

On bifibrations of model categories

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Abstract

In this article, we develop a notion of Quillen bifibration whose purpose is to combine the two notions of Grothendieck bifibration and of Quillen model structure. In particular, given a bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$, we describe when a family of model structures on the fibers $\mathcal{E}_A$ and on the basis category $\mathcal{B}$ combines into a model structure on the total category $\mathcal{E}$, such that the functor p preserves cofibrations, fibrations and weak equivalences. Using this Grothendieck construction for model structures, we revisit the traditional definition of Reedy model structures, and possible generalizations, and exhibit their bifibrational nature.

Keywords: Grothendieck bifibrations, Model categories, Reedy categories

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1. Introduction

In this paper, we investigate how the two notions of *Grothendieck bifibration* and of *Quillen model category* may be suitably combined together. We are specifically interested in the situation of a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ where the basis category $\mathcal{B}$ as well as each fiber $\mathcal{E}_A$ for an object A of the basis category $\mathcal{B}$ is equipped with a Quillen model structure. Our main purpose will be to identify necessary and sufficient conditions on the Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ to ensure that the total category $\mathcal{E}$ inherits a model structure from the model structures assigned to the basis $\mathcal{B}$ and to the fibers $\mathcal{E}_A$'s. More precisely, given a cloven Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$, and provided that the adjoint pairs $(u_!, u^*)$ are Quillen adjunctions for each morphism u of $\mathcal{B}$, one can always equip the category $\mathcal{E}$ with two weak factorization systems $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ where the four classes are described as follows:

- $f : X \rightarrow Y$ is in $\mathfrak{C}_{\mathcal{E}}$ if and only if $u = p(f)$ is a cofibration in $\mathcal{B}$ and in the canonical factorization through a cocartesian morphism λ above u

$$X \xrightarrow{\lambda} u_! X \xrightarrow{f_{\mathcal{B}}} Y \quad (1)$$

the morphism $f_{\mathcal{B}}$ is a cofibration in $\mathcal{E}_{pY}$,

- $f : X \rightarrow Y$ is in $\mathfrak{C}_{\mathcal{E}}^{\sim}$ if and only if $u = p(f)$ is an acyclic cofibration in $\mathcal{B}$ and in the canonical factorization through a cocartesian morphism λ above u

$$X \xrightarrow{\lambda} u_! X \xrightarrow{f_{\mathcal{B}}} Y \quad (2)$$

the morphism $f_{\mathcal{B}}$ is an acyclic cofibration in $\mathcal{E}_{pY}$,

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- $f : X \rightarrow Y$ is in $\mathfrak{F}_\mathcal{E}$ if and only if $u = p(f)$ is a fibration in $\mathcal{B}$ and in the canonical factorization through a cartesian morphism ρ above u

$$X \xrightarrow{f^\triangleleft} u^*Y \xrightarrow{\rho} Y \quad (3)$$

the morphism $f^\triangleleft$ is a cofibration in $\mathcal{E}_{pX}$,

- $f : X \rightarrow Y$ is in $\mathfrak{F}_\mathcal{E}^\sim$ if and only if $u = p(f)$ is an acyclic fibration in $\mathcal{B}$ and in the canonical factorization through a cartesian morphism ρ above u

$$X \xrightarrow{f^\triangleleft} u^*Y \xrightarrow{\rho} Y \quad (4)$$

the morphism $f^\triangleleft$ is an acyclic fibration in $\mathcal{E}_{pX}$,

Observe that $\mathfrak{C}_\mathcal{E}^\sim \subset \mathfrak{C}_\mathcal{E}$ and $\mathfrak{F}_\mathcal{E}^\sim \subset \mathfrak{F}_\mathcal{E}$, and denote $\mathfrak{W}_\mathcal{E}$ for the class $\mathfrak{F}_\mathcal{E}^\sim \circ \mathfrak{C}_\mathcal{E}^\sim$ of compositions of an element of $\mathfrak{C}_\mathcal{E}^\sim$ followed by an element of $\mathfrak{F}_\mathcal{E}^\sim$. Then the three classes $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ is a model structure on $\mathcal{E}$ if and only if $\mathfrak{W}_\mathcal{E}$ has the 2-out-of-3 property (see Section 2.3 for a rigorous proof of this folklore result). However it is far from automatic to have this property on $\mathfrak{W}_\mathcal{E}$. In this paper, we are proposing necessary and sufficient conditions for $\mathfrak{W}_\mathcal{E}$ to have the 2-out-of-3 property.

Related works. The first exploration of *sufficient* conditions for $\mathfrak{W}_\mathcal{E}$ to have the 2-out-of-3 property has been conducted by Roig in his paper [Roi94], with an eye toward homological differentially graded algebra. Roig proposes the following sufficient conditions:

- (R1) the functor u^* preserves weak equivalences for each morphism u in $\mathcal{B}$,
- (R2) the unit of the adjunction $(u_!, u^*)$ is pointwise a weak equivalence whenever u is an acyclic cofibration in $\mathcal{B}$.

Under these assumptions, one can show that the elements of $\mathfrak{W}_\mathcal{E}$ can be described as the arrows $f : X \rightarrow Y$ such that $u = p(f)$ is a weak equivalence in $\mathcal{B}$ and in the canonical factorization through a cartesian morphism ρ above u

$$X \xrightarrow{f^\triangleleft} u^*Y \xrightarrow{\rho} Y$$

the morphism $f^\triangleleft$ is a weak equivalence in $\mathcal{E}_{pX}$. Once presented as such, which is not the case in Roig's original paper, an error appears in Roig's conditions: $\mathfrak{W}_\mathcal{E}$ is certainly closed under composition; if f and g can be composed and both g and gf are in $\mathfrak{W}_\mathcal{E}$, then f also is; however, if f and gf are both in $\mathfrak{W}_\mathcal{E}$, nothing compels g to be also in it. If we detail a little further the last case, we can only prove that $p(f)^*(g^\triangleleft)$ is a weak equivalence and not $g^\triangleleft$ itself.

Stanculescu spotted the mistake in Roig's theorem, and also remarked that it is not necessary for *every* functor u^* to preserve weak equivalences, but only for those u that are weak equivalences. Subsequently he suggests in [Sta12] the following correction:

- (S1) the functor u^* preserves **and reflects** weak equivalences for each **weak equivalence** u in $\mathcal{B}$,
- (S2) the unit of the adjunction $(u_!, u^*)$ is pointwise a weak equivalence whenever u is an acyclic cofibration in $\mathcal{B}$.

Finally, Harpaz and Prasma tackled the problem in [HP15] with a different goal in mind: to reflect at the level of model categories Lurie's unstraightening technique, in order to give a model for lax colimits of diagrams of ∞ -categories. For that purpose, they restrict their interest to Grothendieck bifibrations $p : \mathcal{E} \rightarrow \mathcal{B}$ such that the associated pseudofunctor $\mathcal{B} \rightarrow \mathbf{Mod}$ (valued in the 2-category of model categories and Quillen adjunctions between them) induces a ∞ -functor $\mathcal{B}_\infty \rightarrow \mathbf{Cat}_\infty$ in the sense of Lurie. In other terms, they need the bifibration p to be such that $(u_!, u^*)$ is a Quillen equivalence whenever u is a weak equivalence in $\mathcal{B}$. In that set of mind, they came up with the following sufficient conditions for $\mathfrak{W}_\mathcal{E}$ to have the 2-out-of-3 property:

(HP1) the functors $u_!$ and v^* preserves weak equivalences whenever u is an acyclic cofibration and v and acyclic fibration in $\mathcal{B}$,

(HP2) the adjunction $(u_!, u^*)$ is a Quillen equivalence whenever u is a weak equivalence in $\mathcal{B}$.

It is worth noting that under the second assumption, the first assumption is an improvement over Roig-Stanculescu's conditions, and it is satisfied by more Grothendieck bifibrations. Under both assumptions, one can show that the elements of $\mathfrak{W}_{\mathcal{E}}$ are the morphisms $f : X \rightarrow Y$ such that

(i) $u = p(f)$ is a weak equivalence in $\mathcal{B}$

(ii) and, given the canonical factorization through a cartesian morphism ρ above u

$$X \xrightarrow{f^{\natural}} u^*Y \xrightarrow{\rho} Y,$$

the composite $u^*(j)f^{\natural}$ is a weak equivalence in $\mathcal{E}_{pX}$, where $j : Y \rightarrow Y^f$ is a fibrant replacement of Y inside $\mathcal{E}_{pY}$.

Under Harpaz-Prasma's assumptions, this description is actually symmetrical: through the properties of Quillen equivalences, one shows easily that an element of $\mathfrak{W}_{\mathcal{E}}$ is equivalently described as a morphism $f : X \rightarrow Y$ such that $u = p(f)$ is a weak equivalence in $\mathcal{B}$ and given the canonical factorization through a cocartesian morphism λ above u

$$X \xrightarrow{\lambda} u_!X \xrightarrow{f_{\natural}} Y$$

the composite $f_{\natural}u_!(q)$ is a weak equivalence in $\mathcal{E}_{pX}$, where $q : X^c \rightarrow X$ is a cofibrant replacement of Y inside $\mathcal{E}_{pX}$.

The work presented in this paper is directly in line with, and greatly inspired by, these papers. In our view, both Roig-Stanculescu's and Harpaz-Prasma's results suffer from a common flaw: they are motivated by an upstream description of the elements of $\mathfrak{W}_{\mathcal{E}}$ and then search for conditions which ensure that their definition of $\mathfrak{W}_{\mathcal{E}}$ has the 2-out-of-3 property. We shall take the opposite approach here, and design the minimal conditions that provide the 2-out-of-3 property to the canonical choice of $\mathfrak{W}_{\mathcal{E}}$ as $\mathfrak{F}_{\mathcal{E}} \circ \mathfrak{C}_{\mathcal{E}}$, and only then give an elegant description of its elements (see Proposition 4.12 and the discussion before).

Theorem (Theorem 4.4). *Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck bifibration. Suppose that $\mathcal{B}$, as well as every fiber $\mathcal{E}_A$ for $A \in \mathcal{B}$, are equipped with a model structure, in such a way that the adjoint pairs $(u_!, u^*)$ are Quillen adjunctions. Then the classes $\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}$ form a model structure on $\mathcal{E}$ if and only if the two following conditions are satisfied:*

(hCon) *the functors $u_!$ and v^* preserve and reflect weak equivalences whenever u is an acyclic cofibration and v is an acyclic fibration in $\mathcal{B}$,*

(hBC) *for any commutative square of $\mathcal{B}$*

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ u' \downarrow & & \downarrow u \\ C' & \xrightarrow{v'} & B \end{array}$$

with u, u' acyclic cofibrations and v, v' acyclic fibrations, the mate natural transformation $\mu : (u')_!v^* \rightarrow (v')^*u_!$ is a weak equivalence in $\mathcal{E}_{C'}$ componentwise.

One of our motivations for establishing this theorem is the Reedy construction in homotopical algebra. If $\mathcal{R}$ is a Reedy category and $\mathcal{M}$ is a model category, there is a model structure on the category $[\mathcal{R}, \mathcal{M}]$ of $\mathcal{R}$ -shaped diagrams in $\mathcal{M}$, where the weak equivalences are given pointwise. The standard proof of this fact dates back to Kan, inspired by Reedy's work on simplicial objects in model categories, and goes through a transfinite induction on the degree of the objects in $\mathcal{R}$. The most technical part of the proof is the induction

step for successor cardinals. Denote $\mathcal{R}_\mu$ for the full subcategory of $\mathcal{R}$ spanned by the objects of degree strictly less than μ . Observe now that the restriction functor:

$$[\mathcal{R}_{\mu+1}, \mathcal{M}] \rightarrow [\mathcal{R}_\mu, \mathcal{M}] \quad (5)$$

is a Grothendieck bifibration. The fiber at a diagram A is isomorphic to the product category

$$\prod_{r \in \mathcal{R}, \deg(r) = \mu} L_r A \backslash \mathcal{M} / M_r A \quad (6)$$

where $L_r A$ and $M_r A$ are respectively the latching and matching objects of A at r . Suppose given a map $f : X \rightarrow Y$ in $[\mathcal{R}_{\mu+1}, \mathcal{M}]$ which restricts to $u : A \rightarrow B$ in $[\mathcal{R}_\mu, \mathcal{M}]$. Under the isomorphism with (6), the morphism $f_\triangleright$ in the fiber above B , issued by the factorization of f through a cocartesian morphism above u , can be expressed at component r by the cocartesian gap, in the following way:

$$\begin{array}{ccccc} L_r A & \xrightarrow{L_r u} & L_r B & & \\ \downarrow & & \downarrow & \searrow & \\ X_r & \longrightarrow & X \sqcup_{L_r A} L_r B & \xrightarrow{(f_\triangleright)_r} & Y_r \\ & \searrow & & & \uparrow \\ & & & & f_r \end{array}$$

Dually, the morphism $f^\triangleleft$ in the fiber above A , issued by the factorization of f through a cartesian morphism above u , can be expressed at component r by the cartesian gap described in the following diagram:

$$\begin{array}{ccccc} & & & & f_r \\ & & & & \uparrow \\ X_r & \xrightarrow{(f^\triangleleft)_r} & M_r B \times_{M_r A} Y & \longrightarrow & Y_r \\ & \searrow & \downarrow & & \downarrow \\ & & M_r A & \xrightarrow{M_r u} & M_r B \end{array}$$

Our main observation is that these are exactly the cartesian and cocartesian gaps involved in Kan's definition of the fibrations and cofibrations in the Reedy model structure. More precisely, if $[\mathcal{R}_\mu, \mathcal{M}]$ is equipped with the Reedy model structure and if we denote $\mathcal{E} = [\mathcal{R}_{\mu+1}, \mathcal{M}]$, then $\mathfrak{C}_\mathcal{E}$ and $\mathfrak{F}_\mathcal{E}$, as previously defined in (1) and (3), are exactly the class of cofibrations and the class of fibrations of the Reedy model structure on $[\mathcal{R}_{\mu+1}, \mathcal{M}]$. Unfortunately, the Grothendieck bifibration given by the restriction functor (5) does not enter the hypothesis of either Roig-Stanculescu's theorem (unless the Reedy category is actually a inverse category) or Harpaz-Prasma's theorem (unless $\mathcal{M}$ is both left and right proper). This situation has been an early motivation as well as a constant guide in the elaboration of the necessary and sufficient conditions given in our Theorem 4.4, whose minimal assumptions are indeed satisfied by the restriction functor (5).

Plan of the paper. In Section 2, we recall the basic properties which we will need later about Grothendieck bifibrations and model categories. It also introduces and develops the notion of *Quillen intertwined structures*, which encapsulates the *purely combinatorial* aspects of a model structure. As pointed by a reviewer, the same notion appears under the name of “Quillen structure” in [Lee15]. In addition, we recall in that introductory section a result of [Sta12] which makes our paper self-contained.

Section 3 is devoted to the interplay between Quillen intertwined structures and Grothendieck bifibrations. We build there the easy part of the main theorem to come: namely, we show how one can equip the total

category of a Grothendieck bifibration with a Quillen intertwined structure, provided that the base category and all the fibers already have such structures.

Section 4 contains the main theorem (Thm. 4.4) which we previously discussed in the introduction. The section builds on the results of section 3, and tackles the homotopical content (that is, the 2-out-of-3 property) of the model structure we have in mind on the total category of a Grothendieck bifibration satisfying our assumptions. The proof is organized in two parts: we establish in the first part the necessity of conditions (hCon) and (hBC), and then show in the second part that the two conditions are sufficient as well. The proof of necessity is relatively easy, while the proof of sufficiency is much harder and exposes more crucially how conditions (hCon) and (hBC) play their role in the construction of our model structure on the total category.

In Section 5, we illustrate our main theorem (Thm. 4.4) with applications to classical homotopical algebra. First, we develop an original point of view on Reedy model structures, by reformulating Kan's theorem in a bifibrational fashion. As we already mentioned, this was our motivating example after we realized that neither Roig-Stanculescu's or Harpaz-Prasma's theorem could be applied to explain the Reedy construction.

Convention. All written diagrams commute if not said otherwise. When objects are missing and replaced by a dot, they can be parsed from other informations on the diagram. Gray parts help to understand the diagram's context.

2. Liminaries

2.1. Grothendieck bifibrations

In this section, we recall a number of basic definitions and facts about Grothendieck bifibrations.

Given a functor $p : \mathcal{E} \rightarrow \mathcal{B}$, we shall use the following terminology. The categories $\mathcal{B}$ and $\mathcal{E}$ are called the *basis category* $\mathcal{B}$ and the *total category* $\mathcal{E}$ of the functor $p : \mathcal{E} \rightarrow \mathcal{B}$. We say that an object X of the total category $\mathcal{E}$ is above an object A of the basis category $\mathcal{B}$ when $p(X) = A$ and, similarly, that a morphism $f : X \rightarrow Y$ is above a morphism $u : A \rightarrow B$ when $p(f) = u$. The fiber of an object A in the basis category $\mathcal{B}$ with respect to p is defined as the subcategory of $\mathcal{E}$ whose objects are the objects X such that $p(X) = A$ and whose morphisms are the morphisms f such that $p(f) = \mathbf{1}_A$. In other words, the fiber of A is the category of objects above A , and of morphisms above the identity $\mathbf{1}_A$. The fiber is denoted p_A or $\mathcal{E}_A$ when no confusion is possible.

A morphism $f : X \rightarrow Y$ in a category $\mathcal{E}$ is called *cartesian* with respect to the functor $p : \mathcal{E} \rightarrow \mathcal{B}$ when the commutative diagram

$$\begin{array}{ccc} \mathcal{E}(Z, X) & \xrightarrow{f \circ -} & \mathcal{E}(Z, Y) \\ p \downarrow & & \downarrow p \\ \mathcal{B}(C, A) & \xrightarrow{u \circ -} & \mathcal{B}(C, B) \end{array}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. Here, we write $u : A \rightarrow B$ and C for the images $u = p(f)$ and $C = p(Z)$ of the morphism f and of the object Z , respectively. Unfolding the definition, this means that for every pair of morphisms $v : C \rightarrow A$ and $g : Z \rightarrow Y$ above $u \circ v : C \rightarrow B$, there exists a unique morphism $h : Z \rightarrow X$ above v such that $f \circ h = g$. The situation may be depicted as follows:

$$\begin{array}{ccccc} & & g & & \\ & & \curvearrowright & & \\ Z & \xrightarrow{\quad h \quad} & X & \xrightarrow{\quad f \quad} & Y \\ & \vdots & \vdots & & \vdots \\ C & \xrightarrow{\quad v \quad} & A & \xrightarrow{\quad u \quad} & B \end{array}$$

Dually, a morphism $f : X \rightarrow Y$ in a category $\mathcal{E}$ is called *cocartesian* with respect to the functor $p : \mathcal{E} \rightarrow \mathcal{B}$ when the commutative diagram

$$\begin{array}{ccc} \mathcal{E}(Y, Z) & \xrightarrow{- \circ f} & \mathcal{E}(X, Z) \\ p \downarrow & & \downarrow p \\ \mathcal{B}(B, C) & \xrightarrow{- \circ u} & \mathcal{B}(A, C) \end{array}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. This means that for every pair of morphisms $v : B \rightarrow C$ and $g : X \rightarrow Z$ above $v \circ u : A \rightarrow C$, there exists a unique morphism $h : Y \rightarrow Z$ above v such that $h \circ f = g$. Diagrammatically:

$$\begin{array}{ccccc} & & g & & \\ & & \curvearrowright & & \\ X & \xrightarrow{f} & Y & \xrightarrow{h} & Z \\ & \vdots & \vdots & \vdots & \vdots \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C \end{array}$$

A functor $p : \mathcal{E} \rightarrow \mathcal{B}$ is called a *Grothendieck fibration* when for every morphism $u : A \rightarrow B$ and for every object Y above B , there exists a cartesian morphism $f : X \rightarrow Y$ above u . Symmetrically, a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ is called a *Grothendieck opfibration* when for every morphism $u : A \rightarrow B$ and for every object X above A , there exists a cocartesian morphism $f : X \rightarrow Y$ above u . Note that a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ is a Grothendieck opfibration precisely when the functor $p^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ is a Grothendieck fibration. A *Grothendieck bifibration* is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ which is at the same time a Grothendieck fibration and opfibration.

Definition 2.1. A *cloven Grothendieck bifibration* is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ together with

- for any $Y \in \mathcal{E}$ and $u : A \rightarrow pY$, an object $u^*Y \in \mathcal{E}$ and a cartesian morphism $\rho_{u,Y}^p : u^*Y \rightarrow Y$ above u ,
- for any $X \in \mathcal{E}$ and $u : pX \rightarrow B$, an object $u_!X \in \mathcal{E}$ and a cocartesian morphism $\lambda_{u,X}^p : X \rightarrow u_!X$ above u .

When the context is clear enough, we might omit the index p .

REMARK 2.2. If $\mathcal{E}$ and $\mathcal{B}$ are small relatively to a universe $\mathbb{U}$ in which we suppose the axiom of choice, then a cloven Grothendieck bifibration is exactly the same as the original notion of Grothendieck bifibration. Hence, in this article, we treat the two names as synonyms.

The data of such cartesian and cocartesian morphisms give two factorizations of an arrow $f : X \rightarrow Y$ above some arrow $u : A \rightarrow B$, one denoted $f_{\triangleright}$ in the fiber $\mathcal{E}_B$ and the other denoted $f_{\triangleleft}$ in the fiber $\mathcal{E}_A$: the former goes through $\lambda_{u,X}$ and the latter through $\rho_{u,Y}$. See the diagram below:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{u,X}} & u_!X \\ f_{\triangleleft} \downarrow & \searrow f & \downarrow f_{\triangleright} \\ u^*Y & \xrightarrow{\rho_{u,Y}} & Y \end{array}$$

In turn, this allows $u_!$ and u^* to be extended as *adjoint functors*:

$$u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$$

where the action of $u_!$ on a morphism $k : X \rightarrow X'$ of $\mathcal{E}_A$ is given by $(\lambda_{u,X'} \circ k)_\triangleright$ and the action of u^* on a morphism $\ell : Y' \rightarrow Y$ is given by $(\ell \circ \rho_{u,Y'})^\triangleleft$:

$$\begin{array}{ccc} X & \longrightarrow & u_!X \\ k \downarrow & & \downarrow u_!(k) \\ X' & \longrightarrow & u_!X' \end{array} \qquad \begin{array}{ccc} u^*Y' & \longrightarrow & Y' \\ u^*\ell \downarrow & & \downarrow \ell \\ u^*Y & \longrightarrow & Y \end{array}$$

This gives a mapping $\mathcal{B} \rightarrow \mathbf{Adj}$ from the category $\mathcal{B}$ to the 2-category $\mathbf{Adj}$ of adjunctions: it maps an object A to the fiber $\mathcal{E}_A$, and a morphism u to the push-pull adjunction $(u_!, u^*)$. It is known that this mapping is actually a pseudofunctor, meaning that:

- $(\mathbf{1}_A)_! \simeq \mathbf{1}_{\mathcal{E}_A} \simeq (\mathbf{1}_A)^*$ for any object $A \in \mathcal{B}$,
- $(vu)_! \simeq v_!u_!$ for any $u : A \rightarrow B$ and $v : B \rightarrow C$ in $\mathcal{B}$,
- and so also $(vu)^* \simeq u^*v^*$ for any $u : A \rightarrow B$ and $v : B \rightarrow C$ in $\mathcal{B}$,
- all these isomorphisms are coherent in a precise sense, which is not relevant for this work.

The pseudo functoriality relates through an isomorphism the chosen (co)cartesian morphism above a composite vu with the composite of the chosen (co)cartesian morphisms above u and v . The following lemma (and the dual statement) gives a slight extension of this result that will be needed further in the paper.

Lemma 2.3. *Let $u : A \rightarrow B$, $v : B \rightarrow C$ and $w : C \rightarrow D$ in $\mathcal{B}$. Suppose $f : X \rightarrow Y$ in $\mathcal{E}$ is above the composite wvu . Then for the unique maps $h : v_!u_!X \rightarrow Y$ and $k : (vu)_!X \rightarrow Y$ above w that fill the commutative triangles*

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{u,X}} u_!X & \xrightarrow{\lambda_{v,u_!X}} v_!u_!X \\ & \searrow f & \searrow h \\ & & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_{vu,X}} (vu)_!X & \\ & \searrow f & \searrow k \\ & & Y \end{array}$$

there exists an isomorphism ϕ in the fiber $\mathcal{E}_C$ such that $h\phi = k$.

Proof. We know there is a isomorphism $\phi : (vu)_!X \rightarrow u_!v_!X$ above $\mathbf{1}_C$ such that $\phi\lambda_{vu,X} = \lambda_{v,u_!X} \circ \lambda_{u,X}$. But then $h\phi : (vu)_!X \rightarrow Y$ is above w and fills the same triangle k does in the statement: by unicity, $k = h\phi$.

$$\begin{array}{ccc} & \xrightarrow{\lambda_{vu,X}} & (vu)_!X \\ & \searrow \lambda_{u,X} & \downarrow \phi \\ X & \xrightarrow{\lambda_{u,X}} u_!X & \xrightarrow{\lambda_{v,u_!X}} v_!u_!X \\ & \searrow f & \searrow h \\ & & Y \end{array}$$

□

2.2. Weak factorization systems

In any category $\mathcal{M}$, we denote $j \boxtimes q$, and we say that j has the left lifting property relatively to q (or that q has the right lifting property relatively to j), when for any commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ j \downarrow & & \downarrow q \\ B & \longrightarrow & D \end{array}$$

there exists a morphism $h : B \rightarrow C$, making the two triangles commute in the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & C \\ j \downarrow & \nearrow h & \downarrow q \\ B & \longrightarrow & D \end{array}$$

Such a morphism h is called a *lift* or a *filler* of the original commutative square.

A *weak factorization system* on a category $\mathcal{M}$ is the data of a couple $(\mathfrak{L}, \mathfrak{R})$ of classes of arrows in $\mathcal{M}$ such that

$$\mathfrak{L} = \{j : \forall q \in \mathfrak{R}, j \sqsupset q\} \quad \text{and} \quad \mathfrak{R} = \{q : \forall j \in \mathfrak{L}, j \sqsupset q\}$$

and such that every morphism f of $\mathcal{M}$ may be factored as $f = qj$ with $j \in \mathfrak{L}, q \in \mathfrak{R}$. The elements of $\mathfrak{L}$ are called the *left maps* and the elements of $\mathfrak{R}$ the *right maps* of the factorization system.

Let now $\mathcal{M}$ and $\mathcal{N}$ be categories equipped with a factorization system. An adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is called *wfs-preserving* precisely when the left adjoint L preserves the left maps, or equivalently, when the right adjoint R preserves the right maps.

The following lemma deserves to be stated fully and independently, as it provides a key ingredient in the proof of our main result. We will apply it in particular to construct a weak factorization system on the total category of a Grothendieck bifibration, given that the basis and fibers all have a weak factorization system, in such a way that the adjunctions arising from the bifibration are wfs-preserving (see Prop. 3.7).

Lemma 2.4 (Stanculescu). *Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck bifibration with weak factorization systems $(\mathfrak{L}_C, \mathfrak{R}_C)$ on each fiber $\mathcal{E}_C$ and $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{B}$. If the adjoint pair $(u_!, u^*)$ is a wfs-adjunction for every morphism u of $\mathcal{B}$, then there is a weak factorization system $(\mathfrak{L}_\mathcal{E}, \mathfrak{R}_\mathcal{E})$ on $\mathcal{E}$ defined by*

$$\begin{aligned} \mathfrak{L}_\mathcal{E} &= \{f : X \rightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{L}, f_\flat \in \mathfrak{L}_{pY}\}, \\ \mathfrak{R}_\mathcal{E} &= \{f : X \rightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{R}, f^\flat \in \mathfrak{R}_{pX}\} \end{aligned}$$

The proof in [Sta12, 2.2] is based on a slightly different formulation of weak factorization systems. For that reason, we give a proof of the lemma adapted to the current setting.

Proof. Let us begin with the easy part, which is the factorization property. For a map $f : X \rightarrow Y$ of $\mathcal{E}$, one gets a factorization $p(f) = r\ell$ in $\mathcal{B}$ with $\ell : pX \rightarrow C \in \mathfrak{L}$ and $r : C \rightarrow pY \in \mathfrak{R}$. The bifibrational structure of p induces a morphism $\ell_! X \rightarrow r^* Y$ in the fiber $\mathcal{E}_C$, which we can in turn factor as $r_C \ell_C$ with $\ell_C \in \mathfrak{L}_C$ and $r_C \in \mathfrak{R}_C$. The wanted factorization of f is $\tilde{r}\tilde{\ell}$ where $\tilde{r}$ is the morphism of $\mathcal{E}$ such that $p(\tilde{r}) = r$ and $\tilde{r}^\flat = r_C$, and $\tilde{\ell}$ the one such that $p(\tilde{\ell}) = \ell$ and $\tilde{\ell}_\flat = \ell_C$. This is summed up in the diagram below.

$$\begin{array}{ccc} X & \longrightarrow & \ell_! X \\ & \searrow \tilde{\ell} & \downarrow \ell_C \\ & & \bullet \\ & \swarrow r_C & \searrow \tilde{r} \\ r^* Y & \longrightarrow & Y \end{array}$$

Now that we have shown how to factorize morphisms in the total category, we move on to the lifting properties. Informally speaking, the recipe consists in taking the image by p and doing the job in $\mathcal{B}$, then pushing and pulling in $\mathcal{E}$ so that one ends up in a fiber where the construction can be applied again. Take a map $j : X \rightarrow Y \in \mathfrak{L}_\mathcal{E}$ and let us show that it satisfies the left lifting property with respect to every element of $\mathfrak{R}_\mathcal{E}$. Consider in $\mathcal{E}$ a commutative square with the map q on the right in $\mathfrak{R}_\mathcal{E}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ j \downarrow & & \downarrow q \\ Y & \xrightarrow{g} & W \end{array}$$

By definition, $p(j) \in \mathfrak{L}$ has the left lifting property with respect with $p(q)$, hence the existence of a lift h :

$$\begin{array}{ccc} pX & \xrightarrow{p(f)} & pV \\ p(j) \downarrow & \nearrow h & \downarrow p(q) \\ pY & \xrightarrow{p(g)} & pW \end{array}$$

Now, filling the original square with $\tilde{h} : Y \rightarrow V$ above h is equivalent to filling the following induced solid square in $\mathcal{E}_{pY}$:

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & V \\ & \searrow j & \downarrow j_{\triangleright} & & \downarrow h^*(q^{\triangleleft}) & & \downarrow q^{\triangleleft} \\ & & Y & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & W \\ & & & & \downarrow g & & & & \end{array}$$

Here, one note that $j_{\triangleright} \in \mathfrak{L}_{pY}$, and that $h^*(q^{\triangleleft}) \in \mathfrak{R}_{pY}$ because h is the right adjoint of a wfs-preserving adjunction, hence it transports the right maps of $\mathcal{E}_{pV}$ to right maps in $\mathcal{E}_{pY}$. So, there is such a filler in $\mathcal{E}_{pY}$.

Conversely, we suppose now that $j : X \rightarrow Y$ in $\mathcal{E}$ has the left lifting property relatively to all maps of $\mathfrak{R}_{\mathcal{E}}$, and we show that $j \in \mathfrak{L}_{\mathcal{E}}$, or equivalently that $p(j) \in \mathfrak{L}$ and $j_{\triangleright} \in \mathfrak{L}_{pY}$. We establish first that $j_{\triangleright}$ is in $\mathfrak{L}_{pY}$. Consider in $\mathcal{E}_{pY}$ a commutative square of the following form:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{p(j),X}} & \cdot & \xrightarrow{f} & Y' \\ & \searrow j & \downarrow j_{\triangleright} & & \downarrow q \\ & & Y & \xrightarrow{g} & Y'' \end{array} \quad q \in \mathfrak{R}_{pY}$$

Then, because q is also in $\mathfrak{R}_{\mathcal{E}}$, there is a morphism $h : Y \rightarrow Y'$ such that $g = qh$ and $hj = f\lambda_{p(j),X}$. Note that h is a morphism in the fiber $\mathcal{E}_{pY}$. But then, the two morphisms $hj_{\triangleright}$ and f are solutions to the factorization problem of hj through the cocartesian arrow $\lambda_{p(j),X}$, hence should be equal. This means that h is a filler of the original square in the fiber $\mathcal{E}_{pY}$. We conclude that $j_{\triangleright}$ is a left map in the fiber $\mathcal{E}_{pY}$. Now we show that $p(j) \in \mathfrak{L}$. Consider a commutative square in $\mathcal{B}$:

$$\begin{array}{ccc} pX & \xrightarrow{f} & C \\ p(j) \downarrow & & \downarrow q \\ pY & \xrightarrow{g} & D \end{array} \quad q \in \mathfrak{R}$$

The bifibrational structure induces a commutative square in $\mathcal{E}$:

$$\begin{array}{ccc} X & \longrightarrow & q^*g_!Y \\ j \downarrow & & \downarrow \rho \\ Y & \longrightarrow & g_!Y \end{array}$$

Now, by construction, the arrow ρ on the right is cartesian above a right map, hence is in $\mathfrak{R}_{\mathcal{E}}$ by definition. So j lifts against ρ , giving us a filler $h : Y \rightarrow q^*g_!Y$ whose image $p(h) : pY \rightarrow C$ fills the square in $\mathcal{B}$. We conclude that $p(j)$ is a left map of the basis category $\mathcal{B}$. This conclude the proof that $j \in \mathfrak{L}_{\mathcal{E}}$. We show dually that $\mathfrak{R}_{\mathcal{E}}$ is exactly the class of maps which satisfy the right lifting property against all maps of $\mathfrak{L}_{\mathcal{E}}$. $\square$

2.3. Quillen intertwined structures and model categories

Quillen introduced model categories in [Qui67] as categories with sufficient structural analogies with the category of topological spaces so that a sensible notion of *homotopy between maps* can be defined. There are redundancies in Quillen's definition, that are not necessarily obvious at first sight. These redundancies can be useful for the intuitive understanding of a model category, but the extra checkings required can turn a simple proof into a painful process. To ease things a little bit, this part is dedicated to extract the minimal definition of a model category at the cost of trading topological intuition for combinatorial comfort.

We recall the definition of a model structure.

Definition 2.5. A model structure on a category $\mathcal{M}$ is the data of three classes of maps $\mathfrak{C}$, $\mathfrak{W}$, $\mathfrak{F}$ such that:

- (i) $\mathfrak{W}$ has the 2-out-of-3 property, i.e. if two elements among $\{f, g, gf\}$ are in $\mathfrak{W}$ for composable morphisms f and g , then so is the third,
- (ii) $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ both are weak factorization systems.

The morphisms in $\mathfrak{W}$ are called the *weak equivalences*, those in $\mathfrak{C}$ the *cofibrations* and those in $\mathfrak{F}$ the *fibrations*. Given the role played by the two classes $\mathfrak{C} \cap \mathfrak{W}$ and $\mathfrak{F} \cap \mathfrak{W}$, we also give names to their elements: a fibration (respectively a cofibration) which is also a weak equivalence is called an *acyclic fibration* (respectively an *acyclic cofibration*).

REMARK 2.6. It is well-known that there is some redundancy in the previous definition: in a model structure, any two of the three classes $\mathfrak{C}, \mathfrak{W}, \mathfrak{F}$ determine the third one. In particular, and this will become important later on, the weak equivalences are exactly those morphisms that can be written as qj where j is an acyclic cofibration and q is an acyclic fibration. The first inclusion $\{qj : q \in \mathfrak{F} \cap \mathfrak{W}, j \in \mathfrak{C} \cap \mathfrak{W}\} \subseteq \mathfrak{W}$ is a direct consequence of the 2-out-of-3 property. The reverse inclusion is given by applying one of the two weak factorization systems and by using then the 2-out-of-3 property: if $w \in \mathfrak{W}$, it can be written as $w = qj$ with $q \in \mathfrak{F}, j \in \mathfrak{C} \cap \mathfrak{W}$; but then, since w and j are weak equivalences, q also is.

Recall also the notion of morphism between model structures: a *Quillen adjunction* between two model structures $\mathcal{M}$ and $\mathcal{N}$ is an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ which is wfs-preserving for both the weak factorization system (acyclic cofibrations, fibrations) and the (cofibrations, acyclic fibrations) one.

We will use the following notion on several occasions:

Definition 2.7 (Homotopically conservative functor). A functor $F : \mathcal{M} \rightarrow \mathcal{N}$ between model structures is called *homotopically conservative* when it preserves and reflects weak equivalences.

REMARK 2.8. Let us make two observations to justify the terminology:

- (1) If $\mathcal{M}$ and $\mathcal{N}$ are endowed with the trivial model structure, in which weak equivalences are isomorphisms and cofibrations and fibrations are all morphisms, then the notion boils down to the usual notion of conservative functor.
- (2) Every functor $F : \mathcal{M} \rightarrow \mathcal{N}$ preserving weak equivalences induces a functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbf{Ho}(\mathcal{N})$. Given that weak equivalences are saturated in a model category, homotopically conservative functors are exactly those F such that $\mathbf{Ho}(F)$ is conservative as a usual functor.

We now introduce the notion of *Quillen intertwined structure*.

Definition 2.9. A *Quillen intertwined structure* on a category $\mathcal{C}$ is a quadruple $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$ where $(\mathfrak{C}^\sim, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{F}^\sim)$ are weak factorization systems on a category $\mathcal{C}$ such that

$$\mathfrak{C}^\sim \subseteq \mathfrak{C} \quad \text{and} \quad \mathfrak{F}^\sim \subseteq \mathfrak{F}.$$

The careful reader will notice that the properties $\mathfrak{C}^\sim \subseteq \mathfrak{C}$ and $\mathfrak{F}^\sim \subseteq \mathfrak{F}$ are actually equivalent, but the definition is more naturally stated in this perfectly symmetric way. We are indebted to the anonymous reviewer who pointed out Seunghun Lee's paper [Lee15], in which the author explores the same notion under the name “Quillen structure”¹. Such relations between weak factorization systems also appear in [Col06] and [GS17], although the focus in these articles is somewhat different and their approaches do not require a systematic study of such structures. The similarity between Quillen intertwined structures and Quillen model structures becomes even more apparent when one observes the following two properties:

Proposition 2.10. *Let $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$ be a Quillen intertwined structure on a category $\mathcal{C}$. The following identities hold:*

$$\mathfrak{C}^\sim = \mathfrak{W} \cap \mathfrak{C}, \quad \mathfrak{F}^\sim = \mathfrak{W} \cap \mathfrak{F}.$$

where $\mathfrak{W}$ denotes the class $\mathfrak{F}^\sim \circ \mathfrak{C}^\sim$ of composite morphisms.

Proof. Let us prove the first identity only, as the second one is strictly dual. Suppose $f : A \rightarrow B \in \mathfrak{C}^\sim$, then $f \in \mathfrak{C}$ by the very definition of Quillen intertwined structures, and, at the same time, $f = \mathbf{1}_B f \in \mathfrak{W}$. From this follows the first inclusion: $\mathfrak{C}^\sim \subseteq \mathfrak{W} \cap \mathfrak{C}$.

Conversely, take $f \in \mathfrak{W} \cap \mathfrak{C}$. Then in particular $f \in \mathfrak{W}$, and there exists $j \in \mathfrak{C}^\sim$ and $q \in \mathfrak{F}^\sim$ such that $f = qj$. Put otherwise, the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ f \downarrow & & \downarrow q \\ B & \xlongequal{\quad} & B. \end{array}$$

Since f is in $\mathfrak{C}$ and q is in $\mathfrak{F}^\sim$, there exists a lift $s : B \rightarrow C$ such that $qs = \mathbf{1}_B$ and $sf = j$. Now, for any $p \in \mathfrak{F}$ and any commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & D \\ \downarrow j & & \downarrow p \\ C & & \\ \downarrow q & & \\ B & \xrightarrow{y} & E \end{array}$$

(Note: The diagram in the image is a commutative square with a curved arrow f from A to B and a curved arrow q from C to B . The horizontal arrows are x and y , and the vertical arrows are j and p . The square is $A \xrightarrow{x} D \xrightarrow{p} E \xrightarrow{y} B \xrightarrow{f} A$ with j and q as additional arrows.)

there exists a lift $h : C \rightarrow D$ because j has the left lifting property with respect to p . Then $hs : B \rightarrow D$ provides a lift showing that f has the left lifting property with respect to p : indeed $phs = yqs = y$ and $hsf = hj = x$. This establishes that the morphism f has the left lifting property with respect to any morphism in $\mathfrak{F}$, and thus that f is an element of $\mathfrak{C}^\sim$. This shows the reverse inclusion that $\mathfrak{W} \cap \mathfrak{C} \subseteq \mathfrak{C}^\sim$. $\square$

Corollary 2.11. *Let $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$ be a Quillen intertwined structure on a category $\mathcal{M}$, and denote again $\mathfrak{W} = \mathfrak{F}^\sim \circ \mathfrak{C}^\sim$. The category $\mathcal{M}$ has a model structure with weak equivalences $\mathfrak{W}$, fibrations $\mathfrak{F}$ and cofibrations $\mathfrak{C}$ if and only if $\mathfrak{W}$ has the 2-out-of-3 property.*

Of course in that case, we also get the class of acyclic cofibrations as $\mathfrak{C}^\sim$ and the class of acyclic fibrations as $\mathfrak{F}^\sim$. Even if not stated this way, Lee's paper [Lee15] provides (necessary and) sufficient conditions for $\mathfrak{W}$ to have the 2-out-of-3 property for $\mathfrak{W}$ in the form of Lemma 2.12. Unfortunately his result is proven using pullbacks and pushouts, which are not part of our framework here. We are again indebted to the anonymous reviewer who pointed out how the proof can be adapted without the need for pullbacks and pushouts.

Lemma 2.12. *Let $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$ be a Quillen intertwined structure on a category $\mathcal{C}$. The class $\mathfrak{W} = \mathfrak{F}^\sim \circ \mathfrak{C}^\sim$ has the 2-out-of-3 property if and only if the two following conditions are satisfied:*

¹Because “Quillen structure” is easily confused with “Quillen model structure” when reading quickly, we prefer to stick to our terminology of “Quillen intertwined structure” in the paper.

(i) for every commutative square of $\mathcal{C}$ such as

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ v' \downarrow & & \downarrow v \\ C' & \xrightarrow{u'} & B \end{array} \quad u \in \mathcal{C}, v \in \mathfrak{F}, u' \in \mathcal{C}^\sim, v' \in \mathfrak{F}^\sim$$

it holds that $u \in \mathcal{C}^\sim$ if and only if $v \in \mathfrak{F}^\sim$,

(ii) for every commutative triangle on the left or on the right

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ & \searrow u & \downarrow u' \\ & & B \end{array} \quad \begin{array}{l} f \in \mathcal{C}, \\ u, u' \in \mathcal{C}^\sim \end{array} \quad \begin{array}{ccc} A & & \\ v' \downarrow & \searrow v & \\ C & \xrightarrow{f} & B \end{array} \quad \begin{array}{l} f \in \mathfrak{F}, \\ v, v' \in \mathfrak{F}^\sim \end{array}$$

it holds that $f \in \mathfrak{W}$.

Proof. If $\mathfrak{W}$ has the 2-out-of-3 property, then the two conditions hold trivially. We focus on the converse. Take $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\mathcal{C}$ and let $h = gf$.

Suppose first that f and g are in $\mathfrak{W}$. Then by definition $f = pi$ and $g = qj$ with $p, q \in \mathfrak{F}^\sim$ and $i, j \in \mathcal{C}^\sim$. From this follows that h is equal to the composite morphism

$$A \xrightarrow{i} \bullet \xrightarrow{p} B \xrightarrow{j} \bullet \xrightarrow{q} C$$

Now factorize jp as $p'j'$ according to the weak factorization system $(\mathcal{C}^\sim, \mathfrak{F})$. The morphism p' is in $\mathfrak{F}^\sim$ because of condition (i). As the classes $\mathcal{C}^\sim$ and $\mathfrak{F}^\sim$ are closed under composition, we have a decomposition of $h = (qp')(j'i)$ with $qp' \in \mathfrak{F}^\sim$ and $j'i \in \mathcal{C}^\sim$: we can conclude that h is in $\mathfrak{W}$.

Suppose now that f and h are in $\mathfrak{W}$, and let us show that g also is. By definition $f = pi$ and $h = rk$ for some $i, k \in \mathcal{C}^\sim$ and $p, r \in \mathfrak{F}^\sim$. Factorize also $g = qj$ using the weak factorization system $(\mathcal{C}, \mathfrak{F}^\sim)$. So we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{k} & \bullet \\ i \downarrow & & \downarrow r \\ \bullet & \xrightarrow{p} & B \\ & \downarrow j & \\ & \bullet & \xrightarrow{q} C \end{array}$$

Using condition (i) as before, we find a factorization $p'j'$ of jp with $p' \in \mathfrak{F}^\sim$ and $j' \in \mathcal{C}^\sim$. Hence in the following outer commutative diagram, the composite arrow qp' on the bottom is in $\mathfrak{F}$, so that it has the right lifting property with respect to the map $k \in \mathcal{C}^\sim$ on the top. Hence, there exists a morphism w making the diagram below commute:

$$\begin{array}{ccc} A & \xrightarrow{k} & \bullet \\ i \downarrow & & \downarrow r \\ \bullet & \xrightarrow{j'} & \bullet \\ & \searrow w & \\ & \bullet & \xrightarrow{p'} \bullet \end{array} \quad \begin{array}{l} i, j', k \in \mathcal{C}^\sim \\ p', r \in \mathfrak{F}^\sim \\ q \in \mathfrak{F} \end{array}$$

Now decompose w as $s\ell$ with $s \in \mathfrak{F}$ and $\ell \in \mathcal{C}^\sim$ and denote $r' = qp's \in \mathfrak{F}$. The situation is summed up in

the following commutative diagram:

$$\begin{array}{ccc}
 A & \xrightarrow{k} & \bullet \\
 i \downarrow & & \swarrow \ell \\
 \bullet & & \bullet \\
 j' \downarrow & \swarrow s & \searrow r' \\
 \bullet & \xrightarrow{p'} & \bullet \xrightarrow{q} C
 \end{array}
 \quad
 \begin{array}{l}
 i, j', k, \ell \in \mathfrak{C}^\sim \\
 p', r \in \mathfrak{F}^\sim \\
 q, s \in \mathfrak{F}
 \end{array}$$

Using condition (i) on the (degenerate) square on the left in the diagram below, we end up with $r' \in \mathfrak{F}^\sim$, and using it on the (degenerate) square on the middle, we end up with $s \in \mathfrak{F}^\sim$:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\ell} & \bullet \\
 r \downarrow & & \downarrow r' \\
 \bullet & \xlongequal{\quad} & \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{k\ell} & \bullet \\
 \parallel & & \downarrow s \\
 \bullet & \xrightarrow{j'i} & \bullet
 \end{array}
 \quad
 \begin{array}{ccc}
 \bullet & & \\
 p's \downarrow & \searrow r' & \\
 \bullet & \xrightarrow{q} & C
 \end{array}$$

Then condition (ii) applies to the triangle on the right and establishes $q \in \mathfrak{F}^\sim$. In conclusion, $g = qj$ is in $\mathfrak{W}$. The last case to check (g and h in $\mathfrak{W}$ implies that f also is) is proved similarly. $\square$

Putting together corollary 2.11 and Lemma 2.12, a Quillen model structure is nothing but a Quillen intertwined structure satisfying conditions (i) and (ii). This definition has the advantage of being easy to check, especially in the context of formal constructions, as it is the case in this paper. The definition also emphasizes the fact that Quillen adjunctions are really the right notion of morphisms for Quillen intertwined structures, and have *a priori* nothing to do with weak equivalences. We shall really put that on a stand because everything that follows in the main theorem (Thm. 4.4) can be restated with Quillen intertwined structures in place of Quillen model structures. As a matter of fact, this Quillen intertwined structure part of the theorem is the easy part and all the hard core of Thm. 4.4 resides in establishing the 2-out-of-3 property, as it is usually the case with model structures.

3. The Grothendieck construction for Quillen intertwined structures

Given a category $\mathcal{C}$ equipped with a Quillen intertwined structure $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$, we call “acyclic cofibrations” the elements of $\mathfrak{C}^\sim$, “fibrations” the elements of $\mathfrak{F}$, “cofibrations” the elements of $\mathfrak{C}$, and “acyclic fibrations” the elements of $\mathfrak{F}^\sim$ even when the Quillen intertwined structure is not a model structure. In the same abusive way, given an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ between categories equipped with Quillen intertwined structures, we will call it a Quillen adjunction precisely when L preserves cofibrations and R preserves fibrations.

For this section, let us fix a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ and suppose that both $\mathcal{E}$ and $\mathcal{B}$ are equipped with Quillen intertwined structures, denoted $(\mathfrak{C}_\mathcal{E}^\sim, \mathfrak{F}_\mathcal{E}, \mathfrak{C}_\mathcal{E}, \mathfrak{F}_\mathcal{E}^\sim)$ for $\mathcal{E}$ and $(\mathfrak{C}^\sim, \mathfrak{F}, \mathfrak{C}, \mathfrak{F}^\sim)$ for $\mathcal{B}$. We make in addition the two following assumptions:

- (i) the functor p transports (acyclic) fibrations to (acyclic) fibrations and (acyclic) cofibrations to (acyclic) cofibrations.
- (ii) the Quillen intertwined structure on $\mathcal{E}$ restricts to a Quillen intertwined structure on the fiber $\mathcal{E}_A$, for every $A \in \mathcal{B}$.

In this section, we show that in such a situation, the Quillen intertwined structure on the total category $\mathcal{E}$ is entirely determined by the Quillen intertwined structure on the basis $\mathcal{B}$ together with the Quillen intertwined structures on every fiber $\mathcal{E}_A$.

Lemma 3.1. *For every morphism $u : A \rightarrow B$ in $\mathcal{B}$, the adjunction $u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ is a Quillen adjunction.*

Proof. Let $f : X \rightarrow Y$ be a cofibration in the fiber $\mathcal{E}_A$. We want to show that the morphism $u_!(f)$ of $\mathcal{E}_B$ is a cofibration. Take an arbitrary acyclic fibration $q : W \rightarrow Z$ in $\mathcal{E}_B$ and a commutative square in that fiber:

$$\begin{array}{ccc} u_!X & \xrightarrow{g} & W \\ u_!(f) \downarrow & & \downarrow q \\ u_!Y & \xrightarrow{g'} & Z \end{array}$$

We need to find a lift $h : u_!Y \rightarrow W$ making the diagram commute, i.e. such that $qh = g'$ and $hu_!(f) = g$. Let us begin by precomposing with the square defining $u_!(f)$:

$$\begin{array}{ccccc} X & \xrightarrow{\lambda} & u_!X & \xrightarrow{g} & W \\ f \downarrow & & u_!(f) \downarrow & & \downarrow q \\ Y & \xrightarrow{\lambda} & u_!Y & \xrightarrow{g'} & Z \end{array}$$

As a cofibration, f has the left lifting property with respect to q . This provides a map $k : Y \rightarrow W$ which makes the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{\lambda} & u_!X & \xrightarrow{g} & W \\ f \downarrow & & \downarrow u_!(f) & \nearrow k & \downarrow q \\ Y & \xrightarrow{\lambda} & u_!Y & \xrightarrow{g'} & Z \end{array}$$

Now we use the cocartesian property of $\lambda_{u,Y} : Y \rightarrow u_!Y$ on k , to find a map $h : u_!Y \rightarrow W$ above the identity $\mathbf{1}_B$ such that $h\lambda_{u,Y} = k$. All there remains to show is that $qh = g'$ and $hu_!(f) = g$. Note that both qh and g' are solutions to the problem of finding a map $x : u_!Y \rightarrow Z$ above $\mathbf{1}_B$ such that $x\lambda_{u,Y} = qk$: hence, by the uniqueness property satisfied by the cocartesian morphism $\lambda_{u,Y}$, they must be equal. Similarly, $h \circ u_!(f)$ and g solve the problem of finding $x : u_!X \rightarrow W$ above $\mathbf{1}_B$ such that $x\lambda_{u,X} = kf$: the cocartesian property of $\lambda_{u,X}$ allows us to conclude that they are equal. In the end, $u_!(f)$ has the left lifting property with respect to every acyclic fibration of $\mathcal{E}_B$, so it is a cofibration. We prove dually that the image $u^*(f)$ of a fibration f in $\mathcal{E}_B$ is a fibration in the fiber $\mathcal{E}_A$. $\square$

Lemma 3.2. *A cocartesian morphism in $\mathcal{E}$ above a (acyclic) cofibration of $\mathcal{B}$ is a (acyclic) cofibration.*

Proof. Let $f : X \rightarrow Y$ be cocartesian above a cofibration $u : A \rightarrow B$ in $\mathcal{B}$. Given a commutative square of $\mathcal{E}$

$$\begin{array}{ccc} X & \xrightarrow{g} & W \\ f \downarrow & & \downarrow q \\ Y & \xrightarrow{g'} & Z \end{array} \quad (7)$$

with q an acyclic fibration, we can take its image in $\mathcal{B}$:

$$\begin{array}{ccc} A & \longrightarrow & pW \\ u \downarrow & & \downarrow p(q) \\ B & \longrightarrow & pZ \end{array}$$

Since u is a cofibration and $p(q)$ an acyclic fibration, there exists a morphism $h : B \rightarrow pW$ making the expected diagram commute:

$$\begin{array}{ccc} A & \longrightarrow & pW \\ u \downarrow & \nearrow h & \downarrow p(q) \\ B & \longrightarrow & pZ \end{array}$$

Because f is cocartesian, we know that there exists a (unique) map $\tilde{h} : Y \rightarrow W$ above h making the diagram below commute:

$$\begin{array}{ccc} X & \xrightarrow{g} & W \\ f \downarrow & \nearrow \tilde{h} & \\ Y & & \end{array}$$

For the morphism $\tilde{h}$ to be a lift in the first commutative square (7), there remains to show that $q\tilde{h} = g'$. Because $\tilde{h}$ is above h and $p(q)h = p(g')$, we know that the composite $q\tilde{h}$ is above $p(g')$. Moreover $q\tilde{h}f = qg = g'f$. Using the uniqueness property in the definition of cocartesian maps, we deduce $q\tilde{h} = g'$. We have just shown that the cocartesian morphism f has the left lifting property with respect to every acyclic fibration, and we thus conclude that f is a cofibration. The case of cocartesian morphisms above acyclic cofibrations is treated in a similar way. $\square$

The same argument establishes the dual statement:

Lemma 3.3. *A cartesian morphism in $\mathcal{E}$ above a (acyclic) fibration of $\mathcal{B}$ is a (acyclic) fibration.*

Proposition 3.4. *A map $f : X \rightarrow Y$ in $\mathcal{E}$ is a (acyclic) cofibration if and only if $p(f)$ is a (acyclic) cofibration in $\mathcal{E}$ and $f_{\triangleright}$ is a (acyclic) cofibration in the fiber $\mathcal{E}_{pY}$.*

Proof. One direction of the equivalence is easy: if $p(f) = u : A \rightarrow B$ is a cofibration, then so is the cocartesian morphism $\lambda_{u,X}$ above it by Lemma 3.2; if moreover $f_{\triangleright}$ is a cofibration in the fiber $\mathcal{E}_B$, then $f = f_{\triangleright}\lambda_{u,X}$ is a composite of cofibrations, hence it is a cofibration itself.

Conversely, suppose that $f : X \rightarrow Y$ is a cofibration in $\mathcal{E}$. Then surely $p(f) = u : A \rightarrow B$ is also a cofibration in $\mathcal{B}$, since p preserves cofibrations. Now we want to show that $f_{\triangleright} : u_!X \rightarrow Y$ is a cofibration in the fiber $\mathcal{E}_B$. Consider a commutative square in that fiber

$$\begin{array}{ccc} u_!X & \xrightarrow{g} & W \\ f_{\triangleright} \downarrow & & \downarrow q \\ Y & \xrightarrow{g'} & Z \end{array}$$

where q is an acyclic fibration of the fiber $\mathcal{E}_B$, and g, g' are arbitrary morphisms in that fiber. Since f itself is a cofibration in $\mathcal{E}$, we know that there exists a lift $h : Y \rightarrow W$ for the outer square (with four sides $f, q, g\lambda_{u,X}$ and g') of the following diagram:

$$\begin{array}{ccccc} \lambda_{u,X}X & \xrightarrow{\lambda_{u,X}} & u_!X & \xrightarrow{g} & W \\ & \searrow f & \downarrow f_{\triangleright} & \nearrow h & \downarrow q \\ & & Y & \xrightarrow{g'} & Z \end{array}$$

Now, there remains to show that $hf_{\triangleright} = g$. We already know that $g\lambda_{u,X} = hf_{\triangleright}\lambda_{u,X}$, and taking advantage of the fact that the morphism $\lambda_{u,X}$ is cocartesian, we only need to show that $p(g) = p(hf_{\triangleright})$. Since g and $f_{\triangleright}$ are fiber morphisms, it means we need to show that h also. This follows from the fact that $qh = g'$ and that q and g' are fiber morphisms. $\square$

In the same way, we get the dual statement:

Proposition 3.5. *A map $f : X \rightarrow Y$ in $\mathcal{E}$ is a (acyclic) fibration if and only if $p(f)$ is a (acyclic) fibration in $\mathcal{E}$ and $f^{\triangleleft}$ is a (acyclic) fibration in the fiber $\mathcal{E}_{pY}$.*

In particular, this means that the Quillen intertwined structure on the total category $\mathcal{E}$ is entirely determined by the Quillen intertwined structure on the basis $\mathcal{B}$ and those on each fiber $\mathcal{E}_B$. As the characterization turns out to be important for what follows, we shall give a name to the morphisms of $\mathcal{E}$ appearing in each of the four classes of the Quillen intertwined structure:

Definition 3.6. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck bifibration such that both its basis $\mathcal{B}$ and each of its fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) are equipped with Quillen intertwined structures.

- A *total cofibration* is a morphism $f : X \rightarrow Y$ of $\mathcal{E}$ above a cofibration $u : A \rightarrow B$ of $\mathcal{B}$ such that $f_{\triangleright}$ is a cofibration in the fiber $\mathcal{E}_B$.
- A *total fibration* is a morphism $f : X \rightarrow Y$ of $\mathcal{E}$ above a fibration $u : A \rightarrow B$ of $\mathcal{B}$ such that $f^{\triangleleft}$ is a fibration in the fiber $\mathcal{E}_A$.
- A *total acyclic cofibration* is a morphism $f : X \rightarrow Y$ of $\mathcal{E}$ above an acyclic cofibration $u : A \rightarrow B$ of $\mathcal{B}$ such that $f_{\triangleright}$ is an acyclic cofibration in the fiber $\mathcal{E}_B$.
- A *total acyclic fibration* is a morphism $f : X \rightarrow Y$ of $\mathcal{E}$ above an acyclic fibration $u : A \rightarrow B$ of $\mathcal{B}$ such that $f^{\triangleleft}$ is an acyclic fibration in the fiber $\mathcal{E}_A$.

Using this terminology, the two propositions 3.4 and 3.5 just established come together as follows: the cofibrations, fibrations, acyclic cofibrations and acyclic fibrations in the Quillen intertwined structure on $\mathcal{E}$ are necessarily the total ones. The following proposition gives a converse to this result. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be again a mere Grothendieck bifibration, no longer subject to the conditions stated in the beginning of the section.

Proposition 3.7. *Suppose that $\mathcal{B}$ and all the fibers $\mathcal{E}_A$ are equipped with Quillen intertwined structures, and denote $\mathfrak{C}_{\mathcal{E}}^{\sim}$, $\mathfrak{F}_{\mathcal{E}}$, $\mathfrak{C}_{\mathcal{E}}$, and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ for the total acyclic cofibrations, total fibrations, total cofibrations and total acyclic fibrations respectively. Then $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}}, \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ is a Quillen intertwined structure on $\mathcal{E}$.*

Proof. Obviously $\mathfrak{C}_{\mathcal{E}}^{\sim} \subseteq \mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim} \subseteq \mathfrak{F}_{\mathcal{E}}$. Independently, a direct application of Lemma 2.4 shows that $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ are both weak factorization systems on $\mathcal{E}$. $\square$

Remark then that the bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ respects conditions (i) and (ii) of the beginning of the section. The results of this section can thus be summed up as follows. Let $\mathbf{Qis}$ denote the 2-category of (small) categories equipped with a Quillen intertwined structure and Quillen adjunctions between them (following the direction of the left adjoint functor).

Proposition 3.8. *Given a category $\mathcal{B}$ equipped with a Quillen intertwined structure, the Grothendieck construction on pseudofunctors $\mathcal{B} \rightarrow \mathbf{Qis}$ reaches exactly the Grothendieck bifibrations $\mathcal{E} \rightarrow \mathcal{B}$ with a Quillen intertwined structure on $\mathcal{E}$ satisfying conditions (i) and (ii).*

4. A Grothendieck construction for Quillen bifibrations

In the situation of a pseudofunctor $P : \mathcal{B} \rightarrow \mathbf{Qis}$ as in Proposition 3.8, a natural question is the following: given that the Quillen intertwined structure on $\mathcal{B}$ is defining a genuine Quillen model structure and given that P lands into the full sub-2-category $\mathbf{Mod}$ of $\mathbf{Qis}$ spanned by categories with Quillen model structures, when is the Quillen intertwined structure of total widgets on $\mathcal{E}$ defining a genuine Quillen model structure?

One motivating example for this question is the case of the codomain fibration:

EXAMPLE(s) 4.1. For a category $\mathcal{B}$, one can consider the codomain functor:

$$\text{cod} : [\mathbf{2}, \mathcal{B}] \rightarrow \mathcal{B}, (X \xrightarrow{f} A) \mapsto A$$

Cocartesian morphisms above u relatively to cod are those commutative square of the form

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow f & & \downarrow uf \\ A & \xrightarrow{u} & B \end{array}$$

whereas cartesian morphisms above u are the pullback squares along u . Hence cod is a Grothendieck bifibration whenever $\mathcal{B}$ admits pullbacks.

If moreover $\mathcal{B}$ is a model category, then each fiber $\text{cod}_A \simeq \mathcal{B}_{/A}$ inherits a model structure (namely an arrow is a fibration or a cofibration if it is such as an arrow of $\mathcal{B}$), and the total fibrations and total cofibrations coincide with the fibrations and cofibrations in the injective model structure on $[\mathbf{2}, \mathcal{B}]$: i.e. a cofibration is a commutative square with the top and bottom arrows being cofibrations in $\mathcal{B}$, whereas fibrations are those commutative squares

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ \downarrow f & \searrow h & \nearrow \\ & A \times_{u,g} Y & \\ \downarrow & \nearrow & \downarrow g \\ A & \xrightarrow{\quad u \quad} & B \end{array}$$

where both u and h are fibrations in $\mathcal{B}$. Similarly, the total fibrations and total cofibrations of the Grothendieck bifibration $\text{dom} : [\mathbf{2}, \mathcal{B}] \rightarrow \mathcal{B}$ over a model category $\mathcal{B}$ are exactly those of the projective model structure on $[\mathbf{2}, \mathcal{B}]$.

Definition 4.2. A *Quillen bifibration* is a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ between categories with model structures such that

- p preserves cofibrations, fibrations and weak equivalences,
- the model structure on $\mathcal{E}$ restricts to each fiber, meaning that for each A in $\mathcal{B}$ the morphisms of $\mathcal{E}_A$ that are cofibrations, fibrations and weak equivalences in $\mathcal{E}$ form a model structure on $\mathcal{E}_A$.

The main goal of this section is to determine exactly when a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ can be turned into a Quillen bifibration whenever both the basis category $\mathcal{B}$ and every fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) admit model structures in such a way that all the pairs of adjoint push and pull functors between fibers are “homotopically well-behaved”. To be more precise, we now fix a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ and we suppose that $\mathcal{B}$ is equipped with a model structure $(\mathcal{C}, \mathfrak{W}, \mathfrak{F})$, and that each fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) is also equipped with a model structure $(\mathcal{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. We also make the following **fundamental assumption**:

$$\text{For all } u \text{ in } \mathcal{B}, \text{ the adjoint pair } (u_!, u^*) \text{ is a Quillen adjunction.} \quad (\text{Q})$$

Recall from Definition 3.6 the notions of total cofibrations and total fibrations, as well as their acyclic counterparts. We must insist that in that framework, *total cofibrations* and *total fibrations* are only part of a Quillen intertwined structure, and by no means are they equipping the total category $\mathcal{E}$ with a Quillen model structure just yet. The goal of this section is to provide a complete characterization, under hypothesis (Q), of the Grothendieck bifibrations $p : \mathcal{E} \rightarrow \mathcal{B}$ for which the total cofibrations and total fibrations make p into a Quillen bifibration. For the rest of this section, we shall denote $\mathcal{C}_{\mathcal{E}}$, $\mathfrak{F}_{\mathcal{E}}$, $\mathcal{C}_{\mathcal{E}}^{\sim}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ for the respective classes of total cofibrations, total fibrations, total acyclic cofibrations, and total acyclic fibrations.

4.1. Main theorem

In order to state the theorem correctly, we will need some vocabulary. Recall that the *mate* $\mu : u'_! v^* \rightarrow v'^* u_!$ associated to a commutative square of $\mathcal{B}$

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ u'_! \downarrow & & \downarrow u \\ C' & \xrightarrow{v'} & B \end{array}$$

is the natural transformation constructed at point $Z \in \mathcal{E}_C$ in two steps as follows: the composite

$$v^* Z \xrightarrow{\rho} Z \xrightarrow{\lambda} u'_! Z$$

which is above uv , factors through the cartesian arrow $\rho_{v',u_!Z} : v'^*u_!Z \rightarrow u_!Z$ (because $v'u' = uv$) into a morphism $v^*Z \rightarrow v'^*u_!Z$ above u' , which in turn factors through the cocartesian arrow $\lambda_{u',v_!Z} : u'^*v_!Z \rightarrow v'^*u_!Z$ giving rise to μ_Z , as summarized in the diagram below.

$$\begin{array}{ccc}
 v^*Z & \xrightarrow{\rho} & Z \\
 \downarrow \lambda & & \downarrow \lambda \\
 u'_!v^*Z & \xrightarrow{\mu_Z} & v'^*u_!Z \xrightarrow{\rho} u_!Z
 \end{array}$$

Definition 4.3. A commutative square of $\mathcal{B}$ is said to satisfy the *homotopical Beck-Chevalley condition* if its mate is pointwise a weak equivalence.

Consider then the following properties on the Grothendieck bifibration p :

Every commutative square of $\mathcal{B}$ of the form

$$\begin{array}{ccc}
 A & \xrightarrow{v} & C \\
 u' \downarrow & & \downarrow u \\
 C' & \xrightarrow{v'} & B
 \end{array} \quad \begin{array}{l} u, u' \in \mathfrak{C} \cap \mathfrak{W}, \\ v, v' \in \mathfrak{F} \cap \mathfrak{W} \end{array} \quad (\text{hBC})$$

satisfies the homotopical Beck-Chevalley condition.

and

The functors $u_!$ and v^* are homotopically conservative whenever u is an acyclic cofibration and v an acyclic fibration. (hCon)

The theorem states that this is exactly what it takes to make the names “total cofibrations” and “total fibrations” legitimate, and to turn $p : \mathcal{E} \rightarrow \mathcal{B}$ into a Quillen bifibration.

Theorem 4.4. *Under hypothesis (Q), the total category $\mathcal{E}$ admits a model structure with $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ as cofibrations and fibrations respectively if and only if properties (hBC) and (hCon) are satisfied.*

In that case, the functor $p : \mathcal{E} \rightarrow \mathcal{B}$ is a Quillen bifibration.

The strategy to prove Theorem 4.4 goes as follows:

- first we will show the necessity of conditions (hBC) and (hCon): if $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ are the cofibrations and fibrations of a model structure on $\mathcal{E}$, then hypothesis (hBC) and (hCon) are met,
- next, the harder part is the sufficiency: from Proposition 3.7 we already know that $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}}, \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ is a Quillen intertwined structure on $\mathcal{E}$, hence it is enough to show that the induced class $\mathfrak{W}_{\mathcal{E}} = \mathfrak{F}_{\mathcal{E}}^{\sim} \circ \mathfrak{C}_{\mathcal{E}}^{\sim}$ of *total weak equivalences* has the 2-out-of-3 property to conclude through Corollary 2.11.

4.2. Proof, part I: necessity

In all this section, we suppose that $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ provide respectively the cofibrations and fibrations of a model structure on the total category $\mathcal{E}$. We will denote $\mathfrak{W}_{\mathcal{E}}$ the corresponding class of weak equivalences.

First, we prove a technical lemma, directly following from Proposition 3.7, that will be extensively used in the following. Informally, it states that the name given to the members of $\mathfrak{C}_{\mathcal{E}}^{\sim}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ are not foolish.

Lemma 4.5. $\mathfrak{C}_{\mathcal{E}}^{\sim} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}}$.

Proof. By Proposition 3.7, we know that both $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ are weak factorization systems with the same class of right maps, hence their class of left maps should coincide. Similarly the weak factorization systems $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}})$ have the same class of left maps, hence their class of right maps coincide. $\square$

Corollary 4.6. *For any object A of $\mathcal{B}$, the inclusion functor $\mathcal{E}_A \rightarrow \mathcal{E}$ is homotopically conservative.*

Proof. The preservation of weak equivalences comes from the fact that acyclic cofibrations and acyclic fibrations of $\mathcal{E}_A$ are elements of $\mathfrak{C}_{\mathcal{E}}^{\sim}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ respectively. Thus, by Lemma 4.5, they are elements of $\mathfrak{W}_{\mathcal{E}}$.

Conversely, suppose that f is a map of $\mathcal{E}_A$ which is a weak equivalence of $\mathcal{E}$. We want to show that f is a weak equivalence of the fiber $\mathcal{E}_A$. The map f factors in the fiber $\mathcal{E}_A$ as $f = qj$ where $j \in \mathfrak{C}_A \cap \mathfrak{W}_A$ and $q \in \mathfrak{F}_A$. We just need to show that $q \in \mathfrak{W}_A$. By Lemma 4.5, j is also a weak equivalence of $\mathcal{E}$. By the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$, the map q is a weak equivalence of $\mathcal{E}$. As a fibration of $\mathcal{E}_A$, q is also a fibration of $\mathcal{E}$. This establishes that q is an acyclic fibration of $\mathcal{E}$. By Lemma 4.5, q is thus an element of $\mathfrak{F}_{\mathcal{E}}^{\sim}$. This concludes the proof that $q = q^{\triangleleft}$ is an acyclic fibration, and thus a weak equivalence, in the fiber $\mathcal{E}_A$. $\square$

Proposition 4.7 (Property (hCon)). *If $j : A \rightarrow B$ in an acyclic cofibration in $\mathcal{B}$, then $j_! : \mathcal{E}_A \rightarrow \mathcal{E}_B$ is homotopically conservative.*

If $q : A \rightarrow B$ in an acyclic fibration in $\mathcal{B}$, then $q^* : \mathcal{E}_B \rightarrow \mathcal{E}_A$ is homotopically conservative.

Proof. We only prove the first part of the proposition, as the second one is dual. Recall that the image $j_!(f)$ of a map $f : X \rightarrow Y$ of $\mathcal{E}_A$ is computed as the unique morphism of $\mathcal{E}_B$ making the following square commute:

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & j_!X \\ f \downarrow & & \downarrow j_!(f) \\ Y & \xrightarrow{\lambda} & j_!Y \end{array}$$

The horizontal morphisms in the diagram are cocartesian above the acyclic cofibration j . As such they are elements of $\mathfrak{C}_{\mathcal{E}}^{\sim}$, and thus weak equivalences in $\mathcal{E}$ by Lemma 4.5. By the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$, f is a weak equivalence in $\mathcal{E}$ if and only if $j_!(f)$ is one also in $\mathcal{E}$. Corollary 4.6 allows then to conclude: f is a weak equivalence in the fiber $\mathcal{E}_A$ if and only if $j_!(f)$ is one in the fiber $\mathcal{E}_B$. $\square$

Proposition 4.8 (Property (hBC)). *Commutative squares of $\mathcal{B}$ of the form*

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ u' \downarrow & & \downarrow u \\ C' & \xrightarrow{v'} & B \end{array} \quad u, u' \in \mathfrak{C} \cap \mathfrak{W} \quad v, v' \in \mathfrak{F} \cap \mathfrak{W}$$

satisfy the homotopical Beck-Chevalley condition.

Proof. Recall that for such a square in $\mathcal{B}$, the component of the mate $\mu : u'_!v^* \rightarrow v'^*u_!$ at $Z \in \mathcal{E}_C$ is defined as the unique map of $\mathcal{E}_{Z'}$ making the following diagram commute:

$$\begin{array}{ccc} v^*Z & \xrightarrow{\rho} & Z \\ \downarrow \lambda & & \downarrow \lambda \\ u'_!v^*Z & \xrightarrow{\mu_Z} & v'^*u_!Z \xrightarrow{\rho} u_!Z \end{array}$$

Arrows labelled ρ and λ are respectively cartesian above acyclic fibrations and cocartesian above acyclic cofibrations, hence weak equivalences of $\mathcal{E}$ by Lemma 4.5. By applying the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$ three times in a row, we conclude that the fiber map μ_Z is a weak equivalence of $\mathcal{E}$, hence also of $\mathcal{E}_{C'}$ by Corollary 4.6. $\square$

4.3. Proof, part II: sufficiency

We have established the necessity of (hBC) and (hCon) in Theorem 4.4. We now prove the sufficiency of these conditions. This is the hard part of the proof. Recall that every fiber $\mathcal{E}_A$ of the Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ is equipped with a model structure in such a way that (Q) is satisfied. From now on, we make the additional assumptions that (hBC) and (hCon) are satisfied.

We will use the notation $\mathfrak{W}_{\mathcal{E}} = \mathfrak{F}_{\mathcal{E}}^{\sim} \circ \mathfrak{C}_{\mathcal{E}}^{\sim}$ for the class of maps that can be written as a total acyclic cofibration postcomposed with a total acyclic fibration. The overall goal of this section is to prove that

Claim. $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ defines a Quillen model structure on the total category $\mathcal{E}$.

By Proposition 3.7, we already know that $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}}, \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ is a Quillen intertwined structure on $\mathcal{E}$. From this follows that, by Corollary 2.11, we only need to show that the class $\mathfrak{W}_{\mathcal{E}}$ of *total weak equivalences* satisfies the 2-out-of-3 property. In order to prove that, we shall use Lemma 2.12. We begin by showing Lee's first condition:

Lemma 4.9. *For any commutative square in $\mathcal{E}$ as follows:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ g' \downarrow & & \downarrow g \\ Z' & \xrightarrow{f'} & Y \end{array} \quad \begin{array}{l} f \in \mathfrak{C}_{\mathcal{E}}, g \in \mathfrak{F}_{\mathcal{E}} \\ f' \in \mathfrak{C}_{\mathcal{E}}^{\sim}, g' \in \mathfrak{F}_{\mathcal{E}}^{\sim} \end{array}$$

it holds that $f \in \mathfrak{C}_{\mathcal{E}}^{\sim}$ if and only if $g \in \mathfrak{F}_{\mathcal{E}}^{\sim}$.

Proof. Consider the image of the commutative square through p in $\mathcal{B}$:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ v' \downarrow & & \downarrow v \\ C' & \xrightarrow{u'} & B \end{array} \quad \begin{array}{l} u \in \mathfrak{C}, v \in \mathfrak{F} \\ u' \in \mathfrak{C}^{\sim}, v' \in \mathfrak{F}^{\sim} \end{array} \quad (8)$$

Because $\mathcal{B}$ is a model category, the 2-out-of-3 property implies that u is acyclic if and only if v is acyclic. Hence all we have to show is that, under the hypothesis that u and v are both acyclic, the fiber morphisms $f_{\triangleright}$ and $g^{\triangleleft}$ are weak equivalences together. In order to do that, let us look at the commutative diagram induced inside the fiber at C :

$$\begin{array}{ccc} u_! X & \xrightarrow{f_{\triangleright}} & Z \\ \downarrow u_!(g')^{\triangleleft} & & \downarrow g^{\triangleleft} \\ u_!(v')^* Z' & \xrightarrow{\mu_{Z'}} & v^*(u')_! Z' \xrightarrow{v^*(f')^{\triangleright}} v^* Y \end{array} \quad (9)$$

To see that this diagram indeed commutes, realize that both the top and bottom paths are solutions to the universal problem of finding a morphism h in the fiber at C such that the composite

$$X \xrightarrow{\lambda_{u,X}} u_! X \xrightarrow{h} v^* Y \xrightarrow{\rho_{v,Y}} Y$$

equals $gf = f'g'$, as it is highlighted in the following diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\lambda_{u,X}} & u_!X & \xrightarrow{f_\triangleright} & Z \\
(g')^\triangleleft \downarrow & & \downarrow u_!(g')^\triangleleft & & \downarrow g^\triangleleft \\
(v')^*Z' & \xrightarrow{\lambda_{u,(v')^*Z'}} & u_!(v')^*Z' & \xrightarrow{\mu_{Z'}} & v^*(u')_!Z' \xrightarrow{v^*(f')_\triangleright} v^*Y \\
\rho_{v',Z'} \downarrow & & & \searrow \downarrow \rho_{v,(u')_!Z'} & \downarrow \rho_{v,Y} \\
Z' & \xrightarrow{\lambda_{u',Z'}} & (u')_!(Z') & \xrightarrow{(f')_\triangleright} & Y
\end{array}$$

Now because $g' \in \mathfrak{F}_{\mathcal{E}}^\sim$, the fiber morphism $(g')^\triangleleft$ is a weak equivalence and because $u' \in \mathfrak{C}_{\mathcal{E}}^\sim$ hypothesis (hCon) applies to get that $u_!(g')^\triangleleft$ is a weak equivalence in $\mathcal{E}_C$. Similarly one gets that $v^*(f')_\triangleright$ is also a weak equivalence. Furthermore the square of diagram (8) can be applied hypothesis (hBC), so that $\mu_{Z'}$ in diagram (9) is also a weak equivalence of $\mathcal{E}_C$. Weak equivalences have the 2-out-of-3 property in the model structure on $\mathcal{E}_C$, hence $f_\triangleright$ is a weak equivalence precisely when $g^\triangleleft$ is such. $\square$

Let us now proceed with Lee's second condition:

Lemma 4.10. *For any commutative triangle in $\mathcal{E}$ as follows:*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
h \downarrow & \swarrow g & \\
Y & &
\end{array}$$

it holds that:

- $h, g \in \mathfrak{C}_{\mathcal{E}}^\sim$ and $f \in \mathfrak{C}_{\mathcal{E}}$ implies $f \in \mathfrak{C}_{\mathcal{E}}^\sim$
- $h, f \in \mathfrak{F}_{\mathcal{E}}^\sim$ and $g \in \mathfrak{F}_{\mathcal{E}}$ implies $g \in \mathfrak{F}_{\mathcal{E}}^\sim$

Proof. We only prove the first bullet, the second one is similar. Because $p(g)$ and $p(h)$ are weak equivalences in the model structure on $\mathcal{B}$, the 2-out-of-3 property ensures that the cofibration $p(f)$ is acyclic. So there remains to prove that the cofibration $f_\triangleright$ of $\mathcal{E}_{pZ}$ is acyclic. Denote u, v, w for $p(f), p(g), p(h)$ respectively. Then by Lemma 2.3 there is an isomorphism ϕ in the fiber $\mathcal{E}_{pY}$ that makes the following diagram commutative:

$$\begin{array}{ccccccc}
w_!X & \xrightarrow[\cong]{\phi} & v_!u_!X & \xleftarrow{\lambda} & u_!X & \xleftarrow{\lambda} & X \\
& \searrow h_\triangleright & \downarrow v_!f_\triangleright & & \downarrow f_\triangleright & \swarrow f & \\
& & v_!Z & \xleftarrow{\lambda} & Z & & \\
& & \downarrow g_\triangleright & & \swarrow g & & \\
& & Y & & & &
\end{array}$$

Now in $\mathcal{E}_{pY}$, both $h_\triangleright$ and $g_\triangleright$ are acyclic by definition. The 2-out-of-3 property shows that $v_!f_\triangleright$ also is a weak equivalence. Invoke finally (hCon) on the acyclic cofibration v to get that $f_\triangleright$ is a weak equivalence in $\mathcal{E}_{pZ}$. $\square$

Combining Lemma 4.9 and Lemma 4.10 through Lee's criterion 2.12, we get the wanted corollary:

Corollary 4.11. *The class $\mathfrak{W}_{\mathcal{E}}$ has the 2-out-of-3 property.*

Let us now give a more complete analysis of the total weak equivalences. There is one crucial property which is hidden in Lemma 4.9 and Lemma 4.10 and we shall expose it for future references. First remark that for each $f : X \rightarrow Y$ in $\mathcal{E}$ such that $p(f) = vu$ for two composable morphisms $u : pX \rightarrow C$ and $v : C \rightarrow pY$ of $\mathcal{B}$, there is a unique morphism inside the fiber $\mathcal{E}_C$

$${}_uf^v : u_!X \rightarrow v^*Y$$

such that $f = \rho_{v,Y} \circ {}_uf^v \circ \lambda_{u,X}$. This morphism ${}_uf^v$ can be constructed as $k^\triangleleft$ where k is the unique morphism above v factorizing f through $\lambda_{u,X}$; or equivalently as $\ell_\triangleright$ where ℓ is the unique morphism above u factorizing f through $\rho_{v,Y}$. This is summed up in the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\lambda} & u_!X & & \\ & \searrow \ell & \downarrow \ell_\triangleright = {}_uf^v = k^\triangleleft & \searrow k & \\ & & v^*Y & \xrightarrow{\rho} & Y \end{array}$$

Notice that, in particular, a morphism f of $\mathfrak{W}_{\mathcal{E}}$ is exactly a morphism of $\mathcal{E}$ for which **there exists** a factorization $p(f) = qj$ with $j \in \mathfrak{W} \cap \mathcal{C}$ and $q \in \mathfrak{W} \cap \mathfrak{F}$ such that ${}_jf^q$ is a weak equivalence in the corresponding fiber. We shall now argue that hypothesis (hCon) and (hBC) imply in fact that a morphism f of $\mathfrak{W}_{\mathcal{E}}$ satisfies the same property that ${}_jf^q$ is a weak equivalence **for all** such factorization $p(f) = qj$.

Proposition 4.12. *Let $f : X \rightarrow Y$ in $\mathfrak{W}_{\mathcal{E}}$. If $p(f) = qj$ with $j : pX \rightarrow C \in \mathfrak{W} \cap \mathcal{C}$ and $q : C \rightarrow pY \in \mathfrak{W} \cap \mathfrak{F}$, then ${}_jf^q$ is a weak equivalence.*

Proof. By the very definition of ${}_jf^q$, it holds that $f = \rho \circ {}_jf^q \circ \lambda$. Because j is an acyclic cofibration, the cocartesian morphism $\lambda : X \rightarrow j_!X$ above it is a total acyclic cofibration. Similarly, q being an acyclic fibration, the cartesian morphism $\rho : q^*Y \rightarrow Y$ above it is a total acyclic fibration. Hence f, ρ, λ all are total weak equivalences, so by the 2-out-of-3 property proved in Corollary 4.11, the map ${}_jf^q$ is also a total weak equivalence. $\square$

This result often comes in handy in the following way: to prove that a map f of $\mathcal{E}$ is in $\mathfrak{W}_{\mathcal{E}}$, just follow the definition and find **some** factorization $p(f) = qj$ as an acyclic cofibration followed by an acyclic fibration such that ${}_jf^q$ is acyclic inside its fiber; however to use the fact that some f is a total weak equivalence, one can use that ${}_jf^q$ is a weak equivalence for **every** admissible factorization of $p(f)$.

5. Illustrations

Since the very start, our work is motivated by the idea that the Reedy model structure can be reconstructed by applying a series of Grothendieck constructions of model categories. The key observation is that, when combining them, the two notions of latching and matching functor define a bifibration at each step of the construction of the Reedy model structure. We explain in Section 5.1 how the Reedy construction can be reunderstood from our bifibrational point of view. In Section 5.2, we describe how to express generalized Reedy constructions in a similar fashion.

5.1. A bifibrational view on Reedy model structures

Recall that a *Reedy category* is a small category $\mathcal{R}$ together with two subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ and a degree function $d : \text{Ob } \mathcal{R} \rightarrow \lambda$ for some ordinal λ such that

- every morphism f admits a unique factorization $f = f^+ \circ f^-$ with $f^- \in \mathcal{R}^-$ and $f^+ \in \mathcal{R}^+$,
- non-identity morphisms of $\mathcal{R}^+$ strictly increase the degree and those of $\mathcal{R}^-$ strictly decrease it.

For such a Reedy category, let $\mathcal{R}_\mu$ denote the full subcategory spanned by the objects of degree strictly less than μ . In particular, $\mathcal{R} = \mathcal{R}_\lambda$. Remark also that every $\mathcal{R}_\mu$ inherits a structure of Reedy category from $\mathcal{R}$.

We are interested in the structure of the category of diagrams of shape $\mathcal{R}$ in a complete and cocomplete category $\mathcal{C}$. The category $\mathcal{C}$ is in particular tensored and cotensored over $\mathbf{Set}$, with tensor and cotensor respectively defined as:

$$S \odot C = \coprod_{s \in S} C, \quad S \pitchfork C = \prod_{s \in S} C, \quad S \in \mathbf{Set}, C \in \mathcal{C}.$$

For every $r \in \mathcal{R}$ of degree μ , a diagram $X : \mathcal{R}_\mu \rightarrow \mathcal{C}$ induces two objects in $\mathcal{C}$, called the *latching* and *matching* objects of X at r , and respectively defined as:

$$L_r X = \int^{s \in \mathcal{R}_\mu} \mathcal{R}(s, r) \odot X_s, \quad M_r X = \int_{s \in \mathcal{R}_\mu} \mathcal{R}(r, s) \pitchfork X_s \quad (10)$$

By a slight abuse of notation, we also write $L_r X$ and $M_r X$ for the latching and matching objects of the restriction to $\mathcal{R}_\mu$ of some $X : \mathcal{R}_\kappa \rightarrow \mathcal{C}$ with $\kappa \geq \mu$. In particular, when $\kappa = \lambda$, X is a diagram of shape the entire category $\mathcal{R}$ and we retrieve the usual notions of latching and matching object (see for instance [Hov99]). Universal properties of limits and colimits induce a family of canonical morphisms $\alpha_r : L_r X \rightarrow M_r X$, which can also be understood in the following way. First, one notices that the two functors $L_\mu X, M_\mu X : \mathcal{R}_{\mu+1} \rightarrow \mathcal{C}$ defined as

$$L_\mu X r \mapsto \begin{cases} X_r & \text{if } d(r) < \mu \\ L_r X & \text{if } d(r) = \mu \end{cases}, \quad M_\mu X r \mapsto \begin{cases} X_r & \text{if } d(r) < \mu \\ M_r X & \text{if } d(r) = \mu \end{cases}$$

provide left and right Kan extensions of X along the inclusion $i_\mu : \mathcal{R}_\mu \rightarrow \mathcal{R}_{\mu+1}$. The family of morphisms α_r then describes the unique natural transformation $\alpha : L_\mu X \rightarrow M_\mu X$ which restricts to the identity on $\mathcal{R}_\mu$.

The following property is, in our opinion, the key feature of Reedy categories.

Proposition 5.1. *Extensions of a diagram $X : \mathcal{R}_\mu \rightarrow \mathcal{C}$ to $\mathcal{R}_{\mu+1}$ are in one-to-one correspondence with families of factorizations of the α_r 's*

$$(L_r X \rightarrow \bullet \rightarrow M_r X)_{r \in \mathcal{R}, d(r) = \mu} \quad (11)$$

Proof. The fact that every extension $\hat{X} : \mathcal{R}_{\mu+1} \rightarrow \mathcal{C}$ of X induces such a family of factorizations is easy to establish and does not use the structure of Reedy category. For every r of degree μ in $\mathcal{R}$, one constructs a factorization of α_r

$$L_r X \rightarrow \hat{X}_r \rightarrow M_r X$$

where the arrows $L_r X \rightarrow \hat{X}_r$ and $\hat{X}_r \rightarrow M_r X$ are defined by the following cocone and cone, respectively:

$$\left(\hat{X}(f) : X_s \rightarrow \hat{X}_r \right)_{f: s \rightarrow r, d(s) < \mu} \quad \left(\hat{X}(f) : \hat{X}_r \rightarrow X_s \right)_{f: r \rightarrow s, d(s) < \mu}.$$

The important feature of Reedy categories is that they enable us to establish the converse property. In order to construct an extension $\hat{X}$ of X , one needs to define a morphism $\hat{X}(f)$ for every arrow $f : r \rightarrow r'$ between objects of degree μ , in a functorial way. In the case of a Reedy category, every family (11) of factorizations (with no extra naturality condition required) does define such an extension $\hat{X}$. This is a nice consequence of the unique factorization property of $\mathcal{R}$. Given a family of factorizations

$$\left(L_r X \rightarrow \hat{X}_r \rightarrow M_r X \right)_{r \in \mathcal{R}, d(r) = \mu},$$

one defines $\hat{X}(f^+) : X_s \rightarrow \hat{X}_r$ for $f^+ : s \rightarrow r \in \mathcal{R}^+$ with $d(r) = \mu$ as the composite

$$X_s \rightarrow L_r X \rightarrow \hat{X}_r.$$

where $X_s \rightarrow L_r X$ is the canonical injection associated to f^+ . In the same way, one define $\hat{X}(f^-) : \hat{X}_r \rightarrow X_s$ for $f^- : r \rightarrow s \in \mathcal{R}^-$ with $d(r) = \mu$: f^- as the composite

$$\hat{X}_r \rightarrow M_r X \rightarrow X_s.$$

where $M_r X \rightarrow X_s$ is the canonical projection $M_r X \rightarrow X_s$ associated to f^- . For every other morphism $f : r \rightarrow r'$ with r or r' of degree μ , we first factor f as $f^+ \circ f^-$ with $f^- \in \mathcal{R}^-$ and $f^+ \in \mathcal{R}^+$ and we define $\hat{X}(f)$ as $\hat{X}(f^+) \circ \hat{X}(f^-)$. The fact that $\hat{X}$ defines a functor extending X follows from uniqueness of factorization in the Reedy category $\mathcal{R}$. $\square$

From now on, we fix a model category $\mathcal{M}$, that is a complete and cocomplete category $\mathcal{M}$ with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. The motivation behind Kan's notion of Reedy category is to gives sufficient conditions on $\mathcal{R}$ to equip $[\mathcal{R}, \mathcal{M}]$ with a model structure where weak equivalences are pointwise.

Definition 5.2. Let $\mathcal{R}$ be a Reedy category. The *Reedy triple* on the functor category $[\mathcal{R}, \mathcal{M}]$ is the data of the three following classes

- Reedy cofibrations : those $f : X \rightarrow Y$ such that for all $r \in \mathcal{R}$, the map $L_r Y \sqcup_{L_r X} X_r \rightarrow Y_r$ is a cofibration,
- Reedy weak equivalences : those $f : X \rightarrow Y$ such that for $r \in \mathcal{R}$, $f_r : X_r \rightarrow Y_r$ is a weak equivalence,
- Reedy fibrations : those $f : X \rightarrow Y$ such that for all $r \in \mathcal{R}$, the map $X_r \rightarrow M_r X \times_{M_r Y} Y_r$ is a fibration.

A first reading of this definition is quite facinating: the distinguished morphisms are defined through those latching and matching objects, and it is not clear why one should emphasize these constructions that much. Kan's theorem about Reedy categories states that the Reedy triple turns $[\mathcal{R}, \mathcal{M}]$ into a model category. We show below how this result can be understood as a very natural consequence of our main theorem on Quillen bifibrations (cf. Thm. 4.4).

Recall that an isofibration is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ with the following property: for every $Y \in \mathcal{E}$ and every isomorphism $\phi : A \rightarrow pY$ in $\mathcal{B}$, there exists an isomorphism $\psi : X \rightarrow Y$ in $\mathcal{E}$ such that $p(\psi) = \phi$. The following lemma is folklore but the authors are not aware of any written reference in the literature.

Lemma 5.3. *Suppose given $p : \mathcal{E} \rightarrow \mathcal{B}$ where $\mathcal{E}$ has pullbacks and p preserve them. If p is an isofibration and admits a fully faithful right adjoint then p is a Grothendieck fibration.*

Proof. Denote $R : \mathcal{B} \rightarrow \mathcal{E}$ for the fully faithful right adjoint. The claim is that a morphism $f : X \rightarrow Y$ is cartesian precisely when the following diagram is a pullback square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ RpX & \xrightarrow{Rp(f)} & RpY \end{array} \quad (12)$$

where the vertical arrows are the component at X and Y of the unit η of the adjunction (p, R) . Indeed, such a diagram is a pullback square if and only if the following square is a pullback for all Z :

$$\begin{array}{ccc} \mathcal{E}(Z, X) & \xrightarrow{f \circ -} & \mathcal{E}(Z, Y) \\ \downarrow \eta_X \circ - & & \downarrow \eta_Y \circ - \\ \mathcal{E}(Z, RpX) & \xrightarrow{Rp(f) \circ -} & \mathcal{E}(Z, RpY) \end{array}$$

We take advantage of the adjunction (p, R) and its natural isomorphisms

$$\phi_{Z,A} : \mathcal{E}(Z, RA) \simeq \mathcal{B}(pZ, A)$$

As in any adjunction, this isomorphism is related to the unit by the following identity: for any $g : Z \rightarrow X$, $p(g) = \phi(\eta_X g)$. So in the end, the square in (12) is a pullback if and only if for every Z the outer square of the following diagram is a pullback:

$$\begin{array}{ccc}
 \mathcal{E}(Z, X) & \xrightarrow{f \circ -} & \mathcal{E}(Z, Y) \\
 \downarrow \eta_X \circ - & & \downarrow \eta_Y \circ - \\
 \mathcal{E}(Z, RpX) & \xrightarrow{Rp(f) \circ -} & \mathcal{E}(Z, RpY) \\
 \downarrow \phi \downarrow \eta & & \downarrow \phi \downarrow \eta \\
 \mathcal{B}(pZ, pX) & \xrightarrow{p(f) \circ -} & \mathcal{B}(pZ, pY)
 \end{array}
 \begin{array}{c}
 \leftarrow p \\
 \leftarrow p
 \end{array}$$

This is exactly the definition of a cartesian morphism $f : X \rightarrow Y$.

Now we construct for every $u : A \rightarrow pY$ in $\mathcal{B}$ a cartesian morphism $\rho : X \rightarrow Y$ above u . Consider the unit η of $(p \dashv R)$ at Y and pull it back along $R(u) : RA \rightarrow RpY$:

$$\begin{array}{ccc}
 X & \xrightarrow{\rho} & Y \\
 x \downarrow & \lrcorner & \downarrow \eta_Y \\
 RA & \xrightarrow{R(u)} & RpY
 \end{array} \tag{13}$$

By hypothesis, its image through p is still a pullback in $\mathcal{B}$. Paste it to the naturality square of the counit ε :

$$\begin{array}{ccc}
 pX & \xrightarrow{p(\rho)} & pY \\
 p(x) \downarrow & \lrcorner & \downarrow p(\eta_Y) \\
 pRA & \xrightarrow{pR(u)} & pRpY \\
 \varepsilon_A \downarrow & & \downarrow \varepsilon_{pY} \\
 A & \xrightarrow{u} & pY
 \end{array}$$

Remark that the composite morphism $\varepsilon_{pY} p(\eta_Y)$ on the right equals $\mathbf{1}_{pY}$ by the triangular identities of the adjunction $p \dashv R$. Remark also that because the right adjoint R is fully faithful, the counit is a natural isomorphism. Hence, ε_{pY} is invertible. It makes $p(\eta_Y)$ also invertible. The base change of an invertible morphism being invertible, it follows that $p(x)$ is an isomorphism. We shall denote ϕ for the composite isomorphism $\varepsilon_A p(x)$. In particular, we have $u\phi = p(\rho)$.

We claim that the morphism $\rho : X \rightarrow Y$ is cartesian. Indeed, consider the following composite of squares:

$$\begin{array}{ccc}
 X & \xrightarrow{\rho} & Y \\
 \eta_X \downarrow & & \downarrow \eta_Y \\
 RpX & \xrightarrow{Rp(\rho)} & RpY \\
 R(\phi) \downarrow & & \parallel \\
 RA & \xrightarrow{R(u)} & RpY
 \end{array}$$

The vertical composite on the left is $R(\phi)\eta_X = R(\varepsilon_A)R(p(x))\eta_X = R(\varepsilon_A)\eta_{RA}x = x$. Hence the outer square is the pullback square defined in (13). The bottom square is also a pullback square because its two vertical sides are isomorphisms. Hence, the pasting lemma ensures that the top square is a pullback square as well. By the previous analysis, it means that ρ is cartesian. But ρ is not quite above u yet. We then invoke the isofibration property of p to find an object X' above A together with an isomorphism $\psi : X' \rightarrow X$ above $\phi^{-1} : A \rightarrow pX$. The composite $p\psi$ is still cartesian, and now $p(\rho\psi) = p(\rho)\phi^{-1} = u$. $\square$

Lemma 5.4. *Given an functor $i : A \rightarrow B$ which are injective on objects, and a category $\mathcal{C}$, the restriction functor $i^* : [\mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, \mathcal{C}]$ is an isofibration.*

Proof. Choose a universe $\mathbb{U}$ big enough for $\mathcal{C}$ to be small relatively to $\mathbb{U}$, and consider the canonical model structure on the category $\mathbf{Cat}$ of $\mathbb{U}$ -small categories. Recall that in this model structure, the weak equivalences are the equivalences of categories, the cofibrations are the functors which are injective on objects, and the fibrations are the isofibrations. Hence, it suffices to prove that the contravariant functor $[-, \mathcal{C}]$ maps cofibrations to fibrations. This translates to the usual conditions on pushout products: in a pushout diagram as follows where i and j are cofibrations of $\mathbf{Cat}$, $i \square j$ is a cofibration as well, and it is acyclic whenever j is acyclic.

$$\begin{array}{ccc}
 A \times C & \xrightarrow{A \times j} & A \times D \\
 i \times C \downarrow & & \downarrow \iota_1 \\
 B \times C & \xrightarrow{\iota_2} & (B \times C) \sqcup_{A \times C} (A \times D) \\
 & \searrow i \square j & \downarrow i \times D \\
 & & B \times D
 \end{array}$$

$B \times j$

The fact that $i \square j$ is injective on objects is just a property of functions between sets and is easily verified. There remains only to show that $i \square j$ is an equivalence of categories whenever j is. If j is an equivalence of categories, then $A \times j$ and $B \times j$ also. The functor $i \times C$ is injective on objects, and every object is cofibrant in $\mathbf{Cat}$: hence, ι_2 is the pushout of a weak equivalence along a cofibration between cofibrant object. So ι_2 is an equivalence of categories also. We can now apply the 2-out-of-3 property to the bottom triangle in the diagram above to conclude that $i \square j$ is an equivalence of categories. $\square$

Corollary 5.5. *Let $\mathcal{R}$ be Reedy. The restriction functor $i_\mu^* : [\mathcal{R}_{\mu+1}, \mathcal{M}] \rightarrow [\mathcal{R}_\mu, \mathcal{M}]$ is a Grothendieck bifibration.*

Proof. The category $[\mathcal{R}_{\mu+1}, \mathcal{M}]$ has all pullbacks and the functor i_μ^* preserves them. Moreover, because i_μ is injective on objects, Lemma 5.4 proves that i_μ^* is an isofibration. Hence, through Lemma 5.3 and its dual, it suffices to provide fully faithful left and right adjoints to i_μ^* . We already have left and right adjoints for i_μ^* , respectively L_μ and M_μ , which are both faithful by construction. We establish now that L_μ is also full (the proof for M_μ is similar): given a natural transformation $f : L_\mu X \rightarrow L_\mu Y$, the naturality condition stipulates in particular that for each $k : s \rightarrow r$ with $\deg s < \mu$ and $\deg r = \mu$, the following diagram commutes:

$$\begin{array}{ccc}
 X_s & \xrightarrow{f_s} & Y_s \\
 \iota_k^X \downarrow & & \downarrow \iota_k^Y \\
 L_r X & \xrightarrow{f_r} & L_r Y
 \end{array}$$

where ι_k^Z denotes the canonical injection of Z_s to $L_r Z$. This means that f_r is precisely the map induced by the universal property of the colimit $L_r X$ on the maps $\iota_k^Y f_s : X_s \rightarrow L_r Y$. In other words, $f_r = L_r(i_\mu^*(f))$. This property holds for every r of degree μ , and from this we conclude that $f = L_\mu(i_\mu^*(f))$ and thus that L_μ is full. $\square$

REMARK 5.6. First, we should notice that Proposition 5.1 makes the following multievaluation functor an equivalence:

$$[\mathcal{R}_{\mu+1}, \mathcal{M}]_A \xrightarrow{\sim} \prod_{r \in \mathcal{R}, d(r)=\mu} L_r A \backslash \mathcal{M}_{/M_r A} \quad (14)$$

The notation $L_r A \backslash \mathcal{M}_{/M_r A}$ is slightly abusive and means the coslice category of $\mathcal{M}_{/M_r A}$ under α_r , or equivalently the slice category of $L_r A \backslash \mathcal{M}$ over α_r .

Secondly, we can deduce from the proof of Lemma 5.3 that for a morphism $f : X \rightarrow Y$, the fiber morphisms $f^<$ and $f^>$ are respectively, modulo identification (14), the induced families:

$$(X_r \rightarrow M_r X \times_{M_r Y} Y_r)_{r, d(r)=\mu}, \quad (X_r \sqcup_{L_r X} L_r Y \rightarrow Y_r)_{r, d(r)=\mu}$$

Note that these are precisely the families of morphism appearing in the definition of a Reedy triple. So here it is: the reason behind the *a priori* mysterious morphisms of Def. 5.2, involving latching and matching objects, are nothing else but the witness of a hidden bifibrational structure on the restriction functor i_μ^* . Putting this into light was a tremendous leap forward in our conceptual understanding of Reedy model structures and their generalizations.

The following proposition is the induction step for successor ordinals in the usual proof of the existence of Reedy model structures (see for instance [Hov99]). Our characterization theorem (Thm. 4.4) allows for a very slick and smooth argument.

Proposition 5.7. *If the Reedy triple on $[\mathcal{R}_\mu, \mathcal{M}]$ defines a model structure, then it is also the case of the Reedy triple on $[\mathcal{R}_{\mu+1}, \mathcal{M}]$.*

Proof. We deduce the property from the fact that the Grothendieck bifibration $i_\mu^* : [\mathcal{R}_{\mu+1}, \mathcal{M}] \rightarrow [\mathcal{R}_\mu, \mathcal{M}]$ satisfies the conditions required by Thm. 4.4. By hypothesis, the base category $[\mathcal{R}_\mu, \mathcal{M}]$ has a model structure given by the Reedy triple. Each fiber $(i_\mu^*)_A$ above a diagram $A : \mathcal{R}_\mu \rightarrow \mathcal{M}$ is endowed, via identification (14), with the product model structure. Indeed, if $\mathcal{N}$ is a model category, then its slices $\mathcal{N}_N$ and coslices ${}_{N\backslash}\mathcal{N}$ categories are also model categories, where the cofibrations, a fibrations or a weak equivalences are defined as the morphisms with similar status after projection into $\mathcal{N}$. The product of model categories are then a model category by taking the structure defined pointwise. This can be summarized as follows: the fiber $(i_\mu^*)_A$ is a model category, where a fiber map $f : X \rightarrow X'$ in $(i_\mu^*)_A$ is a cofibration, a fibration or a weak equivalence if and only if $f_r : X_r \rightarrow X'_r$ has the status for every $r \in \mathcal{R}$ of degree μ .

Now, the proof of the statement amounts to showing that conditions (Q), (hCon) and (hBC) are satisfied by the bifibration i_μ^* in this framework. Let us first establish condition (Q). Suppose $u : A \rightarrow B$ in $[\mathcal{R}_\mu, \mathcal{M}]$ and $f : Y \rightarrow Y'$ a fiber morphism above B . The morphism $u^*(f)$ is the unique map above A making the following diagram commute for all r of degree μ :

$$\begin{array}{ccc} (u^*Y)_r & \longrightarrow & Y_r \\ (u^*f)_r \downarrow & & \downarrow f_r \\ (u^*Y')_r & \longrightarrow & Y'_r \\ \downarrow & & \downarrow \\ M_r A & \xrightarrow{M_r u} & M_r B \end{array} \quad (15)$$

where the lower square and outer square are pullback diagrams. By the pasting lemma, the upper square is also a pullback diagram. Hence, $(u^*f)_r$ is a pullback of f_r , and as such is a (acyclic) fibration whenever f_r is such. This proves that u^* is right Quillen for any u , which is precisely the statement of condition (Q).

Let us now establish conditions (hCon) and (hBC). As we will see, they will be handled in pretty much the same way. Indeed, their proof relies on the following well-known fact about Reedy model structures [Hir03, lemma 15.3.9]: for $r \in \mathcal{R}$ of degree μ , the functor $M_r : [\mathcal{R}_\mu, \mathcal{M}] \rightarrow \mathcal{M}$ is right Quillen and in particular preserves acyclic fibrations. This has the wonderful consequence that if u is an acyclic fibration of $[\mathcal{R}_\mu, \mathcal{M}]$, any pullback of $M_r u$ is an acyclic fibration hence a weak equivalence. So the upper square of diagram (15) has acyclic horizontal arrows. By the 2-out-of-3 property, f_r is a weak equivalence if and only if $(u^*f)_r$ is one also. This is true for each $r \in \mathcal{R}$ of degree μ , and so u^* is homotopically conservative whenever u is an acyclic fibration. This validates half of the property (hCon). The other half is proved dually, and deduced on the dual lemma, which states that for any $r \in \mathcal{R}$ of degree μ , the latching functor $L_r : [\mathcal{R}_\mu, \mathcal{M}] \rightarrow \mathcal{M}$ preserves acyclic cofibrations. Then we deduce that pushouts of $L_r u$ are weak equivalences whenever u is an acyclic cofibration, and we conclude in the same way as before that $u_!$ is homotopically conservative.

We still have to show (hBC). Given a commutative square of $[\mathcal{R}_\mu, \mathcal{M}]$

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ u' \downarrow & & \downarrow u \\ C' & \xrightarrow{v'} & B \end{array}$$

with u, u' Reedy acyclic cofibrations and v, v' Reedy acyclic fibrations, and an extension $Z : \mathcal{R}_{\mu+1} \rightarrow \mathcal{M}$ of C , the mate at Z is the unique fiber morphism $\nu_Z : (u'!v^*Z) \rightarrow (v'^*u!Z)$ making the following diagram commute for every $r \in \mathcal{R}$ of degree μ :

$$\begin{array}{ccccc} & & M_r A & \xrightarrow{M_r v} & M_r C \\ & & \uparrow & & \uparrow \\ L_r A & \longrightarrow & (v^*Z)_r & \longrightarrow & Z_r \longleftarrow L_r C \\ \downarrow L_r u' & & \downarrow & & \downarrow L_r v \\ L_r C' & \longrightarrow & (u'!v^*Z)_r & \xrightarrow{(\nu_Z)_r} & (v'^*u!Z)_r \longrightarrow (u!Z)_r \longleftarrow L_r B \\ & & & \downarrow & \downarrow \\ & & & M_r C' & \xrightarrow{M_r v'} M_r B \end{array}$$

where gray squares are either pullbacks (when involving matching objects) or pushouts (when involving latching objects). The same argument as above shows that every horizontal and vertical arrow of the solid pentagone is a weak equivalence. It follows from the 2-out-of-3 property that the r -component of the mate $(\nu_Z)_r$ is also a weak equivalence. This establishes condition (hBC).

Our characterization of Thm. 4.4 thus applies, and yields a model structure on $[\mathcal{R}_{\mu+1}, \mathcal{M}]$ which precisely coincide with the Reedy triple. $\square$

5.2. Notions of generalized Reedy categories

From time to time, people stumble accross almost Reedy categories and build clever workarounds to end up with a model structure “à la Reedy”. The most popular generalizations of that kind were formulated by Cisinski ([Cis06]), and Berger and Moerdijk ([BM11]), in which non trivial automorphisms are allowed. In [Shu15], Shulman establishes a common framework for every such known generalizations of Reedy categories. An *almost-Reedy categories* is defined in his work as a small category $\mathcal{C}$ with a degree function on the objects that satisfies the following property: taking x of degree μ and denoting $\mathcal{C}_\mu$ the full subcategory of $\mathcal{C}$ of objects of degree strictly less than μ , and $\mathcal{C}_x$ the full subcategory of $\mathcal{C}$ spanned by $\mathcal{C}_\mu$ and x , then the diagram category $[\mathcal{C}_x, \mathcal{M}]$ is equivalent to the *bigluing* (to be defined below) of two nicely behaved functors $[\mathcal{C}_\mu, \mathcal{M}] \rightarrow \mathcal{M}$, namely the weighted colimit and weighted limit functors, respectively weighted by $\mathcal{C}(-, x)$ and $\mathcal{C}(x, -)$. In particular, usual Reedy categories are recovered by observing that the formulas of latching and matching objects given in (10) are precisely these weighted colimits and limits.

In order to understand the notion of almost-Reedy category ([Shu15]) in our framework, we formulate below an alternative view of the Reedy construction exposed in detail in Proposition 5.7. We start with a nice consequence of Theorem 4.4:

Lemma 5.8. *Suppose given a strict pullback square of categories*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow q & & \downarrow p \\ \mathcal{C} & \xrightarrow{F} & \mathcal{B} \end{array}$$

in which $\mathcal{C}$ has a model structure and p is a Quillen bifibration. If

(i) $F(u)_!$ and $F(v)^*$ are homotopically conservative whenever u is an acyclic cofibration and v an acyclic fibration in $\mathcal{C}$,

(ii) F transports squares of the form

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ \downarrow u' & & \downarrow u \\ C' & \xrightarrow{v'} & B \end{array}$$

with u, u' acyclic cofibrations and v, v' acyclic fibrations in $\mathcal{C}$ to squares in $\mathcal{B}$ which satisfy the homotopical Beck-Chevalley condition,

then q is also a Quillen bifibration.

Proof. Let $p' : \mathcal{B} \rightarrow \mathbf{Adj}$ denote the pseudo-functor $A \mapsto \mathcal{E}_A$ associated to the Grothendieck bifibration p . Then it is folklore that the pullback q of p along F is a Grothendieck bifibration, moreover obtained by the Grothendieck construction of the pseudo-functor $p'F : \mathcal{C} \rightarrow \mathbf{Adj}$. For every $C \in \mathcal{C}$, the fiber $\mathcal{F}_C = \mathcal{E}_{FC}$ at C of the bifibration q comes equipped with a model structure; moreover for any $u : C \rightarrow D$ in $\mathcal{C}$, the adjunction

$$u_! : \mathcal{F}_C \rightleftarrows \mathcal{F}_D : u^*$$

is given by the adjoint pair $(F(u)_!, F(u)^*)$ defined by p . Hence, Theorem 4.4 asserts that q is a Quillen bifibration as soon as (hBC) and (hCon) are satisfied. The conditions of the lemma are precisely there to ensure that this is the case. $\square$

Let $\Delta[1]$ and $\Delta[2]$ denote the posetal categories associated to $\{0 < 1\}$ and $\{0 < 1 < 2\}$ respectively, and $c : \Delta[1] \rightarrow \Delta[2]$ the functor associated to the mapping $0 \mapsto 0, 1 \mapsto 2$. Given a Reedy category $\mathcal{R}$ and an object r of degree μ , let $i_r : \mathcal{R}_\mu \rightarrow \mathcal{R}_r$ denote the inclusion of the full subcategory of $\mathcal{R}$ spanned by the object of degree strictly less than μ into the one spanned by the same objects plus r . It follows from Proposition 5.1 that the following diagram is a strict pullback square of categories:

$$\begin{array}{ccc} [\mathcal{R}_r, \mathcal{M}] & \longrightarrow & [\Delta[2], \mathcal{M}] \\ i_r^* \downarrow & & \downarrow c^* \\ [\mathcal{R}_\mu, \mathcal{M}] & \xrightarrow{\alpha_r} & [\Delta[1], \mathcal{M}] \end{array}$$

where we write α_r for the functor which maps every diagram $X : \mathcal{R}_\mu \rightarrow \mathcal{M}$ to the canonical arrow $\alpha_r : L_r X \rightarrow M_r X$. The functor c^* is a Grothendieck bifibration, for the following reason: one can easily verify that an arrow in $[\Delta[2], \mathcal{M}]$

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{g} & \bullet \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{h} & \bullet \end{array}$$

is cartesian if and only if the bottom square is a pullback, and is cocartesian if and only if the top square is a pushout. Moreover, for each object $k : A \rightarrow B$ of $[\Delta[1], \mathcal{M}]$, there is a model structure on its fiber $(c^*)_k \simeq {}_A\mathcal{M}_B$ induced by the model structure of $\mathcal{M}$. Stability of cofibrations by pushout and of fibrations by pullback in the model category $\mathcal{M}$ ensures that hypothesis (Q) is satisfied by the Grothendieck bifibration c^* . When the basis category $[\Delta[1], \mathcal{M}]$ is equipped with the trivial model structure, the conditions (hBC) and (hCon) are vacuously satisfied and Theorem 4.4 thus applies, and turns c^* into a Quillen bifibration. The

proof of Proposition 5.7 is precisely showing conditions (i) and (ii) of Lemma 5.8. By applying Lemma 5.8, we can thus conclude that $i_r^* : [\mathcal{R}_r, \mathcal{M}] \rightarrow [\mathcal{R}_\mu, \mathcal{M}]$ is a Quillen bifibration, as we did in Proposition 5.7.

We show now that the result of [Shu15, Theorem 3.11] can be understood in that way, and deduced from Lemma 5.8. Shulman defines the *bigluing* of a natural transformation $\alpha : F \rightarrow G$ between two functors $F, G : \mathcal{M} \rightarrow \mathcal{N}$ as the category $\mathcal{G}\ell(\alpha)$ whose:

- objects are factorizations of the form:

$$\alpha_M : FM \xrightarrow{f} N \xrightarrow{g} GM$$

- morphisms $(h, k) : (f, g) \rightarrow (f', g')$ are commutative diagrams of the form

$$\begin{array}{ccccc} FM & \xrightarrow{f} & N & \xrightarrow{g} & GM \\ F(h) \downarrow & & \downarrow k & & \downarrow G(h) \\ FM' & \xrightarrow{f'} & N' & \xrightarrow{g'} & GM' \end{array}$$

The category $\mathcal{G}\ell(\alpha)$ can be defined more directly as the unique category (up to isomorphism) such that the following diagram is a strict pullback in $\mathbf{Cat}$:

$$\begin{array}{ccc} \mathcal{G}\ell(\alpha) & \longrightarrow & [\Delta[2], \mathcal{N}] \\ \downarrow & & \downarrow c^* \\ \mathcal{M} & \xrightarrow{\alpha} & [\Delta[1], \mathcal{N}] \end{array}$$

where $\alpha : \mathcal{M} \rightarrow [\Delta[1], \mathcal{N}]$ denotes the natural transformation α seen as a functor. In [Shu15], a *couniversal weak equivalence* is defined as a map whose pushouts are all weak equivalences, and a *universal weak equivalence* is defined as a map whose pullbacks are all weak equivalences. In the same fashion as in the proof of Proposition 5.7, we can show that conditions (i) and (ii) are satisfied by the functor α when F maps acyclic cofibrations to couniversal weak equivalences and G maps acyclic fibrations to universal weak equivalences. This establishes that Shulman's theorem below is a natural consequence of Lemma 5.8.

Theorem 5.9 (Shulman). *Suppose $\mathcal{N}$ and $\mathcal{M}$ are both model categories. Let $\alpha : F \rightarrow G$ between $F, G : \mathcal{M} \rightarrow \mathcal{N}$ satisfying that:*

- *F is cocontinuous and maps acyclic cofibrations to couniversal weak equivalences,*
- *G is continuous and maps acyclic fibrations to universal weak equivalences.*

Then $\mathcal{G}\ell(\alpha)$ is a model category whose:

- *cofibrations are the maps (h, k) such that both h and the map $FM' \sqcup_{FM} N \rightarrow N'$ induced by k are cofibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*
- *fibrations are the maps (h, k) such that both h and the map $N \rightarrow GM \times_{GM'} N'$ induced by k are fibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*
- *weak equivalences are the maps (h, k) where both h and k are weak equivalences in $\mathcal{M}$ and $\mathcal{N}$ respectively.*

Maybe the best way to understand the benefits of this construction of a Quillen bifibration $\mathcal{G}\ell(\alpha) \rightarrow \mathcal{M}$ is to see it at play. Recall that a generalized Reedy category in the sense of Berger and Moerdijk (cf. [BM11]) is a kind of Reedy category with degree preserving isomorphisms: precisely it is a category $\mathcal{R}$ with a degree function $d : \text{Ob } \mathcal{R} \rightarrow \lambda$ and wide subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ such that:

- **non-invertible** morphisms of $\mathcal{R}^+$ strictly raise the degree while those of $\mathcal{R}^-$ strictly lower it,

- isomorphisms all preserve the degree,
- the morphisms of $\mathcal{R}^+ \cap \mathcal{R}^-$ are precisely the isomorphisms,
- every morphism f can be factorized as $f = f^+ f^-$ with $f^+ \in \mathcal{R}^+$ and $f^- \in \mathcal{R}^-$, and such a factorization is unique up to isomorphism,
- if θ is an isomorphism and $\theta f = f$ for some $f \in \mathcal{R}^-$, then θ is an identity.

The central result in [BM11] states that:

- (1) the latching and matching objects at $r \in \mathcal{R}$ of some $X : \mathcal{R} \rightarrow \mathcal{M}$ are defined as in the classical case, but now the automorphism group $\text{Aut}(r)$ acts on them, so that $L_r X$ and $M_r X$ are objects of $[\text{Aut}(r), \mathcal{M}]$ rather than mere objects of $\mathcal{M}$.
- (2) suppose $\mathcal{M}$ such that every $[\text{Aut}(r), \mathcal{M}]$ bears the projective model structure, and define Reedy cofibrations, Reedy fibrations and Reedy weak equivalences as usual but considering the usual induced maps $X_r \sqcup_{L_r X} L_r Y \rightarrow Y_r$ and $X_r \rightarrow Y_r \times_{M_r Y} M_r X$ in $[\text{Aut}(r), \mathcal{M}]$, not in $\mathcal{M}$.
- (3) then Reedy cofibrations, Reedy fibrations and Reedy weak equivalences give $[\mathcal{R}, \mathcal{M}]$ a model structure.

The result can be established in that language by constructing a tower of Quillen bifibrations, each of them obtained by applying Theorem 5.9 on the canonical natural transformation α between $L_r, M_r : [\mathcal{R}_\mu, \mathcal{M}] \rightarrow [\text{Aut}(r), \mathcal{M}]$ whenever r is of degree μ . In particular, we see the benefit here of varying the codomain category $\mathcal{N}$ in each successor step.

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