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Quillen bifibrations

(and the Reedy construction)

TYPES 2019 – Oslo

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1. Motivations

2. Grothendieck construction for model categories

3. Illustration



1. Motivations



Categorical modeling of dependency

$$\begin{array}{c} \vdots \\ \downarrow p_3 \\ \mathcal{C}_3 \\ \downarrow p_2 \\ \mathcal{C}_2 \\ \downarrow p_1 \\ \mathcal{C}_1 \\ \downarrow p_0 \\ \mathcal{C}_0 \end{array}$$

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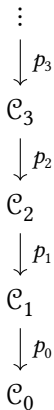
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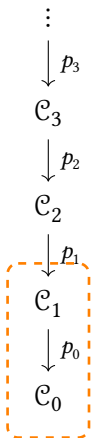
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- **Morphisms** of $\mathcal{C}_i$: **terms** between such contexts.

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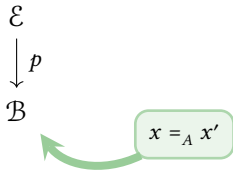
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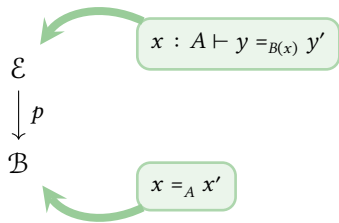
Lifting paths

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- Semantically?



2. Grothendieck construction for model categories



Grothendieck bifibrations

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The functor p is a [Grothendieck bifibration](#), i.e. there is a pseudofunctor

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$$\begin{array}{ccc} X & \xrightarrow{\lambda_u^X} & \Sigma_u X \\ \exists f_{\triangleleft} \downarrow & \searrow \forall f & \downarrow \exists f_{\triangleright} \\ u^* Y & \xrightarrow{\rho_u^Y} & Y \end{array}$$

Quillen bifibrations

Suppose $\mathcal{B}$ is a [model category](#) and that

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0-cells: model categories
1-cells: Quillen adjunctions
2-cells: natural transformations

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Can we equip $\mathcal{E}$ with a model structure?

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A **weak factorization system** (wfs) is the data (L, R) such that

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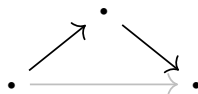
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A **model category** $\mathcal{C}$ consists of two wfs $(\text{WCoF}, \text{WFib})$ and $(\text{CoF}, \text{WFib})$ such that

$$W = \text{WFib} \circ \text{WCoF}$$

has the 2-out-of-3 property.



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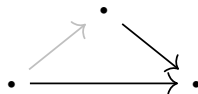
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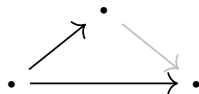
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Suppose $\mathcal{B}$ is a **model category** and that

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Say p is a **Quillen bifibration** if there is one such that

- p preserves **Cof**, **W**, **Fib**
- restriction of it to a fiber matches the fibers model structure.

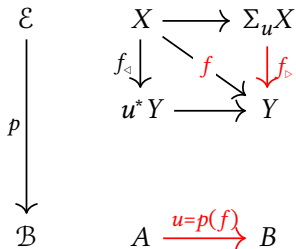
A first step

It is always possible to define two wfs $(\mathbf{WCof}_\mathcal{E}, \mathbf{Fib}_\mathcal{E})$ and $(\mathbf{Cof}_\mathcal{E}, \mathbf{WFib}_\mathcal{E})$:

$$\begin{array}{ccc} \mathcal{E} & X \longrightarrow \Sigma_u X & \\ \downarrow p & \begin{array}{ccc} f_{\triangleleft} \downarrow & \searrow f & \downarrow f_{\triangleright} \\ u^* Y & \longrightarrow & Y \end{array} & \\ \mathcal{B} & A \xrightarrow{u=p(f)} B & \end{array}$$

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- $f \in \mathbf{Fib}_{\mathcal{E}}$ (resp. $\mathbf{WFib}_{\mathcal{E}}$) if and only if:

$$u \in \mathbf{Fib} \text{ (resp. } \mathbf{WFib})$$

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Total weak equivalences

All remains to determine is when

$$W_{\mathcal{E}} \stackrel{\text{def}}{=} \text{WFib}_{\mathcal{E}} \circ \text{WCof}_{\mathcal{E}}$$

has the 2-out-of-3 property?

Necessary conditions

H_1 Σ_u preserves and reflects weak equivalences if $u \in \mathbf{WCof}$.

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$$\begin{array}{ccc} X' & \xrightarrow{\sim} & \Sigma_u X' \\ \uparrow k & & \uparrow \Sigma_u k \\ X & \xrightarrow{\sim} & \Sigma_u X \end{array}$$

A commutative square diagram enclosed in a red rounded rectangle. The diagram consists of four nodes: X' at the top-left, $\Sigma_u X'$ at the top-right, X at the bottom-left, and $\Sigma_u X$ at the bottom-right. Arrows connect the nodes: a horizontal arrow from X' to $\Sigma_u X'$ labeled $\sim$, a horizontal arrow from X to $\Sigma_u X$ labeled $\sim$, a vertical arrow from X to X' labeled k , and a vertical arrow from $\Sigma_u X$ to $\Sigma_u X'$ labeled $\Sigma_u k$.

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$$\begin{array}{ccc} v^*Y' & \xrightarrow{\sim} & Y' \\ \uparrow v^*k & & \uparrow k \\ v^*Y & \xrightarrow{\sim} & Y \end{array}$$

The diagram is a commutative square enclosed in a red rounded rectangle. It consists of four objects arranged in a square: v^*Y' at the top-left, Y' at the top-right, v^*Y at the bottom-left, and Y at the bottom-right. Arrows connect the objects: a horizontal arrow from v^*Y' to Y' labeled $\sim$, a horizontal arrow from v^*Y to Y labeled $\sim$, a vertical arrow from v^*Y to v^*Y' labeled v^*k , and a vertical arrow from Y to Y' labeled k .

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hBC every square of the form

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$$u, u' \in \mathbf{WCof}$$

$$v, v' \in \mathbf{WFib}$$

satisfies the [homotopical Beck-Chevalley](#) condition, i.e. the mate

$$\Sigma_{u'} v^* \rightarrow v'^* \Sigma_u \text{ is pointwise in } \mathbf{W}_{C'}.$$

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b.

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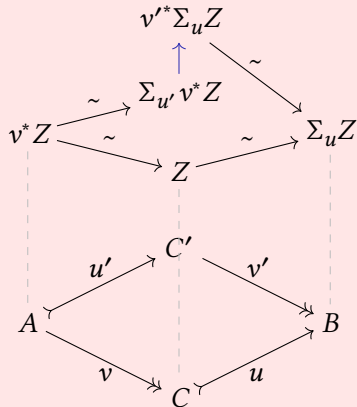
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Characterization of Quillen bifibrations

Theorem (Cagne, Melliès)

Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck bifibration such that (Σ_u, u^*) is a Quillen adjunction for every u in $\mathcal{B}$. Then p is a Quillen bifibration if and only if it satisfies H_1 , H_2 and hBC.

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Related works:

Model category structures in bifibred categories (1994) Roig

Bifibrations and Weak Factorisation Systems (2012) Stanculescu

The Grothendieck construction for model categories (2015) Harpaz, Prasma

3. Illustration

Codomain bifibration

If $\mathcal{C}$ has pullbacks, the codomain functor is a bifibration:

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Resulting structure on [Arr\(\$\mathcal{C}\$ \)](#) is the injective model structure!

Thank you.