Large scale instability of helical flows

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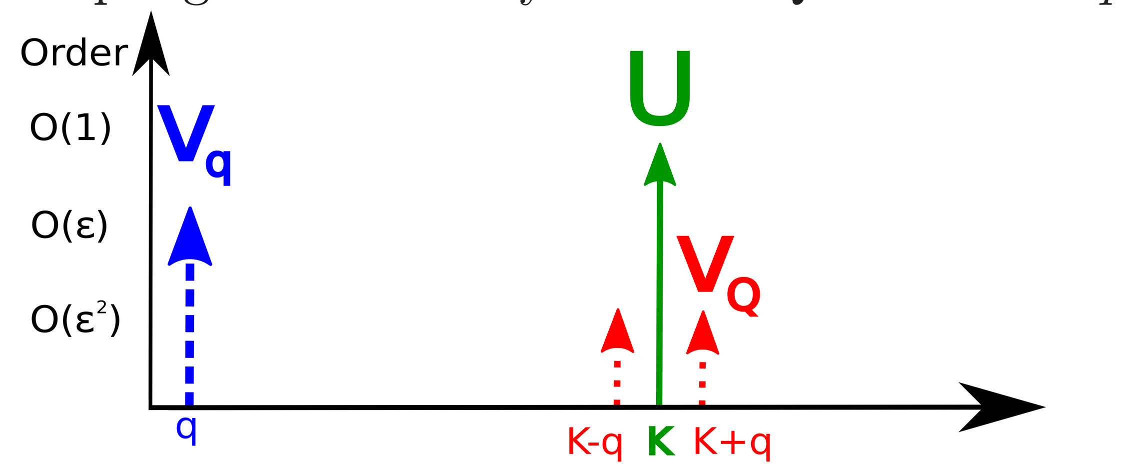
Introduction

We consider the linear instability of a spatial periodic flow $U_K(x)$ with wave number K to large scale modes (small wave numbers q). The instability occurs through the coupling with nearby modes $Q = K \pm q$.



Linearised HD equations

Incompressible Navier-Stokes equations:

$$\partial_t \boldsymbol{U} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla} P_K + \nu\triangle\boldsymbol{U} + \boldsymbol{F}\,, \tag{1}$$

$$\partial_t \boldsymbol{v} = \boldsymbol{U}\times\boldsymbol{\nabla}\times\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{U} - \boldsymbol{\nabla} P + \nu\triangle\boldsymbol{v}\,, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0 \quad , \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,. \tag{3}$$

Floquet framework:

$$\boldsymbol{v}(\boldsymbol{r},t) = \tilde{\boldsymbol{v}}(\boldsymbol{r},t)e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}} + c.c.\,, \tag{4}$$

The perturbation evolution equation (PEE) is:
$\partial_t\tilde{\boldsymbol{v}} = (\boldsymbol{\nabla}\times\boldsymbol{U})\times\tilde{\boldsymbol{v}} + (\imath\boldsymbol{q}\times\tilde{\boldsymbol{v}} + \boldsymbol{\nabla}\times\tilde{\boldsymbol{v}})\times\boldsymbol{U} - \boldsymbol{\nabla}p + \nu(\Delta - \boldsymbol{q}^2)\tilde{\boldsymbol{v}}\,.$

Theory

In [1], Frisch *et al.* showed that an anisotropic kinetic alpha (AKA) effect exists and drives large scale instabilities for flows satisfying certain conditions. It leads to a growth rate:

$$\sigma = \alpha q - \nu q^2\,. \tag{5}$$

In the absence of an AKA-effect, large scale instabilities are expected to be driven by negative eddy viscosity effects [3, 4, 5] and results in:

$$\sigma = \beta q^2 - \nu q^2\,. \tag{6}$$

The coefficients α and β however can only be calculated in certain limits (*e.g.* $Re \to \infty$).

Three modes model

In the limit of small $Re = \frac{U}{K\nu}$, for ABC flows,

$$U_x^{ABC} = C\sin(Kz) + B\cos(Ky)\,, \tag{7}$$

$$U_y^{ABC} = A\sin(Kx) + C\cos(Kz)\,, \tag{8}$$

$$U_z^{ABC} = B\sin(Ky) + A\cos(Ky)\,. \tag{9}$$

the instability can be reduced to three modes:

$$\nu\triangle\boldsymbol{\omega_Q} = -\boldsymbol{\nabla}\times[\boldsymbol{U}\times(\boldsymbol{\omega_q} - K\boldsymbol{v_q})]\,, \tag{10}$$

$$\partial_t\boldsymbol{\omega_q} = \boldsymbol{\nabla}\times[\boldsymbol{U}\times(\boldsymbol{\omega_Q} - K\boldsymbol{v_Q})] + \nu\triangle\boldsymbol{\omega_q}\,, \tag{11}$$

$$\boldsymbol{\omega_q} = \boldsymbol{\nabla}\times\boldsymbol{v_q} \quad , \quad \boldsymbol{\omega_Q} = \boldsymbol{\nabla}\times\boldsymbol{v_Q}\,. \tag{12}$$

For $A = 1 : B = 1 : C = \lambda$ flows:

$$\sigma = \beta q^2 - \nu q^2\;,\; \beta = bRe^2\nu\;,\; b = \frac{1-\lambda^2}{4+2\lambda^2}\,. \tag{13}$$

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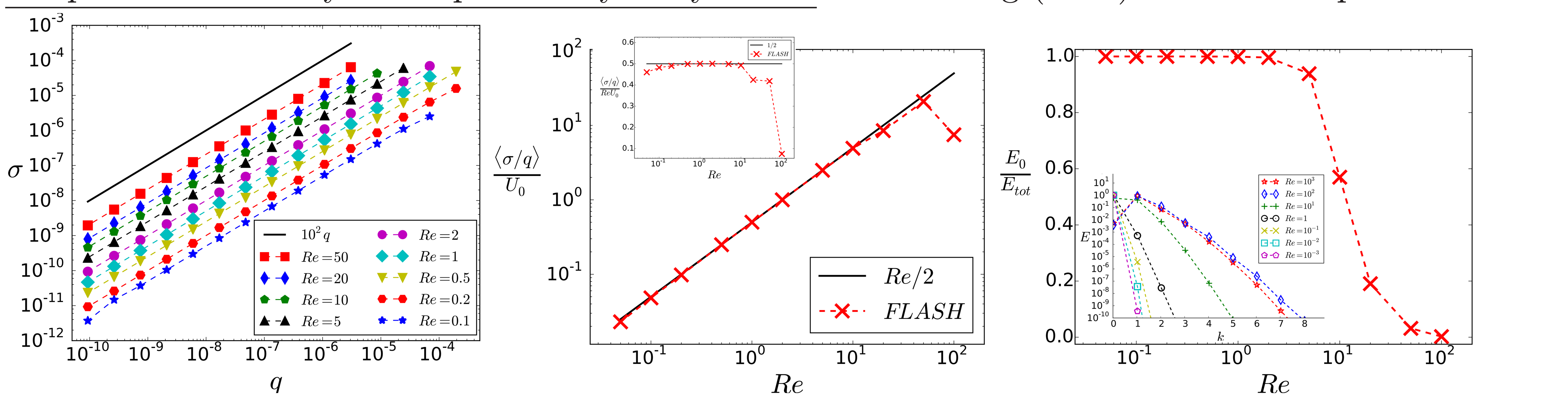
AKA-instability

An example flow that results in an AKA-effect is:

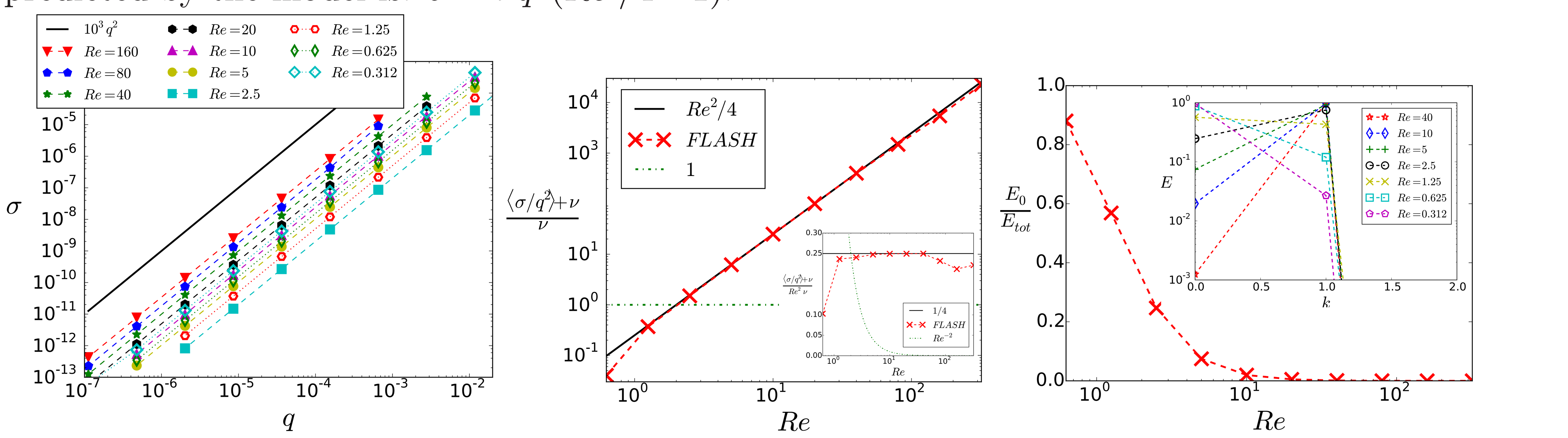
$$U_x = U_0\cos\left(Ky + \nu K^2 t\right)\,,\; U_y = U_0\sin\left(Kx - \nu K^2 t\right)\,,\; U_z = U_x + U_y\,. \tag{14}$$

The growth rate satisfies: $\sigma = \alpha q - \nu q^2 \quad , \quad \alpha = \frac{1}{2}ReU_0$. We confirmed these results using our Floquet Linear Analysis of Spectral Hydrodynamics code solving (PEE) in Fourrier space.



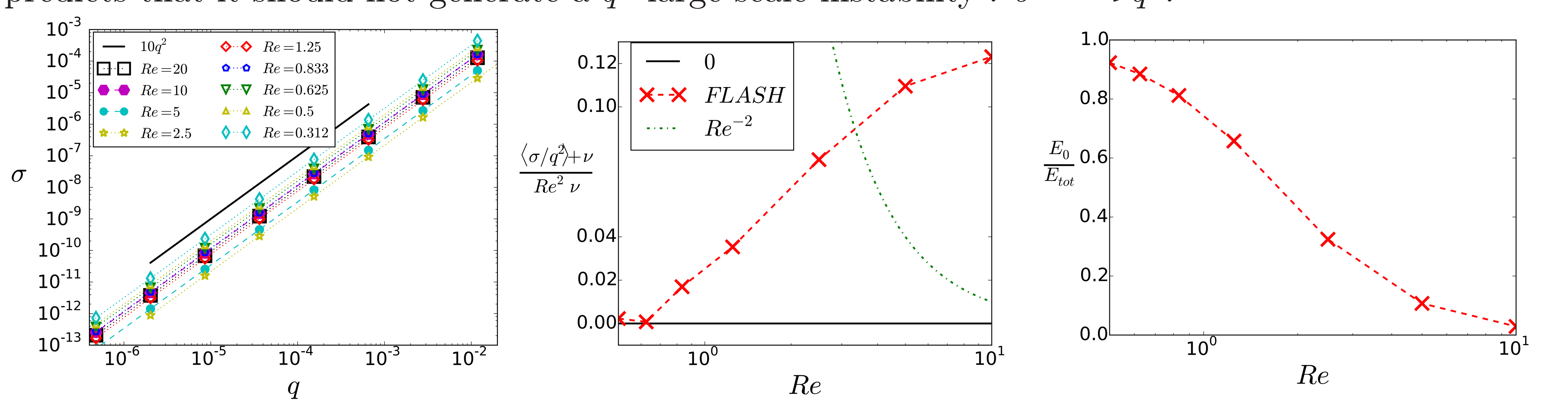
Roberts flow: $\lambda = 0$

The Roberts flow does not satisfy the conditions to generate an AKA-effect. The growth rate predicted by the model is: $\sigma = \nu q^2(Re^2/4 - 1)$.



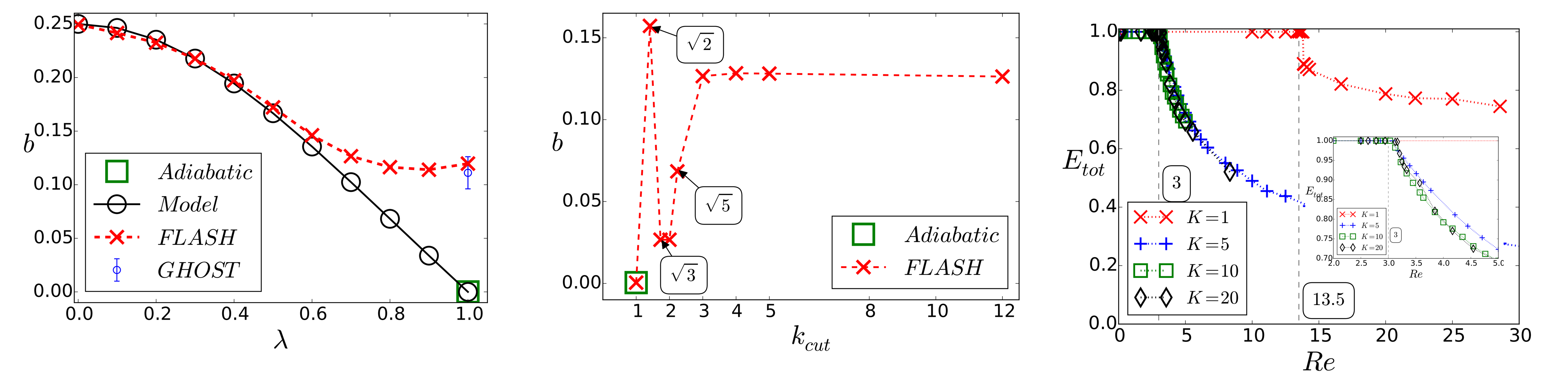
Equilateral ABC flow: $\lambda = 1$

The equilateral ABC flow does not satisfy the conditions to generate an AKA-effect and the theory predicts that it should not generate a q^2-large scale instability : $\sigma = -\nu q^2$.



ABC flows: λ variable

The b-coefficient computed via the FLASH code and the three modes model have a discrepancy for $\lambda \gtrsim 0.5$. As shows the study with variable k_{cut}, the error comes from the impact of modes of higher wave number not taken into account in the model. This is confirmed by the critical Reynolds number found using the GHOST code modelling the full non-linear Navier-Stokes equation. The critical small scale instability matches the results in [2].



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