



Bifibrations of model categories (and the Reedy construction)

Category Theory 2016

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1. Liminaries

2. Bifibration of model categories

3. Comparison and applications



Liminares

Grothendieck bifibrations



A Grothendieck bifibration is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that

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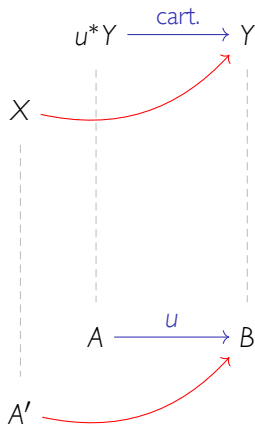
Grothendieck bifibrations

A Grothendieck bifibration is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that

$$\begin{array}{ccc} u^*Y & \xrightarrow{\text{cart.}} & Y \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

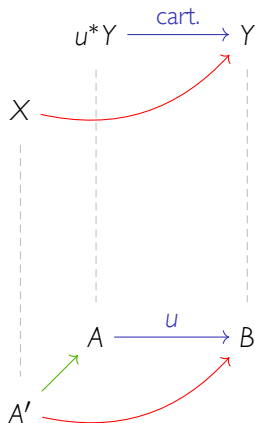
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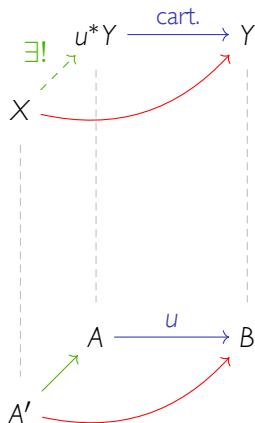
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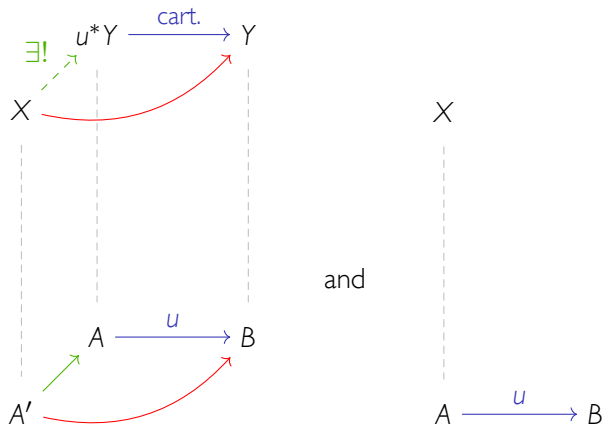
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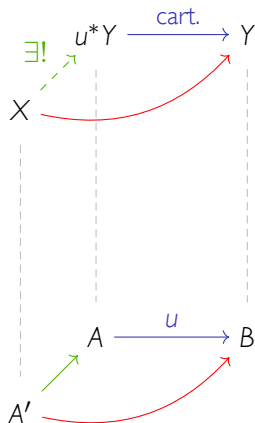
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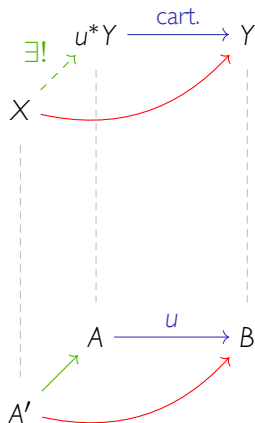


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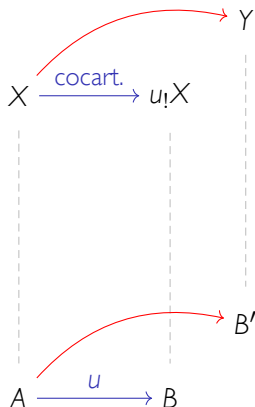


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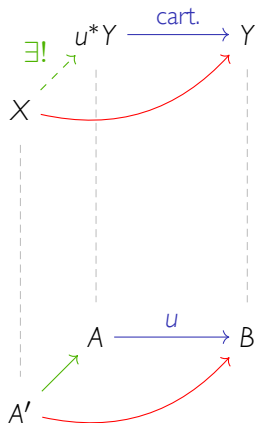


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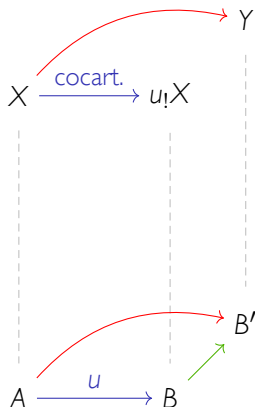


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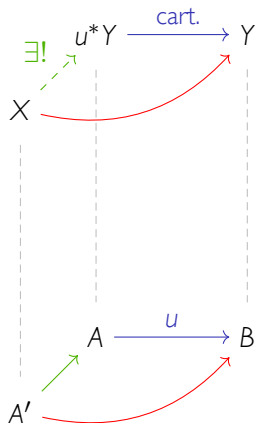


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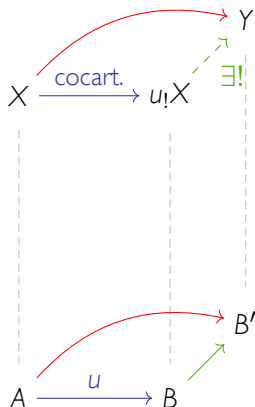


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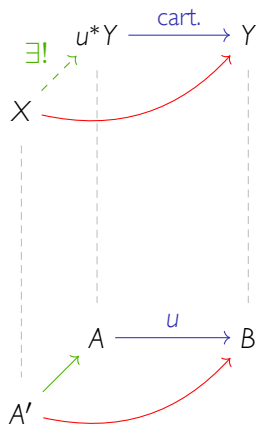


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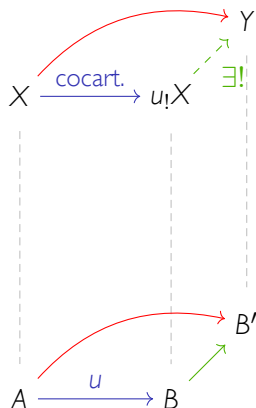


Grothendieck bifibrations

A cloven Grothendieck bifibration is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ together with



and



Push and pull

For any $f : X \rightarrow Y$ in $\mathcal{E}$ over $u : A \rightarrow B$,

$$\begin{array}{ccc} u^*Y & \xrightarrow{\quad} & Y \\ f_{\triangleleft} \uparrow & \nearrow f & \uparrow f_{\triangleright} \\ X & \xrightarrow{\quad} & u_!X \end{array}$$

Push and pull

For any $f : X \rightarrow Y$ in $\mathcal{E}$ over $u : A \rightarrow B$,

A commutative square diagram illustrating the relationship between a morphism f and its pullback and pushforward. The top-left node is u^*Y , the top-right node is Y , the bottom-left node is X , and the bottom-right node is $u_!X$. A blue arrow points from u^*Y to Y . A blue arrow points from X to $u_!X$. A green arrow labeled $f_{\triangleleft}$ points from X to u^*Y . A green arrow labeled $f_{\triangleright}$ points from $u_!X$ to Y . A red diagonal arrow labeled f points from X to Y .

Hence $u_!$ and u^* extend to functors:

A diagram showing a morphism k from X to X' . The node X is at the bottom, and the node X' is at the top. A vertical arrow points from X to X' , with the label k placed to the left of the arrow.

Push and pull

For any $f : X \rightarrow Y$ in $\mathcal{E}$ over $u : A \rightarrow B$,

$$\begin{array}{ccc} u^*Y & \xrightarrow{\quad} & Y \\ f_! \uparrow & \searrow f & \uparrow f_\# \\ X & \xrightarrow{\quad} & u_!X \end{array}$$

Hence $u_!$ and u^* extend to functors:

$$\begin{array}{ccc} X' & \xrightarrow{c_{X'}} & u_!X' \\ \uparrow k & & \\ X & \xrightarrow{c_X} & u_!X \end{array}$$

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$$\begin{array}{ccc}
 X' & \xrightarrow{c_{X'}} & u_!X' \\
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$$\begin{array}{ccc}
 u^*Y' & \xrightarrow{c^{Y'}} & Y' \\
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Hence $u_!$ and u^* extend to functors:

$$\begin{array}{ccc} X' & \xrightarrow{c_{X'}} & u_!X' \\ k \uparrow & \xrightarrow{u_!} & \uparrow (c_{X'}k)_{\triangleright} \\ X & \xrightarrow{c_X} & u_!X \end{array}$$

$$\begin{array}{ccc} u^*Y' & \xrightarrow{c^{Y'}} & Y' \\ (kc^Y)_{\triangleright} \uparrow & \xleftarrow{u^*} & \uparrow k \\ u^*Y & \xrightarrow{c^Y} & Y \end{array}$$

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 \end{array}$$

This produces an adjunction $u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ between fibers.

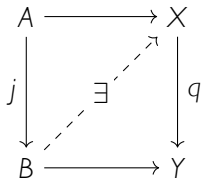
Weak factorization system

In a category $\mathcal{M}$, denote $j \boxdot q$ when for any

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

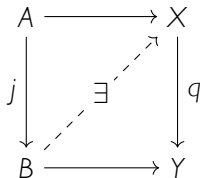
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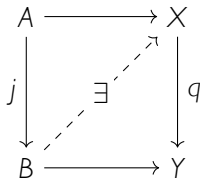


Define $\mathfrak{L}^\boxdot$ as the class of q such that $j \boxdot q$ for all $j \in \mathfrak{L}$.

Define $^\boxdot\mathfrak{R}$ as the class of j such that $j \boxdot q$ for all $q \in \mathfrak{R}$.

Weak factorization system

In a category $\mathcal{M}$, denote $j \sqsupseteq q$ when for any



Define $\mathfrak{L}^\sqsupseteq$ as the class of q such that $j \sqsupseteq q$ for all $j \in \mathfrak{L}$.

Define $\sqsupseteq\mathfrak{R}$ as the class of j such that $j \sqsupseteq q$ for all $q \in \mathfrak{R}$.

Definition

A **weak factorisation system** on a category $\mathcal{M}$ is a couple $(\mathfrak{L}, \mathfrak{R})$ such that

- $\mathfrak{L} = \sqsupseteq\mathfrak{R}$ and $\mathfrak{L}^\sqsupseteq = \mathfrak{R}$
- each morphism factors as $q \circ j$ for $j \in \mathfrak{L}$ and $q \in \mathfrak{R}$.

Definition

A **model category** is a complete and cocomplete category $\mathcal{M}$ together with **Cof**, **W**, **Fib** such that

- **W** has 2-out-of-3,
- $(\mathbf{Cof} \cap \mathbf{W}, \mathbf{Fib})$ and $(\mathbf{Cof}, \mathbf{W} \cap \mathbf{Fib})$ are weak factorisation systems.

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An adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is **Quillen** when:

$$L(\mathbf{Cof}_{\mathcal{M}}) \subseteq \mathbf{Cof}_{\mathcal{N}} \quad R(\mathbf{Fib}_{\mathcal{N}}) \subseteq \mathbf{Fib}_{\mathcal{M}}$$

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Bifibration of model categories

Lifting weak factorisation systems

Define a **wfs-adjunction** as an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ such that

$$L(\mathfrak{L}_{\mathcal{M}}) \subseteq \mathfrak{L}_{\mathcal{N}} \quad R(\mathfrak{R}_{\mathcal{N}}) \subseteq \mathfrak{R}_{\mathcal{M}}$$

Lifting weak factorisation systems

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Proposition (folklore)

Given a bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ with a wfs $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{B}$ and a wfs $(\mathfrak{L}_A, \mathfrak{R}_A)$ on each fiber $\mathcal{E}_A$, if each pair $(u_!, u^*)$ is a wfs-adjunction, the following classes yield a **wfs** on $\mathcal{E}$:

$$\mathfrak{L}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{L}, f_{\triangleright} \in \mathfrak{L}_B\} \quad \mathfrak{R}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{R}, f_{\triangleleft} \in \mathfrak{R}_A\}$$

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$$\begin{array}{ccc} & & Y \\ & \nearrow f & \uparrow \\ X & \rightarrow & u_! X \end{array}$$

$$A \xrightarrow{u} B$$

$$\begin{array}{ccc} u^* Y & \rightarrow & Y \\ \uparrow & \nearrow f & \\ X & & \end{array}$$

$$A \xrightarrow{u} B$$

General principle



From now on, $p : \mathcal{E} \rightarrow \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(u_!, u^*)$ is Quillen.

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Call f in $\mathcal{E}$

- a **total cofibration** if $p(f) \in \mathbf{Cof}$ and $f_{\triangleright} \in \mathbf{Cof}_B$,
- a **total fibration** if $p(f) \in \mathbf{Fib}$ and $f_{\triangleleft} \in \mathbf{Fib}_A$.

Denote $\mathbf{Cof}_{\mathcal{E}}$ and $\mathbf{Fib}_{\mathcal{E}}$ the corresponding classes.

From now on, $p : \mathcal{E} \rightarrow \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(u_!, u^*)$ is Quillen.

Call f in $\mathcal{E}$

- a **total acyclic cofibration** if $p(f) \in \mathbf{Cof} \cap \mathbf{W}$ and $f_{\triangleright} \in \mathbf{Cof}_B \cap \mathbf{W}_B$,
- a **total acyclic fibration** if $p(f) \in \mathbf{Fib} \cap \mathbf{W}$ and $f_{\triangleleft} \in \mathbf{Fib}_A \cap \mathbf{W}_A$.

Denote $\mathbf{Cof}_{\mathcal{E}}$, $\mathbf{WCof}_{\mathcal{E}}$ and $\mathbf{Fib}_{\mathcal{E}}$, $\mathbf{WFib}_{\mathcal{E}}$ the corresponding classes.

From now on, $p : \mathcal{E} \rightarrow \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(u_!, u^*)$ is Quillen.

Call f in $\mathcal{E}$

- a **total acyclic cofibration** if $p(f) \in \mathbf{Cof} \cap \mathbf{W}$ and $f_{\triangleright} \in \mathbf{Cof}_B \cap \mathbf{W}_B$,
- a **total acyclic fibration** if $p(f) \in \mathbf{Fib} \cap \mathbf{W}$ and $f_{\triangleleft} \in \mathbf{Fib}_A \cap \mathbf{W}_A$.

Denote $\mathbf{Cof}_{\mathcal{E}}$, $\mathbf{WCof}_{\mathcal{E}}$ and $\mathbf{Fib}_{\mathcal{E}}$, $\mathbf{WFib}_{\mathcal{E}}$ the corresponding classes.

Key observation

If we find a class $\mathbf{W}_{\mathcal{E}}$ of **total weak equivalences** such that

- $\mathbf{W}_{\mathcal{E}}$ has 2-out-of-3,
- $\mathbf{WCof}_{\mathcal{E}} = \mathbf{Cof}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$ and $\mathbf{WFib}_{\mathcal{E}} = \mathbf{Fib}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$.

then $\mathbf{Cof}_{\mathcal{E}}, \mathbf{W}_{\mathcal{E}}, \mathbf{Fib}_{\mathcal{E}}$ is a model structure on $\mathcal{E}$.

Total weak equivalences



Main Idea

You don't have a choice for the total weak equivalences.

Total weak equivalences



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You don't have a choice for the total weak equivalences.

Call $f : X \rightarrow Y$ **acyclic relatively** to (u, v) if $p(f) = vu$ with $u \in \mathbf{Cof} \cap \mathbf{W}$ and $v \in \mathbf{Fib} \cap \mathbf{W}$, and $u_!X \rightarrow v^*Y \in \mathbf{W}_C$.

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$$\begin{array}{ccc} & v^*Y & \longrightarrow Y \\ & \uparrow & \nearrow \\ X & \longrightarrow & u_!X \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{u} & C & \xrightarrow{v} & B \\ & \searrow & & \nearrow & \\ & & & & \end{array}$$

Total weak equivalences



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$$\begin{array}{ccc} & v^*Y & \longrightarrow Y \\ & \uparrow & \nearrow \\ X & \longrightarrow & u_!X \end{array}$$

Define f to be a **total weak equivalence** if it is acyclic relatively to some pair.

$$\begin{array}{ccccc} A & \xrightarrow{u} & C & \xrightarrow{v} & B \\ & & & & \nearrow \\ & & & & \text{dashed arc from } A \text{ to } B \end{array}$$

H_1 $u_!$ preserves and reflects weak equivalences if $u \in \mathbf{Cof} \cap \mathbf{W}$.

Necessary conditions

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Necessity.

$$\begin{array}{ccc} X' & \longrightarrow & u_! X' \\ \uparrow k & & \uparrow u_! k \\ X & \longrightarrow & u_! X \end{array}$$

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$$\begin{array}{ccc} X' & \longrightarrow & u_! X' \\ \uparrow k & & \uparrow u_! k \\ X & \longrightarrow & u_! X \end{array} \quad \text{with a curved arrow } \mathbf{1} \text{ from } u_! X \text{ to } u_! X$$

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A commutative square diagram illustrating the necessity of the condition. The diagram consists of four nodes arranged in a square:

- Top-left node: X'
- Top-right node: $u_!X'$
- Bottom-left node: X
- Bottom-right node: $u_!X$

The arrows connecting the nodes are:

- A vertical arrow from X to X' labeled k .
- A horizontal arrow from X to $u_!X$ labeled $u_!$ (in blue).
- A vertical arrow from $u_!X$ to $u_!X'$ labeled $u_!k$.
- A horizontal arrow from X' to $u_!X'$.

There is a curved arrow labeled **1** pointing from $u_!X'$ back to $u_!X'$, representing the identity morphism.

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$$\begin{array}{ccc} v^*Y' & \longrightarrow & Y' \\ \uparrow v^*k & & \uparrow k \\ v^*Y & \longrightarrow & Y \end{array}$$

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hBC every square of the form

$$\begin{array}{ccc} A & \xrightarrow{u'} & C' \\ v \downarrow & & \downarrow v' \\ C & \xrightarrow{u} & B \end{array}$$

$$u, u' \in \mathbf{Cof} \cap \mathbf{W}$$

$$v, v' \in \mathbf{Fib} \cap \mathbf{W}$$

satisfies the **homotopical Beck-Chevalley** condition, i.e. the mate $u'_! v^* \rightarrow v'^* u_!$ is pointwise in $\mathbf{W}_{C'}$.

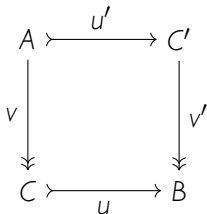
Necessary conditions

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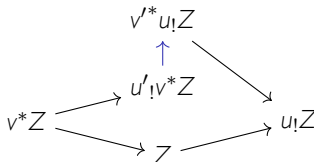


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Necessity.



Theorem

The bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$ satisfies H_1 , H_2 and hBC if and only if $\mathbf{Cof}_{\mathcal{E}}$, $\mathbf{W}_{\mathcal{E}}$, $\mathbf{Fib}_{\mathcal{E}}$ give a model structure on $\mathcal{E}$ where

- $\mathbf{Cof}_{\mathcal{E}} = \{f : p(f) \in \mathbf{Cof}, f_{\triangleright} \in \mathbf{Cof}_{\mathcal{B}}\}$
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What about completeness and cocompleteness of $\mathcal{E}$?

Limits and colimits in $\mathcal{E}$ can be constructed from those in the fibers and in the base.

3

Comparison and applications

Warm up example: the codomain bifibration

15

Given a category $\mathcal{C}$, the functor $\text{cod} : \text{Arr}(\mathcal{C}) \rightarrow \mathcal{C}$, $(A' \rightarrow A) \mapsto A$ is a bifibration where pulls and pushes are given as follow:

$$\begin{array}{ccc} A' & & \\ f \downarrow & & \\ A & \xrightarrow{u} & B \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

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If $\mathcal{C}$ is a model category, each fiber $\mathbf{cod}_A \simeq \mathcal{C}_{/A}$ inherit a canonical model structure. The conditions H_1 , H_2 and $\mathbf{hBC}$ are satisfied, hence we get a model structure on $\mathbf{Arr}(\mathcal{C})$...

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- *Model category structures in bifibred categories* (1994) Roig
Bifibrations and Weak Factorisation Systems (2012) Stanculescu

Defines $\mathbf{W}_{\mathcal{E}}$ as those f such that $p(f) \in \mathbf{W}$ and $f_{\triangleleft} \in \mathbf{W}_A$.

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The Reedy construction...

17

If $\mathcal{R}$ is Reedy, $\mathcal{R}_\lambda$ is the full subcategory with objects of degree $< \lambda$.

The restriction $p_\lambda : [\mathcal{R}_{\lambda+1}, \mathcal{M}] \rightarrow [\mathcal{R}_\lambda, \mathcal{M}]$ is a bifibration where:

$$\begin{array}{ccc} M_\lambda A & \xrightarrow{M_\lambda u} & M_\lambda B \\ \uparrow & & \uparrow \\ & X & \nearrow Y \\ \uparrow & & \uparrow \\ L_\lambda A & \xrightarrow{L_\lambda u} & L_\lambda B \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

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The fiber at A is equivalent to

$$\prod_{r \in \mathcal{R}, \deg(r) = \lambda} L_r A \setminus \mathcal{M} / M_r A$$

hence is a model category.

If $[\mathcal{R}_\lambda, \mathcal{M}]$ has the Reedy model structure, we then obtain a model structure on $[\mathcal{R}_{\lambda+1}, \mathcal{M}]$...

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If $[\mathcal{R}_\lambda, \mathcal{M}]$ has the Reedy model structure, we then obtain a model structure on $[\mathcal{R}_{\lambda+1}, \mathcal{M}]$... which is again the Reedy model structure.

In *Reedy categories and their generalizations* (2015) by Mike Shulman:

Theorem 3.11

If $\mathcal{M}$ and $\mathcal{N}$ model categories, $F, G : \mathcal{M} \rightarrow \mathcal{N}$ and $\alpha : F \rightarrow G$ st

- F cocont. and maps $\mathbf{Cof} \cap \mathbf{W}$ to couniversal weak equivalences,
- G cont. and maps $\mathbf{Fib} \cap \mathbf{W}$ to universal weak equivalences,

then the biglueing $\mathbf{Gl}(\alpha)$ is a model category.

The functor $\mathbf{Gl}(\alpha) \rightarrow \mathcal{M}$ is a bifibration and $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{hBC}$ are satisfied, yielding the same model structure as in the article.

Thank you.

`http://www.normalesup.org/~cagne/`
`https://pierrecagne.github.io`

Sketch of proof: $\mathbf{WCof}_{\mathcal{E}} = \mathbf{Cof}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$

19

A total acyclic cofibration f is a cofibration and is acyclic relatively to $(w, \mathbf{1}_B)$. Hence:

$$\mathbf{WCof}_{\mathcal{E}} \subset \mathbf{Cof}_{\mathcal{E}} \cap \mathbf{W}_{\mathcal{E}}$$

$$\begin{array}{ccc} & Y & \xlongequal{\quad} Y \\ & \uparrow f & \nearrow \\ X & \xrightarrow{\quad} & w!X \end{array}$$

$$A \xrightarrow{\quad w \quad} B \xlongequal{\quad} B$$

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Conversely, if f is a cofibration acyclic relatively to (u, v) , then it is also relatively to $(w, \mathbf{1}_B)$. Hence $f_{\triangleright}$ weak, making f into a total acyclic cofibration:

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$$A \xrightarrow{w} B \xrightarrow{\quad} B$$

$$\begin{array}{ccc} & v^*Y & \xrightarrow{\quad} Y \\ & \uparrow & \uparrow f_{\triangleright} \\ X & \xrightarrow{\quad} & w_!X \\ & \searrow & \\ & u_!X & \end{array}$$

$$\begin{array}{ccc} & A & \xrightarrow{w} B \\ & \nearrow & \searrow \\ A & \xrightarrow{u} & C \xrightarrow{v} B \end{array}$$

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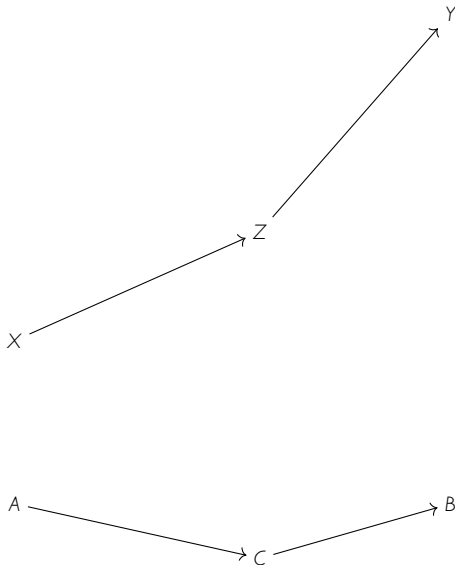
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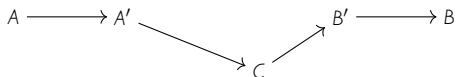
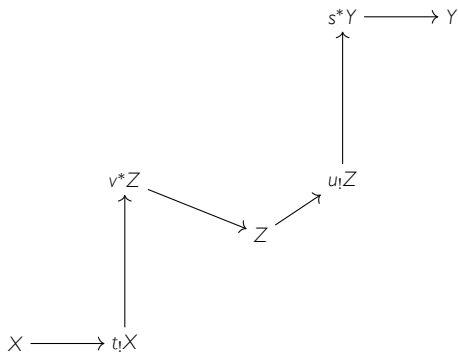
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Sketch of proof: 2-out-of-3



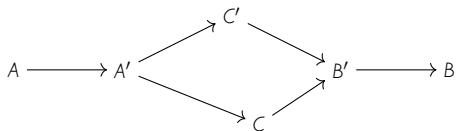
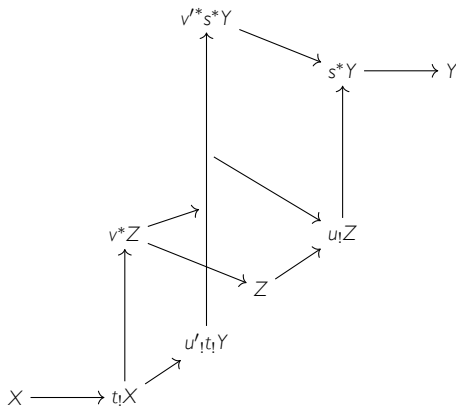
Sketch of proof: 2-out-of-3

19



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19



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19

