

Pierre Cagne

Towards a homotopical algebra of dependent types

Vers une algèbre homotopique des types dépendants

Université Paris Diderot – Paris 7
7 Décembre 2018



1. A gentle introduction

2. Model categories

3. Lawvere's categorical understanding on logic

4. Quillen bifibrations and their localizations

5. Display maps and relative tribes



1. Everything you always wanted to know about what I do^{*}



^{*}But were afraid to ask.

Big picture

Logic

Homotopy

Big picture

Logic

study of the mathematical activity

formal point of view

what is a property?

what is a proof?

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Homotopy

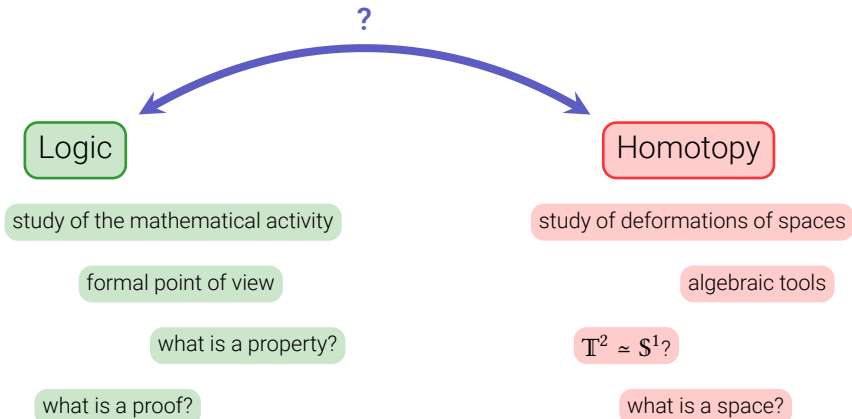
study of deformations of spaces

algebraic tools

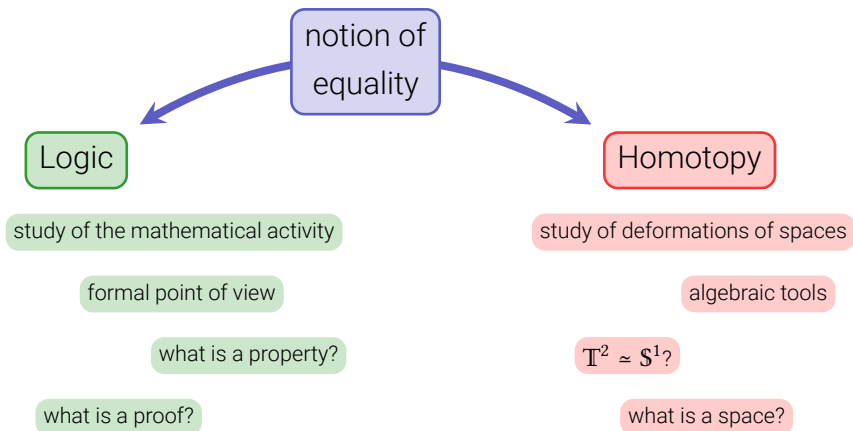
$\mathbb{T}^2 \simeq \mathbb{S}^1?$

what is a space?

Big picture



Big picture



Ontology of equality

Extensional equality

Intensional equality

Ontology of equality

Extensional equality

Equal things are interchangeable.

If x equals y , then a property on x is true **exactly when** it is on y .

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Intensional equality

Proofs of equality are relevant.

A proof that x equals y **translates** meaningful properties on x to meaningful properties on y .

Philosophical (kangaroo-free) detour

Consider the rather **sound** sentence:

Rational beings are crazy about epistemological questions.

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What to **think** then of Plato's substitution:

Featherless bipedes are crazy about epistemological questions.

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Rational beings are crazy about epistemological questions.

What to **think** then of Plato's substitution:

Featherless bipedes are crazy about epistemological questions.

Instead a **proof** that *featherless bipedes* are in fact *rational beings* acts as a translator:

Featherless bipedes, neither unbalanced or handicaped to flight by a superior weight of the head have been able to evolve towards larger and larger brains hence making them capable of reasoning to a point that reasoning itself can be reasoned upon, are crazy about epistemological questions.

Back to math

Fibonacci sequence: $F_0 = 0$ and $F_1 = 1$, and next $F_n = F_{n-1} + F_{n-2}$.

Prove that F_{18} and F_{19} are relatively prime.

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Euclid algorithm!

Proofs with content

Let $a, b \in \mathbb{N}_{>0}$. A good way to compute $\gcd(a, b)$ is Euclid (extended) algorithm, which yield also n, m such that $na + mb = \gcd(a, b)$.

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Limit the search to $n \in \{0, \dots, 4180\}$ and try them all.

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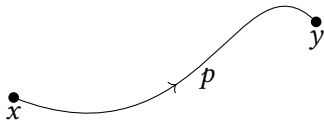
Moral: proofs of equality are relevant mathematical objects.

Enters homotopy

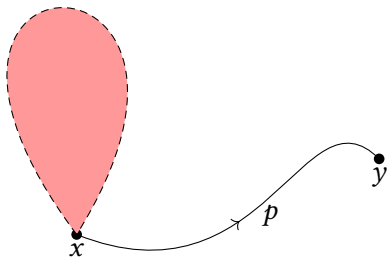
$\bullet$
 x

$\bullet$
 y

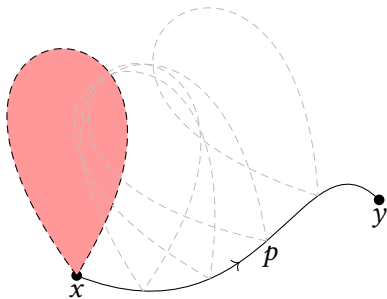
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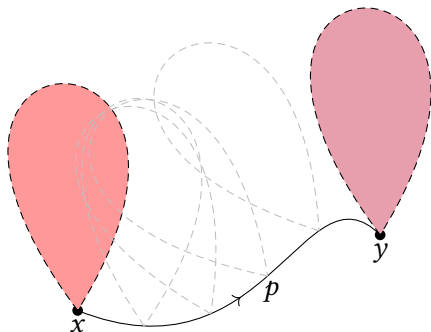
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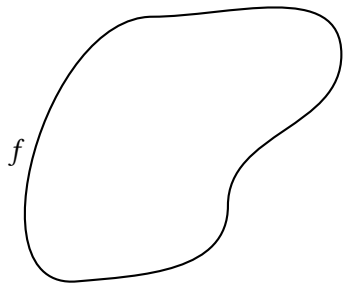




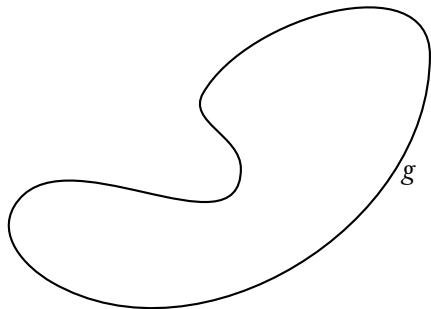
2. Model categories



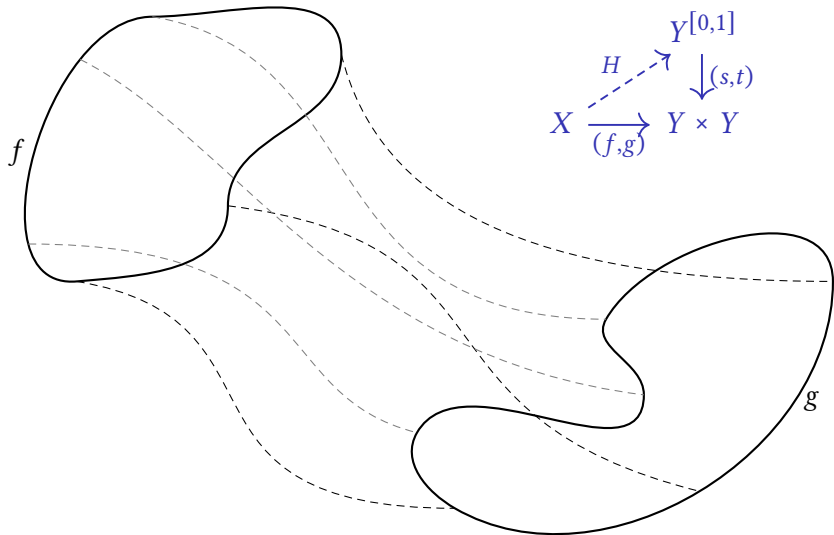
Homotopy in topological spaces



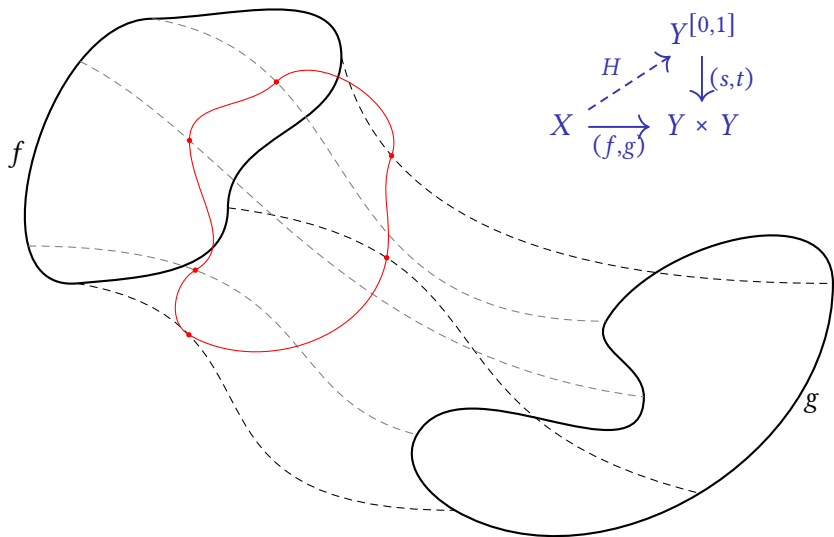
$$X \xrightarrow{(f,g)} Y \times Y$$



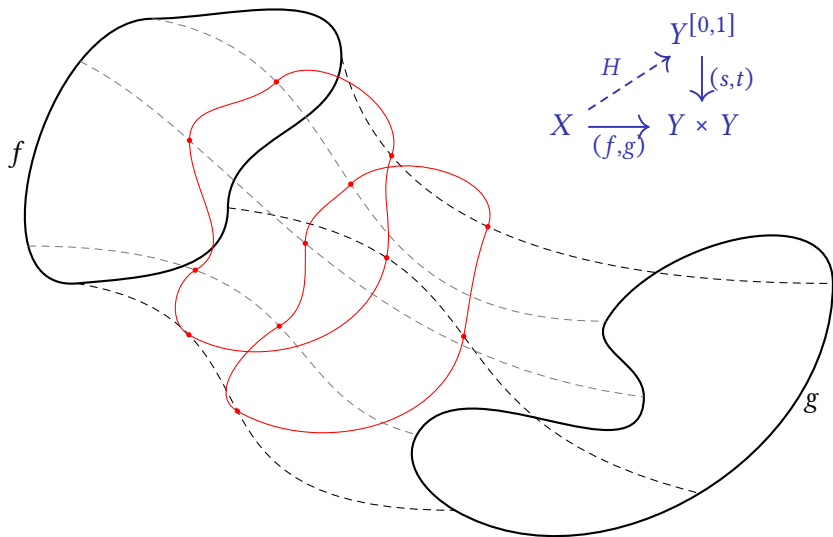
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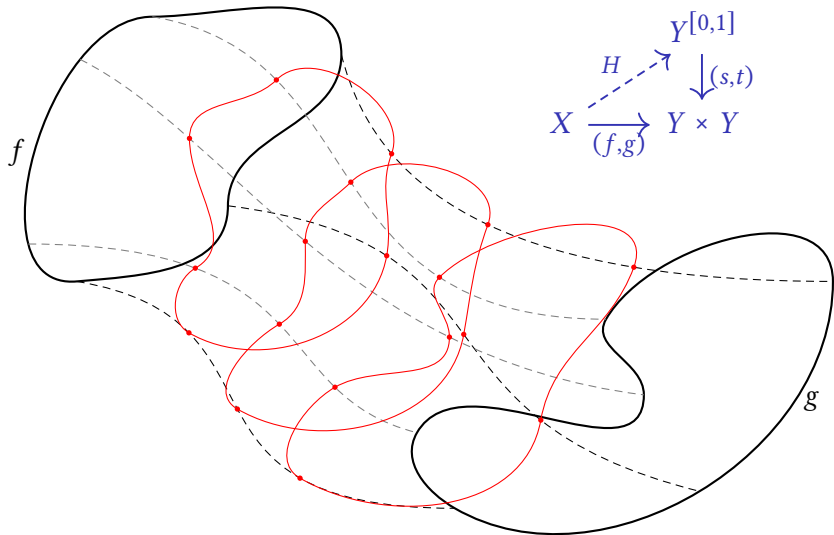
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In categorical terms

The space of paths $Y^{[0,1]}$ in **Top** satisfies the *path lifting property*:

$$\begin{array}{ccc} \{*\} & \xrightarrow{\gamma} & Y^{[0,1]} \\ \downarrow 0 & & \downarrow (s,t) \\ [0,1] & \xrightarrow{(\sigma,\tau)} & Y \times Y \end{array}$$

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Model categories make formal what it means to have such **objects** with such **properties** in any category.

Model categories

In a category $\mathcal{M}$, denote $j \sqsubset q$ when for any

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

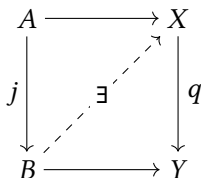
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Model categories

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Define $\mathbf{RLP}(\mathcal{L})$ as the class of q such that $j \sqsupseteq q$ for all $j \in \mathcal{L}$.

Define $\mathbf{LLP}(\mathcal{R})$ as the class of j such that $j \sqsupseteq q$ for all $q \in \mathcal{R}$.

A **weak factorization system** on a category $\mathcal{M}$ is a pair $(\mathcal{L}, \mathcal{R})$ such that

- $\mathcal{L} = \mathbf{LLP}(\mathcal{R})$ and $\mathbf{RLP}(\mathcal{L}) = \mathcal{R}$
- each morphism factors as $q \circ j$ for $j \in \mathcal{L}$ and $q \in \mathcal{R}$.

Model categories

A **model category** is a category $\mathcal{M}$ with finite products and finite sums together with **Cof**, **W**, **Fib** such that

- **W** has 2-out-of-3,
- **(Cof** $\cap$ **W**, **Fib**) and **(Cof**, **W** $\cap$ **Fib**) are weak factorization systems.

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An adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is **Quillen** when:

$$L(\text{Cof}_{\mathcal{M}}) \subseteq \text{Cof}_{\mathcal{N}} \quad R(\text{Fib}_{\mathcal{N}}) \subseteq \text{Fib}_{\mathcal{M}}$$

Path object

How does this give a good notion of **path spaces**?

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Given $Y \in \mathcal{M}$, factor $Y \rightarrow Y \times Y$ into

$$Y \xrightarrow{\text{acyc. cof.}} P_Y \xrightarrow{\text{fibration}} Y \times Y$$

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Now define an [homotopy](#) between $f, g : X \rightarrow Y$ as a map H in

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No need for a whole model structure, good notions of **fibrations** and **acyclic cofibrations** seems enough...

3. (My view of) Lawvere's view on logic



Quantifiers as adjoints

Given formulas $\varphi(x_1, \dots, x_n, y)$ and $\psi(x_1, \dots, x_n)$ in a first-order language:

$$\varphi(\vec{x}, y) \models \psi(\vec{x}) \quad \text{if and only if} \quad \exists y \varphi(\vec{x}, y) \models \psi(\vec{x})$$

and

$$\psi(\vec{x}) \models \varphi(\vec{x}, y) \quad \text{if and only if} \quad \psi(\vec{x}) \models \forall y \varphi(\vec{x}, y)$$

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It feels like adjunctions!

Lawvere's hyperdoctrine

An **hyperdoctrine** is a pseudofunctor $P : \mathcal{C}^\circ \rightarrow \mathbf{Cat}$ such that:

- $\mathcal{C}$ has finite products,
- each $P(f)$ has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$ (with Beck-Chevalley conditions),
- each $P(c)$ is a **Boolean algebra**, and each $P(f)$ a morphism of **Boolean algebras**

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- each $P(c)$ is a **Heyting algebra**, and each $P(f)$ a morphism of **Heyting algebras**.

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- each $P(c)$ is a **CCC**, and each $P(f)$ a morphism of **CCCs**.

Define the **equality predicate** over $c \in \mathcal{C}$ as the direct image of 1_c along the diagonal

$$1_c \xrightarrow{\exists_\Delta} \exists_\Delta(1_c)$$

$$c \xrightarrow{\Delta} c \times c$$

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A Grothendieck *fibration* is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that

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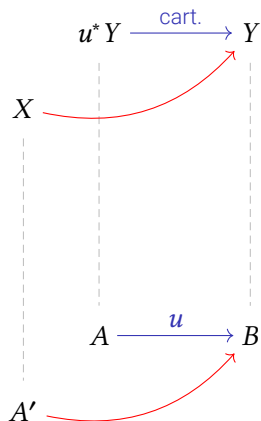
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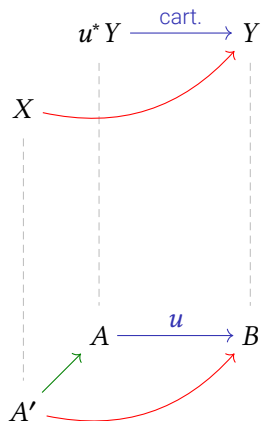
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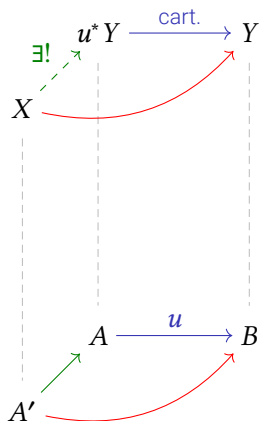
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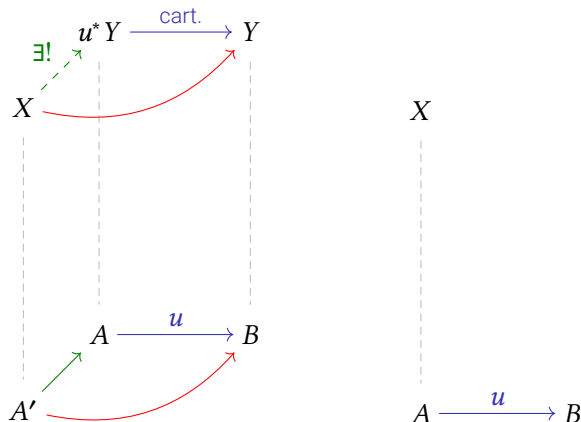
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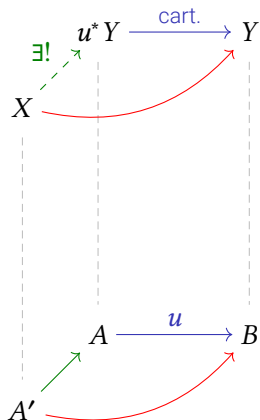
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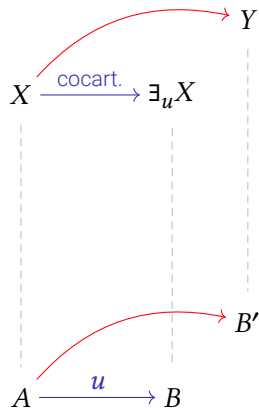
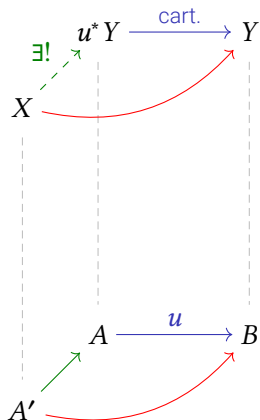
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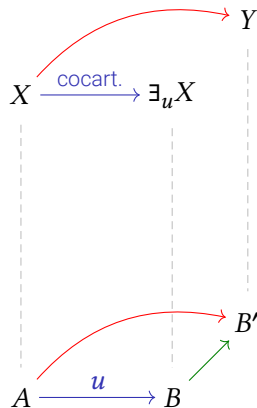
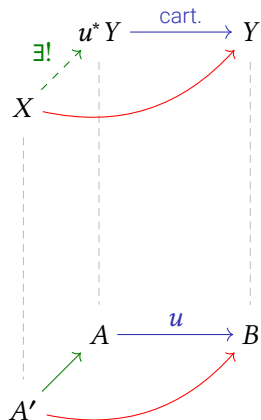
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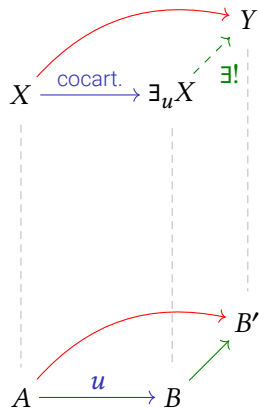
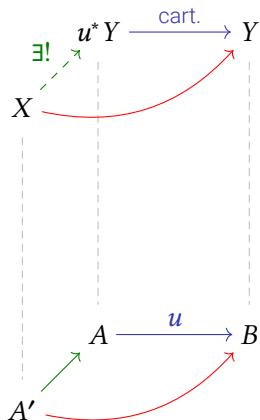
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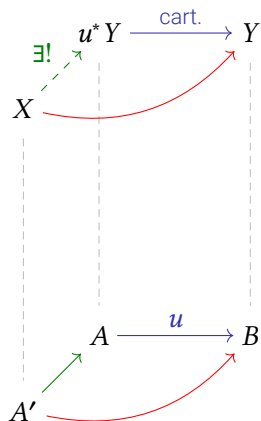
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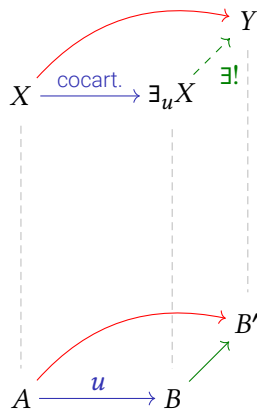


Grothendieck bifibrations

A Grothendieck **bifibration** is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that



and



Push and pull

For any $f : X \rightarrow Y$ in $\mathcal{E}$ over $u : A \rightarrow B$,

A commutative square diagram illustrating the relationship between objects u^*Y , Y , X , and $\exists_u X$. The diagram consists of four nodes arranged in a square. The top-left node is u^*Y , the top-right node is Y , the bottom-left node is X , and the bottom-right node is $\exists_u X$. There are four arrows: a blue arrow from u^*Y to Y labeled c^Y , a blue arrow from X to $\exists_u X$ labeled c_X , a green arrow from X to u^*Y labeled $f_{\triangleleft}$, and a green arrow from $\exists_u X$ to Y labeled $f_{\triangleright}$. A red diagonal arrow labeled f points from X to Y .

$$\begin{array}{ccc} u^*Y & \xrightarrow{c^Y} & Y \\ f_{\triangleleft} \uparrow & f \nearrow & \uparrow f_{\triangleright} \\ X & \xrightarrow{c_X} & \exists_u X \end{array}$$

Push and pull

For any $f : X \rightarrow Y$ in $\mathcal{E}$ over $u : A \rightarrow B$,

A commutative square diagram with vertices u^*Y (top-left), Y (top-right), X (bottom-left), and $\exists_u X$ (bottom-right). The edges are: a blue arrow $c^Y : u^*Y \rightarrow Y$ (top), a blue arrow $c_X : X \rightarrow \exists_u X$ (bottom), a green arrow $f_{\triangleleft} : X \rightarrow u^*Y$ (left), a green arrow $f_{\triangleright} : \exists_u X \rightarrow Y$ (right), and a red diagonal arrow $f : X \rightarrow Y$ (diagonal).

Hence $\exists_u$ and u^* extend to functors:

$$\begin{array}{c} X' \\ \uparrow k \\ X \end{array}$$

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A commutative square diagram with vertices u^*Y (top-left), Y (top-right), X (bottom-left), and $\exists_u X$ (bottom-right). The edges are: a blue arrow c^Y from u^*Y to Y ; a blue arrow c_X from X to $\exists_u X$; a green arrow $f_{\triangleleft}$ from X to u^*Y ; a green arrow $f_{\triangleright}$ from $\exists_u X$ to Y ; and a red diagonal arrow f from X to Y .

Hence $\exists_u$ and u^* extend to functors:

A commutative triangle diagram with vertices X' (top), X (bottom-left), and $\exists_u X$ (bottom-right). The edges are: a black arrow k from X to X' ; a black arrow $c_{X'}$ from X' to $\exists_u X$; and a black arrow c_X from X to $\exists_u X$.

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 \end{array}$$

Hence $\exists_u$ and u^* extend to functors:

$$\begin{array}{ccc}
 X' & \xrightarrow{c_{X'}} & \exists_u X' \\
 \uparrow k & \xrightarrow{\exists_u} & \uparrow \\
 X & \xrightarrow{c_X} & \exists_u X
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 \quad
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 \end{array}$$

This produces an adjunction $\exists_u : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ between fibers.

Grothendieck construction

From a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$, construct:

$$\mathcal{B}^\circ \xrightarrow{\tilde{p}} \mathbf{Cat}$$

$$\begin{array}{ccc} A & \longmapsto & \mathcal{E}_A \\ u \downarrow & & u^* \uparrow \\ B & \longmapsto & \mathcal{E}_B \end{array}$$

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From a pseudofunctor $P : \mathcal{B}^\circ \rightarrow \mathbf{Cat}$ where each $P(u)$ has a left adjoint, construct:

$$\begin{array}{c} \int_{\mathcal{B}} P \\ \downarrow \\ \mathcal{B} \end{array}$$

objects: pairs (X, A) for $X \in P(A)$

morphisms $(X, A) \rightarrow (Y, B)$: pairs (u, f) where $u : A \rightarrow B$ and $f : X \rightarrow P(u)(Y)$.

4. Quillen bifibrations and their localizations



Goal

Given a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$, with $\mathcal{B}$ a **model category** and each fiber $\mathcal{E}_A$ a **model category** also, what are the **conditions** that makes $\mathcal{E}$ into a compatible **model category**?

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Define a **Quillen bifibration** to be a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ between model categories such that:

- p preserves cofibrations, fibrations, and weak equivalences,
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Find a **characterization** of Quillen bifibrations among the Grothendieck bifibrations that comes equipped with a model structure on $\mathcal{B}$ and on each fiber $\mathcal{E}_A$.

General principle

Suppose $p : \mathcal{E} \rightarrow \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(\exists_u, u^*)$ is Quillen.

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Call f in $\mathcal{E}$

- a **total cofibration** if $p(f) \in \mathbf{Cof}$ and $f_{\triangleright} \in \mathbf{Cof}_B$,
- a **total fibration** if $p(f) \in \mathbf{Fib}$ and $f_{\triangleleft} \in \mathbf{Fib}_A$.

$$\begin{array}{ccc} & & Y \\ & \nearrow f & \uparrow \\ X & \rightarrow & \exists_u X \end{array}$$

Denote $\mathbf{Cof}_{\mathcal{E}}$ and $\mathbf{Fib}_{\mathcal{E}}$ the corresponding classes.

$$A \xrightarrow{u} B$$

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Call f in $\mathcal{E}$

- a total acyclic cofibration if $p(f) \in \mathbf{Cof} \cap \mathbf{W}$ and $f_{\triangleright} \in \mathbf{Cof}_B \cap \mathbf{W}_B$,
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Fact: in a Quillen bifibration, the (acyclic) cofibrations and (acyclic) fibrations are the total ones.

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Fact: $(\mathbf{Cof}_{\mathcal{E}}, \mathbf{WFib}_{\mathcal{E}})$ and $(\mathbf{WCof}_{\mathcal{E}}, \mathbf{Fib}_{\mathcal{E}})$ are always WFS.

Remains to find $\mathbf{W}_{\mathcal{E}}$ such that

- $\mathbf{W}_{\mathcal{E}}$ has 2-out-of-3,
- $\mathbf{WCof}_{\mathcal{E}} = \mathbf{Cof}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$ and $\mathbf{WFib}_{\mathcal{E}} = \mathbf{Fib}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$.

Total weak equivalences

Main Idea

You don't have a choice for the total weak equivalences:

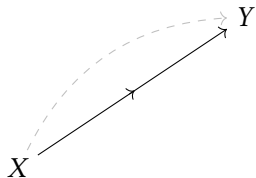
$$\mathbf{W}_{\mathcal{E}} \stackrel{\text{def}}{=} \mathbf{WFib}_{\mathcal{E}} \circ \mathbf{WCof}_{\mathcal{E}}$$

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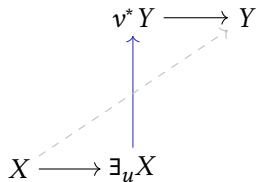
$$A \xrightarrow{u} C \xrightarrow{v} B$$

Total weak equivalences

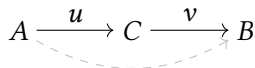
Main Idea

You don't have a choice for the total weak equivalences:

$$W_{\mathcal{E}} \stackrel{\text{def}}{=} \mathbf{WFib}_{\mathcal{E}} \circ \mathbf{WCoF}_{\mathcal{E}}$$



Call $f : X \rightarrow Y$ **acyclic relatively** to (u, v) if $p(f) = vu$ with $u \in \mathbf{CoF} \cap \mathbf{W}$ and $v \in \mathbf{Fib} \cap \mathbf{W}$, and $\exists_u X \rightarrow v^* Y \in W_C$.



Define f to be a **total weak equivalence** if it is acyclic relatively to some pair.

Necessary conditions

H_1 $\exists u$ preserves and reflects weak equivalences if $u \in \text{Cof}_n \mathbf{W}$.

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$$\begin{array}{ccc} v^*Y' & \xrightarrow{\sim} & Y' \\ \uparrow v^*k & & \uparrow k \\ v^*Y & \xrightarrow{\sim} & Y \end{array}$$

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hBC every square of the form

$$\begin{array}{ccc} A & \xrightarrow{u'} & C' \\ \downarrow v & & \downarrow v' \\ C & \xrightarrow{u} & B \end{array}$$

$$u, u' \in \mathbf{Cof} \cap \mathbf{W}$$

$$v, v' \in \mathbf{Fib} \cap \mathbf{W}$$

satisfies the [homotopical Beck-Chevalley](#) condition, i.e. the mate

$$\exists_{u'} v^* \rightarrow v'^* \exists_u \text{ is pointwise in } \mathbf{W}_{C'}.$$

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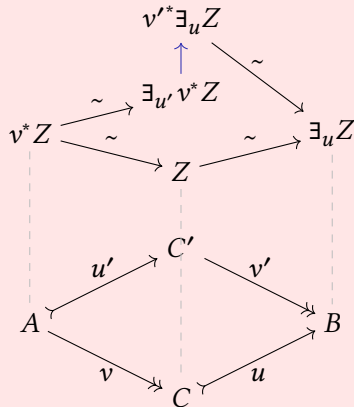
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Characterization of Quillen bifibrations

Theorem (Cagne, Melliès)

Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck bifibration such that $(\exists_u, u^*)$ is a Quillen adjunction for every u in $\mathcal{B}$. Then p is a Quillen bifibration if and only if it satisfies H_1 , H_2 and hBC.

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Related works:

Model category structures in bifibred categories (1994) Roig

Bifibrations and Weak Factorisation Systems (2012) Stanculescu

The Grothendieck construction for model categories (2015) Harpaz, Prasma

Warm up example: the codomain bifibration

Given a category $\mathcal{C}$ with pullbacks, the functor

$$\text{cod} : \text{Arr}(\mathcal{C}) \rightarrow \mathcal{C}, (A' \rightarrow A) \mapsto A$$

is a bifibration where $\exists_u$ and u^* are given by:

$$\begin{array}{ccc} A' & & \\ f \downarrow & & \\ A & \xrightarrow{u} & B \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

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$$\text{cod} : \text{Arr}(\mathcal{C}) \rightarrow \mathcal{C}, (A' \rightarrow A) \mapsto A$$

is a bifibration where $\exists_u$ and u^* are given by:

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The Reedy construction

Consider the category of standard simplices: Δ .

Denote Δ_n for the full subcategory with objects $[k]$ s.t. $k \leq n$.

The restriction $p_n : [\Delta_{n+1}, \mathcal{M}] \rightarrow [\Delta_n, \mathcal{M}]$ is a bifibration where:

$$\begin{array}{ccc} M_n A & \xrightarrow{M_n u} & M_n B \\ \uparrow & & \uparrow \\ & \nearrow Y & \\ X & & \\ \uparrow & & \\ L_n A & \xrightarrow{L_n u} & L_n B \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

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If $[\Delta_n, \mathcal{M}]$ has the Reedy model structure, we then obtain a model structure on $[\Delta_{n+1}, \mathcal{M}]$... which is again the [Reedy model structure](#).

Localization of Quillen bifibrations

A Quillen bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ preserves weak equivalences:

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Also each adjunction $\exists_u : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ is Quillen, so there is a derived adjunction

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Which relationship does $\mathbf{Ho}(p)$ sustain with the collection of all those derived adjunctions $\mathbf{L}(\exists_u) \dashv \mathbf{R}(u^*)$?

Fiberwise localization

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Construct it as follows:

- Consider $\mathcal{B}$ with the **trivial model structure**: denote it $\mathcal{B}_t$.
- Use the theorem to get a model structure on $\mathcal{E}$: denote it $\mathcal{E}_t$.

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Then $\mathbf{Ho}(p_t) : \mathbf{Ho}(\mathcal{E}_t) \rightarrow \mathbf{Ho}(\mathcal{B}_t) \cong \mathcal{B}$ is still a **Grothendieck bifibration**: its fiber at A is $\mathbf{Ho}(\mathcal{E}_A)$ and the adjunction above $u : A \rightarrow B$ is the derived adjunction $\mathbf{L}(\exists_u) : \mathbf{Ho}(\mathcal{E}_A) \rightleftarrows \mathbf{Ho}(\mathcal{E}_B) : \mathbf{R}(u^*)$.

Theorem (Cagne, Melliès)

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & \mathbf{Ho}(\mathcal{E}_t) & \longrightarrow & \mathbf{Ho}(\mathbf{Ho}(\mathcal{E}_t)) \cong \mathbf{Ho}(\mathcal{E}) \\ p \downarrow & & \downarrow \mathbf{Ho}(p_t) & & \downarrow \mathbf{Ho}(p) \\ \mathcal{B} & \longrightarrow & \mathbf{Ho}(\mathcal{B}_t) \cong \mathcal{B} & \longrightarrow & \mathbf{Ho}(\mathcal{B}) \end{array}$$

How so?

- Give back to $\mathcal{B}$ its [original](#) model structure.
- Use again the theorem to get the model structure on $\mathbf{Ho}(\mathcal{E}_t)$.

5. Display maps and relative tribes



Given a **dependent type theory** $\mathbb{T}$, build a category **ctx**:

- the **objects** are the dependent contexts

$$x_1 : A_1, x_2 : A_2(x_1), \dots, x_n : A_n(x_1, \dots, x_{n-1})$$

- the **morphisms** $(\vec{x} : \vec{A}) \rightarrow (\vec{y} : \vec{B})$ are the $|B|$ -dependent tuples of terms

$$\vec{x} : \vec{A} \vdash t_1(\vec{x}) : B_1,$$

$$\vec{x} : \vec{A} \vdash t_2(\vec{x}) : B_2(t_1),$$

...

$$\vec{x} : \vec{A} \vdash t_n(\vec{x}) : B_n(t_1, \dots, t_{n-1})$$

Martin Löf's Type Theory (cont'd)

Given a type A dependent on $\vec{A}$, there is a *dependent projection* in **ctx**:

$$\text{pr}_B : (\vec{x} : \vec{A}, y : B) \rightarrow (\vec{x} : \vec{A})$$

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The codomain functor **Fib** $\rightarrow$ **ctx** is a Grothendieck fibration.

Associated pseudofunctor:

$$\mathbb{T} : \text{ctx}^\circ \rightarrow \text{Cat}, \quad \Gamma \mapsto (\text{cxt}_{/\Gamma})^{\text{Fib}}$$

Dependent pairs and dependent functions

The substitution functor $\mathbf{pr}_B^*$ has both left and right adjoint, given by dependent pair and dependent function types:

$$\frac{\vec{x} : \vec{A}, y : B(\vec{x}), z : C(\vec{x}, y) \vdash t(\vec{x}, y, z) : C'(\vec{x})}{\vec{x} : \vec{A}, w : \sum_{y:B(\vec{x})} C(\vec{x}, y) \vdash t(\vec{x}, w_1, w_2) : C'(\vec{x})}$$

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$\mathbb{T} : \mathbf{ctx}^\circ \rightarrow \mathbf{Cat}$ admits left/right adjoints for the functors of the form $\mathbb{T}(\mathbf{pr}_B)$.

Identity types

A way to treat **proofs of equality** as mathematical entities:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A, y : A \vdash \text{Id}_A(x, y) \text{ type}}$$

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How to **use** such proofs of equality?

$$\frac{\Gamma, x : A, y : A, p : \text{Id}_A(x, y) \vdash C(x, y, p) \text{ type} \quad \Gamma, x : A \vdash c(x) : C(x, x, \text{refl}_x)}{\Gamma, x : A, y : A, p : \text{Id}_A(x, y) \vdash j(c, x, y, p) : C(x, y, p)}$$
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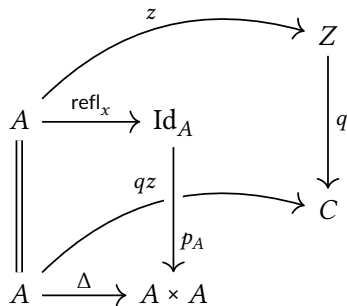
Identity types as an equality predicates?

The previous rules induces:

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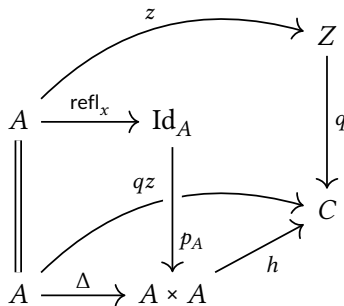
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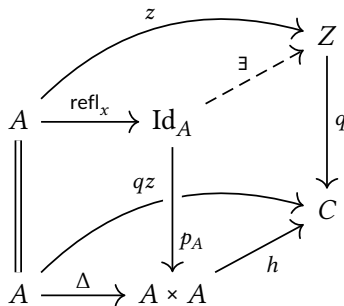
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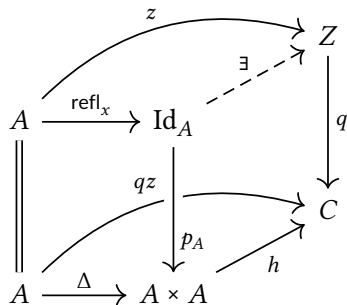
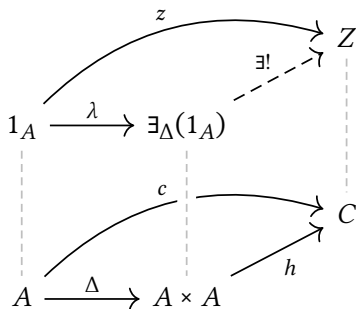
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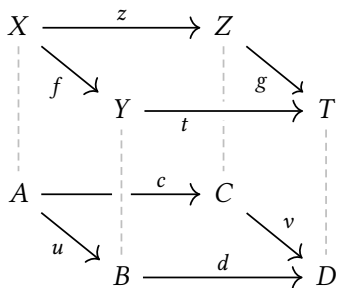
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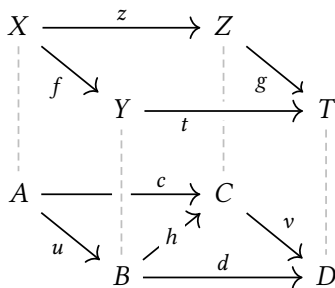
Relative lifting property

Given a functor $p : \mathcal{E} \rightarrow \mathcal{B}$, say that a map f in $\mathcal{E}$ has the **weak left lifting property** relative to p against g when:



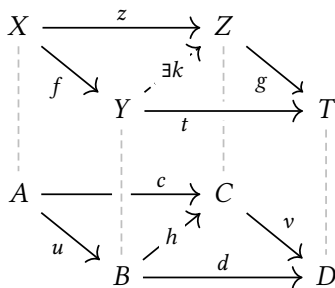
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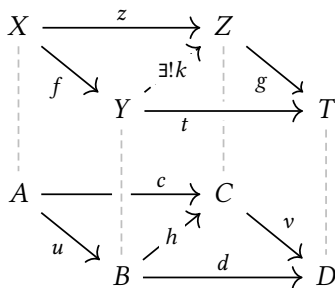
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Relative factorization systems

Define a right weak factorization system relative to $p : \mathcal{E} \rightarrow \mathcal{B}$ to consist of

- two classes $\mathcal{L}_{\mathcal{E}}, \mathcal{R}_{\mathcal{E}}$ of morphisms of $\mathcal{E}$,
- and two classes $\mathcal{L}_{\mathcal{B}}, \mathcal{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

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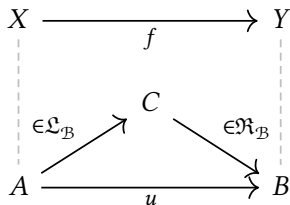
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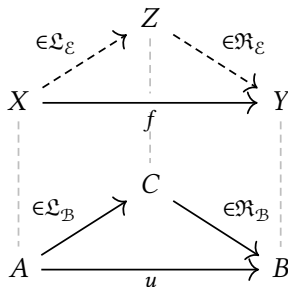
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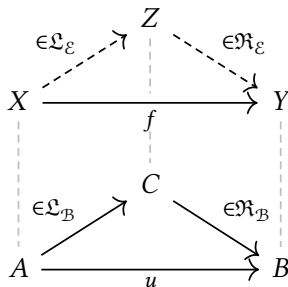
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Define a **right strong factorization system** relative to $p : \mathcal{E} \rightarrow \mathcal{B}$ to consist of

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- and two classes $\mathcal{L}_{\mathcal{B}}, \mathcal{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathcal{L}_{\mathcal{E}}) \subseteq \mathcal{L}_{\mathcal{B}}$ and $p(\mathcal{R}_{\mathcal{E}}) \subseteq \mathcal{R}_{\mathcal{B}}$,
- $\mathcal{L}_{\mathcal{E}} = \text{LLP}_p^{\perp}(\mathcal{R}_{\mathcal{E}})$
- for every f in $\mathcal{E}$:



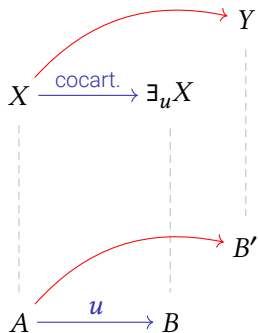
Cocartesian morphism as lifting problems



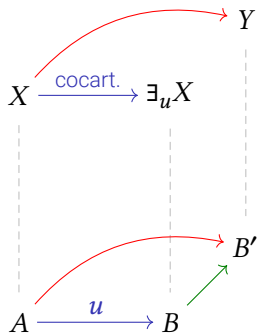
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$$\begin{array}{ccc} X & \xrightarrow{\text{cocart.}} & \exists_u X \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

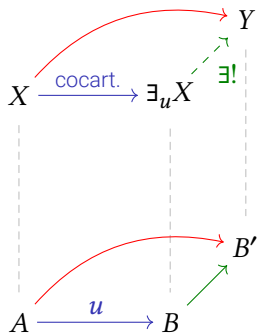
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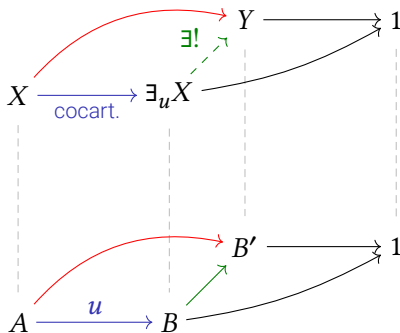
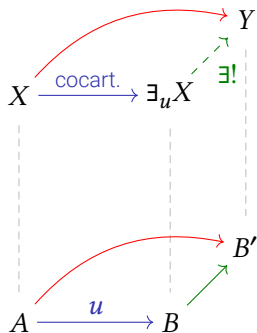
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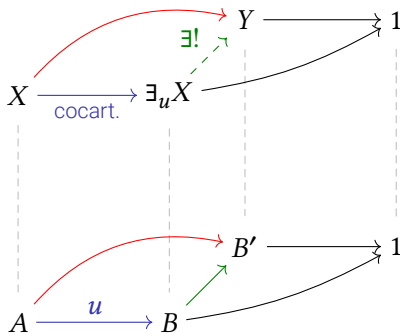
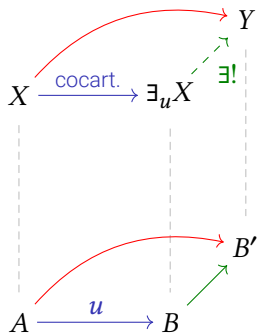


Cocartesian morphism as lifting problems



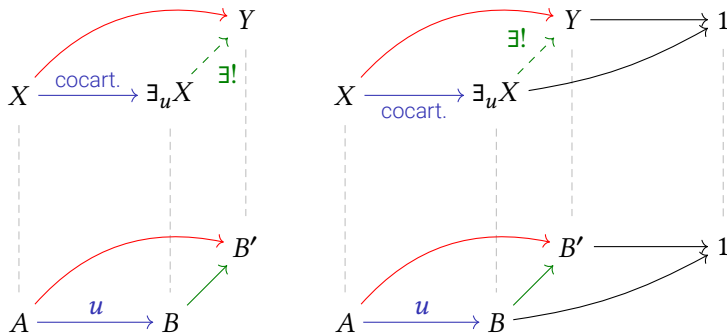
Cocartesian morphism as lifting problems

f is **cocartesian** if and only if $f \in \text{LLP}^\perp(\text{any} \rightarrow 1)$



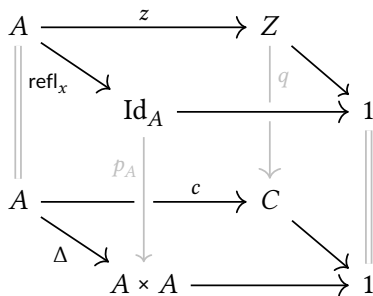
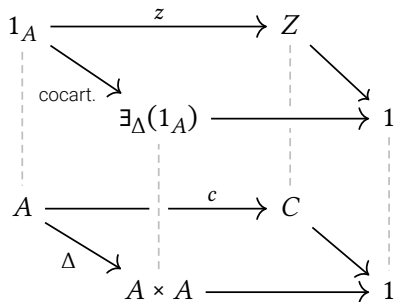
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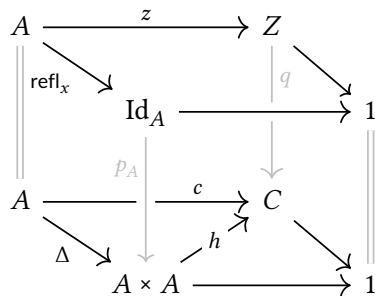
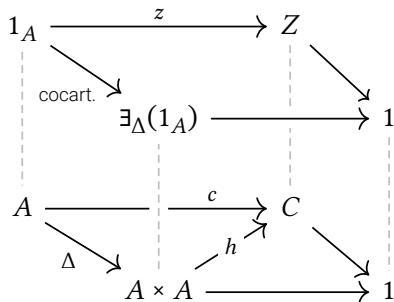


$p : \mathcal{E} \rightarrow \mathcal{B}$ is a **Grothendieck opfibration** if and only if there is a **strong RFS** relative to p with $\mathfrak{R}_{\mathcal{E}} = \{\text{any} \rightarrow 1\}$ and $\mathfrak{L}_{\mathcal{B}} = \{\text{any}\}$

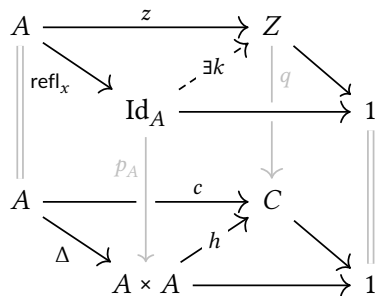
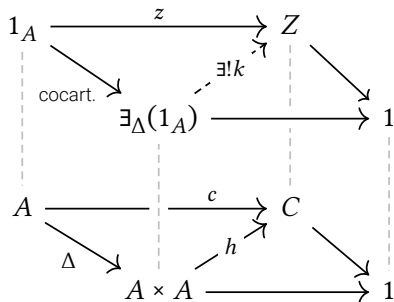
Reconcile Lawvere's equality and identity types



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Models of MLTT in Lawvere's style

Definition

A **relative tribe** is a Grothendieck fibration $p : \mathcal{E} \rightarrow \mathcal{B}$:

- mapping $\mathbf{1}$ to $\mathbf{1}$,
- with left and right adjoints Σ_B and Π_B to the substitution $\mathbf{pr}_B^*$ for a specific class of morphisms $\mathbf{pr}_B$ (with Beck-Chevalley conditions),
- with a right weak factorization system relative to p where:

$$\begin{aligned}\mathfrak{L}_{\mathcal{E}} &= \{\text{total anodyne}\}, & \mathfrak{R}_{\mathcal{E}} &= \{\text{total fibration}\} \\ \mathfrak{L}_{\mathcal{B}} &= \{\text{any}\}, & \mathfrak{R}_{\mathcal{B}} &= \{\text{fibrations}\}\end{aligned}$$

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The functor $\mathbf{Fib} \rightarrow \mathbf{ctx}$ is a relative tribe.

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- Integrate **univalent universes** in that framework.
- **Simplicial structure** of the syntax might be reflected in the models.
- **Substructural** MLTT through semantics?

Thank you.

<http://www.normalesup.org/~cagne/>