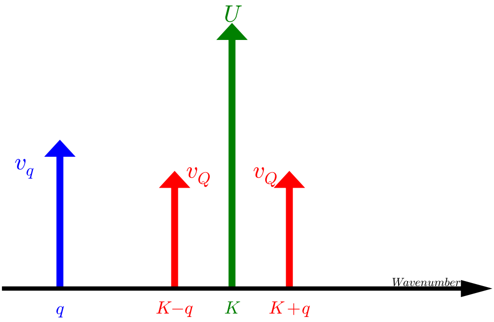


Three-mode model

Underlying parameters

- q : wavenumber at the largest scale.
- K : wavenumber at the forcing scale.
- $q/K < 1$: scale separation.

Sketch



The Fr87 flow

Large-scale flow driven by the Anisotropic Kinetic Alpha-effect, U. FRISCH _et al._, Physica D 28, 382–392 (1987)

Flows generating AKA-instabilities at $Re \to 0$ should not be:

- time-independent
- delta-correlated in time
- isotropic
- parity-invariant

Abbreviation

The flow used in the article of U. FRISCH *et al.* 1987 to generate an AKA effect will be referred to as the **Fr87** flow.

The Fr87 flow (non-helical)

Abbreviation

The flow used in the article of U. Frisch *et al.* 1987 to generate an AKA effect will be referred to as the **Fr87** flow.

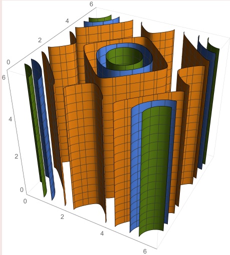
Expression

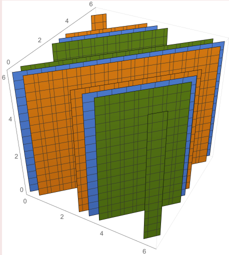
$$u^x = \cos(ky + \nu k^2 t) \quad (13)$$

$$u^y = \cos(kx - \nu k^2 t) \quad (14)$$

$$u^z = u^x + u^y \quad (15)$$

Isosurface: u^2 and $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$ at $t = 0$





Floquet framework

Hypothesis

$$Re \ll 1 \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v} \quad \text{with} \quad \|\boldsymbol{v}\| \ll \|\boldsymbol{U}\|, \tag{16}$$

- $\boldsymbol{u}$: total flow.
- $\boldsymbol{U}$: laminar solution.
- $\boldsymbol{v}$: perturbation to the laminar solution.

Ansatz

$$\boldsymbol{v}(\boldsymbol{r}) = \widetilde{\boldsymbol{v}}(\boldsymbol{r})e^{\imath \boldsymbol{q}\boldsymbol{r}} + c.c. \tag{17}$$

- $\boldsymbol{q}$: parameter accounting for the scale separation.

Evolution equation

Use *ansatz* in linearized Navier-Stokes equation.

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Anisotropic Kinetic Alpha (AKA) effect

Growth rate γ

$$\|\boldsymbol{v}(t)\| = \|\boldsymbol{v}(0)\| e^{\gamma t} \quad \text{and} \quad \frac{\gamma}{\nu K^2} = aRe\left(\frac{q}{K}\right) - \left(\frac{q}{K}\right)^2 \tag{18}$$

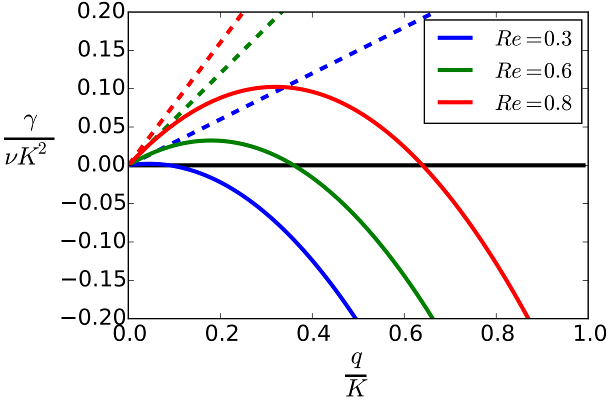
- γ : growth rate.
- U : intensity of the laminar flow.
- K : wavenumber of the laminar flow.
- q : wavenumber of the largest scale of the system.
- Re : Reynolds number.
- a : constant dependent on the geometry of the flow. In the case of **Fr87**, $a = 1/2$.

AKA growth rate

Expression

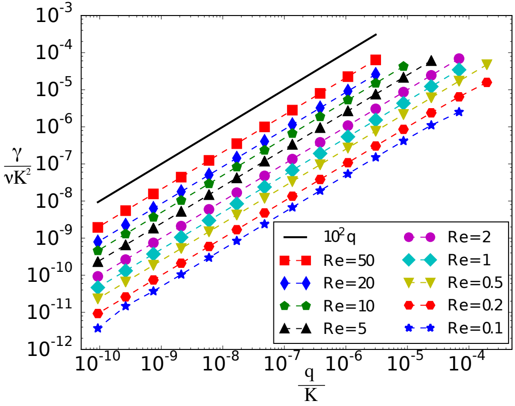
$$\|\boldsymbol{v}(t)\| = \|\boldsymbol{v}(0)\| e^{\gamma t} \quad \text{and} \quad \frac{\gamma}{\nu K^2} = aRe\left(\frac{q}{K}\right) - \left(\frac{q}{K}\right)^2 \tag{19}$$

Sketch: $a = 1$



Fr87 flow growth rate

Floquet Linear Analysis of Spectral Hydrodynamics results



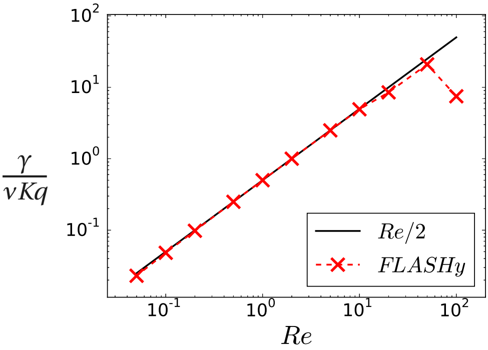
FLASHy results

When $(q/K) \to 0$, $\gamma \propto (q/K)$. The instability has a AKA-behavior.

Extension to $Re \sim 1$

Does the prefactor match ?

$$\frac{\gamma}{\nu K^2} \underset{\frac{q}{K}\to 0}{\sim} aRe\left(\frac{q}{K}\right) \quad \text{thus} \quad \frac{\gamma}{\nu K q} \underset{\frac{q}{K}\to 0}{\sim} aRe. \tag{20}$$



FLASHy results

U. Frisch *et al.* Multi-scale expansion is even valid at $Re \sim 10$.

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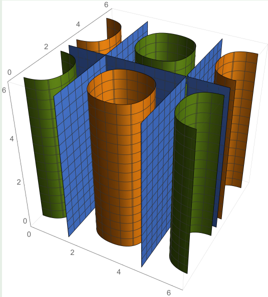
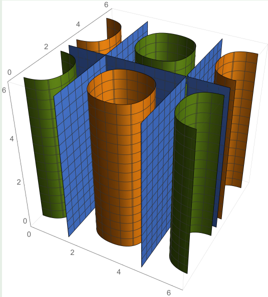
The Roberts flow (helical)

Expression

$$u^x = \cos(ky) + 0 \tag{21}$$
$$u^y = 0 \qquad + \sin(kx) \tag{22}$$
$$u^z = \cos(kx) + \sin(ky) \tag{23}$$

Isosurface: u^2 and $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$





AKA-stable flows: negative viscosity instability

Growth rate of $\lambda - ABC$ flows

$$\frac{\gamma}{\nu K^2} = bRe^2\left(\frac{q}{K}\right)^2 - \left(\frac{q}{K}\right)^2 . \tag{24}$$

- γ : growth rate.
- U : intensity of the laminar flow.
- K : wavenumber of the laminar flow.
- q : wavenumber of the largest scale of the system.
- Re : Reynolds number.
- b : constant dependent on the geometry of the flow.
  In the case of the Roberts flow, we derived a three-mode model predicting that $b = 1/4$.

Threshold of the instability

When does the instability occur for AKA-stable flows?

$$\frac{\gamma}{\nu K^2} = \left(bRe^2 - 1\right)\left(\frac{q}{K}\right)^2 . \tag{25}$$

For all $\underline{\frac{q}{K}}$, the instability occurs when $Re^2 > 1/b$.

Comparison with AKA

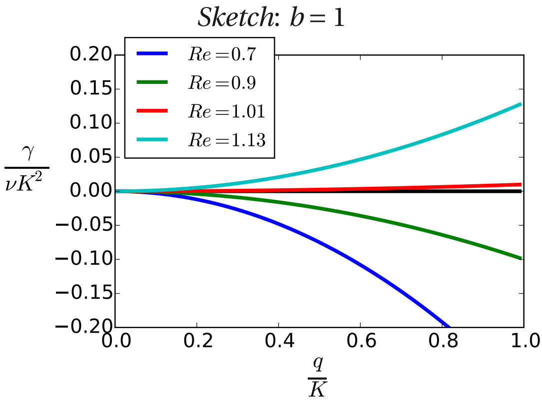
$$\frac{\gamma}{\nu K^2} = aRe\left(\frac{q}{K}\right) - \left(\frac{q}{K}\right)^2 . \tag{26}$$

For all $\underline{Re}$, the instability occurs when $\frac{q}{K} < aRe^2$.

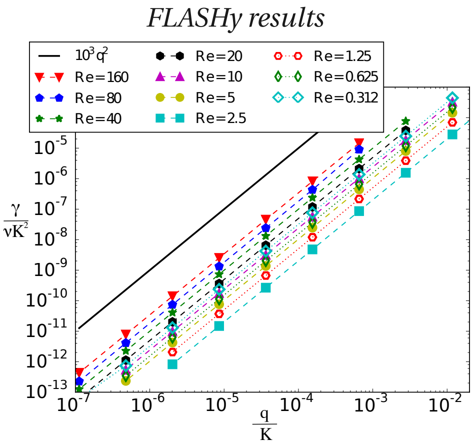
AKA-stable flows: growth rate

Expression

$$\|\boldsymbol{v}(t)\| = \|\boldsymbol{v}(0)\| e^{\gamma t} \quad \text{and} \quad \frac{\gamma}{\nu K^2} = \left(bRe^2 - 1\right)\left(\frac{q}{K}\right)^2 \tag{27}$$



Growth rate of the Roberts flow



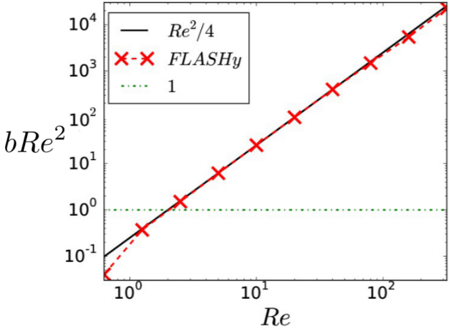
FLASHy results

When $(q/K) \to 0$, $\gamma \propto (q/K)^2$. The instability has a viscous behavior.

Roberts flow negative viscosity instability

Measuring the b growth rate

$$\frac{\gamma}{\nu K^2} = \left(bRe^2 - 1\right)\left(\frac{q}{K}\right)^2 \quad \text{thus} \quad bRe^2 = \frac{\gamma}{\nu q^2} + 1\,. \tag{28}$$



FLASHy results

$b = 1/4$ prediction is even valid at $Re \sim 1$.

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The ABC flow (helical)

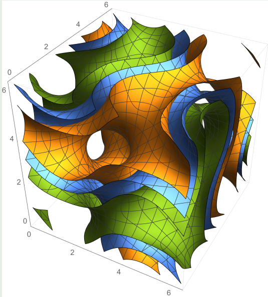
Expression

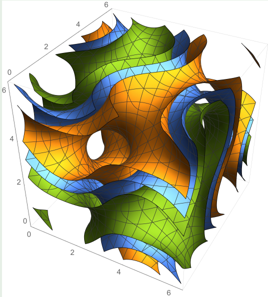
$$u^x = \cos(ky) + \sin(kz) \tag{29}$$

$$u^y = \cos(kz) + \sin(kx) \tag{30}$$

$$u^z = \cos(kx) + \sin(ky) \tag{31}$$

Isosurface: u^2 and $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$

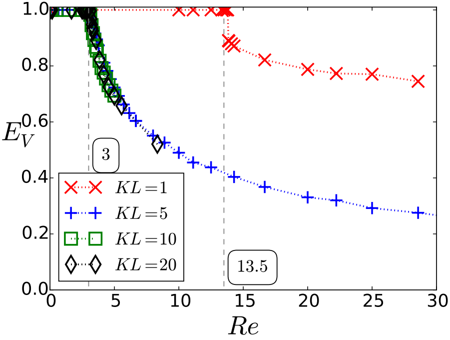




Non-linear system: negative-viscosity instability (ABC flow)

Variables

- E_V : energy of the total velocity field $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v}$.
- KL : box size / forcing scale, scale separation.



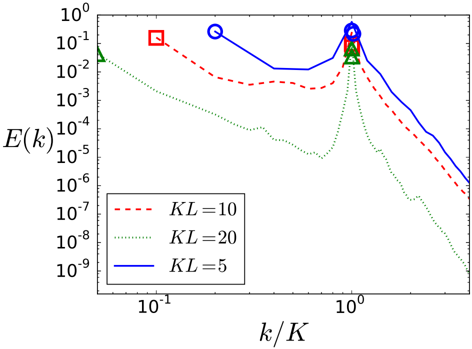
Direct numeric simulations: non-linear results

Large scale instability observed and independent of KL.

Non-linear consequence (ABC flow)

Variables

- KL : scale separation between the forcing and the box size.
- k/K : normalized wavenumber.



Non-linear results

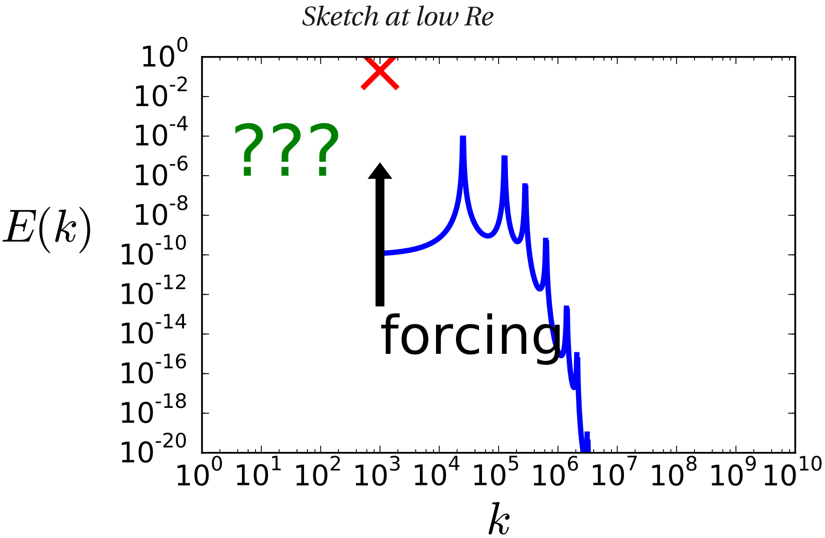
Peak in the large scales ($k/K \to 0$).

Recapitulation

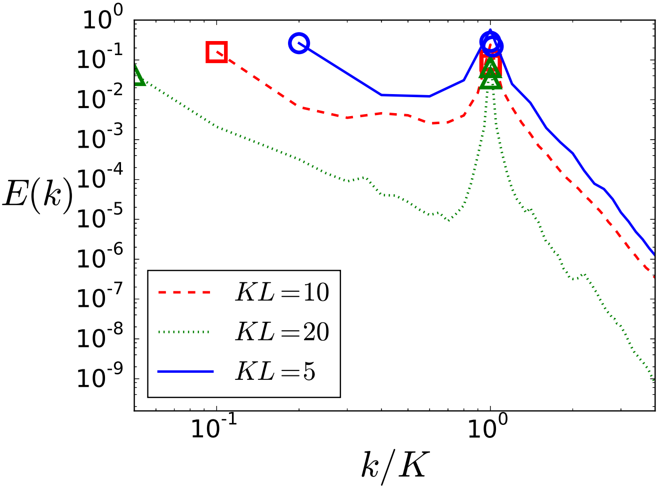
Conclusion

- The AKA instability can be observed at $Re \sim 1$.
- AKA-stable flows can generate negative-viscosity instabilities.
- Instability effects are observed in the non-linear system.

Back to the initial question



Possible effect



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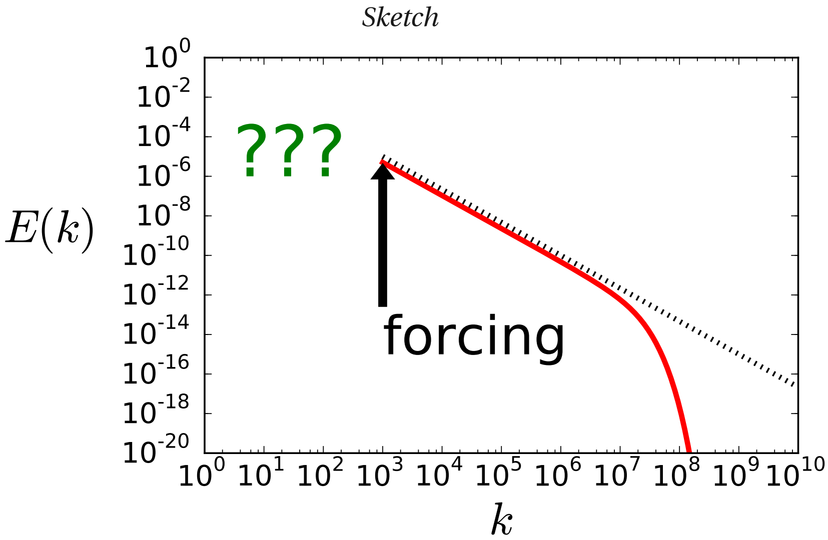
At high *Re*, what happens before the forcing scale?



Absolute equilibrium analogy

Turbulence: The Legacy of A. N. Kolmogorov, U. FRISCH

"**Absolute equilibrium solutions** seem highly unphysical in view of the approximately $k^{-5/3}$ spectrum of the three-dimensional turbulence. Actually, they are appropriate at the very smallest wavenumbers of **turbulent flows** maintained by **forcing at intermediate wavenumbers** (Forster, Nelson and Stephen 1977)."

What is an absolute equilibrium solution ?

It's a the statistically stationary solution of the **truncated Euler equation**.

What is the truncated Euler equation ?

It's a high wavenumber filtered version of the **Euler equation**.

What is the truncated Euler equation ?

Navier-Stokes equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} P + \boxed{\nu \Delta \boldsymbol{u}} + \boxed{\boldsymbol{F}} \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0. \tag{32}$$

Truncated Euler equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} P \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{33}$$

$$\text{and} \qquad \boxed{\boldsymbol{u}(k > k_M) = 0}. \tag{34}$$

k_M is the maximal wavenumber.

What is the equilibrium distribution ?

Truncated Euler equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla P \quad \text{with} \quad \nabla \cdot \boldsymbol{u} = 0 \tag{35}$$

$$\text{and} \quad \boxed{\boldsymbol{u}(k > k_M) = 0}. \tag{36}$$

Absolute equilibrium solution

The statistically stationary solution of the system follow

$$P(\boldsymbol{u}) \propto e^{-(\alpha|\boldsymbol{u}|^2 + \beta \boldsymbol{u} \cdot \nabla \times \boldsymbol{u})} \quad \text{and} \quad \mathcal{K}r = -\frac{\beta k_M}{\alpha}. \tag{37}$$

- $\boldsymbol{u}$: velocity field.
- $P(\boldsymbol{u})$: probability of a velocity configuration.
- $\mathcal{K}r$: the Kraichnan number $|\mathcal{K}r| \leq 1$.

What is the relation with equipartition ?

Absolute equilibrium solution

The statistically stationary statistics of the system follow

$$P(\boldsymbol{u}) \propto e^{-(\alpha|\boldsymbol{u}|^2+\beta \boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{u})} \quad \text{and} \quad \mathcal{K}r = -\frac{\beta k_M}{\alpha}. \tag{38}$$

Properties

- The probability of the components of the velocity field follow independent Gaussian distributions.
- The standard deviation of the variables depends on $\mathcal{K}r$.
- The energy spectrum is given by:

$$E(k) \propto \frac{k^2}{1-\left(\mathcal{K}r\,\frac{k}{k_M}\right)^2} \tag{39}$$

Conjecture

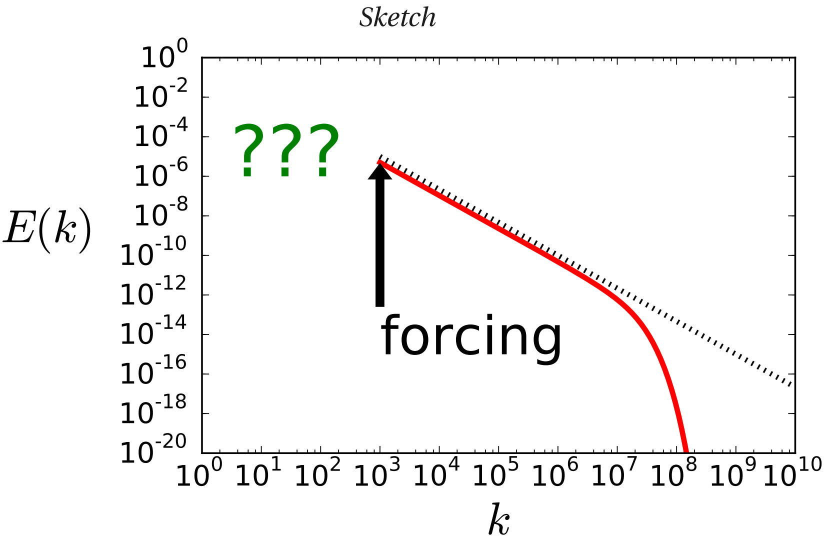
Parameter

- k_f : forcing wavenumber in the Navier-Stokes equation.
- k_M : maximum wavenumber in the truncated Euler equation.

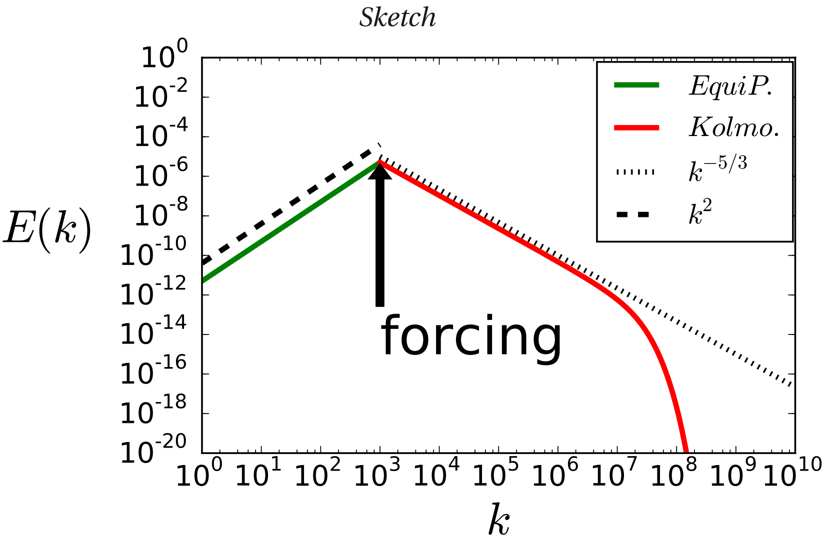
Equipartition prediction on the energy spectrum for $Re \gg 1$

- $k < k_f : E(k) = 4\pi k^2 e(k) \propto\ k^2$.
- $k_f < k : E(k) \propto\ k^{-5/3}$.

At high *Re*, what happens in the largest scales ?



Absolute equilibrium conjecture



To what extent is this conjecture true ?

Analysis of the temporal statistics

Can the absolute equilibrium theory describe the temporal statistic of the system ?

Temporal properties

- What is the correlation time of an absolute equilibrium ?
- What is the correlation time in the large scale for solutions of the Navier-Stokes equation ?
- Do they match ?

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Sweeping effect: scales smaller than the forcing

Definition

$$\Gamma_k(t) = \frac{1}{\overline{|u_k|(s)}} \overline{(u_k)^*(s)\, u_k(t+s)} \quad \text{with} \quad \overline{G(s)} \underset{T\to\infty}{=} \frac{1}{2T}\int_{-T}^{T} G(s)\, ds \quad (40)$$

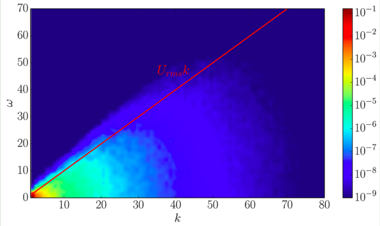
Dimensional analysis U_{rms}, k

$$\partial_t \boldsymbol{u} \simeq (\boldsymbol{U}\cdot\nabla)\boldsymbol{u} \quad (41)$$

$$\partial_t \boldsymbol{u}_k \simeq (\imath \boldsymbol{k}\cdot\boldsymbol{U})\boldsymbol{u}_k \quad (42)$$

$$\tau \propto \frac{1}{kU_{rms}} \quad (43)$$

DNS spatio-temporal spectrum



Analytic derivation of the correlation time

Hypothesis

- Independent Gaussian variables (Gibbs ensemble).
- Ergodicity.
- Parabolic short-time approximation.

Expression

$$\Gamma_{\boldsymbol{k}}(t) \underset{t\to 0}{=} \frac{1}{|\boldsymbol{u}|^2} \int \boldsymbol{u}_{\boldsymbol{k}}(s) \left(\boldsymbol{u}^*_{\boldsymbol{k}}(-s) + t\partial_t \boldsymbol{u}^*_{\boldsymbol{k}}(-s) + \frac{t^2}{2} \partial_t^2 \boldsymbol{u}^*_{\boldsymbol{k}}(-s) \right) ds, \tag{44}$$

$$\tau_{\boldsymbol{k}} = \sqrt{\frac{|\langle \boldsymbol{u}_{\boldsymbol{k}} \rangle|^2}{|\langle \partial_t \boldsymbol{u}_{\boldsymbol{k}} \rangle|^2}}. \tag{45}$$

Possible time scales

Quantities conserved during the truncated Euler evolution

$$E = \frac{1}{L^3}\int \left(|\boldsymbol{u}|^2\right) d^3\boldsymbol{r} \quad \text{and} \quad H = \frac{1}{L^3}\int (\boldsymbol{u}\cdot\nabla\times\boldsymbol{u})\, d^3\boldsymbol{r} \tag{46}$$

$$\text{with} \quad L^3 = \int d^3\boldsymbol{r}. \tag{47}$$

Energy scaling ($|\mathscr{H}r| \lesssim 0.8$)

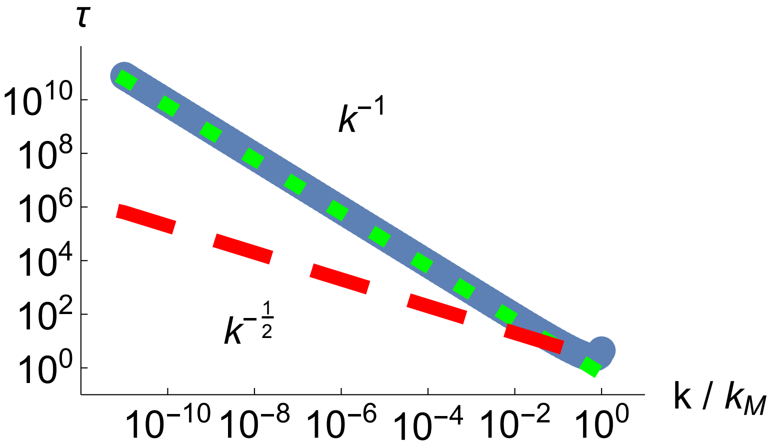
$$\tau_k^E \propto \frac{1}{kU_{rms}} \propto \frac{1}{\sqrt{k^2 E}} \tag{48}$$

Helicity scaling ($|\mathscr{H}r| \to 1$)

$$\tau_k^H \propto \frac{1}{\sqrt{kH}} \tag{49}$$

Analytic prediction of the correlation time :

$1 - \mathcal{K}r = \{10^{-1}; 10^{-2}; 10^{-3}; 10^{-4}; 10^{-5}; 10^{-6}; 10^{-7}; 10^{-8}\}$



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What do we want to do ?

Objective

- Check equipartition : $E(k) \propto k^2$
- Validation correlation time ($|\mathcal{H}r| \lesssim 0.8$): $\tau_k^E \propto \frac{1}{\sqrt{k^2 E}}$
- Validation correlation time ($|\mathcal{H}r| \to 1$): $\tau_k^H \propto \frac{1}{\sqrt{kH}}$

Taylor Green (TYGRES)

- Less degrees of freedom than general periodic flow.
- Faster to compute with optimized code.
- $\mathcal{H}r = 0$.

General periodic (GHOST)

- Slower to compute than TG code.
- $\mathcal{H}r$ can vary.

Taylor-Green symmetric flow (non-helical)

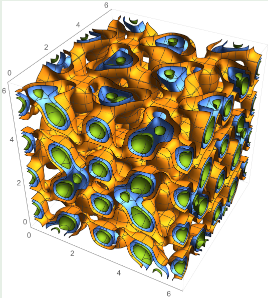
Example

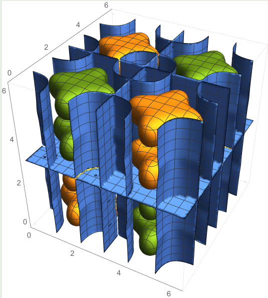
$$u^x = \sin(kx)\cos(ky)\cos(kz) + \sin(2kx)\cos(2ky)\cos(2kz) \quad (50)$$

$$u^y = \cos(2kx)\sin(2ky)\cos(2kz) - \cos(kx)\sin(ky)\cos(kz) \quad (51)$$

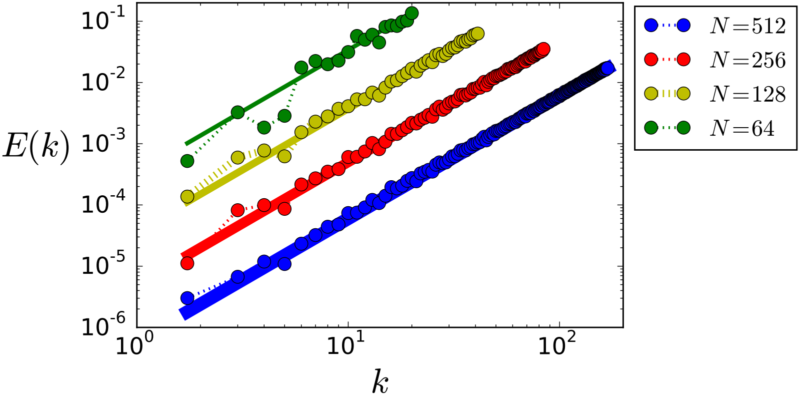
$$u^z = -2\cos(2kx)\cos(2ky)\sin(2kz) \quad (52)$$

Isosurface: u^2 and $\boldsymbol{u}\cdot\nabla\times\boldsymbol{u}$





TG Spectrum



DNS results

The energy spectrum follows the k^2-law.

Correlation time

Aim

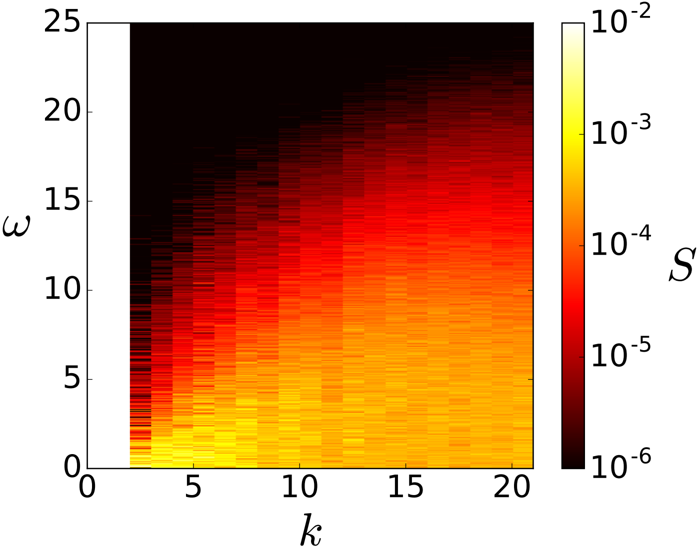
$$\Gamma_k(t) \underset{T\to\infty}{=} \frac{1}{2T|\boldsymbol{u}|^2}\int_{-T}^{T} \boldsymbol{u}_{\boldsymbol{k}}(s)\,\boldsymbol{u}^*_{\boldsymbol{k}}(t-s)\,ds \tag{53}$$

Find $\tau_{1/2}$ when $\Gamma_k(\tau_{1/2}) = 1/2$.

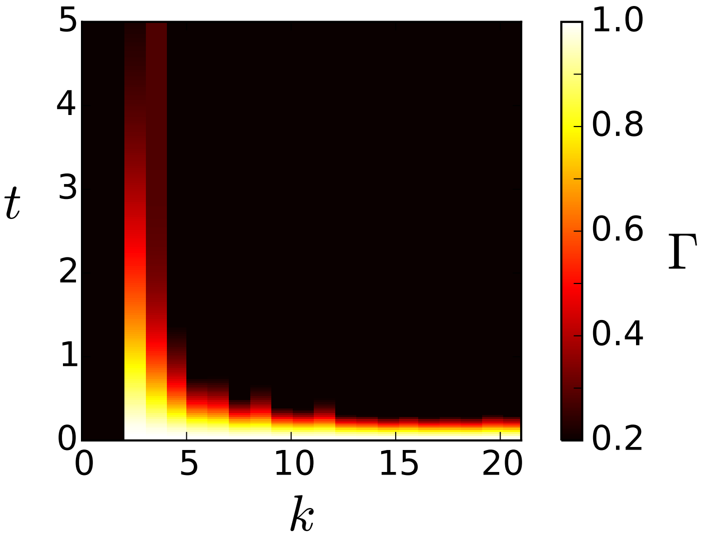
Method

1. Output $\boldsymbol{u}(k_x, k_y, k_z)$ on the planes $k_x = 0$, $k_y = 0$ and $k_z = 0$.
2. Compute the power spectrum.
3. Compute the correlation function and evaluate $\tau_{1/2}$.

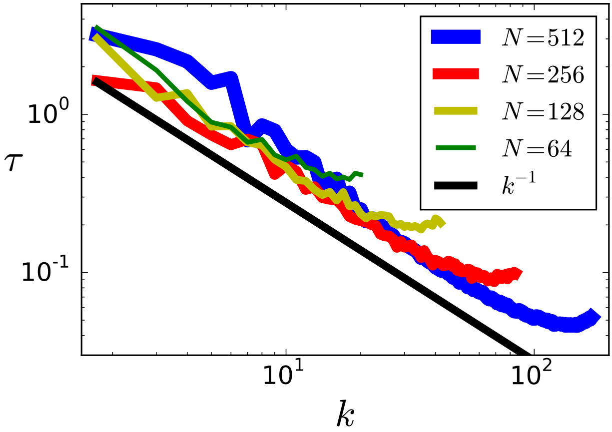
TG Spatio-temporal Power Spectrum



TG Correlation Spectrum



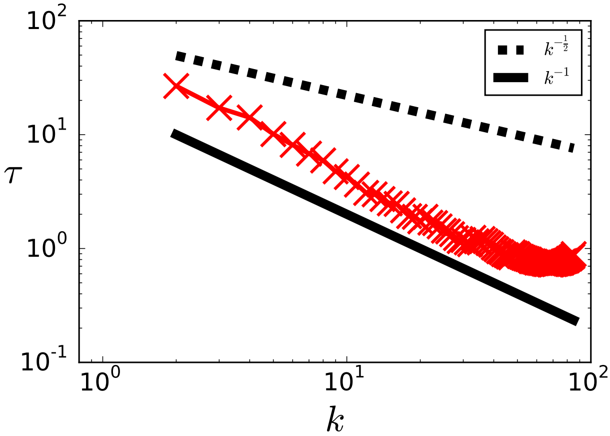
TG Correlation time



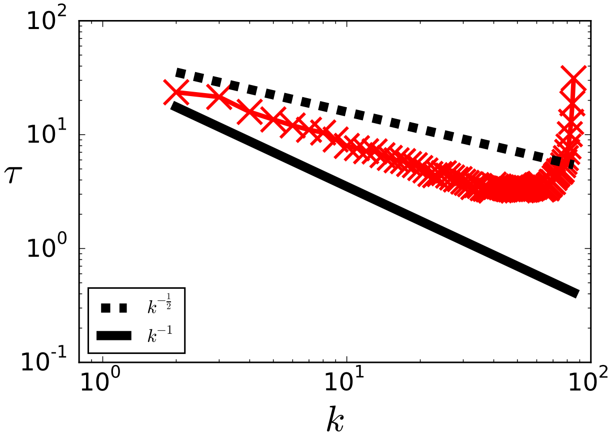
DNS results

The correlation time follows the $\tau(k) \propto k^{-1}$-law.

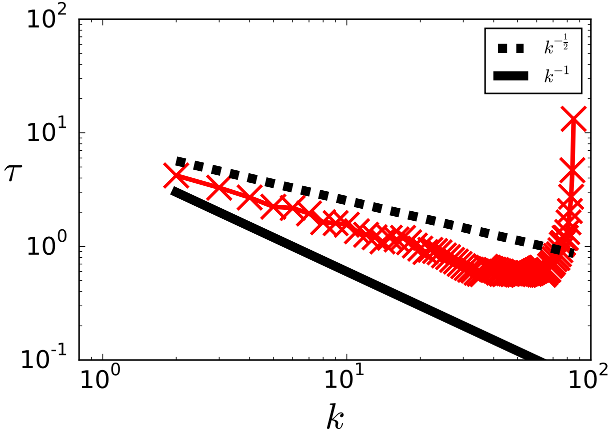
No helicity $\mathcal{H}r = 0$



Some helicity $\mathcal{H}r = 0.8$



A lot of helicity $\mathscr{H}r = 0.99$



DNS results

As $\mathscr{H}r \to 1$, the τ transitions from k^{-1}-law to $\tau \propto k^{-1/2}$-law.

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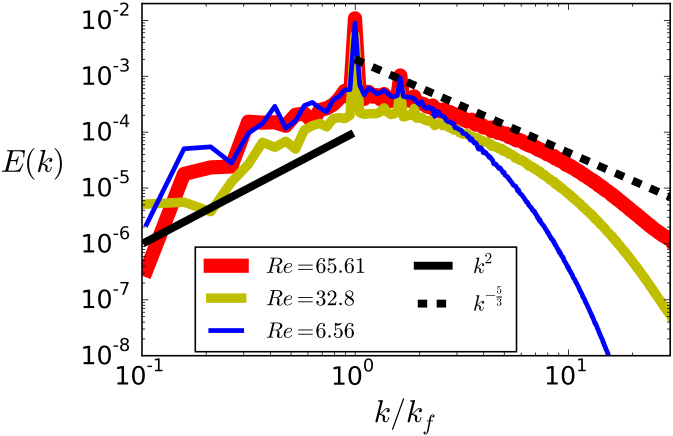
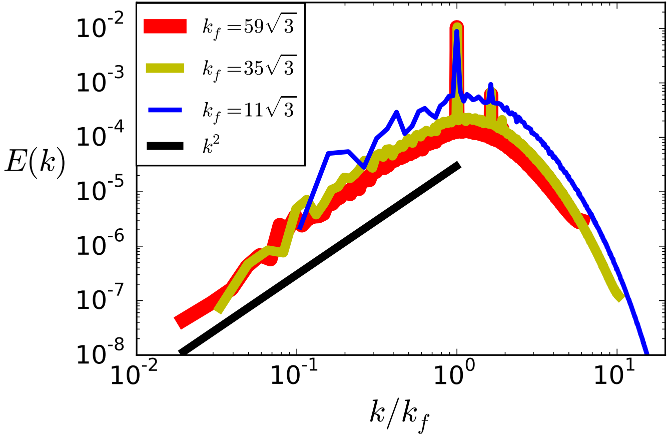
TG Spectrum varying Re, fixed $k_f L$



Observation

The equipartition spectrum can be observe even without the Kolmogorov's scaling in the inertial range.

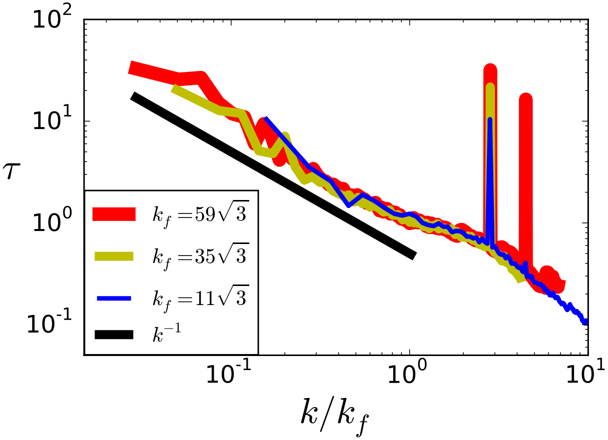
TG Spectrum fixed *Re*, varying $k_f L$



Results

The energy spectrum follows a k^2-law.

TG correlation time



Results

The correlation time follows a k^{-1}-law.

The ABC flow (helical)

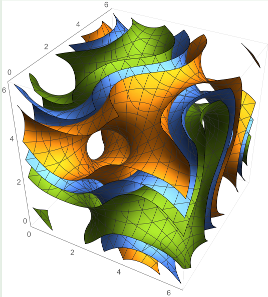
Expression

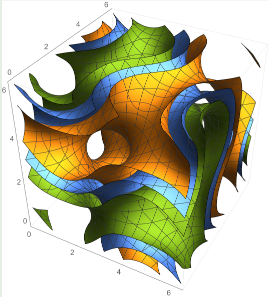
$$u^x = \cos(ky) + \sin(kz) \tag{54}$$

$$u^y = \cos(kz) + \sin(kx) \tag{55}$$

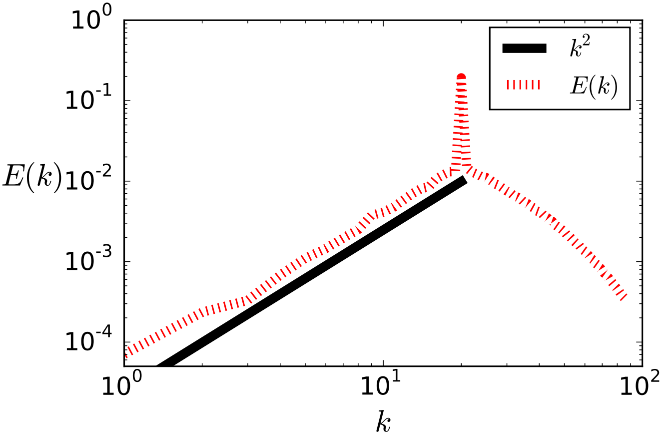
$$u^z = \cos(kx) + \sin(ky) \tag{56}$$

Isosurface: u^2 and $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$





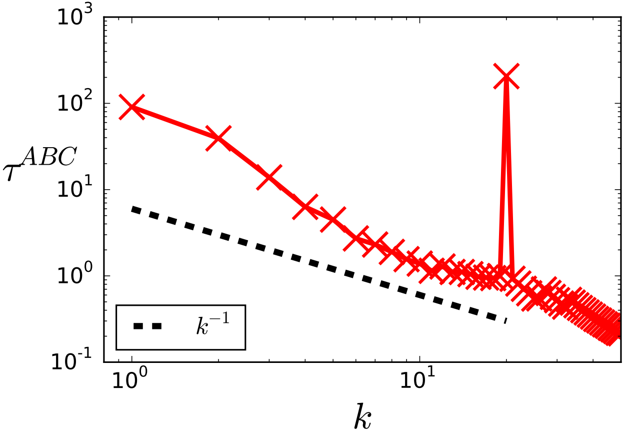
ABC energy spectrum



Results

The energy spectrum spectrum does not show deviation from k^2.

ABC correlation time



Results

The correlation time deviates from the k^{-1}-law.

Recapitulation

New results

- Analytic computation of the correlation time of absolute equilibrium.
- Numeric measurements of the correlation time in truncated Euler solutions.
- Numeric measurements of the correlation time in Navier-Stokes solutions.

Conclusions

- Absolute equilibrium describes well statistically stationary solution of the truncated Euler equation.
- Large scale modes of TG flow behave like absolute equilibrium.
- Large scale modes of ABC flow deviate from absolute equilibrium statistics.

Perspectives

Three open questions

- What is the origin of the differences between TG and ABC ?
- How is energy transferred to large scale ?
- Do low *Re* instabilities still affect the flow ?

Thanks you all for your attention

Thank you for your attention

Thank you for your supervision

4/5-law

Expression

$$(\delta v_{\parallel})^3(r,\ell) = -\frac{4}{5}\epsilon\ell \tag{57}$$

Dimension

$$[[\delta v]] = (\epsilon[[\ell]])^{1/3} = \epsilon^{1/3}k^{-1/3} \tag{58}$$

Energy spectrum

$$[[E(k)]] = [[v^2k^{-1}]] = \epsilon^{2/3}k^{-5/3} \tag{59}$$

Two-mode model

Large scales (u_q) and small scales (u_K) interaction

$$\left\{ \begin{array}{ccccccc} \partial_t u_q & = & -\eta q^2 & u_q & + & \alpha q & u_K \\ \partial_t u_K & = & \alpha K & u_q & + & \gamma_K & u_K \end{array} \right. \quad (60)$$

$$\partial_t \begin{bmatrix} u_q \\ u_K \end{bmatrix} = M \begin{bmatrix} u_q \\ u_K \end{bmatrix} \quad \text{with} \quad M = \begin{bmatrix} -\eta q^2 & \alpha q \\ \alpha K & \gamma_K \end{bmatrix} \quad (61)$$

Expression of γ_K

$$\left\{ \begin{array}{cl} \gamma_K < 0 & \text{before small scale instability threshold} \\ \gamma_K = \gamma_{\mathcal{G}} > 0 & \text{after small scale instability threshold} \end{array} \right. \quad (62)$$

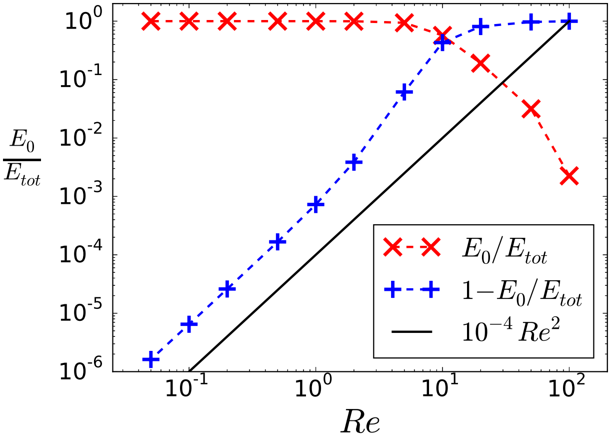
Expression of the energy ratio

Small scale instability threshold	Before	After
Growth rate:	$\frac{\gamma}{KU} \simeq Re\left(\frac{q}{K}\right) - \frac{1}{Re}\left(\frac{q}{K}\right)^2$	$\gamma \simeq \gamma_{\mathscr{G}}$
Amplitude ratio:	$\frac{u_q}{u_K} \simeq \frac{1}{Re}$	$\frac{u_q}{u_K} \simeq q\frac{U}{\gamma_{\mathscr{G}}}$
Energy ratio:	$\frac{E_0}{E_{tot}} \underset{Re \ll 1}{\simeq} 1 - Re^2$	$\frac{E_0}{E_{tot}} \simeq \left(q\frac{U}{\gamma_{\mathscr{G}}}\right)^2$

Energy ratio related to the Fr87 flow

Prediction

$$1 - \frac{E_0}{E_{tot}} \underset{Re \ll 1}{\simeq} Re^2 \qquad (63)$$



Equivalence

Navier-Stokes

$$\partial_t \boldsymbol{u} = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{\nabla} P + \frac{1}{Re}\Delta \boldsymbol{u} \quad \text{with} \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{64}$$

Note: $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$.

Vorticity equation

$$\partial_t \omega = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\omega) + \frac{1}{Re}\Delta\omega \quad \text{with} \quad \boldsymbol{\nabla}\times\boldsymbol{u} = \omega \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{65}$$

Induction equation

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) + \frac{1}{Rm}\Delta\boldsymbol{B} \quad \text{with} \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{66}$$

Inductive flows

General expression

$$\boldsymbol{u} = U \begin{bmatrix} \sin(ky+\phi_2) & + & \cos(kz+\psi_3) \\ \sin(kz+\phi_3) & + & \cos(kx+\psi_1) \\ \sin(kx+\phi_1) & + & \cos(ky+\psi_2) \end{bmatrix}. \tag{67}$$

Specific flows

A Helical ABC flow : $\phi_i = \psi_i = 0$ for $i \in \{1,2,3\}$.

B Non-helical ABC flow : $\phi_i = 0$ and $\psi_i = \frac{\pi}{2}$ for $i \in \{1,2,3\}$.

C Random ABC flow : $\phi_i = \psi_i = random$ for $i \in \{1,2,3\}$.

Growth rate

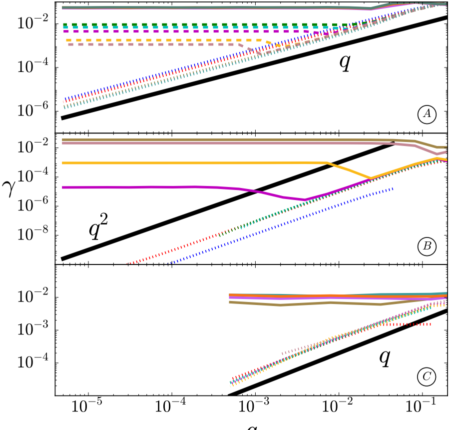
Prediction

$$\frac{\gamma}{KU} \underset{Rm \ll 1}{\simeq} Rm \left(\frac{q}{K}\right) \quad \text{and} \quad \gamma \underset{1 \ll Rm}{\simeq} \gamma_{\mathcal{G}} \,. \tag{68}$$

A Helical ABC.

B Non-helical ABC.

C Random helical ABC.

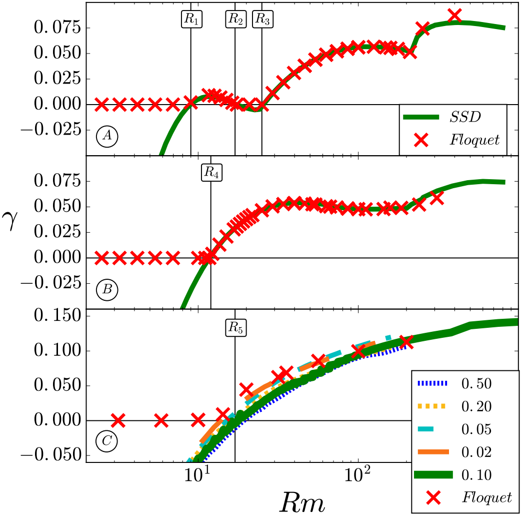


Instability map

A Helical ABC.

B Non-helical ABC.

C Random helical ABC.



Energy Ratio

Prediction

$$\frac{E_0}{E_{tot}} \underset{Rm \ll 1}{\simeq} 1 - Rm^2 \quad \text{and} \quad \frac{E_0}{E_{tot}} \underset{1 \ll Rm}{\simeq} \left(q \frac{U}{\gamma_{\mathcal{G}}} \right)^2 \tag{69}$$

- **A** Helical ABC.
- **B** Non-helical ABC.
- **C** Random helical ABC.