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Towards a homotopical algebra of dependent types

par Pierre Cagne

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Abstract

Towards a homotopical algebra of dependent types

This thesis is concerned with the study of the interplay between homotopical structures and categorical model of Martin-Löf's dependent type theory. The memoir revolves around three big topics: Quillen bifibrations, homotopy categories of Quillen bifibrations, and generalized tribes. The first axis defines a new notion of bifibrations, that classifies correctly behaved pseudo functors from a model category to the 2-category of model categories and Quillen adjunctions between them. In particular it endows the Grothendieck construction of such a pseudo functor with a model structure. The main theorem of this section acts as a characterization of the well-behaved pseudo functors that tolerates this "model Grothendieck construction". In that respect, we improve the two previously known theorems on the subject in the literature that only give sufficient conditions by designing necessary and sufficient conditions. The second axis deals with the functors induced between the homotopy categories of the model categories involved in a Quillen bifibration. We prove that this localization can be performed in two steps, by means of Quillen's construction of the homotopy category in an iterated fashion. To that extent we need a slightly larger framework for model categories than the one originally given by Quillen: following Egger's intuitions we chose not to require the existence of equalizers and coequalizers in our model categories. The background chapter makes sure that every usual fact of basic homotopical algebra holds also in that more general framework. The structures that are highlighted in that chapter call for the design of notions of "homotopical pushforward" and "homotopical pullback". This is achieved by the last axis: we design a structure, called relative tribe, that allows for a homotopical version of cocartesian morphisms by reinterpreting Grothendieck (op)fibrations in terms of lifting problems. The crucial tool in this last chapter is given by a relative version of orthogonal and weak factorization systems. This allows for a tentative design of a new model of intentional type theory where the identity types are given by the exact homotopical counterpart of the usual definition of the equality predicate in Lawvere's hyperdoctrines.

Keywords : model category, homotopical algebra, Grothendieck bifibration, dependent type theory, homotopy type theory.

Résumé

VERS UNE ALGÈBRE HOMOTOPIQUE DES TYPES DÉPENDANTS

Cette thèse est consacrée à l'étude des interactions entre les structures homotopiques en théorie des catégories et les modèles catégoriques de la théorie des types de Martin-Löf. Le mémoire s'articule selon trois axes : les bifibrations de Quillen, les catégories homotopiques des bifibrations de Quillen, et les tribus généralisées. Le premier axe définit une nouvelle notion de bifibration classifiant les pseudo foncteurs avec de bonnes propriétés depuis un catégorie de modèles et à valeurs dans la 2-catégorie des catégories de modèles et adjonctions de Quillen entre elles. En particulier on montre comment équiper d'une structure de catégorie de modèles la construction de Grothendieck d'un tel pseudo foncteur. Le théorème principal de cette partie est une caractérisation des bonnes propriétés qu'un pseudo foncteur doit posséder pour supporter cette structure de catégorie de modèles sur sa construction de Grothendieck. En ce sens, on améliore les deux théorèmes précédemment existants dans la littérature qui ne donnent que des conditions suffisantes alors que nous donnons des conditions nécessaires et suffisantes. Le second axe se concentre sur le foncteur induit entre les catégories homotopiques des catégories de modèles mises en oeuvre dans une bifibration de Quillen. On y prouve que cette localisation peut se faire en deux étapes au moyen d'un quotient homotopique à la Quillen itéré. De manière à rendre cette opération rigoureuse, on a besoin de travailler dans un cadre légèrement plus large que celui imaginé par Quillen initialement : en se basant sur le travail d'Egger, on utilise des catégories de modèles sans nécessairement tous les (co)égalisateurs. Le chapitre de pré-requis sert précisément à reconstruire la théorie basique de l'algèbre homotopique à la Quillen dans ce cadre élargi. Les structures mises à nues dans cette partie imposent de considérer des versions "homotopique" des poussés en avant et des tirés en arrière qu'on trouve habituellement dans les (op)fibrations de Grothendieck. C'est le point de départ pour le troisième axe, dans lequel on définit une nouvelle structure, appelée tribu relative, qui permet d'axiomatiser des versions homotopiques de la notion de flèche cartésienne et cocartésienne. Cela est obtenu en réinterprétant les (op)fibrations de Grothendieck en termes de problèmes de relèvement. L'outil principal dans cette partie est une version relative des systèmes de factorisation stricts ou faibles usuels. Cela nous permet en particulier d'expérimenter un nouveau modèle de la théorie des types dépendants intentionnelle dans lequel les types identités sont donnés par l'exact analogue homotopique du prédicat d'égalité dans les hyperdoctrines de Lawvere.

Mots-clefs : catégorie de modèles, algèbre homotopique, bifibration de Grothendieck, théorie des types dépendants, théorie des types homotopique.

À ma mamie

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Comme de coutume, je tiens à commencer par des remerciements. Une manière de préface au manuscrit[1], une façon d'épilogue à mes années d'étudiant. Ces quelques paragraphes, outre qu'ils seront probablement la partie la plus lue de mon mémoire, scellent le caractère profondément humain du travail de recherche. J'y délaisse quelque peu la solennité habituelle de la littérature scientifique et y adopte une prose plus adaptée au message que j'aimerais y faire passer : la mathématique est avant tout une activité sociale. Dissocier l'individu qui la pratique des interactions qu'il entretient à l'intérieur et à l'extérieur de la recherche, c'est le diminuer de moitié.

Il me faut en premier lieu remercier mes directeurs de thèse sans qui ces trois années de recherche n'auraient pu exister. Merci Paul-André pour ton enthousiasme scientifique qui m'a poussé plus d'une fois à explorer des notions que je pensais naïvement trop pauvres en structures pour pouvoir être intéressantes. J'espère pouvoir prolonger à l'avenir nos « disputes » mathématiques, qui m'ont apporté jusqu'alors mille et une petites lumières que je peux désormais insérer dans l'éclairage que j'essaie de former des différents concepts de nos domaines de recherche. Si je ne devais néanmoins retenir qu'un seul enseignement, ce serait à coup sûr celui de la recherche de la *structure juste* : non pas en accumulant sans arrêt sur les structures déjà existantes mais au contraire en les déconstruisant pour n'en retenir que l'essentiel. Merci Clemens pour la gentillesse avec laquelle tu as su me guider. Tu m'as certes encadré de plus loin, mais tes remarques scientifiques sont toujours arrivées à point nommé et leur pertinence m'ont évité bien des errances. Les qualités que tu développes à travers ta recherche sont de celles que j'aimerais un jour pouvoir dire qu'elles sont miennes, et tes travaux sont une grande source d'inspiration pour l'apprenti-chercheur que je suis.

J'aimerais poursuivre en remerciant mes rapporteurs, qui ont eu le travail ingrat de relire minutieusement mon mémoire. D'aucun en recherche fondamentale sait quel

[1]Anachronisme auquel le thésard semble tenir. Probablement pour souligner que le document final est tout autant sang et sueur aujourd'hui qu'hier, quand bien même le crissement de la plume sur le papier a été remplacé par la danse éthérée des doigts sur le clavier.

travail cela implique, de relire preuves et arguments pour en vérifier la véracité et la rigueur là où il est tellement tentant de faire confiance à l'auteur en se concentrant sur le résultat. Un grand merci à André Joyal qui m'a fait l'honneur d'accepter de rapporter ce mémoire. Let me temporarily switch to english to thank Thomas Streicher in a way that my french might not convey properly to him. Your dedication in the reading of my memoir was a very big help to me and I deeply appreciate the amount of work that it represents.

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Quel est ce phénomène étrange qui pousse le thésard en informatique théorique, cette bête bizarre qui pourrait se suffire d'une connexion internet, à venir tous les jours s'asseoir au coin de bureau qui lui est alloué dans le laboratoire ? Bien sûr la bande passante du réseau RENATER est probablement un facteur, mais il y a fort à parier qu'il recherche également la présence de ses pairs ! Les fameux « compagnons de galère » avec qui il peut partager ses peurs et ses espérances, ses colères et ses angoisses, ses utopies et ses désillusions. Un grand merci à tous ceux qui se sont relayés dans ce rôle au fur et à mesure. Tout d'abord le « bureau des autistes » dont j'ai fièrement fait parti dès mon stage de master : Amina, Charles, Clément, Marie et Yann ont du supporter dès cette époque mes calembours vaseux et mes sautes d'humeur politiques. Merci en particulier à Charles et Clément, « grands frères » de thèse on ne peut plus attentionnés. Merci à Amina pour ses rêves fous de devenir un baron colombien de la drogue (à défaut tu pourras toujours être une chercheuse à succès). Merci Yann pour les soirées (trop ?) arrosées de l'ÉPIT et ton houmous à tomber. Merci Marie d'avoir eu la bonne idée de soutenir moins de 2 mois avant moi et de m'avoir ainsi évité toutes les galères administratives en me guidant pas à pas. Merci aux autres qui se sont succédés dans ce bureau : Gabriel, le magicien du GIT qui m'a sorti nombre de fois une *branch* du pied ; Léo, dont le flegme incroyable n'a d'égal que la faiblesse de son volume sonore (tu me diras à nous deux on faisait une moyenne) ; Rémi, qui est sans doute en train de réinstaller Debian ; Paulina, qui a eu la patience infinie de nous supporter. Merci au « bureau des matheux » d'avoir apporter toujours un peu plus de catégories dans ma vie : Maxime, pour tes questions qui au final m'en apprenaient plus que ce que je pouvais apporter de réponses ; Cyril, pour avoir élargi mon spectre musical ; Léonard, pour les « Eh, tu connais ce petit lemme ? » qui finissent en discussion de deux heures ; Zeinab, pour avoir repris peu ou proue le sujet de thèse que je n'ai pas eu le courage d'affronter ; j'annexe à ce bureau Daniel, thésard par procuration, l'étrangeté de nos échanges a toujours été une bouffée d'oxygène bien *réelle* (je te le laisse prouver en COQ). Merci aussi à mes « petits frères de thèse » Léo, Axel et Chaitanya : nos discussions sont parties intégrantes de mes travaux à bien des égards. J'ajoute volontiers Kenji à cette catégorie (sic), membre honoraire de l'IRIF qui *pop* ici et là lors de la présence de gâteaux et/ou de théories de Lawvere. Merci aussi à la CAKE team pour son souci de la nutrition des doctorants le jeudi après-midi. Enfin, merci au reste des thésards et post-docs, en particulier à ceux qui prennent volontairement la charge d'organiser le séminaire thésards. Une mention spéciale pour Sacha, qui m'a permis à quelques reprises d'être un transfuge du 3ème pour retrouver les matheux du 6ème et 7ème étage.

Let's go back to english for a minute and allow me to thanks all the participants of the AARMS Summer School in Halifax and the CT conference that has followed. I have spent a marvelous month there, both mathematically and socially. Special thanks to Ramón, Branko (the *professional*), Giulia, Eva, Chris and the famous interpol-wanted Alexander. I also want to thank the participants of the online Kan extension seminar II: David, Ze, Daniel, Maru, Eva (again!), Simon and José. Besides the amazing talks you presented, I feel that I have learned a great deal from our "reading responses". That was a fabulous experience.

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[2]Sérieux ? On l'avait jamais faite celle-là ?

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[3] « Mais tu sais déjà pas trouver tes chaussettes dans ton placard mon fils… »

Contents

Prenons par exemple la tâche de démontrer un théorème qui reste hypothétique (à quoi, pour certains, semblerait se réduire le travail mathématique). Je vois deux approches extrêmes pour s'y prendre. L'une est celle du *marteau et du burin*, quand le problème posé est vu comme une grosse noix, dure et lisse, dont il s'agit d'atteindre l'intérieur, la chair nourricière protégée par la coque. Le principe est simple : on pose le tranchant du burin contre la coque, et on tape fort. Au besoin, on recommence en plusieurs endroits différents, jusqu'à ce que la coque se casse – et on est content. Cette approche est surtout tentante quand la coque présente des aspérités ou protubérances, par où « la prendre ». Dans certains cas, de tels « bouts » par où prendre la noix sautent aux yeux, dans d'autres cas, il faut la retourner attentivement dans tous les sens, la prospecter avec soin, avant de trouver un point d'attaque. Le cas le plus difficile est celui où la coque est d'une rotondité et d'une dureté parfaite et uniforme. On a beau taper fort, le tranchant du burin patine et égratigne à peine la surface – on finit par se lasser à la tâche. Parfois quand même on finit par y arriver, à force de muscle et d'endurance.

Je pourrais illustrer la deuxième approche, en gardant l'image de la noix qu'il s'agit d'ouvrir. La première parabole qui m'est venue à l'esprit tantôt, c'est qu'on plonge la noix dans un liquide émollient, de l'eau simplement pourquoi pas, de temps en temps on frotte pour qu'elle pénètre mieux, pour le reste on laisse faire le temps. La coque s'assouplit au fil des semaines et des mois – quand le temps est mûr, une pression de la main suffit, la coque s'ouvre comme celle d'un avocat mûr à point ! Ou encore, on laisse mûrir la noix sous le soleil et sous la pluie et peut-être aussi sous les gelées de l'hiver. Quand le temps est mûr c'est une pousse délicate sortie de la substantifique chair qui aura percé la coque, comme en se jouant – ou pour mieux dire, la coque se sera ouverte d'elle-même, pour lui laisser passage.

Alexander Grothendieck
Récoltes et Semailles

Résumé de la thèse

Ce chapitre constitue un résumé substantiel en français de ma thèse. Il permet de présenter les principaux résultats du mémoire aux lecteurs non anglophones.

Ma thèse est consacrée à l'élaboration et à l'étude de nouveaux cadres sémantiques pour la théorie des types dépendants intentionnelle de Martin-Löf (abrégée en MLTT par la suite). L'interprétation homotopique de MLTT par Voevosdky et les intuitions développées par Awodey et Warren sur le type identité vu comme objet en chemin ont porté la communauté à interpréter les types dépendants comme des fibrations dans une catégorie portant quelque structure homotopique (catégorie de modèles, catégorie à fibrations de Brown, tribu, etc.). Ces modèles ont l'avantage de former une continuité immédiate avec l'interprétation de Seely de la théorie des type dépendants extensionnelle dans les catégories localement cartésiennes fermées, et de ce fait profite de tous les travaux menés sur les catégories à *display maps*. En revanche, les règles sous-structurelles de la théorie des types sont chevillées au corps de ces modèles : ils sont donc inadaptés à l'étude (et à la remise en question) de règles telles que l'affaiblissement, la contraction, etc.

Dans le cadre de la logique du premier ordre, la même obstruction apparaît quand on décide d'interpréter les prédicats comme des sous-objets dans une catégorie de Heyting $\mathcal{C}$. La notion d'hyperdoctrine de Lawvere lève cette obstruction : au lieu d'interpréter les prédicats comme des sous-objets, Lawvere propose d'utiliser n'importe quel pseudo foncteur $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ qui arbore les même propriétés fondamentales que le pseudo foncteur des sous-objets

$$\mathrm{Sub}(-) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat},\ A \mapsto \mathrm{Sub}(A)$$

En précisant quelles sont les propriétés fondamentales qui nous intéressent, on peut libérer la logique de certaines règles ou au contraire en entériner d'autres. En particulier, si l'on requiert que $\mathcal{C}$ admet les produits finis, que chaque $P(A)$ admet un objet terminal 1_A et que $P(f)$ admet un adjoint à gauche $\exists^P_f$ pour chaque flèche f de $\mathcal{C}$, alors on peut définir pour P le prédicat d'égalité sur $A \in \mathcal{C}$ comme l'image $\exists^P_f(1_A)$ dans $P(A \times A)$ où $\Delta_A : A \longrightarrow A \times A$ est la diagonale de A dans $\mathcal{C}$. Dans le cas particulier de $P = \mathrm{Sub}(-)$, ce prédicat d'égalité est tautologiquement le monomorphisme (ou plutôt sa classe d'équivalence) $\Delta_A : A \longrightarrow A \times A$.

Ma thèse est fondée sur la volonté de réconcilier les modèles de MLTT à base de fibrations avec la sémantique Lawverienne des hyperdoctrines. Plus précisément, on cherche à effectuer un saut conceptuel similaire à celui de Lawvere dans le cadre de la sémantique de MLTT avec l'objectif d'exprimer le type identité intentionnel de manière similaire au prédicat d'égalité explicité ci-dessus. Le mémoire s'organise en trois temps que l'on décrit plus en détails ci-après :

- une étude de l'interaction entre catégories de modèles (et en particulier leurs objets en chemin) et bifibrations de Grothendieck (une autre manière de voir les pseudo foncteurs $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ avec adjoint à gauche pour chaque $P(f)$),

- une poursuite de l'étude précédente à travers les structures dérivées induites faisant apparaître des analogues "à homotopie près" de l'interprétation catégorique des quantificateurs,

- enfin une tentative de réconcilier les tribus de Joyal (une instance de modèles de MLTT à base de fibrations) avec les idées de Lawvere exposées plus haut.

Bifibrations de Quillen

Le chapitre 3 développe une structure nouvelle : les bifibrations de Quillen. Inspiré par les travaux de Roig ([Roi94], Stanculescu ([Sta12]) et Harpaz et Prasma ([HP15]), cette notion répond à la l'interrogation suivante : étant donnée une bifibration de Grothendieck $p : \mathcal{E} \longrightarrow \mathcal{B}$ dont la base $\mathcal{B}$ admet une structure de catégorie de modèles ainsi que toutes les fibres $\mathcal{E}_A$ ($A \in \mathcal{B}$), à quelles conditions peut-on relever ces données en une structure de catégorie de modèles sur $\mathcal{E}$? Les travaux cités précédemment proposent des conditions suffisantes pour une telle construction. Nous proposons un raffinement de ces résultats en fournissant des conditions nécessaires et suffisantes.

Définition (Definition 3.1.2)**.** Une bifibration de Quillen est une bifibration de Grothendieck $p : \mathcal{E} \longrightarrow \mathcal{B}$ entre catégories munie de structures de catégorie de modèles telle que :

(i) la structure de catégorie de modèles de $\mathcal{E}$ se restreint à chacune des fibres, c'est-à-dire que pour un objet $A \in \mathcal{B}$ les fibrations, cofibrations et équivalences faibles de $\mathcal{E}$ qui sont également des morphismes de la fibre $\mathcal{E}_A$ forment une structure de catégorie de modèles sur $\mathcal{E}_A$,

(ii) le foncteur p préserve les fibrations, les cofibrations et les équivalences faibles.

Quitte à cliver nos bifibrations de Grothendieck, toute flèche $u : A \longrightarrow B$ de la base $\mathcal{B}$ induit une adjonction $(u_!, u^*)$ entre les fibres $\mathcal{E}_A$ et $\mathcal{E}_B$. On montre facilement que dans le cas d'une bifibration de Quillen, ces adjonctions sont des adjonctions de Quillen. Autrement dit, une bifibration de Quillen $p : \mathcal{E} \longrightarrow \mathcal{B}$ peut alternativement être vue en particulier comme un pseudo foncteur $\mathcal{B} \longrightarrow \mathsf{Quil}$ depuis une catégorie

$\mathcal{B}$ avec structure de catégorie de modèles vers la 2-catégorie Quil des catégories avec structure de catégories de modèles, adjonctions de Quillen entre celles-ci (dans le sens de l'adjoint à gauche) et transformations naturelles. Le théorème principal du chapitre 3 est une caractérisation des bifibrations de Quillen parmi (les constructions de Grothendieck) de tels pseudo foncteurs.

Théorème (Theorem 3.4.2). *Soit $p : \mathcal{E} \longrightarrow \mathcal{B}$ une bifibration de Grothendieck avec une structure de catégorie de modèles sur $\mathcal{B}$ et sur chacune des fibres $\mathcal{E}_A, A \in \mathcal{B}$. Sous l'hypothèse que les adjonctions $(u_!, u^*)$ soient toutes des adjonctions de Quillen, le foncteur p est une bifibration de Quillen si et seulement si les conditions suivantes sont satisfaites :*

(hCon) *le foncteur $u_!$ préserve et reflète les équivalences faibles pour toute cofibration acyclique u ; dualement le foncteur v^* préserve et reflète les équivalences faibles pour toute fibration acyclique v,*

(hBC) *pour tout carré commutatif de $\mathcal{B}$ de la forme suivante*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"'] & B \end{tikzcd}$$

où u, u' sont des cofibrations acycliques et v, v' sont des fibrations acycliques, la transformation naturelle compagnon $(u')_! v^* \longrightarrow (v')^* u_!$ est une équivalence faible en toutes composantes dans la fibre $\mathcal{E}_{C'}$.

Le chapitre 3 poursuit avec une présentation de la construction de Reedy et de ses généralisations à la lumière de ce théorème. Étant donnée une catégorie de Reedy $\mathcal{R}$, Kan démontre dans un mémoire non publié que la catégorie $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ de diagrammes de forme $\mathcal{R}$ à valeurs dans une catégorie de modèle $\mathcal{M}$ admet une structure de catégorie de modèles où les équivalences faibles sont les transformations naturelles qui le sont en toutes composantes. La preuve classique de ce fait passe par une récurrence transfinie sur le degré des objets de la catégorie de Reedy $\mathcal{R}$. C'est l'étape d'induction entre un cardinal μ et un cardinal $\mu + 1$ qui demande le plus de travail et c'est sur celle-ci qu'on se concentre ici. Dénotons $\mathcal{R}_\mu$ pour la sous-catégorie pleine de $\mathcal{R}$ engendrée par les objets de degré strictement inférieur à μ. La proposition 3.5.4 montre que le foncteur de restriction $\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right) \longrightarrow \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$ est une bifibration de Grothendieck, et si l'on munit le codomaine de la structure de catégorie de modèles de Reedy alors cette bifibration satisfait aux hypothèses du théorème 3.4.2 : on en déduit une structure de catégorie de modèles sur $\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)$ qui est bien la structure introduite par Kan. Cette reconstruction du théorème de Kan a l'avantage de faire émerger naturellement la description des cofibrations et fibrations de la structure de catégorie de modèles sur $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$: les sommes amalgamées et produits fibrés d'espaces *latching* et *matching* qui peuvent sembler un peu ad hoc à première vue dans la preuve originale apparaissent ici comme des *poussés en avant* et *tirés en arrière* dans la bifibration de restriction. Il faut ici remarquer que cette relecture de l'œuvre de Reedy et Kan n'est pas possible à travers les travaux de Roig-Stanculescu et Harpaz-Prasma sans renforcer les hypothèses de façon drastique (en supposant par exemple que $\mathcal{M}$ est propre à droite et à gauche). La volonté d'adapter les résultats de [Roi94], [Sta12] et [HP15] au cadre de la construction de Reedy a été le fil d'Ariane qui nous a mené au final au théorème 3.4.2.

Catégories homotopiques des bifibrations de Quillen

Le chapitre 4 pousse plus loin l'étude des bifibrations de Quillen introduites au chapitre précédent. Une telle bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ entre catégorie de modèles respecte les équivalences faibles par définition et induit donc par localisation de Gabriel-Zisman un foncteur

$$\mathbf{Ho}\left(p\right) : \mathbf{Ho}\left(\mathcal{E}\right) \to \mathbf{Ho}\left(\mathcal{B}\right)$$

La question centrale du chapitre se pose alors en ces termes : comment reconstruire le foncteur $\mathbf{Ho}\left(p\right)$, et donc la catégorie homotopique $\mathbf{Ho}\left(\mathcal{E}\right)$ en particulier, à partir des catégories homotopiques $\mathbf{Ho}\left(\mathcal{B}\right)$ et $\mathbf{Ho}\left(\mathcal{E}_A\right)$ pour tout $A \in \mathcal{B}$ et des adjonctions localisées $\mathbf{L}(u_!) \dashv \mathbf{R}(u^*)$ pour tout $u : \mathcal{A} \longrightarrow \mathcal{B}$ de $\mathcal{B}$?

Pour comprendre la réponse apportée par le théorème suivant, il nous faut introduire un foncteur intermédiaire faisant le lien entre p et $\mathbf{Ho}\left(p\right)$. Oublions pour un moment la structure de catégorie de modèles de $\mathcal{B}$ et munissons la de la structure triviale (où les équivalences faibles sont réduites aux isomorphismes), mais gardons pour chaque fibre la structure de catégorie de modèles héritée de $\mathcal{E}$. Alors la bifibration de Grothendieck p rentre encore dans les hypothèses du théorème 3.4.2, permettant ainsi de munir $\mathcal{E}$ d'une nouvelle structure de catégorie de modèles. Nous appelons *fibre à fibre* cette structure de catégorie de modèles et on note $\mathcal{E}_{\mathrm{fw}}$ pour désigner la catégorie $\mathcal{E}$ munie de la structure fibre à fibre. On note également $p_{\mathrm{fw}} : \mathcal{E}_{\mathrm{fw}} \longrightarrow \mathcal{B}_{\mathrm{triv}}$ pour référer à la bifibration de Quillen ainsi créée. Le foncteur p_{fw} est le même que le foncteur p, seul leur contenu homotopique change ! Le foncteur d'intérêt est

$$\mathbf{Ho}\left(p_{\mathrm{fw}}\right) : \mathbf{Ho}\left(\mathcal{E}_{\mathrm{fw}}\right) \longrightarrow \mathbf{Ho}\left(\mathcal{B}_{\mathrm{triv}}\right) \cong \mathcal{B}$$

Ce foncteur est précisément la construction de Grothendieck du pseudo foncteur $\mathcal{B} \longrightarrow \mathsf{Adj}$ qui associe à un objet A la catégorie $\mathbf{Ho}\left(\mathcal{E}_A\right)$ et à une flèche $u : A \longrightarrow B$ l'adjonction $\mathbf{L}(u_!) \dashv \mathbf{R}(u^*)$. Afin d'établir la pseudo fonctorialité de cette relation, la section 4.1 est consacrée à la preuve de pseudo 2-fonctorialité de la relation $\mathsf{Quil} \longrightarrow \mathsf{Adj}$ qui associe à toute catégorie de modèles sa catégorie homotopique et à toute adjonction de Quillen son adjonction dérivée. Ce fait semble absent de la littérature, à l'exception du livre de Hovey ([Hov99]) qui utilise cependant le cadre plus restrictifs des catégories de modèles avec factorisations fonctorielles.

Redonnons alors à $\mathcal{B}$ sa structure de catégorie de modèles initiale et munissons chaque fibre de $\mathbf{Ho}\left(p_{\mathrm{fw}}\right)$ de la structure triviale : le théorème 3.4.2 s'applique de nouveau et on obtient une structure de catégorie de modèles sur $\mathbf{Ho}\left(\mathcal{E}_{\mathrm{fw}}\right)$ qui fait de $\mathbf{Ho}\left(p_{\mathrm{fw}}\right)$ une bifibration de Quillen. On est maintenant en mesure d'énoncer le théorème principal du chapitre 4 :

Théorème (Proposition 4.4.2)**.** *En tant que foncteur,* $\mathbf{Ho}\left(p\right)$ *est isomorphe à la localisation itérée de p, à droite dans le diagramme suivant :*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, equal] \arrow[d, "p"'] & \mathcal{E}_{\mathrm{fw}} \arrow[r] \arrow[d, "p_{\mathrm{fw}}"] & \mathbf{Ho}\left(\mathcal{E}_{\mathrm{fw}}\right) \arrow[r] \arrow[d, "\mathbf{Ho}(p_{\mathrm{fw}})"] & \mathbf{Ho}\left(\mathbf{Ho}\left(\mathcal{E}_{\mathrm{fw}}\right)\right) \arrow[d, "\mathbf{Ho}(\mathbf{Ho}(p_{\mathrm{fw}}))"] \\
\mathcal{B} \arrow[r, equal] & \mathcal{B}_{\mathrm{triv}} \arrow[r, "\cong"] & \mathcal{B} \arrow[r] & \mathbf{Ho}\left(\mathcal{B}\right)
\end{tikzcd}$$

Pour que cette énoncé soit parfaitement rigoureux, on ne peut pas utiliser la théorie des catégories de modèles telle que Quillen l'a introduite. En effet, Quillen requiert qu'une catégorie de modèles ait toutes les petites limites et toutes les petites colimites. Or une catégorie telle que $\mathbf{Ho}\left(\mathcal{E}_{\mathrm{fw}}\right)$, bien qu'elle possède les petits produits et petites sommes, n'admet que très rarement les égalisateurs et coégalisateurs. En s'appuyant sur une idée lumineuse d'Egger ([Egg16]), le chapitre préliminaire 2 reprouve les bases de la théorie de Quillen dans le cadre des catégories de modèles sans recourir aux (co)égalisateurs : l'essentiel de la théorie se transporte facilement à ce cadre un peu plus large et le chapitre 2 insiste sur les points de divergences.

MLTT et tribus relatives

L'étude des structures introduites aux chapitres 3 et 4 fait apparaître des phénomènes de *poussés en avant homotopique* et *tirés en arrière homotopique*, qui semblent entretenir la même relation avec les poussés en avant et tirés en arrière usuels d'une bifibration que celle entretenu par les produits fibrés homotopiques avec les produits fibrés usuels. Cela fait écho à notre volonté originelle de réconcilier les modèles existants de MLTT avec les hyperdoctrines de Lawvere.

En effet, dans une hyperdoctrine le prédicat d'égalité "$x =_A y$" entre deux variables de sortes A est défini comme le poussé en avant du prédicat `true` au-dessus de la diagonale $A \longrightarrow A \times A$. Cela traduit la conséquence logique qu'étant donnés un terme $t(x, y)$ de sorte B (x, y étant de sorte A) et une formule $\phi(z)$ sur la sorte B, si $\phi(t(x, x))$ est universellement vraie, alors $x =_A y \longrightarrow \phi(t(x, y))$ aussi. Ou plus précisément, il y a une correspondance entre les preuves de $\phi(t(x, x))$ dans le contexte (x^A) et celle de de $x =_A y \longrightarrow \phi(t(x, y))$ dans le contexte (x^A, y^A). Dans une MLTT, le type identité (intentionnel) se comporte de manière similaire grâce à la règle j qui impose qu'étant donné un type $C(x, y, p)$ dépendant d'une preuve p de l'identification de x et y, l'existence d'un terme de type $C(x, x, \mathsf{refl}_x)$ induit l'existence d'un terme de $C(x, y, p)$. En revanche ce terme induit n'est unique que *propositionnellement*, pas *définitionnellement*. C'est-à-dire que pour deux tels termes j et j', le type $\mathrm{Id}\,(\mathsf{j}, \mathsf{j}')$ est habité, mais le jugement $\mathsf{j} \equiv \mathsf{j}'$ n'est pas forcément dérivable. Autrement dit, le type identité $\mathrm{Id}_A\left(x, y\right)$ est un *poussé en avant homotopique* du type terminal 1_A au-dessus de la diagonale $A \longrightarrow A \times A$.

Dans le chapitre 5, on décide de se concentrer sur les modèles de MLTT introduits par Joyal sous le nom de *tribu*. Une tribu est un raffinement sur la notion de catégorie à *display maps* étudiées dans le cadre de la théorie des types dépendants extensionnelle. Elle est inspiré de l'analogie d'Awodey et Warren entre type identité et objet en chemin dans une catégorie de modèles, et apparaît de manière indépendante dans d'autres travaux (notamment dans le travail de Shulman sur les *type-theoretic fibrations categories*.

Après des rappels extensifs sur MLTT et sur les tribus de Joyal, on élabore dans la section la partie 5.3 la notion de système de factorisation relatif à un foncteur.

Définition (Definition 5.3.3)**.** Un *système de factorisation à droite faible relatif* au foncteur $p : \mathcal{E} \longrightarrow \mathcal{B}$ est la donnée de deux classes $\mathfrak{L}$ et $\mathfrak{R}$ de flèches de $\mathcal{E}$ et de deux classes $\mathfrak{l}$ et $\mathfrak{r}$ de flèches de $\mathcal{B}$ telles que :

(i) $p(\mathfrak{L}) \subseteq \mathfrak{l}$ et $p(\mathfrak{R}) \subseteq \mathfrak{r}$,

(ii) les éléments de $\mathfrak{L}$ sont exactement ceux ayant la propriété de relèvement à gauche relative à p contre tous les éléments de $\mathfrak{R}$; c'est-à-dire que $f \in \mathfrak{L}$ si et seulement si pour tout carré commutatif de $\mathcal{E}$ comme suit

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "\in\mathcal{R}"] \\ Y \arrow[r] & T \end{tikzcd}$$

si l'image dans $\mathcal{B}$ admet un relevé h comme suit

$$\begin{tikzcd} pX \arrow[r] \arrow[d] & pZ \arrow[d] \\ pY \arrow[r] \arrow[ur, dashed, "h"] & pT \end{tikzcd}$$

alors il existe $k : Y \to Z$ relevant le premier carré tel que $p(k) = h$.

(iii) pour toute flèche $f : X \to Y$ dans $\mathcal{E}$ et toute factorisation de son image $p(f)$ en rl avec $r \in \mathfrak{r}$ et $l \in \mathfrak{l}$, il existe $g \in \mathfrak{L}$ et $h \in \mathfrak{R}$ telle que $f = hg$ et $p(g) = l$ et $p(h) = r$.

On obtient la notion de *système de factorisation à droite strict relatif à* p en demandant que k soit unique dans le deuxième point. On retrouve les notions de systèmes de factorisation strict et faible classiques en prenant $\mathcal{B}$ la catégorie terminale.

L'observation clé du chapitre 5 peut se résumer aux deux résultats ci-dessous :

Théorème (Proposition 5.3.7). *Soit $p : \mathcal{E} \to \mathcal{B}$ un foncteur entre catégories avec objets terminaux. Si p préserve les objets terminaux, alors p est une opfibration de Grothendieck si et seulement s'il existe un système de factorisation à droite strict relatif à p où*

(i) $\mathfrak{R}$ contient tous les morphismes de $\mathcal{E}$,

(ii) $\mathfrak{l}$ contient tous les morphismes de $\mathcal{B}$.

Théorème (Proposition 5.3.10). *Soit $\mathcal{C}$ un clan de Joyal et $p : \mathfrak{F}_{\mathcal{C}} \to \mathcal{C}$ le foncteur codomaine associé restreint aux fibrations. Le clan $\mathcal{C}$ est une tribu si et seulement si il y a un système de factorisation à droite faible relatif à p tel que*

(i) $\mathfrak{R}$ contient les fibrations totales, définies comme les morphismes de $\mathfrak{F}_{\mathcal{C}}$ comme suit

$$\begin{tikzcd} X \arrow[r, "u'"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

où u et l'écart cartésien $X \to A \times_B Y$ sont des fibrations de $\mathcal{C}$,

(ii) $\mathfrak{l}$ contient tous les morphismes de $\mathcal{C}$,

(iii) les morphismes cartésiens au-dessus des morphismes anodins de $\mathcal{C}$ sont dans $\mathfrak{L}$.

Si l'on oublie le dernier point dans ce deuxième théorème (qui peut être en effet omis en présence de Π-types), la similarité entre les deux résultats est une motivation suffisante pour prendre comme généralisation des tribus les fibrations de Grothendieck $p : \mathcal{E} \to \mathcal{B}$ avec une classe distinguée de flèches, appelées fibrations, dans $\mathcal{B}$ et dans chacune des fibres $\mathcal{E}_A$, et avec un système de factorisation à droite faible relatif à p tel que :

(i) $\mathfrak{R}$ contient les fibrations totales, définies comme les morphismes $f : X \longrightarrow Y$ de $\mathcal{E}$ où $u = p(f)$ et l'écart cartésien $X \longrightarrow u^* Y$ sont des fibrations de $\mathcal{C}$,

(ii) $\mathfrak{l}$ contient tous les morphismes de $\mathcal{C}$,

Incorporée aux intuitions de *catégories à compréhension* de Jacobs et aux $\mathbb{D}$-*catégories* de Ehrhard, cette généralisation trouve sa forme rigoureuse dans la définition 5.4.9 des *tribus relatives*. Cette généralisation se veut l'analogue dans le cadre des tribus au saut conceptuel effectué par Lawvere lors du passage du foncteur $\mathrm{Sub}(-) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ pour $\mathcal{C}$ une catégorie de Heyting à la notion d'hyperdoctrine générale.

Le chapitre 5 se conclut sur une ouverture directement motivée par les lacunes des tribus relatives. On y propose un premier pas vers des modèles de MLTT prenant la forme d'une catégorie simpliciale $\mathcal{C}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ où les contextes de taille n sont interprétés dans la catégorie $\mathcal{C}_{n-1}$. Le prototype d'un tel modèle est le "nerf fibreux" $\mathrm{N}\,\mathcal{C}$ d'une tribu $\mathcal{C}$ où $\mathrm{N}\,\mathcal{C}_n$ est la sous-catégorie pleine de $\mathrm{Fun}\,(\Delta[n], \mathcal{C})$ engendrée par les chemins composés de fibrations de $\mathcal{C}$ uniquement.

Chapter 1
Introduction

This thesis is a quest for structures. It falls within the field of *categorical semantics of logic*, a mathematical area devoted to the construction of suitable objects that allow for a meaningful interpretation of a given *calculus*. Roughly put a calculus is composed of

- a collection of symbols (the *alphabet*),
- with rules that control what strings of symbols can be written in the calculus (the *grammar*)
- and rules that explain how to deduce new strings of symbols from existing ones (the *inference rules*).

A *semantics* for a calculus will provide a consistent framework in which to actually compute the calculations described by the calculus. A far-fetched analogy, that has the advantage to be understandable also by the non mathematicians, can be made with *numerals* (that plays here the role of the calculus) versus *numbers* (that plays the role of the semantics). Each numeral, whatever basis it is written in, is a *denotation* for a concrete number. Or symmetrically, numbers are *interpretations* of numerals. Numerals are *syntactic* gadgets necessary for people to communicate about the number, that are the *true* objects of interest. Changing the numeral presentation does not change the properties that hold for numbers. For example, the property "there is a unique even number that is also prime" is true regardless of the fact that this property is written with numbers represented in binary or in decimal notation.

Most working mathematicians are acquainted with calculi, even if sometimes unknowingly. Indeed, the usual algebraic objects define calculi. For example, to a monoid M with presentation $\langle S \mid R \rangle$ is associated a calculus with four main symbols, namely "(", ")", "1", "$\cdot$", and a letter symbol "x" for any $x \in S$. The grammar of this calculus allows to write exactly strings of the form

$$(x \cdot y) \cdot (1 \cdot y), \quad 1 \cdot 1, \quad x \cdot z, \quad y, \quad (((y \cdot x) \cdot z) \cdot t), \quad ((y \cdot x) \cdot (z \cdot t))$$

that is correctly parenthesized strings of symbol using "·" as a binary operations. The inference rules are given by the axioms of monoid theory:

$$\begin{gathered} x \rightsquigarrow x \cdot 1, \quad x \rightsquigarrow 1 \cdot x, \quad x \cdot 1 \rightsquigarrow x, \quad 1 \cdot x \rightsquigarrow x, \\ (x \cdot y) \cdot z \rightsquigarrow x \cdot (y \cdot z), \quad x \cdot (y \cdot z) \rightsquigarrow (x \cdot y) \cdot z \end{gathered}$$

and the relations in R: for each related pairs $(\vec{x}, \vec{y}) \in R$,

$$(\dots(x_1 \cdot x_2) \cdot \dots) \cdot x_n) \rightsquigarrow (\dots(y_1 \cdot y_2) \cdot \dots) \cdot y_m), \quad (\dots(y_1 \cdot x_2) \cdot \dots) \cdot y_m) \rightsquigarrow (\dots(x_1 \cdot x_2) \cdot \dots) \cdot x_n)$$

where the sign $\rightsquigarrow$ just means that the string on the left can be rewritten as the string on the right.

Computer scientist also are acquainted with calculus, often more knowingly. Indeed, every programming language can be thought of as a calculus where the symbols are the ASCII character, the grammar is constituted of the syntactic rules of the programming language and where the inference rules are determined by the intended behavior of the code once interpreted or compiled. If one were to be a little more formal, a good abstraction of a programming language is given by Church's λ-calculus, designed in the 30's long before the actual first programming language. The λ-calculus operates on four symbols, namely "(", ")", "λ", "." and an infinite countable amount of letter symbols "x", "y", etc. The grammar restricts the writable strings to the one of the form

$$x \quad \lambda x.(\lambda y.yy)x \quad (\lambda z.z)(\lambda x.(xx)x) \quad \lambda y.(\lambda x.y)$$

that is either *application* of a correct string to another, or *abstraction* of a correct string through the λ. There are three kinds of inferences rules:

$$\begin{aligned} &\lambda x.M \rightsquigarrow \lambda y.M\,[x \leftarrow y]\,, \\ &(\lambda x.M)N \rightsquigarrow M\,[x \leftarrow N]\,, \\ &\lambda x.(Mx) \rightsquigarrow M, \quad M \rightsquigarrow \lambda x.(Mx) \end{aligned}$$

where M and N stands for any writable strings allowed by the grammar, and $M\,[x \leftarrow N]$ means the string M where every (free) occurrences of x are replaced by the string N, without binding of free variables in N.

Both examples before are very simple and the more we want to express with a calculus, the more it complexifies. The role of the semantics is to find mathematical entities in which the symbols can be instantiated in a way that the strings created by the grammar make sense and that the inference rules are respected (by which is usually meant that the relation $\rightsquigarrow$ becomes equality). *Categorical* semantics is the art to craft such entities from a very structural point of view, using the array of tools that category theory makes available. In this introductory chapter, we shall first give a quick retrospective of the beginnings of categorical semantics. Next we will review Lawvere's proposal for categorical semantics of first-order logic. While keeping the exposition relatively short, we shall nevertheless make it sufficiently precise as it will guide our intuition for a good part of this memoir. Next, the reader will find a introductory section on the homotopy type theory that emerged in the 2000s under the aegis of Vladimir Voevodsky, and which early traces of can be found already in Lawvere's work. Finally, this chapter ends on a quick overview of the remaining chapters of this thesis.

Historical retrospective

The birth of categorical semantics can be traced back to Lawvere's PhD thesis [Law63] in 1963. In his manuscript, Lawvere presents a way to craft a category $\mathcal{T}$ with finite products for any algebraic theory $\mathbb{T}$, in such a way the models of $\mathbb{T}$ can be identified with finite products preserving functors out of $\mathcal{T}$. By an algebraic theory here is meant a first-order theory on a language without relation symbols and axiomatizable only by universally quantified equation. Group theory, abelian group theory, ring theory, vector space theory over a fixed field, monoid theory, module theory over a fixed ring, algebra theory, etc. are all of this kind. The study of algebraic theories through traditional means was then already well developed under the name of *universal algebra*. Roughly put, universal algebra is the same as model theory restricted to algebraic theories: it is the study of a theory through its *models*. Given a algebraic theory $\mathbb{T}$ over a language with a set of constant symbols C and a family of sets $(F_n)_{n\geq 1}$ of functions symbol of arity $n \geq 1$, a model in the traditional sense is a set M together with the data of

$$\begin{aligned} &c^M \in M \quad \forall c \in C \\ &f^M : M^n \longrightarrow M \quad \forall f \in F_n \end{aligned} \tag{1.1}$$

that satisfy the axioms of $\mathbb{T}$ in the obvious way. The data of (1.1) is usually called the *interpretations* of the symbol in the language of the theory $\mathbb{T}$, and it extends straightforwardly to an interpretation of *terms*. Rigorously a term is inductively either a variable, or a constant symbol or a string $f(t_1, \dots, t_n)$ for $f \in F_n$ and each t_i a previously constructed term. Variables are interpreted as the function id_M, constant symbols c already have interpretations that can be identified with constant function $c^M : 1 \longrightarrow M$ when needed, and if $t_1, \dots, t_n$ are terms with interpretation $t_1^M, \dots, t_n^M$, then for $f \in F_n$ the interpretation of $f(t_1, \dots, t_n)$ in M is the composite function $f^M(t_1^M, \dots, t_n^M)$. Axioms of $\mathbb{T}$ are by definition of the form $t = u$ for some terms t and u. The model M satisfies the axiom $t = u$ in the sense that the function defined by the term t equals (pointwise) the function defined by the term u. As an example, take the $\mathbb{T}$ of (1.2) in the language with a constant symbol e and a function symbol m of arity 2.

$$\begin{aligned} m(e, x) &= x \\ m(x, e) &= x \\ m(x, m(y, z)) &= m(m(x, y), z) \end{aligned} \tag{1.2}$$

A model of this theory is a set M together with a constant e^M and a binary operation m^M. The interpretation of the term $m(e, x)$ is $a \mapsto m^M(e^M, a)$ and the interpretation of the term x is just $a \mapsto a$, so that saying that M satisfies the first axiom amounts to enforce that $m^M(e^M, a) = a$ for each element $a \in M$. In the same manner, the second axiom enforces that e^M is neutral on the right for m^M and the third one that m^M is associative. Hence a model M of this theory is precisely a monoid as one would expect.

The farsighted idea of Lawvere was to reckon that the compositionality of the interpretations in a model M as well as the neutrality of the interpretations of variables were part of a functor structure on the mapping $t \mapsto t^M$. And indeed he constructs in his thesis a small category $\mathcal{T}$ associated with any algebraic theory $\mathbb{T}$ as follow: the objects of the category are the natural numbers $n \in \mathbb{N}$, the morphism $n \longrightarrow m$ are the m-tuples of terms with less than n variables modulo the congruence generated by

$(t_1, \dots, t_n) \sim (u_1, \dots, u_n)$ whenever $t_i = u_i$ is derivable from $\mathbb{T}$ for each $i = 1, \dots, n$. This category $\mathcal{T}$ has finite products, n being the n-fold product of 1 (so in particular 0 is terminal). A model M is then easily identified with a finite products preserving functor $\mathcal{T} \longrightarrow \mathsf{Set}$ where 1 is mapped to the set M (hence n is mapped to M^n) and a morphism $n \longrightarrow m$ with a representative $(t_1 \dots, t_n)$ is mapped to the map $M^n \longrightarrow M^m$ induced by the t_i^M. Such a category $\mathcal{T}$ is nowadays called a *Lawvere theory*. An immediate consequence of this definition is that models are no longer constrained to being sets: for any category $\mathcal{C}$, a $\mathcal{C}$-model of $\mathbb{T}$ can be defined as a finite product preserving functor $\mathcal{T} \longrightarrow \mathcal{C}$. Given the algebraic theory of groups for example, one recovers topological groups as model in the category Top of topological spaces, whereas Lie groups are the models in the category Mnfd of manifolds, and sheaves in groups over a space X are the models in the category $\mathrm{Sh}(X)$ of usual sheaves over the space X. Hence Lawvere theories give a unified definition of different flavour of the same algebraic operations, and offers a usable framework to derive uniformly properties that does not depend on the flavour. Moreover, if we denote $\mathrm{Mod}_{\mathcal{T}}(\mathcal{C})$ the category of model of $\mathcal{T}$ in the category $\mathcal{C}$, then the category of models in sets comes with a *forgetful* functor

$$U_{\mathcal{T}} : \mathrm{Mod}_{\mathcal{T}}(\mathsf{Set}) \longrightarrow \mathsf{Set} \tag{1.3}$$

that maps a model $M : \mathcal{T} \longrightarrow \mathsf{Set}$ to its underlying set $M(1)$. This can be showed to be monadic, and the small category $\mathcal{T}$ cat be reconstructed from it (at least up to equivalence), by taking the opposite of the full subcategory of $\mathrm{Mod}_{\mathcal{T}}(\mathsf{Set})$ spanned by the objects of the form $F_{\mathcal{T}}(n)$, $n \geq 0$ where $F_{\mathcal{T}}$ is left adjoint to $U_{\mathcal{T}}$ and n designates the finite set $\{0, \dots, n-1\}$ with n elements. Furthermore, this construction of a Lawvere theory from a category of models together with a forgetful functor to Set defines a functor

$$\mathcal{S} : \mathcal{K} \longrightarrow \mathsf{Law}^{\mathrm{op}} \tag{1.4}$$

from the category $\mathcal{K}$ of the so called *algebraic categories* to the category Law of Lawvere theories. More precisely, the category $\mathcal{K}$ is the full subcategory of $\mathsf{CAT}/\mathsf{Set}$ spanned by those functor isomorphic to some $U_{\mathcal{T}}$, while Law is the full subcategory of ${\aleph_0}^{\mathrm{op}}\backslash\mathsf{Cat}$ spanned by identity on objects finite products preserving functor where $\aleph_0$ is the category of finite ordinals and set-theoretic mapping between them. This functor $\mathcal{S}$ is called the *syntax functor*[1], and Lawvere shows that it is left adjoint to the functor

$$\mathfrak{S} : \mathsf{Law}^{\mathrm{op}} \longrightarrow \mathcal{K}, \quad \mathcal{T} \mapsto U_{\mathcal{T}} \tag{1.5}$$

This last functor deserves the name *semantics* as it associates to any theory its category of models (in Set). This gives rise the following observation:

> Syntax and semantics are adjoint on the right.

This became a slogan, that one wants to obtain for any categorical framework of logic: where one has in one hand a category encoding all the theories given by a syntax of a logical system, and in the other hand the category of those mathematical objects that supposedly interpret the syntax. It conveys a powerful intuition, even in situation not as formally nice as the situation of Lawvere theories.

A few years later and quite independently, Dana Scott started a similar process for λ-calculus. Picking up on the early ideas of Strachey's influential paper [Str66], Scott introduces *domains* and the now called *Scott-continuous* functions between them in an

[1] In his thesis Lawvere names it the algebraic structure functor.

attempt to fix the issues encountered when trying to deal with λ-terms as set-theoretic functions. Rigorously a λ-term can be defined as a word on the alphabet $\{(, \lambda, .,)\} \cup \mathsf{Var}$, where Var is a (naive) uncountable set, of the following inductive form:

- if $x \in \mathsf{Var}$, then x is a λ-terms,
- if M is a λ-term and $x \in \mathsf{Var}$, then $\lambda x.M$ is a λ-term, called the *abstraction* of M bounding x,
- if M and N are λ-terms, then also $(M)N$ is a λ-term, called the *application* of M to N.

Define an occurrence of a variable x in M to be bound, if is appears under an abstraction. When an occurrence is not bound, it is said to be free. Ans write $M\,[x \leftarrow N]$ to denote the term M in which the free occurrences of x are (simultaneously) replaced by the term N. The rewriting relation, denoted $\longrightarrow^*_\beta$, for λ-terms is the transitive closure of the relation $\longrightarrow_\beta$ defined by:

$$
\begin{aligned}
&(\lambda x.M)N \longrightarrow_\beta M\,[x \leftarrow N] \\
&M \longrightarrow_\beta N \implies (M)L \longrightarrow_\beta (N)L \\
&M \longrightarrow_\beta N \implies (L)M \longrightarrow_\beta (L)N
\end{aligned}
\tag{1.6}
$$

Church initially introduced λ-terms to model computable functions. He first encodes the natural number $n \in \mathbb{N}$ as the λ-terms $\underline{n}$ that applies its first bounded variable n times to its second bounded variables. For example, 0, 1, 2 and 3 are respectively encoded as:

$$\underline{0} = \lambda f.\lambda x.x, \quad \underline{1} = \lambda f.\lambda x.(f)x, \quad \underline{2} = \lambda f.\lambda x.(f)(f)x, \quad \underline{3} = \lambda f.\lambda x.(f)(f)(f)x \tag{1.7}$$

Then one can call a function $\varphi : \mathbb{N}^k \longrightarrow \mathbb{N}$ a *λ-definable function* if there exists a λ-term Φ such that for every tuple $(n_1, \dots, n_k) \in \mathbb{N}^k$,

$$(\dots(((\Phi)\underline{n_1})\underline{n_2})\dots\underline{n_k}) \longrightarrow^*_\beta \underline{\varphi(\vec{n})} \;\wedge\; \underline{\varphi(\vec{n})} \longrightarrow^*_\beta (\dots(((\Phi)\underline{n_1})\underline{n_2})\dots\underline{n_k}) \tag{1.8}$$

For example, the function $\varphi : (n_1, n_2) \longmapsto n_1 + n_2$ is λ-definable: a corresponding λ-term Φ is given by

$$\lambda n_1.\lambda n_2.\lambda f.\lambda x.((n_1)f)((n_2)f)x \tag{1.9}$$

It can be shown that λ-definable functions are exactly the Gödel's recursive function, or equivalently Turing's computable functions. The issue rises when one wants to consider *any* λ-term M as function between sets. The problem is that λ-terms are both the function acting on a set of arguments and the arguments themselves. So to have a meaningful interpretation of λ-terms as set-functions, it would require a set D such that $D^D \cong D$. For sets have cardinalities, this can only be if D is a singleton… Scott's idea is to try and find a solution to this equation in a different cartesian closed category, where obstructions as cardinalities of sets does not occur! He turned to what is now known as *Scott domains* which are posets that are quite similar to accessible categories when regarded as such. A natural notion of morphism between such domains emerges as non-increasing functors that preserves the extra property of a Scott domain: these morphisms are called *Scott-continuous* and given two Scott domains A and B, the set $[A, B]$ of Scott-continuous functions inherits the structure of a Scott domain. More precisely, the domain $[A, B]$ is exponential object B^A in the category Scott of

Scott domains. Then Scott designed a domain D_∞ as follow : take D_0 to be the poset $\{0 < 1\}$ and define inductively D_{n+1} as the exponential $[D_n, D_n]$; it comes easily with projections $p_n : D_{n+1} \longrightarrow D_0$ that actually forms a diagram

$$D : \mathbb{N}^{\mathrm{op}} \longrightarrow \mathsf{Scott} \tag{1.10}$$

where $\mathbb{N}$ is the thin category generated by $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \dots$. The domain D_∞ is then defined as the limit of this diagram, and Scott proves that it solves the domain equation $X \cong [X, X]$. It provides a non trivial model of Church's λ-calculus where every λ-term denotes a Scott-continuous $D_\infty \longrightarrow D_\infty$. Remark that Scott domains can be considered as a specific kind of T_0-topological spaces and topological intuition might be applied here and there. Already the idea of type(s) as spaces is emerging.

Categorical logic and semantics

The categorical understanding of logic initiated by Lawvere in his thesis probably culminated in his notion of *hyperdoctrine*. The way we see it, it is a very structural point of view on the broadly known subject of first-order logic and model theory. It provides intuition and understanding that could not have been achieved with the traditional view, and it surely had a great impact on the modern investigations of logic (like HoTT which we talk about in the next section).

Recall that a first-order (intuitionistic) language $\mathscr{L}$ is comprised of:

- a (naive) set S of sorts,
- a (naive) set C_s of constant symbols for each $s \in S$,
- a (naive) set $F_n^{\vec{s}}$ of function symbols of arity n for each $\vec{s} \in S^{n+1}$ and $n \in \mathbb{N}^*$,
- a (naive) set $R_n^{\vec{s}}$ of relation symbols of arity n for each $\vec{s} \in S^n$ and $n \in \mathbb{N}^*$.

We consider that there is a (naive) uncountable set Var of variables at disposal and that $R_2^{(s,s)}$ always contain a symbol "$\doteq_s$". A *term* of sort $s \in S$ in the language $\mathscr{L}$ is defined inductively as either a constant symbol in C_s, a variable or a string of the form $f(t_1, \dots, t_n)$ where $f \in F_n^{\vec{s}}$, $\vec{s} = (s_1, \dots, s_n, s)$ and each t_i is a term of sort s_i. A well-formed *formula* is now inductively defined as either string of the following forms:

- $r(t_1, \dots, t_n)$ where $r \in R_n^{\vec{s}}$, $s = (s_1, \dots, s_n)$ and each t_i is a term of sort s_i,
- $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \longrightarrow \psi$ for formulae φ and ψ,
- $\exists x^s \varphi$ and $\forall x^s \varphi$ for $x \in \mathsf{Var}$, $s \in S$ and φ a formula.

Free occurrences of a variable are defined as usual for a formula as the occurrences that are not appearing under one of the binding operators $\exists$ and $\forall$. Also we write the more common "$t_1 \doteq_s t_2$" instead of the rigorous "$\doteq_s (t_1, t_2)$". There are rules that govern the possibility to *deduce* a formula from others. More explicitly, for a tuple $\vec{x}$ of variables of given sorts (formally a function $S \longrightarrow \mathsf{Var}$ with finite support), let us denote $\mathscr{F}(\vec{x})$ for the set of formulae that have their free variables among the x_j's. Define now a relation $\vec{\varphi} \vdash_{\vec{x}} \psi$ where ψ and all the φ_i have their free variables among the x_j's. We shall require that $\wedge$, $\vee$ and $\longrightarrow$ makes $\mathscr{F}(\vec{x})$ a complete Heyting algebra. The rules that are of interest to us here are the ones that govern the behaviour of the quantifiers: these rules explain how the different order relations $\vdash_{\vec{x}}$ behave one relatively to another when $\vec{x}$ vary. They are given by

$$\begin{array}{lll} \varphi(\vec{y},z) \vdash_{\vec{y},z} \psi^*(\vec{y}) & \text{if and only if} & \exists z^s \varphi(\vec{y},z) \vdash_{\vec{y}} \psi(\vec{y}) \\ \psi^*(\vec{y}) \vdash_{\vec{y},z} \varphi(\vec{y},z) & \text{if and only if} & \psi(\vec{y}) \vdash_{\vec{y}} \forall z^s \varphi(\vec{y},z) \end{array} \tag{1.11}$$

where s is the sort associated to z in the tuple $(\vec{y}, z)$ with given sorts. Here ψ^* denotes the same formula as $\psi \in \mathscr{F}(\vec{y})$ but considered with free variables among $(\vec{y}, z)$ and not only among $\vec{y}$. Also we have written $\chi(\mathrm{var}z)$ for a formula $\chi \in \mathscr{F}(\vec{z})$ to indicates more clearly what are the potential free variables of χ. There is nothing mysterious about those rules if we read the formulae with their intended meaning: the first rules says that if one finds some z that makes ψ true from φ and that ψ does not depend of this z, then ψ is a logical consequence of $\exists z\varphi$; the second rules says that if one can derive φ from ψ for a generic z and that ψ does not depend on z, then ψ can derive $\forall z\varphi$. The rules of (1.11) furiously resemble each to the property defining a categorical adjunction. Lawvere's work on hyperdoctrines can be understood as a effort to make this resemblance an actual fact. Or maybe more accurately it is a will to design a structure so fundamental for logic that these two rules of (1.11) are just consequences of living in such a structure. Actually the rules of intuitionistic logic permits a little more: for any tuples $\vec{x}$ and $\vec{y}$, if $\vec{t}$ is of the same length as $\vec{y}$ and if t_i is a term of the same sort as y_i that has its free variables among $\vec{x}$ for each i, then it holds that

$$\begin{array}{lll} \varphi(\vec{x}) \vdash_{\vec{x}} \psi(\vec{t}(\vec{x})) & \text{if and only if} & \exists\vec{x}\left(\varphi(\vec{x}) \wedge \vec{y} \doteq \vec{t}(\vec{x})\right) \vdash_{\vec{y}} \psi(\vec{y})e \\ \psi(\vec{t}(\vec{x})) \vdash_{\vec{x}} \varphi(\vec{x}) & \text{if and only if} & \psi(\vec{y}) \vdash_{\vec{y}} \forall\vec{x}\left(\vec{y} \doteq \vec{t}(\vec{x}) \longrightarrow \varphi(\vec{x})\right) \end{array} \tag{1.12}$$

Here $\vec{t}(\vec{x})$ is just the tuple $\vec{t}$ on which we insist that each component has free variables among $\vec{x}$. The expression $\psi(\vec{t}(\vec{x}))$ is a shortcut for the simultaneous substitution $\psi\left[\vec{y} \leftarrow \vec{t}\right]$ of each y_i by t_i in ψ. Finally $\vec{y} \doteq \vec{t}(\vec{x})$ is a short version of the more rigorous:

$$\bigwedge_i y_i \doteq_{s_i} t_i(\vec{x}) \tag{1.13}$$

Similarly, $\forall\vec{x}$ and $\exists\vec{x}$ are short version of the full version of (1.14) where each s_i is the sort associated with x_i:

$$\forall x_1^{s_1}, \dots, \forall x_n^{s_n} \quad \exists x_1^{s_1}, \dots, \exists x_n^{s_n} \tag{1.14}$$

The statements of (1.11) then arise as an instance of (1.12) with $\vec{x} = (\vec{y}, z)$ and $t_i(\vec{x}) = y_i$.

Suppose given a theory $\mathbb{T}$ over $\mathscr{L}$, that is a set of formulae without free variables. Define the category $\mathcal{C}_{\mathscr{L}}$ as the category whose objects are the tuples $\vec{x}$ of variables with given sorts and whose morphisms $\vec{x} \longrightarrow \vec{y}$ (the tuples might have different lengths) are the tuples $\vec{t}$ of terms such that t_i is a term whose sort is the same as y_i and that it has free variables among $\vec{x}$. The composition is given by substitution. Similarly as before, define $\mathscr{F}^{\mathbb{T}}(\vec{x})$ as the set of formulae with free variables among $\vec{x}$ and make it a poset with the relation $\varphi \vdash_{\vec{x}}^{\mathbb{T}} \psi$ defined as $\{\mathbb{T}, \varphi\} \vdash_{\vec{x}} \psi$ (meaning that φ entails ψ under the further assumption that formulae of $\mathbb{T}$ are axioms). As a poset, it is in particular a category. Define now for each morphism $\vec{t} : \vec{x} \longrightarrow \vec{y}$ of $\mathcal{C}_{\mathscr{L}}$ a functor:

$$\mathbb{T}(\vec{t}) : \mathscr{F}^{\mathbb{T}}(\vec{y}) \longrightarrow \mathscr{F}^{\mathbb{T}}(\vec{x}) \tag{1.15}$$

that maps a formula $\varphi(\vec{y})$ to the formula $\varphi(\vec{t}(\vec{x}))$. It produces a (pseudo) functor, abusively denoted $\mathbb{T} : \mathcal{C}_{\mathcal{L}}{}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$, that maps each *context* $\vec{x}$ to the category $\mathcal{F}^{\mathbb{T}}(\vec{x})$ of formulae in context $\vec{x}$ modulo the theory $\mathbb{T}$, and that maps a morphism $\vec{t}$ to the functor $(\vec{t})$. The equations of (1.12) still hold when replacing the order relations $\vdash$ with $\vdash^{\mathbb{T}}$ (by the weakening rules of the intuitionistic logic). So they state that $\mathbb{T}(\vec{t})$ admits both a left and right adjoint.

Alternatively it can be rephrased in the language of Grothendieck bifibration. More precisely, consider the functors

$$p^{\exists}_{\mathbb{T}} : \mathcal{E}^{\exists}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathcal{L}}, \qquad p^{\forall}_{\mathbb{T}} : \mathcal{E}^{\forall}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathcal{L}} \tag{1.16}$$

obtained respectively as the Grothendieck constructions of the pseudo functors $\mathbb{T}$ and $-^{\mathrm{op}} \circ \mathbb{T}$. Then the previous statement on the existence of adjoints for $\mathbb{T}(\vec{t})$ can be rephrased as follow: both $p^{\exists}_{\mathbb{T}}$ and $p^{\forall}_{\mathbb{T}}$ are Grothendieck bifibrations. This point of view is developed in details in [MZ16].

The same phenomenon about quantifiers being embodied by adjoint functors can be observed when dealing with subsets of sets. Denote $\mathrm{Sub}(X)$ for the lattice of subsets of the set X. As before we shall see this lattice as a category when needed. Now given a map $f : X \longrightarrow Y$, there is certainly an induced functor $\mathrm{Sub}(f) : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ that is given by the "preimage" operation: it maps a subset $B \subseteq Y$ to the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X \tag{1.17}$$

It produces a pseudo functor $\mathrm{Sub}(-) : \mathsf{Set} \longrightarrow \mathsf{Cat}$. The functors $\mathrm{Sub}(f)$ all have both a left adjoint and a right adjoint. The left adjoint is given by the "image" operation on the left of (1.18), and the right adjoint is given by the operation on the right of (1.18):

$$\begin{aligned} A &\longmapsto \{y \in Y \mid \exists x \in X, f(x) = y \wedge x \in A\}, \\ A &\longmapsto \{y \in Y \mid \forall x \in X, f(x) = y \longrightarrow x \in A\} \end{aligned} \tag{1.18}$$

There is a clear similarity with (1.12). And there is a reason behind this. Recall that a structure M of the first-order language $\mathcal{L}$ is a family of sets M_s for each $s \in S$ together with *interpretations* of the symbols:

- for each $c \in C_s$, an element $c^M \in M_s$,
- for each $f \in F^{\vec{s}}_n$, an function $f^M : M_{s_1} \times \cdots \times M_{s_n} \longrightarrow M_{s_{n+1}}$,
- for each $r \in R^{\vec{s}}_n$, a relation $r^M \subseteq M_{s_1} \times \cdots \times M_{s_n}$.

Moreover, one always require that $\doteq_s$ is interpreted by the diagonal $\{(x, y) \in M_s \mid x = y\}$ for all sorts. It gives, as in the case of algebraic theory, an extended interpretation $t(\vec{x})^M : M_{s_1} \times \cdots \times M_{s_n} \longrightarrow M_{s_{n+1}}$ for each term t of sort s_{n+1} that have free variables among the tuple $\vec{x}$ of variables with given sorts $(s_1, \dots, s_n)$. A *model* of the theory $\mathbb{T}$ is a structure M such that every formula of $\mathbb{T}$ is satisfied by M, in the sense of the following inductive definition:

- $M \vDash r(t_1(\vec{m}), \dots, t_n(\vec{m}))$ if and only if $(t_1^M(\vec{m}), \dots, t_n^M(\vec{m}))$ is an element of r^M,
- $M \vDash (\varphi \wedge \psi)(\vec{m})$ if and only if $M \vDash \varphi(\vec{m})$ and $M \vDash \psi(\vec{m})$,
- $M \vDash (\varphi \vee \psi)(\vec{m})$ if and only if $M \vDash \varphi(\vec{m})$ or $M \vDash \psi(\vec{m})$,

- $M \vDash (\varphi \longrightarrow \psi)(\vec{m})$ if and only if $M \vDash \varphi(\vec{m})$ implies $M \vDash \psi(\vec{m})$,
- $M \vDash \exists x^s \varphi(\vec{m}, x)$ if and only if there is an element $n \in M_s$ such that $M \vDash \varphi(\vec{m}, n)$,
- $M \vDash \forall x^s \varphi(\vec{m}, x)$ if and only if for every element $n \in M_s$ it holds that $M \vDash \varphi(\vec{m}, n)$.

From the categorical point of view, a structure M (if we forgot about the interpretation of relation symbols) is a finite products preserving functor:

$$M : \mathcal{C}_{\mathcal{L}} \longrightarrow \mathsf{Set} \tag{1.19}$$

Indeed, $\vec{x}$ in $\mathcal{C}_{\mathcal{L}}$ is a product $x_1 \times \cdots \times x_n$ of length 1 tuple (x_i) whose chosen sort is the one for x_i in $\vec{x}$. Hence the functor M chooses a set M_s for each sort s and then maps $\vec{x}$ to $\prod_i M_{s_i}$ where s_i is the sort of x_i. To be completely rigorous, we shall require that the functor M maps isomorphisms to identities so that $\vec{x}$ and $\vec{y}$ are mapped to the same set when they have same length and that each x_i has the same chosen sort that y_i. A model is the data of such a functor M together with a natural transformation μ

$$\begin{tikzcd} {\mathcal{C}_{\mathcal{L}}}^{\mathrm{op}} \arrow[r, "\mathbb{T}"] \arrow[d, "M^{\mathrm{op}}"'] & \mathsf{Cat} \arrow[dl, phantom, "\Downarrow \mu"] \\ \mathsf{Set}^{\mathrm{op}} \arrow[ur, bend right, "\mathrm{Sub}(-)"'] & \end{tikzcd} \tag{1.20}$$

such that $\mu_{\vec{x}} : \mathscr{F}^{\mathbb{T}}(\vec{x}) \longrightarrow \mathrm{Sub}(\prod_i M_{s_i})$ is a morphism of Heyting algebras that commutes with the left and right adjoints described at (1.12) and (1.18) as illustrated in diagrams (1.21) where the L's stand for left adjoint of their indices and R's for the right adjoints of their indices.

$$\begin{tikzcd} \mathscr{F}^{\mathbb{T}}(\vec{x}) \arrow[r, "L_{\mathbb{T}(\vec{t})}"] \arrow[d, "\mu_{\vec{x}}"'] & \mathscr{F}^{\mathbb{T}}(\vec{y}) \arrow[d, "\mu_{\vec{y}}"] \\ \mathrm{Sub}(\prod_i M_{s_i}) \arrow[r, "L_{\mathrm{Sub}(M(\vec{t}))}"'] & \mathrm{Sub}(\prod_j M_{s_j}) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}^{\mathbb{T}}(\vec{x}) \arrow[r, "R_{\mathbb{T}(\vec{t})}"] \arrow[d, "\mu_{\vec{x}}"'] & \mathscr{F}^{\mathbb{T}}(\vec{y}) \arrow[d, "\mu_{\vec{y}}"] \\ \mathrm{Sub}(\prod_i M_{s_i}) \arrow[r, "R_{\mathrm{Sub}(M(\vec{t}))}"'] & \mathrm{Sub}(\prod_j M_{s_j}) \end{tikzcd} \tag{1.21}$$

More explicitly, $\mu_{\vec{x}}$ corresponds to the morphism of Heyting algebras that maps a formula to the subset in the model that satisfies the formula.

Benefiting from this observation, one can now define more general models of a theory by simply modifying the pseudo functor Sub(−). Or conversely one can consider more general theory to interpret inside a pseudo functor like Sub(−). We shall in particular define what is a suitable replacement for $\mathbb{T}$. Lawvere proposes the notion of hyperdoctrines. There are several variations on this notion, but they all boil more or less to the following definition.

Definition. An hyperdoctrine is a pseudo functor $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ such that:

(i) $\mathcal{C}$ is a (small) category with finite products,

(ii) $P(f)$ has both a left adjoint $\exists_f$ and right adjoint $\forall_f$ for every f in $\mathcal{C}$,

(iii) for each pullback square of $\mathcal{C}$, the Beck-Chevalley condition holds for both kind of adjunctions.

We might add more or less structures in each of the category $P(x)$ for $x \in \mathcal{C}$, for example asking that they are cartesian closed, and then one requires that the functors $P(f)$ preserves that additional structure. Taking the point of view of the Grothendieck construction $p : \mathcal{E} \longrightarrow \mathcal{C}$, it amounts to put more or less structures on the functor p. Taken to the extreme, one can even think of any functor as presenting a kind of logic: this is argued by Melliès and Zeilberger in [MZ15].

The objects of the category $\mathcal{C}$ are called *contexts* and its morphisms are called morphisms of contexts and are to be thought as "term-tuples". The objects of the category $P(x)$ for some $x \in \mathcal{C}$ are called *types* or *predicates* over the context x, and the morphisms are called *proofs*. One could also just ask for a left adjoint $\exists_f$ for each $P(f)$ without necessarily requiring the right adjoint $\forall_f$ if we are interested only in the property of the existential quantifier. Lawvere does such a thing in his paper [Law70], only asking for a left adjoint and for the existence of a terminal object $\star_x$ in each $P(x)$[2]. And he calls this variation an *elementary existential doctrine*, or EED. This is all he need to reason on the equality predicates. Indeed, in a EED $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$, the equality predicate on a context x is defined as the predicate $\exists_{\Delta_x}(\star_x)$, where $\Delta_x : x \longrightarrow x \times x$ is the diagonal for x in $\mathcal{C}$. Alternatively, we can consider the Grothendieck construction $p : \mathcal{E} \longrightarrow \mathcal{C}$ of P. This is a Grothendieck bifibration by assumption, and the equality predicates is an object Id_x of $\mathcal{E}$ above $x \times x$ such that there is a cocartesian morphism $\star_x \longrightarrow \mathrm{Id}_x$ above $\Delta_x : x \longrightarrow x \times x$. It seems quite a reasonable definition when considering the previous example from which the motivation of hyperdoctrines come from. Indeed, in the case of the hyperdoctrine $\mathbb{T}$, the terminal object $\star_{\vec{x}}$ is the tautology $\vec{x} \doteq \vec{x}$ and its image through $\exists_{\Delta_{\vec{x}}}$ is precisely the statement $\vec{x} = \vec{y}$ if we have chosen $\Delta_{\vec{x}} : \vec{x} \longrightarrow \vec{x} \times \vec{y}$ with $\vec{y}$ a tuple of same length and same pointwise sorts as $\vec{x}$. Similarly, for the hyperdoctrine $\mathrm{Sub}(-)$, the terminal predicate over X is X itself and its image through the diagonal is exactly the equality relation on X: $\{(x, y) \in X \times X \mid x = y\}$.

Incidentally, Lawvere questions the definition of the equality in his paper [Law70]. Indeed, for the EED that have small categories as contexts and presheaves over $\mathcal{A}$ as types over $\mathcal{A}$, the adjoints $\exists_f$ are given by left Kan extensions, and the equality predicates $\mathrm{Id}_{\mathcal{A}}$ as defined above is far from meaningful. Already at that time, Lawvere has the intuition that a good equality predicate for $\mathcal{A}$ in this EED should be more in the lines of $\mathcal{A}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{A}$. Of course, this is not a presheaf on $\mathcal{A} \times \mathcal{A}$, so it is not encompassed by the framework developed above. However, Lawvere has this formidable observation:

> This should not to be taken as indicative of a lack of vitality of [this] hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. Equality should be the "graph" of the identity term. But present categorical conceptions indicates that, in the context of set-valued attributes, the graph of a functor $f : \mathcal{B} \longrightarrow \mathcal{C}$ should be [...] rather the corresponding "profunctor", a binary attribute of *mixed* variance [...]

Replacing small categories with small groupoids, the question of mixed variance is ancillary, and the intuition above is realized more formally in the '90s by Hofmann and Streicher in their model of type theory in the category of small groupoids (see [HS96]).

[2] Actually Lawvere asks for more structure on each $P(x)$, but this will be enough for what we want to present here.

The case of small categories and the problem of mixed variance has been tackled by Melliès and Zeilberger in their paper [MZ16].

Homotopy of type theory

In the early years of the 21$^{\text{th}}$, Vladimir Voevodsky started a program to develop strong foundations for proof assistance in abstract algebra. He was motivated by recent setbacks in his research in algebraic geometry when he spotted, with the help of the community, several errors in his papers that have been considered correct for years. By its own admission, he "got scared" and started focusing his work towards a mean to alleviate these kinds of situation.

From our understanding, his program picked up on the work in [HS96] where the identity proofs are interpreted as paths in groupoids. Unfortunately, the identity proofs between proofs does not appear in this model because the groupoids are "1-truncated". Voevodsky was coming from a field of mathematics connected to abstract homotopy theory. He was very acquainted with higher homotopy types and made the transition from groupoids to Grothendieck ∞-groupoids. He advocated the view of types as homotopy types, where inhabitants of a type are points in this space, and a proof of equality between two inhabitants is represented by a path between the two corresponding points. Equality proofs between such proofs can then be interpreted as homotopies between these paths and so on.

Formally speaking, the homotopy type theory as proposed by Voevodsky, and developed in full in 2012 by researchers from all around the globe during a semester at IAS devoted to this particular subject, is based on Martin-Löf type theory (MLTT for short) that we present formally in section 5.1. Informally speaking, MLTT is a logical framework based on the usual simply typed λ-calculus, but where *types* can depend on *values* of other types. This is something that the working mathematicians do all the time without looking for a formal justification of such a process. For example, when writing a statement such as

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^n, \exists y \in x^\perp, ||y|| = 1$$

one makes use of dependent type twice in a row. Once when writing $y \in x^\perp$ where the type $x^\perp$ of y depends of the value of the vector x. This is easily overcome without any use of dependent types by writing instead $y \in \mathbb{R}^n$ such that $y \perp x$. And actually saying the sentence out loud in English and asking two students to write it down in symbols, one might use the second possibility. However there is another use of dependent types in this sentence: the type $\mathbb{R}^n$ of x depends on the value of the natural number n. And this times, there is no easy way out. Of course it would still be possible to declare x in the big set $\coprod_{k \in \mathbb{N}} \mathbb{R}^k$ and then specifying that the length of x is indeed n, but it is quite cumbersome, and no student or working mathematician would naturally write the statement this way. MLTT is packed with constructors Σ, Π and Id that are to be understood roughly as follow:

- For a type $B(x)$ that depends on the value x which is of type A, one can construct the type $\Sigma_{x:A} B(x)$ whose elements are construed as the "pairs" (a, b) of an element a of A together with an element b of $B(a)$.

- For a type $B(x)$ that depends on the value x which is of type A, one can construct the type $\Pi_{x:A}B(x)$ whose elements are construed as the "functions" f that returns an element $f(a)$ of type $B(a)$ for each input a of type A.
- For two elements x and y of a same type A, one can construct a type $\mathrm{Id}_A(x,y)$ that can be construed as the proofs of equality of x and y.

These three constructions allow to consider propositions directly as types, where the Σ-construction embodies the existential quantifier, the Π-construction embodies the universal quantifier, and the Id-construction embodies the equality statement. The elements of a type are then construed as the different proofs that a proposition is true, and the absence of such elements indicates that the proposition is false. For example, given a monoid M with unit 1, the following type

$$\Pi_{x:M}\left(\Pi_{y:M}\left(\mathrm{Id}_M(y, x\cdot y)\times \mathrm{Id}_M(y, y\cdot x)\right)\longrightarrow \mathrm{Id}_M(x,1)\right) \tag{1.22}$$

ought to be non empty. Indeed, it is read "for all x in M, if for all y in M one has proofs that $y = xy$ and that $y = yx$ then one can craft a proof that $x = 1$". Or a little more type-theoretically, it should be possible to find a function that takes input x in M and returns a function that itself takes as input a function that maps every y to a pair of proofs of equality, one between y and $x\cdot y$ and the other one between y and $y\cdot x$, and that returns a proof of equality between x and 1.

From there, there are two separated ways to go, and they are mutually exclusive. The first possibility is to consider the *extensional* version of MLTT, in which x really is equal to y whenever $\mathrm{Id}_A(x,y)$ has an element. It is enticing but it rises a major problem from the proof assistant point of view as it renders type-checking non computable. However, from a mathematical standpoint, it is a perfectly sound thing to do. Seely has shown how to give semantics for such a type theory in any locally cartesian closed category $\mathcal{C}$, where a type $B(x)$ depending on a value x of a plain type A is interpreted as a morphism $B\longrightarrow A$ of $\mathcal{C}$. This reconciles very easily with Lawvere's view on logic by considering the pseudo functor $\mathcal{C}^{\mathrm{op}}\longrightarrow \mathsf{Cat}$ that associates to every object of $\mathcal{C}$ its slice $\mathcal{C}/A$, and to any morphism $f : A\longrightarrow B$, the *substitution functor* $f^* : \mathcal{C}/B\longrightarrow\mathcal{C}/A$ that is defined by a choice of pullback. These functors always have a left adjoint given by the postcomposition and the assumption of local cartesian closedness exactly states the existence of a right adjoint for those functors f^*. In the language of Grothendieck fibrations it says that the functor $\mathrm{cod} : \mathrm{Fun}(\mathbf{2},\mathcal{B})$ is a bifibration, as well as the Grothendieck construction of the pseudo functor $A\mapsto \mathcal{C}/A^{\mathrm{op}}$.

The other way is called *intentional* MLTT, where nothing of the sort as $x = y$ is required when $\mathrm{Id}_A(x,y)$ is inhabited. In that case, Seely's interpretation of MLTT cannot be used as such: one shall restrain the maps $B\longrightarrow A$ of the category $\mathcal{C}$ that can interpret dependent types. This restricted structure is known under the name of *display maps category* or *full comprehension category* or *full D-categories*. It does not reconcile that much with Lawvere's view of logic through hyperdoctrines: there still is a pseudo functor $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathsf{Cat}$ that maps an object A to its slice category $\mathcal{C}/A$, but the functor f^* will not have adjoints on the left and on the right for every morphism f; only the maps f chosen to interpret dependent types will. In particular, the diagonal $\Delta_A : A\longrightarrow A\times A$ is generally not part of the chosen maps, for if it was we get back basically to the extensional case. Hence there is no way to "push" the terminal predicate of A over Δ_A. As indicated by Lawvere's quote of [Law70], this should not be taken as a lack of vitality of these models, but rather as an excess of naivety from our part when trying to model equality in a manner too closely suggested by classical conceptions. In this thesis, we tackle this issue in particular, and we try to design a framework limber enough to incorporate "classical conceptions" of equalities as well as newer weaker ones.

On top of intentional MLTT, Voevodsky enforces an axiom now known as *univalence*. More precisely, univalence is a property of *universe types* and Voevodsky asks for the existence of a univalent universe. Without getting very formal, this axiom is tricky to state and one quickly falls into the trap of saying something quite inaccurate, or even non sensical. For the axiom quickly reduces to a tautology in the extensional case that our intuition is usually based on. Also we delay our discussion on the univalence axiom to section 5.1.5 where all the formal material will have been given and the axiom will be able to be stated in a correct manner. For now, let us say that univalence would have been hard to even find as axiom without the topological intuition put on proofs of equalities, now seen as paths, so that functions $f, g : A \longrightarrow B$ between types can be considered homotopic when they are "pointwise equal", meaning that the type $\Pi_{x,y:A} \mathrm{Id}_B(f(x), g(x))$ is inhabited. All the juice of intentional MLTT and Voevodsky's homotopical interpretation of it comes from the difference between such an inhabited type and an identification of the functions f and g itself.

Overview of this thesis

This thesis is organized around three main topics: Quillen bifibrations, homotopy categories of Quillen bifibrations and generalized tribes. Quillen bifibrations is kind of an indexed version of Quillen model categories. The study of their homotopy categories counterpart is quite natural, and it leads to a structure with some sort of "homotopy cartesian" and "homotopy cocartesian" morphisms. The intuition developed around this structure is a big motivation for the generalized of Joyal tribes we propose afterwards, in which we try and give a formal version of these "homotopy (co)cartesian" morphisms.

Chapter 2 first discusses the foundational settings of this memoir, and then gives a complete introduction to model categories. The framework in which this theory is developed is slightly more general than usual: our model categories need not have all pushouts or all pullbacks. This larger notion of a model category has been introduced by Jeff Egger in [Egg16] and we replay most of Quillen classical result on closed model categories in Egger's framework. This larger framework is needed in chapter 4 where we will consider localization of homotopy categories by means of quotient by an homotopy relation (see more specifically section 4.4. We have decided to relegate the introduction material on type theory to chapter 5 in order to create a more pleasant experience for the reader: indeed chapter 3 and chapter 4 can be read with no prior knowledge on type theory.

Chapter 3 introduces a new notion: Quillen bifibrations. We prove there a fundamental theorem for those structures that can be thought a Grothendieck construction for pseudo functors from a model category to the 2-category of model structures and Quillen adjunctions and natural transformations. It gives a criterion for a Grothendieck bifibration that have a model structure on the basis and in each fibers to glue these together into a model structure in the total category. These sufficient conditions are quite light and elegant, and it shed a new light on the more cumbersome (sufficient) conditions that can be found in the literature for the same problem. It generalizes in particular part of the work presented in [Roi94], [Sta12] and [HP15]. This chapter also gives a detailed reconstruction of Kan's theorem about Reedy model structures along the line of the main theorem presented above.

Chapter 4 copes with localizations of Quillen bifibrations. Any Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ induces a functor $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \longrightarrow \mathbf{Ho}(\mathcal{B})$ between the homotopy categories. We show how one can use the main theorem of chapter 3 to presents the homotopy category $\mathcal{E}$ as the homotopy category of a model structure on $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$ for a carefully constructed model category $\mathcal{E}_{\mathrm{fw}}$. This is where the framework redeveloped in chapter 2 from Egger's results comes into play: the category $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$ does not have pullbacks and pushouts *a priori*; however it bears a model structure which we would like to take the homotopy quotient from. Egger's definition of model categories solves the problem, and by this construction we provide the first example of iterated homotopy categories, as anticipated by Egger at the end of its paper [Egg16]. This chapter is also the place where our intuition meets for the first time a kind of "push functor" only defined up to homotopy. Nothing of this kind is formally stated but section 4.2 in particular have been crucial in the process of conceiving a notion resembling an homotopy push and designing the structures found in chapter 5.

Chapter 5 starts with a big overview of MLTT. We try to be as precise as possible without bothering too much with syntactical subtleties. This overview follow an increasing-in-structures presentation that is meant to match the increase in structures on the semantics side that we present later in the chapter. The chapter continue with a presentation of Joyal's notion of tribes. We put a heavy emphasis on the type theory that it is supposed to interpret. Once all that has been introduced we are ready to present our newly developed notion of *relative factorization system*. This is the structure that can both incorporate an equality predicates as in Lawvere's work and and path object as in the tribal interpretation of MLTT. It leads us to a notion of relative tribe in which we try to put together all the structure needed on a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ to give a meaningful interpretation of an MLTT without giving up on Lawvere's approach to semantics. Finally the chapter opens up on an appealing simplicial enhancement of the classical semantics, usually confined to a unique functor. This last part is still work in progress.

Chapter 2

Background material on model categories

This chapter is devoted to develop the material on homotopical algebra needed in the rest of the memoir. First we give a quick account on the foundational setting we are working in. Next we cover the theory of model categories on a slightly more general framework than usual. No prior knowledge about this subject is required, but the reader is assumed to be at ease with category theory.

2.1 Foundational issues

Our work is mostly foundation independent, but we will use terms as *small* and *large*, hence we shall fix conventions in this section. Here are two possible foundations among which the reader is free to choose the one that suits best their needs.

The first possible ambient theory is a set theory with Grothendieck universes. Grothendieck universes are defined to be sets $\mathbb{U}$ satisfying

(i) if $s \in \mathbb{U}$ and $t \in s$, then $t \in \mathbb{U}$,

(ii) if $s \in \mathbb{U}$, then its powerset (s) is in $\mathbb{U}$,

(iii) for a family $(s_i)_{i \in I}$ of sets in $\mathbb{U}$ indexed by $I \in \mathbb{U}$, then $\bigcup_{i \in I} s_i \in \mathbb{U}$,

(iv) $\mathbb{N} \in \mathbb{U}$

On top of ZFC we add the following axiom, called *axiom of universes*,

$$\forall s, \exists \mathbb{U}, (s \in \mathbb{U}) \wedge \mathbb{U} \text{ is a Grothendieck universe} \tag{2.1}$$

Iterating this axiom from the empty set $\varnothing$, we end up with a tower of universes

$$\mathbb{U}_0 \in \mathbb{U}_1 \in \mathbb{U}_2 \in \dots \tag{2.2}$$

Each of the $\mathbb{U}_i$ being universe, it is in particular a model of ZFC. Hence, most of the construction made from within $\mathbb{U}_i$ stays in $\mathbb{U}_i$, but sometimes you have to *jump one universe up*. Elements of $\mathbb{U}_0$ will be called *small sets*, while those of $\mathbb{U}_1$ will be called *large sets*. The transitivity of universes assures that any small sets is a large one. Large sets that are not small will be called *properly large*. By analogy, elements of $\mathbb{U}_i$ are sometimes called $\mathbb{U}_i$-small. Most of the time, small sets, large sets and the knowledge that the collection of large sets is $\mathbb{U}_2$-small for some universe $\mathbb{U}_2$ is enough.

Definition 2.1.1. Given universes $\mathbb{U} \in \mathbb{V}$, a locally $\mathbb{U}$ $\mathbb{V}$-category $\mathcal{C}$ is the data of sets

$$\operatorname{Ob}\mathcal{C} \in \mathbb{V}, \quad \mathcal{C}(x,y) \in \mathbb{U}\ \forall x,y \in \operatorname{Ob}\mathcal{C} \tag{2.3}$$

and of functions

$$\begin{aligned} \mathrm{id} \,:\, \operatorname{Ob}\mathcal{C} &\longrightarrow \bigcup_{x\in\operatorname{Ob}\mathcal{C}} \mathcal{C}(x,x), \\ - \circ - \,:\, \mathcal{C}(y,z) \times \mathcal{C}(x,y) &\longrightarrow \mathcal{C}(x,z)\ \ \forall x,y,z \in \operatorname{Ob}\mathcal{C} \end{aligned} \tag{2.4}$$

subject to the usual axioms of category theory:

(i) $h \circ (g \circ f) = (h \circ g) \circ f$ for all f, g, h for which it makes sense,

(ii) $\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$ for all $f \in \mathcal{C}(x,y)$.

Locally $\mathbb{U}$ $\mathbb{U}$-categories are simply called $\mathbb{U}$-categories. In particular any locally $\mathbb{U}$ $\mathbb{V}$-category is a $\mathbb{V}$-category. *Small categories* will refer to $\mathbb{U}_0$-categories, *locally small categories* (or simply *categories*) will refer to locally $\mathbb{U}_0$ $\mathbb{U}_1$-categories, and *large categories* will refer to $\mathbb{U}_1$-categories.

The second foundation that is suitable for our needs is an *extensional* dependent type theory with Π- and Σ-types and with a hierarchy of universe types. We insist here on the *extensional* part : the witnesses of equality between terms of the same type are not to be taken too much seriously; either there is such a witness and the two terms are actually the same, or there is not and they are distinct. Keeping serious track of witnesses of equalities leads to the homotopical interpretations of type theory, a task that this memoir tries to tackle. But the foundational basis in which this study takes place (which is the matter at hand in this section) need not cope with such hassles. Let us give a quick presentation of such a foundational type theory; more will be said about the exact rules of such type theories in the intentional framework in section 5.1, so for now we shall opt for a less formal approach that underline better the similarities with a material set theory as presented before. Describe types inductively as

- either *base types* such as the empty type $\varnothing$, the unit type $\mathbf{1}$ and the type of natural numbers $\mathbb{N}$,
- or *constructed types* as product types $A \times B$, a sum types $A + B$, a function types $A \longrightarrow B$ where A, B are previously well-formed type

We declare an *inhabitant* a of a type A by the syntax $x : A$. The empty type has no inhabitant, the unit type has a unique one $* : \mathbf{1}$. The inhabitants of $\mathbb{N}$ are the natural numbers. The inhabitants of $A \times B$ are the pairs (a, b) for $a : A$ and $b : B$, those of $A + B$ are both those of A and those of B. The inhabitants $f : A \longrightarrow B$ of a function type can be *applied* to any inhabitant $a : A$ to yield an inhabitant $f(a) : B$; and given inhabitants $f_a : B$ for every $a : A$, we can craft $\lambda x.f_x : A \longrightarrow B$ with the property that $(\lambda x.f_x)(a)$ is the inhabitant $f_a : B$.

On top of this simple type theory, we require a *cumulative hierarchy* of universe types:

$$\mathbb{U}_0 : \mathbb{U}_1 : \mathbb{U}_2 : \ldots \tag{2.5}$$

where each $\mathbb{U}_i$ is a type *and* a inhabitant of $\mathbb{U}_{i+1}$. We require for any of the types A recursively defined before, $A : \mathbb{U}_0$. These universes are cumulative in the sense that every inhabitant $x : \mathbb{U}_i$ of the level i is also an inhabitant of the universe above $x : \mathbb{U}_{i+1}$. These universe types are required to have the property that given a inhabitant $B : A \longrightarrow \mathbb{U}_i$, called a *dependent type*, there are types

$$\Sigma_A B : \mathbb{U}_i, \qquad \Pi_A B : \mathbb{U}_i \tag{2.6}$$

respectively called *dependent pairs type* and *dependent function type*. The inhabitant of $\Sigma_A B$ are the pairs (a, b_a) with $a : A$ and $b_a : B(a)$. Hence, if $B : \mathbb{U}_i$, the type $\Sigma_A(\lambda x.B)$ is just the type of all pairs and we ask it coincides with $A \times B$. The inhabitant $f : \Pi_A B$ can be *applied* to any inhabitant $a : A$ to yield an inhabitant $f(a) : B(a)$; and given $f_a : B(a)$ for each $a : A$, we can craft $\lambda x.f_x : \Pi_A B$ with the property that $(\lambda x.f_x)(a)$ is the inhabitant $f_a : B(a)$. In particular, for $B : \mathbb{U}_i$, the type $\Pi_A(\lambda x.B)$ has the same properties as $A \longrightarrow B$ and we actually ask that they are the same type. To make the notation less cumbersome, we might write $\Sigma_{x:A}B(x)$ and $\Pi_{x:A}B(x)$ instead of their rigorous form $\Sigma_A(\lambda x.B(x))$ and $\Pi_A(\lambda x.B(x))$.

Finally, we require that for each $A : \mathbb{U}_i$, there is a dependent type $\mathrm{Id}_A : A \times A \longrightarrow \mathbb{U}_i$ with the special property that the type $\mathrm{Id}_A(a, a')$ is inhabited if and only if a is the same inhabitant of A than a'[1]. In order to make such a type theory has more or less as expressive as a set theory with Grothendieck universes, we need to add an *axiom*, in the form of a inhabitant as follow for each $\mathbb{U}_i$, called function extensionality,

$$\mathrm{ext}_{A,B} : \Pi_{f,g:A\to B}\left((\Pi_{x:A}\,\mathrm{Id}_B(f(x), g(x))) \longrightarrow \mathrm{Id}_{A\to B}(f, g)\right) \tag{2.7}$$

It witnesses the equality of f and g as inhabitant of $A \longrightarrow B$ whenever their are pointwise equal in B.

We shall say that inhabitants of $\mathbb{U}_0$ are *small*, and those of $\mathbb{U}_1$ are *large*. By the cumulative property any small type is large. We shall use *properly large* to describe inhabitants of $\mathbb{U}_1$ which are not inhabitants of $\mathbb{U}_0$. By analogy, types inhabiting $\mathbb{U}_i$ will sometimes be referred as $\mathbb{U}_i$-*small*.

Definition 2.1.2. Given $i \leq j$, a locally $\mathbb{U}_i$ $\mathbb{U}_j$-category $\mathcal{C}$ is the data of

$$\mathrm{Ob}\,\mathcal{C} : \mathbb{U}_j, \qquad \mathcal{C}(-,-) : \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathbb{U}_i \tag{2.8}$$

together with inhabitants

$$\begin{aligned} \mathrm{id} &: \Pi_{x:\mathrm{Ob}\,\mathcal{C}}\,\mathcal{C}(x,x), \\ -\circ- &: \Pi_{x,y,z:\mathrm{Ob}\,\mathcal{C}}\left(\mathcal{C}(y,z) \times \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z)\right) \end{aligned} \tag{2.9}$$

satisfying the usual axioms, meaning the following types are inhabited

$$\begin{aligned} &\Pi_{x,y:\mathrm{Ob}\,\mathcal{C}}\Pi_{f:\mathcal{C}(x,y)}\,\mathrm{Id}_{\mathcal{C}(x,y)}\left((\mathrm{id}_y \circ f), f\right) \\ &\Pi_{x:\mathrm{Ob}\,\mathcal{C}}\Pi_{f:\mathcal{C}(x,x)}\,\mathrm{Id}_{\mathcal{C}(x,y)}\left(f, (\mathrm{id}_x \circ f)\right) \\ &\Pi_{x,y,z,t:\mathrm{Ob}\,\mathcal{C}}\Pi_{f:\mathcal{C}(x,y)}\Pi_{g:\mathcal{C}(y,z)}\Pi_{h:\mathcal{C}(z,t)}\,\mathrm{Id}_{\mathcal{C}(x,t)}\left(h \circ (g \circ f), (h \circ g) \circ f\right) \end{aligned} \tag{2.10}$$

[1]This is where the *extensionality* is kicking off.

Locally $\mathbb{U}_i$ $\mathbb{U}_i$-categories are simply called $\mathbb{U}_i$-categories. In particular any locally $\mathbb{U}_i$ $\mathbb{U}_j$-category is a $\mathbb{U}_j$-category. *Small categories* will refer to $\mathbb{U}_0$-categories, *locally small categories* (or simply *categories*) will refer to locally $\mathbb{U}_0$ $\mathbb{U}_1$-categories, and *large categories* will refer to $\mathbb{U}_1$-categories.

Whatever foundation is chosen, smallness of sets/types will always be assumed when not stated otherwise. In particular, the category Set of sets is actually denoting the locally small category of small sets. The same goes for the category Grp of groups, the category Top of topological spaces, or any kind of category of sets endowed with some structure. Among those is the category Cat of small categories. In contrast, the locally large $\mathbb{U}_2$-category of locally small categories will be denote CAT.

2.2 Model categories

Model categories were first introduced by Quillen in [Qui67] as a framework to perform *non-abelian homology*. Homological constructions, like derived functors, are usually done starting from abelian categories $\mathcal{A}$ and $\mathcal{B}$ with nice properties; additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ induce functors $\mathrm{Ch}(\mathcal{A}) \longrightarrow \mathrm{Ch}(\mathcal{B})$ between the categories of chain complexes, which in turn induce functors $(\mathcal{A}) \longrightarrow (\mathcal{B})$ between the categories of chain complexes modulo chain homotopies. For example, when $\mathcal{A}$ has enough projectives, the nth left derived functor of $F : \mathcal{A} \longrightarrow \mathcal{B}$ is defined to be the functor $\mathcal{A} \longrightarrow \mathcal{B}$ obtained on an object X by first taking its projective resolution in $\mathrm{Ch}(\mathcal{A})$, viewing it in $(\mathcal{A})$, applying (F) to land in $(\mathcal{B})$ and finally taking the nth homology of this chain complexes. By considering simplicial objects instead of chain complexes, Dold and Puppe were able to define derived functors for a non-additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ when $\mathcal{A}$ is an abelian category with enough projectives. Building on these ideas, Quillen designed the general framework of model categories to remove the abelian condition on $\mathcal{A}$. Under mild conditions on a category $\mathcal{A}$ with enough projectives, Quillen is able to endow the category $s\mathcal{A}$ of simplicial objects in $\mathcal{A}$ with a (simplicial) model structure. Whenever $\mathcal{A}$ is abelian, the Dold-Kan normalization equivalence

$$N : s\mathcal{A} \xrightarrow{\sim} \mathrm{Ch}(\mathcal{A})$$

allows to transport this structure to make $\mathrm{Ch}(\mathcal{A})$ a model category in which the homotopies coincide with the usual chain homotopies, and where cofibrant replacements are projective resolutions. The usefulness of such a shift in perspective appears in the following situation: given a category $\mathcal{C}$ with finite products and its associated category $\mathcal{C}_{\mathrm{ab}}$ of abelian group objects, suppose there is an adjunction

$$\mathcal{C} \underset{i}{\overset{\mathrm{ab}}{\rightleftarrows}} \mathcal{C}_{\mathrm{ab}}$$

where the right adjoint i is the forgetful functor. If $\mathcal{C}$ and $\mathcal{C}_{\mathrm{ab}}$ both are model categories in such a way that this adjunction is a Quillen adjunction (to be defined), then cohomology groups $\mathrm{H}^{\bullet}_{M}(X, A)$ of objects $X \in \mathcal{C}$ with coefficients $A \in \mathcal{C}_{\mathrm{ab}}$ can be defined as particular hom-sets of $\mathbf{Ho}(\mathcal{C}_{\mathrm{ab}})$. This settings is fit for the case where $\mathcal{C} = s\mathcal{A}$ with a category $\mathcal{A}$ satisfying the mild conditions of before, for which the category of abelian group objects can be expressed as $\mathcal{C}_{\mathrm{ab}} = s(\mathcal{A}_{\mathrm{ab}})$. If $\mathcal{A}_{\mathrm{ab}}$ happens to be abelian, then the Dold-Kan normalization eases the computation of the groups $\mathrm{H}^{\bullet}_{M}(X, A)$, in particular for $X \in \mathcal{A}$ and $A \in \mathcal{A}_{\mathrm{ab}}$ both viewed as constant simplicial objects. In the case where $\mathcal{A} = \mathsf{Set}$, we get back the usual cohomology of simplicial sets. In the case where $\mathcal{A}$ is the category of groups, we get back the usual cohomology of groups. But the case of interest for Quillen was originally where $\mathcal{A}$ is the category Alg_R/X of commutative R-algebras over a fixed one X, for which $\mathcal{A}_{\mathrm{ab}}$ is equivalent to the category of X-modules. The machinery described above allows the definition of cohomology groups of (simplicial) R-algebras over X with coefficients in X-modules, now called Quillen cohomology, which has since proved to be important in algebraic geometry.

Model categories have since made themselves essential way beyond Quillen's original motivations. They are a tool of choice to work with higher structures and have been advertised as *presentation of homotopy theories*. Much like a presentation of a group is a practical way to compute elements of the group, a model category $\mathcal{C}$ with weak equivalence $\mathfrak{W}$ offers effective procedures to compute data associated with the homotopy theory underlying the couple $(\mathcal{C}, \mathfrak{W})$. By homotopy theory here is actually meant the $(\infty, 1)$-category that is produced from $(\mathcal{C}, \mathfrak{W})$ through Dwyer-Kan localization (see [DK80]). Having many models of a same homotopy theory is a clear advantage: to each application a suitable model can be used. An example of the power of such a technique is given by the homotopy theory of *spaces*, which counts among its models:

- The category Top of topological spaces with the Serre model structure, which defines most easily the notion of *homotopy type* from the historical definition of homotopies,
- The category $\mathbf{S}$ of simplicial sets with the Quillen model structures, which ease the calculations by giving a combinatorial machinery that implements Whitehead's presentation of homotopy types as CW-complexes.
- The category $\mathrm{Psh}(\mathcal{A})$ of presheaves over any test category $\mathcal{A}$ with the Cisinski model structure. This is analogous to the previous presentation, which is the case $\mathcal{A} = \Delta$. Various $\mathcal{A}$ allows to support computations by various geometric intuitions (simplicial, cubical, globular, etc.).
- The category Cat with the Thomason model structure, which reflects Grothendieck's conceptual view of homotopy types through small categories.

A second example of the flexibility given by the existence of many model categories for a same homotopy theory is the theory of $(\infty, 1)$-categories themselves, which is presented equivalently by Joyal's quasicategories, Rezk's complete Segal spaces, Bergner's simplicially enriched categories, Dwyer and Kan's relative categories, and Pelissier's Segal categories. Depending on the field of studies, one model can be better suited to actually compute things, but in the end the homotopy-invariant statements in each of these will be about the $(\infty, 1)$-category of $(\infty, 1)$-categories.

2.2.1 Weak factorization systems and model structures

A class of maps $\mathfrak{W}$ in a category $\mathcal{C}$ is said to have the *2-out-of-3 property* when for every commutative triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{2.11}$$

if two of the three arrows are in $\mathfrak{W}$ then so is the third.

An arrow $f : A \longrightarrow B$ is said to have the left lifting property relatively to $g : X \longrightarrow Y$, or equivalently g is said to have the right lifting property relatively to f, and denote $f \boxslash g$, when the following map is surjective:

$$\mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X) \times_{-\circ f, g\circ -} \mathcal{C}(B, Y) \tag{2.12}$$

More explicitly it means that for any $x : A \longrightarrow X$ and any $y : B \longrightarrow Y$ such that commutes the following diagram:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \tag{2.13}$$

there exists a map $h : B \longrightarrow X$ such that commutes the two triangles in

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "y"'] \arrow[ur, "h"] & Y \end{tikzcd} \tag{2.14}$$

Sometimes h is called a filler for the diagram (2.13).

If f has the left lifting property relatively to every map in a class $\mathfrak{C}$, we shall say that f has the left lifting property relatively to $\mathfrak{C}$, and we denote it $f \in \mathfrak{C}^{\boxslash}$. Similarly, if g has the right lifting property relatively to every element of a class $\mathfrak{C}$ of maps, then we shall say that g has the right lifting property relatively to $\mathfrak{C}$ and we denote $g \in \mathfrak{C}^{\boxslash}$.

Definition 2.2.1. A *weak factorization system* on $\mathcal{C}$ is the data of two classes $(\mathfrak{L}, \mathfrak{R})$ of maps such that:

(i) $\mathfrak{L} = {}^{\boxslash}\mathfrak{R}$ and $\mathfrak{L}^{\boxslash} = \mathfrak{R}$,

(ii) for any $f : A \longrightarrow B$ in $\mathcal{C}$, there exists $j \in \mathfrak{L}$ and $q \in \mathfrak{R}$ such that $f = qj$.

Lemma 2.2.2. *Let $(\mathfrak{L}, \mathfrak{R})$ be a weak factorization system on a category $\mathcal{C}$. For any commutative square in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \tag{2.15}$$

(i) If (2.15) is a pullback square and $g \in \mathfrak{R}$, then $f \in \mathfrak{R}$.

(ii) If (2.15) is a pushout square and $f \in \mathfrak{L}$, then $g \in \mathfrak{L}$.

Proof. Actually, this lemma is even true if $(\mathfrak{L}, \mathfrak{R})$ is only meeting condition (i) of the definition of weak factorization systems. Suppose (2.15) is a pullback square and $g \in \mathfrak{R}$. To prove f is also in $\mathfrak{R}$, we should show that f as the right lifting property relatively to elements of $\mathfrak{L}$. Consider then a commutative square:

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "b"'] & B \end{tikzcd} \quad \text{with } f' \in \mathfrak{L} \tag{2.16}$$

Then by pasting the square (2.15) on the right, one gets a new commutative square:

$$\begin{tikzcd} A' \arrow[r, "xa"] \arrow[d, "f'"'] & X \arrow[d, "g"] \\ B' \arrow[r, "yb"'] & Y \end{tikzcd} \tag{2.17}$$

We can use the right lifting property of g relatively to $f' \in \mathfrak{L}$ to get $h : B' \longrightarrow X$ filling (2.17). In particular $gh = yb$, and the universal property of the pullback square (2.15) gives a (unique) morphism $h' : B' \longrightarrow A$ making the following commute:

$$\begin{tikzcd} & A \arrow[r, "x"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B' \arrow[ur, "h'"] \arrow[urr, "h"] \arrow[r, "b"'] & B \arrow[r, "y"'] & Y \end{tikzcd} \tag{2.18}$$

Then h' is a filler of (2.16). Indeed, we already have $fh' = b$ by definition of h'. Also $h'f'$ and a both are solution to the universal problem of finding $\xi : A' \longrightarrow A$ such that $x\xi = xa$ and $f\xi = bf'$: such a ξ is unique, hence $h'f' = a$.

Proof of the second property is dual. □

Lemma 2.2.3. *Let $(\mathfrak{L}, \mathfrak{R})$ be a weak factorization system on a category $\mathcal{C}$. Then both $\mathfrak{L}$ and $\mathfrak{R}$ are stable under retracts. Meaning that for any commutative diagram in $\mathcal{C}$ on the following form:*

$$\begin{tikzcd} A \arrow[r, "s_0"'] \arrow[d, "f"'] \arrow[rr, bend left, equal] & X \arrow[r, "r_0"'] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "s_1"] \arrow[rr, bend right, equal] & Y \arrow[r, "r_1"] & B \end{tikzcd} \tag{2.19}$$

(i) If $g \in \mathfrak{R}$, then $f \in \mathfrak{R}$.

(ii) If $g \in \mathfrak{L}$, then $f \in \mathfrak{L}$.

Proof. Suppose $g \in \mathfrak{R}$. To prove that $f \in \mathfrak{R}$, we will show that f has the right lifting property relatively to any elements of $\mathfrak{L}$. Take a commutative square in $\mathcal{C}$:

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "b"'] & B \end{tikzcd} \quad \text{with } f' \in \mathfrak{L} \tag{2.20}$$

Pasting the left half of (2.19) on the right, we get a commutative square with $f' \in \mathfrak{L}$ on the right and $g \in \mathfrak{R}$ on the right, hence there exists a filler $h : A' \longrightarrow X$ making the following diagram commute:

$$\begin{tikzcd} A' \arrow[r, "s_0 a"] \arrow[d, "f'"'] & X \arrow[d, "g"] \\ B' \arrow[r, "s_1 b"'] \arrow[ur, "h"] & Y \end{tikzcd} \tag{2.21}$$

Now paste the right half of (2.19) on the right of this diagram to get:

$$\begin{tikzcd} A' \arrow[r, "s_0 a"] \arrow[d, "f'"'] & X \arrow[r, "r_0"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B' \arrow[r, "s_1 b"'] \arrow[ur, "h"] & Y \arrow[r, "r_1"'] & B \end{tikzcd} \tag{2.22}$$

It presents $r_0 h$ as a filler for (2.20). Indeed $r_0 h f' = r_0 s_0 a = a$ and $f r_0 h = r_1 s_1 b = b$. $\square$

Definition 2.2.4. A *model structure* on a category $\mathcal{C}$ is the data of three classes $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ such that

(i) $\mathfrak{W}$ satisfies the 2-out-of-3 property,

(ii) $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ both are weak factorization systems.

Elements of $\mathfrak{C}$ are called *cofibrations*, elements of $\mathfrak{F}$ are called *fibrations*, and elements of $\mathfrak{W}$ are called *weak equivalences*

Remark 2.2.5. The data of a model structure is redundant: if $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ forms a model structure on $\mathcal{C}$, the knowledge of two of the classes determines the third one. Indeed, if $\mathfrak{C}$ and $\mathfrak{W}$ are given then $\mathfrak{F}$ is recovered as $(\mathfrak{C} \cap \mathfrak{W})^{\boxslash}$. Similarly, if $\mathfrak{F}$ and $\mathfrak{W}$ are given then $\mathfrak{C}$ is recovered as ${}^{\boxslash}(\mathfrak{F} \cap \mathfrak{W})$. Finally if $\mathfrak{C}$ and $\mathfrak{F}$ are given, then elements of $\mathfrak{W}$ are exactly those qj with $q \in \mathfrak{C}^{\boxslash}$ and $j \in {}^{\boxslash}\mathfrak{F}$: indeed all these composite are in $\mathfrak{W}$ by the 2-out-of-3 property; and conversely, every element of $\mathfrak{W}$ can be written as qj with $q \in \mathfrak{F}$ and $j \in \mathfrak{C} \cap \mathfrak{W}$ and we deduce $q \in \mathfrak{W}$ also by the 2-out-of-3 property.

2.2.2 Model categories: definitions

Definition 2.2.6. A *Quillen model category* is a finitely complete and finitely cocomplete category $\mathcal{C}$ together with a model structure.

Quillen originally introduces model categories in another equivalent way that we recall here because it makes more clear some of the points we ought to discuss below.

Definition 2.2.7 (Quillen's original definition)**.** A *Quillen model category* is a category $\mathcal{C}$ with three classes of morphisms $\mathfrak{C}, \mathfrak{W}, \mathfrak{F}$ such that:

M0 $\mathcal{C}$ admits all finite limits and all finite colimits.

M1 $\mathfrak{C} \cap \mathfrak{W} \subseteq {}^{\boxslash}\mathfrak{F}$ and $\mathfrak{F} \cap \mathfrak{W} \subseteq \mathfrak{C}^{\boxslash}$.

M2 Every map f can be factored as $f = qj$ with $q \in \mathfrak{F}$ and $j \in \mathfrak{C} \cap \mathfrak{W}$. Every map f can be factored as $f = qj$ with $q \in \mathfrak{F} \cap \mathfrak{W}$.

M3 $\mathfrak{F}$ contains all isomorphisms and is stable under composition and base change. $\mathfrak{C}$ contains all isomorphisms and is stable under composition and co-base change.

M4 Base changes of elements of $\mathfrak{F} \cap \mathfrak{W}$ is in $\mathfrak{W}$. Co-base changes of elements of $\mathfrak{C} \cap \mathfrak{W}$ is in $\mathfrak{W}$.

M5 $\mathfrak{W}$ contains all isomorphisms and has the 2-out-of-3 property.

M6 Each of the classes $\mathfrak{F}, \mathfrak{C}, \mathfrak{W}$ is stable under retracts.

In Quillen's monograph [Qui67], axiom **M6** only refers to *closed model category* that Quillen chose to differentiate from model categories. This property happens to be crucial to the *saturation* of the weak equivalences (on which we will come back later in this chapter), and modern textbooks does not even consider non closed model categories. We shall follow this convention also and just include axiom **M6** in the definition of a Quillen model category. Let us insist on the fact that definition 2.2.7 is strictly equivalent to definition 2.2.6 (a careful proof is given in [MP12, 14.1]). In particular, and to make easier the exposure of some technical points below, here is the proof of the fact that the modern definition given in 2.2.6 entails axiom **M6**. As any left or right class of a weak factorization system is closed under retracts, $\mathfrak{F}$ and $\mathfrak{C}$ are taken care of and we shall concentrate on weak equivalences.

Lemma 2.2.8. *In a Quillen model category $\mathcal{C}$ in the sense of 2.2.6, if $g \in \mathfrak{C}$ is a retract of $f \in \mathfrak{W}$, then $g \in \mathfrak{W}$.*

Proof. Let the following diagram present g as a retract of f

$$\begin{tikzcd} A' \arrow[r, "s_0"] \arrow[d, "g"'] \arrow[rr, bend left, equal] & A \arrow[r, "r_0"] \arrow[d, "f"] & A' \arrow[d, "g"] \\ B' \arrow[r, "s_1"'] \arrow[rr, bend right, equal] & B \arrow[r, "r_1"'] & B' \end{tikzcd} \tag{2.23}$$

Factor f as pi where $p \in \mathfrak{F} \cap \mathfrak{W}$ and $i \in \mathfrak{C}$. By the 2-out-of-3 property, i actually is an acyclic cofibration. Now using the left lifting property of g relatively to p, we obtain a commutative diagram as follow:

$$\begin{tikzcd} A' \arrow[r, "s_0"] \arrow[dd, "g"'] \arrow[rr, bend left, equal] & A \arrow[r, "r_0"] \arrow[d, "i"] & A' \arrow[dd, "g"] \\ & C \arrow[d, "p"] \arrow[dr, "r_1 p"] & \\ B' \arrow[r, "s_1"'] \arrow[ur, "h"] \arrow[rr, bend right, equal] & B \arrow[r, "r_1"'] & B' \end{tikzcd} \tag{2.24}$$

It presents g as a retract of i. But $\mathfrak{C} \cap \mathfrak{W}$, as the left class of a weak factorization system, is stable under retract (see lemma 2.2.3). Hence $g \in \mathfrak{W}$. $\square$

Lemma 2.2.9. *In a Quillen model category $\mathcal{C}$ in the sense of 2.2.6, the class $\mathfrak{W}$ of weak equivalences is closed under retracts.*

Proof. Let $f : A \longrightarrow B \in \mathfrak{W}$ and a retract g of f as follow:

$$\begin{tikzcd}
A' \arrow[r, "s_0"] \arrow[d, "g"'] \arrow[rr, bend left, equal] & A \arrow[r, "r_0"] \arrow[d, "f"] & A' \arrow[d, "g"] \\
B' \arrow[r, "s_1"'] \arrow[rr, bend right, equal] & B \arrow[r, "r_1"'] & B'
\end{tikzcd} \tag{2.25}$$

Factor g as $q'j'$ with $q' \in \mathfrak{F} \cap \mathfrak{W}$ and $j' \in \mathfrak{C}$. All we need to show is that j' is a weak equivalence then. Consider the base change of q' along r_1, which is know to exist:

$$\begin{tikzcd}
C \arrow[r, "r"] \arrow[d, "q"'] & C' \arrow[d, "q'"] \\
B \arrow[r, "r_1"'] & B'
\end{tikzcd} \tag{2.26}$$

Applying the universal property of the pullback twice, we obtain $s : C' \longrightarrow C$ and $j : A \longrightarrow C$ making the following commute:

$$\begin{tikzcd}
A' \arrow[r, "s_0"] \arrow[d, "j'"'] \arrow[rr, bend left, equal] & A \arrow[r, "r_0"] \arrow[d, "j"] \arrow[d, bend right, "f"', to=3-2] & A' \arrow[d, "j'"] \\
C' \arrow[r, dash, "s"] \arrow[d, "q'"'] \arrow[rr, bend left, equal] & C \arrow[r, "r"] \arrow[d, "q"] & A' \arrow[d, "q'"] \\
B' \arrow[r, "s_1"'] \arrow[rr, bend right, equal] & B \arrow[r, "r_1"'] & B'
\end{tikzcd} \tag{2.27}$$

It presents j' as retract of j. Yet q is a pullback of $q' \in \mathfrak{F} \cap W$, so q is in $\mathfrak{F} \cap \mathfrak{W}$ (see lemma 2.2.2) and in particular in $\mathfrak{W}$; by the 2-out-of-3 property of $\mathfrak{W}$, and because f also is in $\mathfrak{W}$, we get that $j \in \mathfrak{W}$. So in the end j' is a cofibration which is a retract of a weak equivalence, hence $j' \in \mathfrak{W}$ by the previous lemma 2.2.8. Hence g is a weak equivalence as it is a composite of such. □

The emphasis on lemma 2.2.9 points out an important property of both definitions 2.2.6 and 2.2.7: the existence of pullbacks and pushouts[2]. We consider as a big limitation the ability to allow only finitely complete and cocomplete categories. Reasons behind this will become clearer in chapter 4. Fortunately, model categories without pullbacks or pushouts have already been introduced and studied by Jeff Egger in [Egg16], in which he supports the idea that Quillen's theory survive by passing to categories with finite products and finite coproducts only. Very little changes in the theory and every main result of Quillen's framework remains valid with this weaker definition. Only a couple of lemmas and technicalities have to be treated with more care. We shall take that as a token of the quality of Egger's work. What follow is a review of this generalized version of model categories, for which we claim no originality other than the way we present it.

[2] The previous proof only requires pullbacks (of fibrations) to exists. But using the dual argument would have used pushouts (of cofibrations).

Definition 2.2.10. A *model category* is a category $\mathcal{C}$ with finite products and finite coproducts together with a model structure.

Remark 2.2.11. In the rest of the memoir, *model category* will refer to definition 2.2.10 if nothing indicates otherwise, while *Quillen model category* will refer to definition 2.2.6 assuming all finite limits and finite colimits. Otherwise put, a Quillen model category is a model category with equalizers and coequalizers.

Remark 2.2.12. If $\mathcal{C}$ is a model category with weak equivalences $\mathfrak{W}$, fibrations $\mathfrak{F}$, and cofibrations $\mathfrak{C}$, then $\mathcal{C}^{\mathrm{op}}$ is also a model category with weak equivalences $\mathfrak{W}^{\mathrm{op}}$, fibrations $\mathfrak{C}^{\mathrm{op}}$ and cofibrations $\mathfrak{F}^{\mathrm{op}}$. This allows to dualize a fair amount of properties and their proofs. We will below take advantage of this fact, only writing "the proof is dual" when needed.

We shall emphasize that lemma 2.2.9 is a priori **no longer** true for model categories: at least we can not replay the proof given above as pullbacks of fibrations are no longer required to exists. However, and this is crucial, lemma 2.2.8 (and its dual version) remains valid in the context of Egger's definition. For the sake of clarity, and for future references in the memoir, let us state that properly.

Lemma 2.2.13. *Let $\mathcal{C}$ be a model category. Every cofibration which is a retract of a weak equivalence is acyclic. Every fibration which is a retract of a weak equivalence is acyclic.*

Proof. Same as lemma 2.2.8. □

Remark 2.2.14. While all pullbacks and pushouts are not required to exist in a model category, we can still take advantage of the ones that are there. In particular, lemma 2.2.2 still applies to both factorization system if a model category.

2.2.3 Homotopy in model categories

Definition 2.2.15. Let X be an object in a model category $\mathcal{C}$. A *cylinder object* for X is an object C together with weak equivalences $i_0, i_1 : X \longrightarrow C$ and $p : C \longrightarrow X$ such that $pi_0 = \mathrm{id}_X = pi_1$. The cylinder is *good* when the induced map $\langle i_0, i_1\rangle : X + X \longrightarrow C$ is a cofibration. It is *very good* if in addition p is an acyclic fibration.

Remark 2.2.16. By the 2-out-of-3 property, it is sufficient that only one of i_0, i_1 and p is a weak equivalence to imply that they all are. We should often rely on that fact implicitly.

Definition 2.2.17. A *left homotopy* between $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$ is a choice of cylinder C for X together with a map $h : X + X \longrightarrow Y$ such that commutes the following diagram:

$$\begin{tikzcd} X + X \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle i_0,i_1\rangle"'] & Y \\ C \arrow[ur, "h"'] & \end{tikzcd} \tag{2.28}$$

If the C is good (respectively very good), the homotopy h is said to be *good* (respectively *very good*).

The morphisms f, g are said *left homotopic* is there exists a left homotopy between them.

Definition 2.2.18. Let X be an object in a model category $\mathcal{C}$. A *path object* for X is an object P together with weak equivalences $p_0, p_1 : P \longrightarrow X$ and $i : X \longrightarrow P$ such that $p_0 i = \mathrm{id}_X = p_1 i$. The path object is *good* when the induced map $(p_0, p_1) : P \longrightarrow X \times X$ is a fibration. It is *very good* if in addition i is an acyclic cofibration.

Definition 2.2.19. A *right homotopy* between $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$ is a choice of cylinder P for X together with a map $h : X \longrightarrow Y \times Y$ such that commutes the following diagram:

$$\begin{tikzcd} & P \arrow[d, "{(p_0,p_1)}"] \\ X \arrow[ur, "h"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd} \tag{2.29}$$

If the C is good (respectively very good), the homotopy h is said to be *good* (respectively *very good*).

The morphisms f, g are said *right homotopic* is there exists a right homotopy between them.

Lemma 2.2.20. *Let f, g be left (respectively right) homotopic. Then f is a weak equivalence if and only if g is so.*

Proof. Suppose $f : X \longrightarrow Y$ is left homotopic to g. Then there is a cylinder C and a left homotopy h such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ C \arrow[r, "h"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{2.30}$$

Recall that i_0 and i_1 are weak equivalence. Hence by the 2-out-of-3 property, f is a weak equivalence if and only if h is a weak equivalence if and only if g is a weak equivalence.

The proof for right homotopic map is dual. □

Definition 2.2.21. Let $\mathcal{C}$ be a model category. An object Y is *fibrant* is the unique map $Y \longrightarrow 1$ to the terminal object is a fibration. An object X is *cofibrant* if the unique map $0 \longrightarrow X$ from the initial object is a cofibration.

An object that are fibrant and cofibrant is called *bifibrant*.

Proposition 2.2.22. *Let $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$.*

(i) There is a left homotopy between f and g if and only if there is a good left homotopy between them.

(ii) If Y if fibrant, there is a good left homotopy between f and g if and only if there is a very good left homotopy between them.

(iii) If f and g are left homotopic, then so are yf and yg for any $y : Y \longrightarrow B$.

(iv) If Y is fibrant and f and g are left homotopic, then so are fx and gx for any $x : A \longrightarrow X$.

Proof. (i) Of course, a good left homotopy is in particular a good homotopy. Conversely, suppose $h : C \longrightarrow Y$ is a left homotopy between f and g. The cylinder C comes with maps $i_0, i_1 : X \longrightarrow C$ and a retraction $p : C \longrightarrow X$ of both i_0 and i_1, and the following diagram commute:

$$\begin{tikzcd} X + X \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle i_0,i_1\rangle"'] & Y \\ C \arrow[ur, "h"'] & \end{tikzcd} \tag{2.31}$$

We can consider the following factorization of $\langle i_0, i_1\rangle$:

$$X + X \xrightarrow{j} C' \xrightarrow{q} C \qquad \text{with } j \in \mathfrak{C}, q \in \mathfrak{F} \cap \mathfrak{W} \tag{2.32}$$

Precomposing j with both inclusions $X \longrightarrow X + X$ yield $j_0, j_1 : X \longrightarrow C'$ and both admit $pq : C' \longrightarrow X$ as a retract: indeed $pqj_0 = pi_0 = \mathrm{id}_X$ and similarly for j_1. As p and q are both weak equivalences, so is pq. Hence C' is a good cylinder for X and $hq : C' \longrightarrow Y$ is a left good homotopy between f and g.

(ii) Any very good left homotopy is a good left homotopy. Conversely suppose $h : C \longrightarrow Y$ is a good left homotopy between f and g. The good cylinder comes equipped with maps $i_0, i_1 : X \longrightarrow C$ and a shared retraction $p : C \longrightarrow X$. Factor p as

$$C \xrightarrow{j} C' \xrightarrow{q} X \qquad \text{with } j \in \mathfrak{C}, q \in \mathfrak{F} \cap \mathfrak{W} \tag{2.33}$$

Denote $j_0 = ji_0$ and $j_1 = ji_1$. Then the weak equivalence q is a retraction of both j_0 and j_1, making C' a cylinder for X. Moreover $\langle j_0, j_1\rangle$ is the composite $j\langle i_0, i_1\rangle$ of two cofibrations, hence is a cofibration itself. Finally, as q is an acyclic fibration, it makes C' a very good cylinder. Because Y is fibrant and j an acyclic cofibration, we can find h' such that $h'j = h$. Such an h' is then a very good left homotopy between f and g:

$$\begin{tikzcd} X + X \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle i_0,i_1\rangle"'] & Y \\ C \arrow[ur, "h"'] \arrow[d, "j"'] & \\ C' \arrow[uur, dashed, bend right, "h'"'] & \end{tikzcd} \tag{2.34}$$

(iii) Recall that $\langle yf, yg\rangle$ is just $y\langle f, g\rangle$. Then the following diagram exhibits yh as a left homotopy between f and g:

$$\begin{tikzcd} X + X \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle i_0,i_1\rangle"'] & Y \arrow[r, "y"] & B \\ C \arrow[ur, "h"'] & & \end{tikzcd} \tag{2.35}$$

(iv) Because Y is fibrant, we can take a very good left homotopy $h : C \longrightarrow Y$ between f and g. The very good cylinder C comes with structural maps $i_0, i_1 : X \longrightarrow C$ and $p : C \longrightarrow X$. Now choose any very good cylinder C' for A with structural maps $j_0, j_1 : A \longrightarrow C'$ and $q : C' \longrightarrow A$. The following outer square commutes, and using the left lifting property of the cofibration $\langle j_0, j_1\rangle$ relatively to the acyclic fibration p, we can find $\bar{x} : C' \longrightarrow C$ as follow:

$$\begin{tikzcd}
A + A \arrow[r, "\langle x,x\rangle"] \arrow[d, "\langle j_0,j_1\rangle"'] & X + X \arrow[d, "\langle i_0,i_1\rangle"] \\
C' \arrow[r, dashed, "\bar{x}"] \arrow[d, "q"'] & C \arrow[d, "p"] \\
A \arrow[r, "x"'] & X
\end{tikzcd} \tag{2.36}$$

Now the following diagram exhibits $h\bar{x}$ as an homotopy between fx and gx:

$$\begin{tikzcd}
A + A \arrow[r, "\langle x,x\rangle"] \arrow[d, "\langle j_0,j_1\rangle"'] & X + X \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle i_0,i_1\rangle"'] & Y \\
C' \arrow[r, dashed, "\bar{x}"] & C \arrow[ur, "h"'] &
\end{tikzcd} \tag{2.37}$$

□

The dual statements hold automatically.

Proposition 2.2.23. *Let $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$.*

(i) There is a right homotopy between f and g if and only if there is a good right homotopy between them.

(ii) If X if cofibrant, there is a good right homotopy between f and g if and only if there is a very good right homotopy between them.

(iii) If f and g are right homotopic, then so are fx and gx for any $x : A \longrightarrow X$.

(iv) If X is cofibrant and f and g are right homotopic, then so are yf and yg for any $y : Y \longrightarrow B$.

Notation 2.2.24. Denote $f \sim_\ell g$ when f is left homotopic to g and $f \sim_r g$ when f is right homotopic to g. Also denote $f \sim g$ when f is both left and right homotopic to g.

Proposition 2.2.25. *Let $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$.*

(i) If X is cofibrant and $f \sim_\ell g$ then $f \sim_r g$.

(ii) If Y is fibrant and $f \sim_r g$ then $f \sim_\ell g$.

In particular, $\sim_\ell$ and $\sim_r$ and $\sim$ coincide on the hom-sets $\mathcal{C}(X, Y)$ when X is cofibrant and Y is fibrant.

Proof. Suppose X is cofibrant and f is left homotopic to g. By proposition 2.2.22 (i), there is a good left homotopy $h : C \longrightarrow Y$, where the cylinder has structural maps $i_0, i_1 : X \longrightarrow C$ and $p : C \longrightarrow X$. Because X is cofibrant and the following is a pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[d, "\iota_1"] \\
X \arrow[r, "\iota_0"'] & X + X
\end{tikzcd} \tag{2.38}$$

it follows that ι_0 and ι_1 are cofibrations. Because C is a good cylinder for X, also $\langle i_0, i_1\rangle$ is a cofibration. Hence both $i_0 = \langle i_0, i_1\rangle\, \iota_0$ and $i_1 = \langle i_0, i_1\rangle\, \iota_1$ are cofibrations. They already are weak equivalences, so they are acyclic cofibration in the end. Now choose a good path object P for Y, with maps $p_0, p_1 : P \longrightarrow Y$ and $i : Y \longrightarrow P$, so that $(p_0, p_1) : P \longrightarrow Y \times Y$ is a fibration. Then a filler k exists in the following square:

$$\begin{tikzcd} X \arrow[r, "if"] \arrow[d, "i_0"'] & P \arrow[d, "(p_0,p_1)"] \\ C \arrow[r, "(fp,h)"'] \arrow[ur, dashed, "k"] & Y \times Y \end{tikzcd} \tag{2.39}$$

That is k is a right homotopy between fp and h. By proposition 2.2.23 (iv), it gives a right homotopy between $fpi_1 = f$ and $hi_1 = g$.

The second property is dual. □

Remark 2.2.26. Let $f, g : X \longrightarrow Y$ be homotopic between X cofibrant and Y fibrant. The proof of the previous proposition allow to choose **any good path object** to witness the right homotopy between f and g. Indeed, $f \sim g$ so in particular $f \sim_\ell g$. Now play the proof from the previous proposition: in the process of showing $f \sim_r g$ we allow the good path object P for Y. Similarly, starting from $f \sim_r g$ and playing the second part of the proof (left implicit by duality), we can choose any good cylinder object to witness $f \sim_\ell g$.

The key ingredient hidden in the proof of proposition 2.2.25 can be reduce to the following lemma.

Lemma 2.2.27. *(i) Let $j : A \longrightarrow X$ be an acyclic cofibration and $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$. Then $fj \sim_r gj$ if and only if $f \sim_r g$.*

(ii) Let $q : Y \longrightarrow B$ be an acyclic fibration and $f, g : X \longrightarrow Y$ in a model category $\mathcal{C}$. Then $qf \sim_\ell qg$ if and only if $f \sim_\ell g$.

Proof. If $f \sim_r g$ then we already know that $fj \sim_r gj$ by proposition 2.2.23 without any assumption on j.

Conversely, suppose there is a right homotopy $h : A \longrightarrow P$ from fj to gj. By proposition 2.2.23, P can be chosen to be a good path object for Y, with structural maps $p_0, p_1 : P \longrightarrow Y$ and $i : Y \longrightarrow P$. Then the following square commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "j"'] & P \arrow[d, "(p_0,p_1)"] \\ X \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd} \tag{2.40}$$

The map j is an acyclic cofibration, so it has the left lifting property relatively to the fibration (p_0, p_1). Hence there is a right homotopy $k : X \longrightarrow P$ from f to g.

The proof of the second property is dual. □

Usually at this point, a textbook on homotopical algebra would show that:

- if X is cofibrant the relation $\sim_\ell$ is an equivalence relation on $\mathcal{C}(X, Y)$,
- if Y is fibrant the relation $\sim_r$ is an equivalence relation on $\mathcal{C}(X, Y)$.

This is not true anymore in Egger's settings! Indeed, transitivity of $\sim_\ell$ require the existence of a (weak) pushout square

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i'_0"'] & C \arrow[d] \\ C' \arrow[r] & C'' \end{tikzcd} \tag{2.41}$$

with C'' being a cylinder object for X, for any given good cylinders C and C' for X. Model categories with pushouts (of acyclic cofibrations at least) have such a C'', but general model categories need not to. Similarly the transitivity of $\sim_r$ goes through a weak pullback condition on given path objects, that model categories with pullbacks (of acyclic fibrations) satisfy automatically. This obstruction could seems quite important as this is usually the way one proves that the relation $\sim$ is a congruence of the subcategory of bifibrant objects.

But this is overcome quite elegantly by Egger in [Egg16]. A fine tuned analysis of left and right homotopy relations yield the following statement.

Proposition 2.2.28. *Let $f, e, g : X \longrightarrow Y$ in a model category $\mathcal{C}$.*

(i) If X is cofibrant and $f \sim_r e \sim_\ell g$, then $f \sim_r g$.

(ii) If Y is fibrant and $f \sim_\ell e \sim_r g$, then $f \sim_\ell g$.

Proof. Again, we prove only (i) as (ii) is dual. The idea of the proof is to recast the key diagram (2.39) of the previous proof, replacing the trivial right homotopy if of the top by a non-trivial homotopy: indeed proposition 2.2.25 is found back when inputting $e = f$ (and obviously $f \sim_r f$).

Suppose that X is cofibrant and that there is a right homotopy $k : X \longrightarrow P$ between f and e and a left homotopy $h : C \longrightarrow Y$ between e and g. Recall from propositions 2.2.22 and 2.2.23 that we can choose C to be a good cylinder object for X with structural maps $i_0, i_1 : X \longrightarrow C$ and $p : C \longrightarrow X$, and similarly we can choose P to be a good path object for Y with structural maps $p_0, p_1 : P \longrightarrow Y$ and $i : Y \longrightarrow P$. Moreover, because X is cofibrant, it makes i_0 and i_1 acyclic cofibrations. Now there is two ways of writing $(f, e) : X \longrightarrow Y \times Y$, making the following outer square commutes:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "i_0"'] & P \arrow[d, "(p_0,p_1)"] \\ C \arrow[r, "(fp,h)"'] \arrow[ur, dashed, "\ell"] & Y \times Y \end{tikzcd} \tag{2.42}$$

The left lifting property of i_0 relatively to the fibration (p_0, p_1) assures that a filler $\ell : C \longrightarrow P$ exists. This is a right homotopy between fp and h. We can precompose by i_1 to obtain a right homotopy from $fpi_1 = f$ to $hi_1 = g$. □

Corollary 2.2.29. *Let X be cofibrant and Y be fibrant in a model category $\mathcal{C}$. The relation $\sim$ is an equivalence relation on $\mathcal{C}(X, Y)$.*

Proof. The relation is reflexive: for any $f : X \longrightarrow Y$, Y itself is a path object for Y and f is then seen as an homotopy from f to f.

The relation is symmetric: for any $f, g : X \longrightarrow Y$, if $h : X \longrightarrow P$ is a right homotopy between f and g, then swapping the structural maps p_0, p_1 of P gives another path object structure on P; as such h is now a right homotopy between g and f.

The relation is transitive: given $f, g, e : X \longrightarrow Y$, if f is homotopic to g which is homotopic to e, then by proposition 2.2.28 f is both left and right homotopic to e. □

NOTATION 2.2.30. We shall write $\pi(X, Y)$ for the quotient $\mathcal{C}(X, Y)/\sim$.

Corollary 2.2.31. *The relation $\sim$ is a congruence on the full subcategory $\mathcal{C}^{\mathrm{cf}}$ of bifibrant objects of $\mathcal{C}$. In particular, there is a category $\pi\,\mathcal{C}$ with the same object as $\mathcal{C}^{\mathrm{cf}}$ and with hom-sets:*

$$\pi\,\mathcal{C}(X, Y) = \pi(X, Y) \tag{2.43}$$

There is a functor $\pi : \mathcal{C}^{\mathrm{cf}} \longrightarrow \pi\,\mathcal{C}$, identity on objects, and mapping a morphism f to its homotopy class $[f]$.

Proof. We already proved that $\sim$ is an equivalence relation on every set $\mathcal{C}(X, Y)$ with X and Y bifibrant. It remains only to show that it is congruent relatively to composition. Let $f, g : X \longrightarrow Y$ in $\mathcal{C}^{\mathrm{cf}}$ be homotopic:

- there are in particular left homotopic, so yf and yg are left homotopic for any $y : Y \longrightarrow B$ in $\mathcal{C}^{\mathrm{cf}}$ (see proposition 2.2.22 (iii), hence yf and yg are homotopic by proposition 2.2.25,
- there are also in particular right homotopic, so fx and gx are right homotopic for any $x : A \longrightarrow X$ in $\mathcal{C}^{\mathrm{cf}}$ (see proposition 2.2.23 (iii), hence fx and gx are homotopic by proposition 2.2.25.

□

Definition 2.2.32. Let $\mathcal{C}$ be a model category. An *homotopy equivalence* in $\mathcal{C}$ is a map $f : X \longrightarrow Y$ such that there exists $g : Y \longrightarrow X$ that satisfies

$$fg \sim \mathrm{id}_Y \quad \text{and} \quad gf \sim \mathrm{id}_X \tag{2.44}$$

Such a g is called a *pseudo inverse* to f.

REMARK 2.2.33. If X and Y are bifibrant, then f is an homotopy equivalence if and only if its image $[f]$ is an isomorphism in $\pi\,\mathcal{C}$. In particular, homotopy equivalences in C^{cf} compose.

The following proposition is the heart of the *saturation* property of $\mathfrak{W}$, on which more will be told later.

Proposition 2.2.34. *Let $f : X \longrightarrow Y$ in a model category $\mathcal{C}$ with X, Y bifibrant. Then f is a weak equivalence if and only if f is an homotopy equivalence.*

Proof. Suppose f is an acyclic cofibration. Because X is fibrant, there exists $r : Y \longrightarrow X$ as in:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \\ Y \arrow[ur, "r"'] & \end{tikzcd} \tag{2.45}$$

Moreover $frf = f$, so that $fr \sim_r \mathrm{id}_X$ from lemma 2.2.27. Dually, if f is an acyclic fibration, and because Y is cofibrant, then it is an homotopy equivalence. For weak equivalences are composite of acyclic cofibrations and acyclic fibrations and homotopy equivalences compose between bifibrant objects, we can deduce that a weak equivalence between bifibrant objects is an homotopy equivalence.

Conversely, if f is an homotopy equivalence, then it admits a pseudo inverse $g : Y \longrightarrow X$. Factor f as qj with $j : X \longrightarrow Z \in \mathfrak{C} \cap \mathfrak{W}$ and $q : Z \longrightarrow Y \in \mathfrak{F}$. All we need to show is that q is a weak equivalence. This is already a fibration, so through lemma 2.2.13, it is enough to show that q is a retract of a weak equivalence. Here is the strategy to do so:

- find a section s of q, so that q is a now a retract of sq, as presented in the following diagram:

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "q"'] & Z \arrow[r, equal] \arrow[d, "sq"] & Z \arrow[d, "q"] \\ Y \arrow[r, "s"'] & Z \arrow[r, "q"'] & Y \end{tikzcd} \tag{2.46}$$

- prove that $sq \sim \mathrm{id}_Z$, hence making sq a weak equivalence by lemma 2.2.20

Such a s can be found by considering a left good homotopy $h : C \longrightarrow Y$ from fg to id_Y from a good cylinder C for Y with structural maps $i_0, i_1 : Y \longrightarrow C$ and $p : C \longrightarrow Y$. Then the following outer square commutes:

$$\begin{tikzcd} Y \arrow[r, "jg"] \arrow[d, "i_0"'] & Z \arrow[d, "q"] \\ C \arrow[r, "h"'] \arrow[ur, dashed, "k"] & Y \end{tikzcd} \tag{2.47}$$

It yields a filler $k : C \longrightarrow Z$ which can be consider as a good left homotopy from jg to $s = ki_1$. The map s is a section of q by construction: $qs = qki_1 = hi_1 = \mathrm{id}_Y$. Every object here, X, Y, and Z, are bifibrant, so the distinction between left and right homotopy are not relevant and we can just use $\sim$. So in one hand $s \sim gj$, hence $sq \sim jgq$. In the other hand $\mathrm{id}_Z \sim jr$ when r is constructed as in diagram (2.45). So $q \sim qjr = fr$. In the end,

$$sq \sim jgq \sim jgfr \sim j\mathrm{id}_X r = jr \sim \mathrm{id}_Z \tag{2.48}$$

□

To end this section, we recall Ken Brown's lemma and its consequence. The usual proofs make no use of non trivial pullbacks or pushouts, so we can write them directly as such in Egger's settings.

Lemma 2.2.35 (Ken Brown). *Let $f : X \longrightarrow Y$ be a weak equivalence in a model category $\mathcal{C}$ between cofibrant objects X, Y. Then there exists acyclic cofibrations j, j' and an acyclic fibration q such that $f = qj$ and $qj' = \mathrm{id}_Y$.*

Proof. This is a variation on the existence of good cylinder objects with cofibrant inclusions for cofibrant objects. Consider the induced map $\langle f, \mathrm{id}_Y \rangle : X + Y \longrightarrow Y$ and factor it as

$$X + Y \xrightarrow{i} C \xrightarrow{q} Y \qquad i \in \mathfrak{C},\ q \in \mathfrak{F} \cap \mathfrak{W} \tag{2.49}$$

The object C can be thought as the mapping cylinder of f. Denote $\iota_0 : X \longrightarrow X + Y$ and $\iota_1 : Y \longrightarrow X + Y$ the canonical injections. Because X and Y are cofibrant, ι_0, ι_1 are cofibrations. So is the composites $j = i\iota_0$ and $j' = i\iota_1$. By definition, $qj = \langle f, \mathrm{id}_Y \rangle \iota_0 = f$ and $qj' = \langle f, \mathrm{id}_Y \rangle \iota_1 = \mathrm{id}_Y$. Finally, j and j' are weak equivalences by the 2-out-of-3 property. □

Corollary 2.2.36. *Let $F : \mathcal{C} \longrightarrow \mathcal{A}$ a functor from a model category $\mathcal{C}$ to any category $\mathcal{A}$. If F maps acyclic cofibrations between cofibrant objects to isomorphisms, then it maps weak equivalences between cofibrant objects to isomorphisms.*

Proof. Let $f : X \longrightarrow Y$ be a weak equivalence with X, Y cofibrant. Use 2.2.35 to get q, j, j' such that $f = qj$, $qj' = \mathrm{id}_Y$ and j, j' acyclic cofibrations. Remark that the shared codomain of j and j' is also cofibrant, so that $F(j)$ and $F(j')$ are isomorphism. Because $qj' = \mathrm{id}_Y$, then $F(q)F(j') = \mathrm{id}_{FY}$ and then $F(q)$ is the inverse isomorphism of $F(q')$. In the end, $F(f) = F(q)F(j)$ is a composite of isomorphism, so is an isomorphism itself. □

Remark 2.2.37. We will often need variations on the following argument. It is not worth making a general statement, useful enough to be applied directly. Therefore we present the trick here and will refer to it later on when needed.

Suppose a cylinder C is given for an object X in a model category $\mathcal{C}$, with structural maps $i_0, i_1 : X \longrightarrow C$ and $p : C \longrightarrow X$. If $F : \mathcal{C} \longrightarrow \mathcal{A}$ maps i_0, i_1 and p to isomorphisms in $\mathcal{A}$ then for any morphisms $f, g : X \longrightarrow Y$ that are left homotopic through the cylinder C, $F(f) = F(g)$. Indeed $F(i_0), F(i_1), F(p)$ are isomorphisms and, because $pi_0 = \mathrm{id}_X = pi_1$, $F(i_0)$ and $F(i_1)$ are both inverse of $F(p)$, hence $F(i_0) = F(i_1)$. It follows that for any homotopy $h : C \longrightarrow Y$ from f to g, one gets $F(f) = F(h)F(i_0) = F(h)F(i_1) = F(g)$.

Of course, there is a dual remark to do on functors that maps structural maps of path objects to isomorphisms.

2.2.4 The homotopy category of a model category

Definition 2.2.38. Let A be an object in a model category $\mathcal{C}$. A *fibrant replacement* of A is a fibrant object A' together with a weak equivalence $A \longrightarrow A'$. A *cofibrant replacement* of A is a cofibrant object A' together with a weak equivalence $A' \longrightarrow A$.

Remark 2.2.39. Every object A admits a fibrant replacement: it only takes to factor $A \longrightarrow 1$ as an acyclic cofibration followed by a fibration. Similarly, factorizing $0 \longrightarrow A$ as a cofibration followed by an acyclic fibration, we find a cofibrant replacement of A.

For the following of this section, a model category $\mathcal{C}$ is given, together with a choosing of a cofibrant replacement and a fibrant replacement for every object A:

$$A^{\mathrm{c}} \xrightarrow{\mathfrak{q}_A} A \xrightarrow{\mathfrak{j}_A} A^{\mathrm{f}} \qquad \begin{array}{c} A^{\mathrm{c}} \text{ cofibrant, } A^{\mathrm{f}} \text{ fibrant} \\ \mathfrak{j}_A \in \mathfrak{C} \cap \mathfrak{W},\ \mathfrak{q}_A \in \mathfrak{F} \cap \mathfrak{W} \end{array} \tag{2.50}$$

Moreover, if A is cofibrant then we suppose $A^{\mathrm{c}} = A$ and $\mathfrak{q}_A = \mathrm{id}_A$. Similarly, if A is fibrant then we suppose that $A^{\mathrm{f}} = A$ and $\mathfrak{j}_A = \mathrm{id}_A$. When the object A is inferable from context we might simply write $\mathfrak{q}$ and $\mathfrak{j}$ without subscript. We shall also write A^{cf} as a shorthand for $(A^{\mathrm{c}})^{\mathrm{f}}$. For any $f : A \longrightarrow B$ in $\mathcal{C}$, one can find morphisms f^{c} and f^{f} making the following squares commute:

$$\begin{tikzcd} A^{\mathrm{c}} \arrow[r, "\mathfrak{q}_A"] \arrow[d, "f^{\mathrm{c}}"'] & A \arrow[r, "\mathfrak{j}_A"] \arrow[d, "f"] & A^{\mathrm{f}} \arrow[d, "f^{\mathrm{f}}"] \\ B^{\mathrm{c}} \arrow[r, "\mathfrak{q}_B"'] & B \arrow[r, "\mathfrak{j}_B"'] & B^{\mathrm{f}} \end{tikzcd} \tag{2.51}$$

These morphisms are chosen as liftings in the commutative squares:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & B^{\mathrm{c}} \arrow[d, "\mathrm{q}_B"] \\
A^{\mathrm{c}} \arrow[r, "f\mathrm{q}_A"'] \arrow[ur, dashed, "f^{\mathrm{c}}"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{j}_B f"] \arrow[d, "\mathrm{j}_A"'] & B^{\mathrm{f}} \arrow[d] \\
A^{\mathrm{f}} \arrow[r] \arrow[ur, dashed, "f^{\mathrm{f}}"'] & 1
\end{tikzcd}
\tag{2.52}
$$

As such they are a priori not unique. However, two choices of f^{c} will be left homotopic, while two choices of f^{f} will be right homotopic through lemma 2.2.27. In particular, we deduce that the bifibrant replacement of a morphism is well-defined up to homotopy:

Lemma 2.2.40. *Suppose given morphisms $f : A \longrightarrow B$, $g_1, g_2 : A^{\mathrm{c}} \longrightarrow B^{\mathrm{c}}$ and $h_1, h_2 : A^{\mathrm{cf}} \longrightarrow B^{\mathrm{cf}}$ such that the following diagram commutes for $i \in \{1, 2\}$:*

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
A^{\mathrm{c}} \arrow[u, "\mathrm{q}_A"] \arrow[r, "g_i"] \arrow[d, "\mathrm{j}_{A^{\mathrm{c}}}"'] & B^{\mathrm{c}} \arrow[u, "\mathrm{q}_B"'] \arrow[d, "\mathrm{j}_{B^{\mathrm{c}}}"] \\
A^{\mathrm{cf}} \arrow[r, "h_i"] & B^{\mathrm{cf}}
\end{tikzcd}
\tag{2.53}
$$

Then h_1 is (left and right) homotopic to h_2.

Proof. From what precedes, $g_1 \sim_\ell g_2$. Hence $\mathrm{j}_{B^{\mathrm{c}}} g_1 \sim_\ell \mathrm{j}_{B^{\mathrm{c}}} g_2$ by left-congruity. As A^{c} is cofibrant, proposition 2.2.25 shows that they are also right homotopic and lemma 2.2.27 can be applied with the acyclic cofibration $\mathrm{j}_{A^{\mathrm{c}}}$ to conclude that $h_1 \sim_r h_2$. Between the bifibrant objects A^{cf} and B^{cf}, the left and right homotopy relations coincide so that $h_1 \sim h_2$. □

Denote by $\mathbf{Ho}(\mathcal{C})$ the category whose objects are the same as those of $\mathcal{C}$ and where the hom-sets are defined as:

$$
\mathbf{Ho}(\mathcal{C})(A, B) = \pi\left(A^{\mathrm{cf}}, B^{\mathrm{cf}}\right) \tag{2.54}
$$

Composition is given as in $\pi\,\mathcal{C}$. There is a functor $\gamma : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C})$, sending an object A to itself and a map f to the homotopy class of f^{cf}. Lemma 2.2.40 ensures the functoriality of γ: indeed, given $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we have diagrams as follow

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
A^{\mathrm{c}} \arrow[u, "\mathrm{q}_A"] \arrow[r, "f^{\mathrm{c}}"] \arrow[d, "\mathrm{j}_{A^{\mathrm{c}}}"'] & B^{\mathrm{c}} \arrow[u, "\mathrm{q}_B"'] \arrow[r, "g^{\mathrm{c}}"] \arrow[d, "\mathrm{j}_{B^{\mathrm{c}}}"] & C^{\mathrm{c}} \arrow[u, "\mathrm{q}_C"'] \arrow[d, "\mathrm{j}_{C^{\mathrm{c}}}"] \\
A^{\mathrm{cf}} \arrow[r, "f^{\mathrm{cf}}"] & B^{\mathrm{cf}} \arrow[r, "g^{\mathrm{cf}}"] & C^{\mathrm{cf}}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "gf"] & C \\
A^{\mathrm{c}} \arrow[u, "\mathrm{q}_A"] \arrow[r, "(gf)^{\mathrm{c}}"] \arrow[d, "\mathrm{j}_{A^{\mathrm{c}}}"'] & C^{\mathrm{c}} \arrow[u, "\mathrm{q}_C"'] \arrow[d, "\mathrm{j}_{C^{\mathrm{c}}}"] \\
A^{\mathrm{cf}} \arrow[r, "(gf)^{\mathrm{cf}}"] & C^{\mathrm{cf}}
\end{tikzcd}
\tag{2.55}
$$

Hence lemma 2.2.40 shows that $\left[(gf)^{\mathrm{cf}}\right] = \left[g^{\mathrm{cf}}\right]\left[f^{\mathrm{cf}}\right]$.

Remark 2.2.41. For any object A, the image of the cofibrant replacement of A and of the fibrant replacement of A^{c} are respectively given by:

$$
\gamma(\mathrm{q}_A) = \left[\mathrm{id}_{A^{\mathrm{cf}}}\right] : A^{\mathrm{c}} \longrightarrow A, \qquad \gamma(\mathrm{j}_{A^{\mathrm{c}}}) = \left[\mathrm{id}_{A^{\mathrm{cf}}}\right] : A^{\mathrm{c}} \longrightarrow A^{\mathrm{cf}} \tag{2.56}
$$

In particular they are isomorphisms with respective inverse $\left[\mathrm{id}_{A^{\mathrm{cf}}}\right] : A \longrightarrow A^{\mathrm{c}}$ and $\left[\mathrm{id}_{A^{\mathrm{cf}}}\right] : A^{\mathrm{cf}} \longrightarrow A^{\mathrm{c}}$. They give a canonical presentation of each arrow of $\mathbf{Ho}(\mathcal{C})$ as a zig-zag: for any morphism $[f] : A \longrightarrow B$ with representative $f : A^{\mathrm{cf}} \longrightarrow B^{\mathrm{cf}}$, we get obviously a map $\gamma(f) : A^{\mathrm{cf}} \longrightarrow B^{\mathrm{cf}}$ in $\mathbf{Ho}(\mathcal{C})$. From this map $\gamma(f)$ we can craft the following composite:

$$A \xrightarrow{\gamma(\mathrm{q})^{-1}} A^{\mathrm{c}} \xrightarrow{\gamma(\mathrm{j})} A^{\mathrm{cf}} \xrightarrow{\gamma(f)} B^{\mathrm{cf}} \xrightarrow{\gamma(\mathrm{j})^{-1}} B^{\mathrm{c}} \xrightarrow{\gamma(\mathrm{q})} B \tag{2.57}$$

Such a composite we can actually compute: this is equal to

$$\left[\mathrm{id}_{B^{\mathrm{cf}}} \circ \mathrm{id}_{B^{\mathrm{cf}}} \circ f \circ \mathrm{id}_{A^{\mathrm{cf}}} \circ \mathrm{id}_{A^{\mathrm{cf}}}\right] = [f] \tag{2.58}$$

Proposition 2.2.42. *The functor $\gamma : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C})$ is a localization, meaning that γ maps weak equivalences to isomorphisms and for every functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ with the same property, there exists a unique $\tilde{F} : \mathbf{Ho}(\mathcal{C}) \longrightarrow \mathcal{A}$ such that $\tilde{F}\gamma = F$.*

Proof. This is contained in [Qui67, Theorem 1 § 1] but it is mixed with other assertions about categories that do not necessarily exist in Egger's framework (e.g. the category of cofibrant objects with morphisms up to left homotopy). Let us then quickly rephrase this argument here.

By the two-out-of-three property, if f is weak equivalence in $\mathcal{C}$, then f^{cf} also. Weak equivalences between bifibrant objects are homotopy equivalences, hence $\gamma(f)$ is invertible. Let now $F : \mathcal{C} \longrightarrow \mathcal{A}$ be a functor mapping weak equivalences to isomorphisms, and define $\tilde{F}$ on objects as $\tilde{F}(X) = F(X)$. To define $\tilde{F}$ on a morphism $[f] : A \longrightarrow B$, consider the following zig-zag in $\mathcal{C}$:

$$\begin{tikzcd} A & B \\ A^{\mathrm{c}} \arrow[u, "\mathrm{q}"] \arrow[d, "\mathrm{j}"'] & B^{\mathrm{c}} \arrow[u, "\mathrm{q}"] \arrow[d, "\mathrm{j}"'] \\ A^{\mathrm{cf}} \arrow[r, "f"] & B^{\mathrm{cf}} \end{tikzcd} \tag{2.59}$$

Then define $\tilde{F}([f])$ as the composite

$$F(A) \xrightarrow{F(\mathrm{q})^{-1}} F(A^{\mathrm{c}}) \xrightarrow{F(\mathrm{j})} F(A^{\mathrm{cf}}) \xrightarrow{F(f)} F(B^{\mathrm{cf}}) \xrightarrow{F(\mathrm{j})^{-1}} F(B^{\mathrm{c}}) \xrightarrow{F(\mathrm{q})} B \tag{2.60}$$

This composite is independent of the choice of f: indeed, remark 2.2.37 applies here as F maps every weak equivalence (especially the structural maps of cylinder) to isomorphisms. Such a functor $\tilde{F}$ is unique: the action on objects is necessarily defined as $\tilde{F}(A) = \tilde{F}(\gamma(A)) = F(A)$; and by the canonical zig-zag described in (2.57), the action on morphisms is necessarily defined as

$$\begin{aligned} \tilde{F}\left([f] : A \longrightarrow B\right) &= \tilde{F}\left(\gamma(\mathrm{q}_B)\gamma(\mathrm{j}_{B^{\mathrm{c}}})^{-1}\gamma(f)\gamma(\mathrm{j}_{A^{\mathrm{c}}})\gamma(\mathrm{q}_A)^{-1}\right) \\ &= F(\mathrm{q}_B)F(\mathrm{j}_{B^{\mathrm{c}}})^{-1}F(f)F(\mathrm{j}_{A^{\mathrm{c}}})F(\mathrm{q}_A)^{-1} \end{aligned} \tag{2.61}$$

□

REMARK 2.2.43. The category $\mathbf{Ho}(\mathcal{C})$ is *a priori* dependent on the choices of fibrant and cofibrant replacements: different choices yields isomorphic categories, not strictly equal ones. The previous theorem ensures that it is not harmful: the only thing that really matters is the universal property that $\mathbf{Ho}(\mathcal{C})$ satisfies, which defines it anyway only up to (unique) isomorphism.

Remark 2.2.44. Any map $f : X \longrightarrow Y$ is a weak equivalence if and only if $f^{\mathrm{cf}} : X^{\mathrm{cf}} \longrightarrow Y^{\mathrm{cf}}$ is such, by two successive applications of the 2-out-of-3 property in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A^{\mathrm{c}} \arrow[u, "\mathrm{q}_A"] \arrow[r, "f^{\mathrm{c}}"] \arrow[d, "\mathrm{j}_{A^{\mathrm{c}}}"'] & B^{\mathrm{c}} \arrow[u, "\mathrm{q}_B"'] \arrow[d, "\mathrm{j}_{B^{\mathrm{c}}}"] \\ A^{\mathrm{cf}} \arrow[r, "f^{\mathrm{cf}}"] & B^{\mathrm{cf}} \end{tikzcd} \tag{2.62}$$

In turn, by proposition 2.2.34, because X and Y are bifibrant, f^{cf} is a weak equivalence if and only if it is an homotopy equivalence. So in the end f is a weak equivalence if and only if $\gamma(f)$ is invertible. This property is called the *saturation* of $\mathfrak{W}$.

An unexpected consequence of the saturation of weak equivalence is that $\mathfrak{W}$ is closed under retracts: given a retract g of a weak equivalence f, $\gamma(g)$ is a retract of $\gamma(f)$; but $\gamma(f)$ is an isomorphism and isomorphisms are always closed under retracts, so $\gamma(g)$ is an isomorphism; by saturation, it follows that $g \in \mathfrak{W}$. As we argued in the discussion preceding lemma 2.2.9, the usual proof that $\mathfrak{W}$ is closed by retract does not hold in Egger's framework. Yet it turns out that the outcome is still valid! (Please note that in the process of proving saturation, lemma 2.2.9 is crucial and we cannot do without it.)

Definition 2.2.45. Let $F : \mathcal{C} \longrightarrow \mathcal{A}$. (Absolute) Left Kan extensions of F along γ are called (absolute) *right derived* functors of F. (Absolute) Right Kan extensions of F along γ are called (absolute) *left derived* functors of F.

As left Kan extensions are always naturally isomorphic, we make the abuse to talk about *the* right derived functor of F when it exists, and we denote it $\mathbf{R}\,F$. Dually, we will talk about *the* left derived functor of F, and denote it $\mathbf{L}\,F$.

Proposition 2.2.46. *Let $F : \mathcal{C} \longrightarrow \mathcal{A}$ that maps weak equivalences between cofibrant objects to isomorphisms. Then the left derived functor of F exists and is absolute.*

Proof. This is [Qui67, Proposition 1 §I.4], which goes roughly as follow. Remark that

$$A \longmapsto F(A^{\mathrm{c}}), \qquad f \longmapsto F(f^{\mathrm{c}}) \tag{2.63}$$

defines a functor $F^{\mathrm{c}} : \mathcal{C} \longrightarrow \mathcal{A}$. Although f^{c} is *a priori* only well-defined up to left homotopy, consider g left homotopic to it. Every homotopy can be promoted to a good left homotopy (see proposition 2.2.22) and, because the source of f^{c} is cofibrant, the structural maps of its good cylinders are weak equivalences between cofibrant objects, that are mapped to isomorphism through F; remark 2.2.37 applies then and $F(f^{\mathrm{c}}) = F(g)$. Hence, $F(f^{\mathrm{c}})$ only depends on the class of left homotopy of f^{c}. Now apply proposition 2.2.42 to F^{c} to get $\tilde{F} : \mathbf{Ho}\,(\mathcal{C}) \longrightarrow \mathcal{A}$.

Because $\tilde{F}\gamma = F^{\mathrm{c}}$, there is an obvious natural transformation $\alpha : \tilde{F}\gamma \longrightarrow F$ with component at $A \in \mathcal{C}$ given by:

$$\alpha_A : \quad F(A^{\mathrm{c}}) \xrightarrow{F(\mathrm{q}_A)} F(A) \tag{2.64}$$

Naturality of α is directly given by the functoriality of F. We shall now prove that $(\tilde{F}, \alpha)$ provides a right Kan extension of F along γ. In order to do so, take $G : \mathbf{Ho}\,(\mathcal{C}) \longrightarrow \mathcal{A}$

together with a natural transformation $\beta : G\gamma \longrightarrow F$, and let us prove that β can factor through α. Surely we can use the naturality of β to obtain commutative squares:

$$\begin{tikzcd} G\gamma(A^c) \arrow[r, "G\gamma(\mathfrak{q}_A)"] \arrow[d, "\beta_{A^c}"] & G\gamma(A) \arrow[d, "\beta_A"] \\ F(A^c) \arrow[r, "F(\mathfrak{q}_A)"] & F(A) \end{tikzcd} \tag{2.65}$$

Now remark that $\mathfrak{q}_A$ is a weak equivalence, making $\gamma(\mathfrak{q}_A)$ an isomorphism. Hence,

$$\beta_A = \alpha_A \circ \beta_{A^c} \circ (G\gamma(\mathfrak{q}_A))^{-1} \tag{2.66}$$

The absolute property of $(\tilde{F}, \alpha)$ as Kan extension is quite straightforward: given a functor $H : \mathcal{A} \longrightarrow \mathcal{B}$, HF also maps weak equivalences between cofibrant objects to isomorphisms, so the construction we just exhibited furnishes a right Kan extension $(\mathbf{L}\,HF, \eta)$ of HF along γ in which we recognize immediately $(H \circ \mathbf{L}\,F, \mathrm{id}_H \cdot \alpha)$. □

Remark 2.2.47. The proof shows in particular that the factorization $\delta : G \longrightarrow \mathbf{L}\,F$ of a natural isomorphism $\beta : G\gamma \longrightarrow F$ is also a natural isomorphism.

2.2.5 Quillen adjunctions

Recall that given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with unit η and counit ε, then for any category $\mathcal{A}$,

$$\mathrm{Fun}\,(\mathcal{A}, \mathcal{C}) \underset{G\circ -}{\overset{F\circ -}{\rightleftarrows}} \mathrm{Fun}\,(\mathcal{A}, \mathcal{D}) \tag{2.67}$$

is again an adjunction with $F \circ -$ on the left and $G \circ -$ on the right. Indeed, the unit and counit are respectively given by

$$\begin{gathered} (\eta \cdot \mathrm{id}_X : X \longrightarrow G \circ F \circ X)_{X \in \mathrm{Fun}(\mathcal{A},\mathcal{C})} \\ \text{and} \\ (\varepsilon \cdot \mathrm{id}_Y : F \circ G \circ Y \longrightarrow Y)_{Y \in \mathrm{Fun}(\mathcal{A},\mathcal{D})} \end{gathered} \tag{2.68}$$

Lemma 2.2.48. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Suppose there are weak factorizations systems $(\mathfrak{L}_\mathcal{C}, \mathfrak{R}_\mathcal{C})$ on $\mathcal{C}$ and $(\mathfrak{L}_\mathcal{D}, \mathfrak{R}_\mathcal{D})$ on $\mathcal{D}$. Then F maps elements of $\mathfrak{L}_\mathcal{C}$ to elements of $\mathfrak{L}_\mathcal{D}$ if and only if G maps elements of $\mathfrak{R}_\mathcal{D}$ to elements of $\mathfrak{R}_\mathcal{C}$.*

Proof. Actually this lemma is true for systems that only satisfy condition (i) of definition 2.2.1.

Denote $\mathbf{2}$ the walking arrow. Then given $f : A \longrightarrow B$ in $\mathcal{C}$ and $g : X \longrightarrow Y$ in $\mathcal{D}$, we have a commutative square as follow:

$$\begin{tikzcd} \mathcal{D}\,(FB, X) \arrow[r, "\cong", "\phi"'] \arrow[d] & \mathcal{D}\,(B, GX) \arrow[d] \\ \mathrm{Fun}\,(\mathbf{2}, \mathcal{D})\,(Ff, g) \arrow[r, "\cong"] & \mathrm{Fun}\,(\mathbf{2}, \mathcal{C})\,(f, Gg) \end{tikzcd} \tag{2.69}$$

The vertical maps are induced by composition in $\mathcal{C}$ and $\mathcal{D}$. The top isomorphism is given by the adjoint pair (F, G), and the bottom isomorphism is given by the previous remark that $F \circ -$ is left adjoint to $G \circ -$, instantiated at $\mathcal{A} = \mathbf{2}$. Commutativity of (2.69) is a direct consequence of the naturality of ϕ. In particular, the left vertical arrow is a surjection if and only if the right one is. Put otherwise, g has the right lifting property relatively to Ff if and only if f has the left lifting property relatively to Gg. The lemma follows immediately. □

Definition 2.2.49. An adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between model categories is called a *Quillen adjunction* if it satisfies the following equivalent properties:

(i) F maps cofibrations to cofibrations and G maps fibrations to fibrations,

(ii) F maps cofibrations to cofibrations and acyclic cofibrations to acyclic cofibrations,

(iii) G maps fibrations to fibrations and acyclic fibrations to acyclic fibrations.

Proof. The three properties are equivalent because of lemma 2.2.48 applied to either the weak factorization systems (acyclic cofibrations, fibrations) or (cofibrations, acyclic fibrations) in both categories $\mathcal{C}$ and $\mathcal{D}$. □

Terminology 2.2.50. A *left Quillen functor* is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between model categories satisfying item (ii) in definition 2.2.49. A *right Quillen functor* is a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ between model categories satisfying item (iii) in definition 2.2.49.

Corollary 2.2.51. *Given a left Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, the composite*

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longrightarrow \mathbf{Ho}(\mathcal{D}) \tag{2.70}$$

admits a left derived functor, abusively denoted $\mathbf{L}F : \mathbf{Ho}(\mathcal{C}) \longrightarrow \mathbf{Ho}(\mathcal{D})$.

Proof. It is a direct application of proposition 2.2.46 after using Ken Brown's lemma. □

Dual statements of proposition 2.2.46 and corollary 2.2.51 also hold, eventually leading to the great property of Quillen adjunctions.

Proposition 2.2.52. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a Quillen adjunction. the left derived functor of F and the right derived functor of G form an adjunction*

$$\mathbf{L}F : \mathbf{Ho}(\mathcal{C}) \rightleftarrows \mathbf{Ho}(\mathcal{D}) : \mathbf{R}G \tag{2.71}$$

Proof. We refer to [Mal07]. □

Chapter 3

Quillen bifibrations

In this chapter, we investigate how the two notions of *Grothendieck bifibration* and of *Quillen model category* may be suitably combined together. We are specifically interested in the situation of a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ where the basis category $\mathcal{B}$ as well as each fiber $\mathcal{E}_A$ for an object A of the basis category $\mathcal{B}$ is equipped with a Quillen model structure. The main purpose of this work presented here is to identify necessary and sufficient conditions on the Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ to ensure that the total category $\mathcal{E}$ inherits a model structure from the model structures assigned to the basis $\mathcal{B}$ and to the fibers $\mathcal{E}_A$'s.

3.1 Introduction

This section is devoted to give a broad overview of the content of this chapter. Apart from presenting the motivations behind the results hereafter, it can serve both as an informal introduction to various of the notions needed in this work for the unfamiliar reader and as a guide to fast-forward to the key points of the chapter for the expert. The basic theory of Grothendieck fibrations are assumed to be known in this introduction. But the unfamiliar reader should be reassured, everything needed in this chapter is recalled in section 3.2.1.

Grothendieck fibrations and indexed categories. Let us start our inquiry by recalling the fundamental relationship between bifibrations and adjunctions. This connection will be a guide for the rest of the chapter. Our purpose is indeed to carve out a notion of *Quillen bifibration* playing the same role for Grothendieck bifibrations as the notion of *Quillen adjunction* plays today for the notion of adjunction.

Given a Grothendieck fibration, a *cleavage* is a choice, for every morphism $u : A \longrightarrow B$ and for every object Y above B, of a cartesian morphism $\rho_{u,Y} : u^*Y \longrightarrow Y$ above u. Dually, a *cleavage on a Grothendieck opfibration* is a choice, for every morphism $u : A \longrightarrow B$ and for every object X above A, of a left cartesian morphism $\lambda_{u,X} : X \longrightarrow u_!X$ above u. In a cloven Grothendieck fibration, every morphism $u : A \longrightarrow B$ in the basis category $\mathcal{B}$ induces a functor

$$u^* : \mathcal{E}_B \longrightarrow \mathcal{E}_A \tag{3.1}$$

Symmetrically, in a cloven Grothendieck opfibration, every morphism $u : A \longrightarrow B$ in the basis category $\mathcal{B}$ induces a functor

$$u_! : \mathcal{E}_A \longrightarrow \mathcal{E}_B \tag{3.2}$$

A *cloven bifibration* is both a Grothendieck fibration and opfibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ equipped with a cleavage on both sides.

Formulated in this way, a cloven bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ is simply the "juxtaposition" of Grothendieck fibration and opfibration cleavages, with no apparent connection between the two structures. Hence, a remarkable phenomenon is that the two fibrational structures are in fact strongly interdependent. Indeed, it appears that in a bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, the pair of functors (3.1) and (3.2) associated to a morphism $u : A \longrightarrow B$ defines an adjunction between the fiber categories

$$u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^* \tag{3.3}$$

where the functor $u_!$ is left adjoint to the functor u^*.

The bond between bifibrations and adjunctions is even tighter when one looks at it from the point of view of indexed categories. Recall that a (covariantly) *indexed category* of basis $\mathcal{B}$ is defined as a pseudofunctor

$$P : \mathcal{B} \longrightarrow \mathsf{Cat} \tag{3.4}$$

where Cat denotes the 2-category of categories, functors and natural transformations. Every cloven Grothendieck opfibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ induces an indexed category P which transports every object A of the basis $\mathcal{B}$ to the fiber category $\mathcal{E}_A$, and every morphism $u : A \longrightarrow B$ of the basis to the functor $u_! : \mathcal{E}_A \longrightarrow \mathcal{E}_B$. Conversely, the Grothendieck construction enables one to construct a cloven Grothendieck opfibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ from an indexed category P. This back-and-forth translation defines a equivalence of 2-categories

$$\mathrm{PFun}\,(\mathcal{B}, \mathsf{Cat}) \rightleftarrows \mathrm{OpFib}\,(\mathcal{B}) \tag{3.5}$$

the 2-category on the left having: pseudofunctors $\mathcal{B} \longrightarrow \mathsf{Cat}$ has objects, natural transformations as morphisms, and modifications as 2-cells; and the 2-category on the right having: Grothendieck opfibrations with basis $\mathcal{B}$ as objects, commutative triangles

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{E}' \arrow[dl, "p'"] \\ \mathcal{B} \end{tikzcd} \tag{3.6}$$

with f preserving cocartesian maps as morphisms, and natural transformations above $\mathrm{id}_{\mathcal{B}}$ as 2-cells.

All this is well-known. What is a little bit less familiar (possibly) and which matters to us here is that this correspondence may be adapted to Grothendieck bifibrations, in the following way. Consider the 2-category Adj with

- categories as objects,
- adjunctions
$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R \tag{3.7}$$
as morphisms from $\mathcal{C}$ to $\mathcal{D}$ (L being the left adjoint to R), and with natural transformations
$$\theta : L_1 \longrightarrow L_2 : \mathcal{C} \longrightarrow \mathcal{D} \tag{3.8}$$
between the left adjoint functors as 2-cells $\theta : (L_1, R_1) \longrightarrow (L_2, R_2)$.

In the same way as we have done earlier, an *indexed category-with-adjunctions* with basis category $\mathcal{B}$ is defined as a pseudofunctor
$$P : \mathcal{B} \longrightarrow \mathsf{Adj} \tag{3.9}$$
For the same reasons as in the case of Grothendieck opfibrations, there is an equivalence of 2-categories
$$\mathrm{PFun}\,(\mathcal{B}, \mathsf{Adj}) \rightleftarrows \mathrm{BiFib}\,(\mathcal{B}) \tag{3.10}$$
where the 2-category on the left consists of: pseudofunctors $\mathcal{B} \longrightarrow \mathsf{Adj}$ as objects, natural transformations as morphisms, and modifications as 2-cells; while the 2-category on the right has: Grothendieck bifibrations with basis $\mathcal{B}$ as objects, commutative triangles
$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "L"] \arrow[d, "p"'] & \mathcal{E} \arrow[l, shift left, "R"] \arrow[dl, "p'"] \\ \mathcal{B} & \end{tikzcd} \tag{3.11}$$
with L preserving cocartesian maps (equivalently R preserving cartesian maps) as morphisms, and natural transformations between left adjoint above $\mathrm{id}_{\mathcal{B}}$ as 2-cells.

From this follows, among other consequences, that a cloven bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ is the same thing as a cloven right fibration where the functor $u^* : \mathcal{E}_B \longrightarrow \mathcal{E}_A$ comes equipped with a left adjoint $u_! : \mathcal{E}_A \longrightarrow \mathcal{E}_B$ for every morphism $u : A \longrightarrow B$ of the basis category $\mathcal{B}$.

By way of illustration, consider the ordinal category $\mathbf{2}$ with two objects 0 and 1 and a unique non-identity morphism $u : 0 \longrightarrow 1$. By the discussion above, a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ on the basis category $\mathcal{B} = \mathbf{2}$ is the same thing as an adjunction (3.7). The correspondence relies on the observation that every adjunction (3.7) can be turned into a bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ where the category $\mathcal{E}$ is defined as the category of *collage* associated to the adjunction (L, R), with fibers $\mathcal{E}_0 = \mathcal{C}$, $\mathcal{E}_1 = \mathcal{D}$ and mediating functors $u^* = R$ and $u_! = L$ (see [Str80] for the notion of collage). For that reason, the Grothendieck construction for bifibrations may be seen as a generalized and fibrational notion of collage.

Quillen bifibrations. Seen from that angle, the notion of Grothendieck bifibration provides a fibrational counterpart (and also a far-reaching generalization) of the fundamental notion of adjunction between categories. Our goal is to transfer this view into the realm of modern homotopy theory, thanks to the notion of *Quillen adjunction* between model categories. Recall from definition 2.2.4 that a *model structure* on a category $\mathcal{C}$ delineates three classes $\mathfrak{C}$, $\mathfrak{W}$, $\mathfrak{F}$ of maps called *cofibrations*, *weak equivalences* and *fibrations* respectively ; these classes of maps are moreover required to satisfy some properties. A fibration or a cofibration which is at the same time a weak equivalence is called *acyclic*.

Remark 3.1.1. We find sometimes convenient to call *model structure* a category $\mathcal{C}$ *together* with its model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$. It only differs from a model category, as defined in 2.2.10, by the existence of finite products and finite coproducts. However, in this chapter, there is no issue working with either notions, the extra completeness assumptions being independent of the relationship between Grothendieck bifibrations and model structures.

More precisely, whenever the basis and all the fibers of a given Grothendieck bifibration have all limits of a certain shape, then the total category also has all limits of this shape. The same goes for colimits. For that reason, existence of some limits and colimits in the total category will be treated separately.

Recall from 2.2.50 the notion of left and right Quillen functors and let us call $F : \mathcal{C} \longrightarrow \mathcal{D}$ a *Quillen functor* when it is both left and right Quillen at the same time. A simple argument shows that a Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ transports every weak equivalence of $\mathcal{C}$ to a weak equivalence of $\mathcal{D}$. For that reason, a Quillen functor is the same thing as a functor which transports every cofibration, weak equivalence or fibration $f : A \longrightarrow B$ of $\mathcal{C}$ to a map $Ff : FA \longrightarrow FB$ with the same status in the model structure of $\mathcal{D}$.

Recall from 2.2.49 that the notion of *Quillen adjunction* relies on the observation that in an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ between model categories, the left adjoint L is left Quillen if and only if the right adjoint R is right Quillen.

At this stage, we are ready to introduce the notion of *Quillen bifibration* which we will study in the rest of the chapter. We start by observing that whenever the total category $\mathcal{E}$ of a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is equipped with a model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$, every fiber $\mathcal{E}_A$ associated to an object A of the basis category $\mathcal{B}$ comes equipped with three classes of maps noted $\mathfrak{C}_A$, $\mathfrak{W}_A$ and $\mathfrak{F}_A$, called cofibrations, weak equivalences and fibrations above the object A, respectively. The classes are defined in the expected way:

$$\mathfrak{C}_A = \mathfrak{C}_{\mathcal{E}} \cap \mathrm{Mor}_A \qquad \mathfrak{W}_A = \mathfrak{W}_{\mathcal{E}} \cap \mathrm{Mor}_A \qquad \mathfrak{F}_A = \mathfrak{F}_{\mathcal{E}} \cap \mathrm{Mor}_A \tag{3.12}$$

where Mor_A denotes the class of maps f of the category $\mathcal{E}$ above the object A, that is, such that $p(f) = \mathrm{id}_A$. We declare that the model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ on the total category $\mathcal{E}$ *restricts* to a model structure on the fiber $\mathcal{E}_A$ when the three classes $\mathfrak{C}_A$, $\mathfrak{W}_A$, $\mathfrak{F}_A$ satisfy the properties required of a model structure on the category $\mathcal{E}_A$.

This leads us to the main concept of the chapter:

Definition 3.1.2. A *Quillen bifibration* $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a Grothendieck bifibration where the basis category $\mathcal{B}$ and the total category $\mathcal{E}$ are equipped with a model structure, in such a way that

(i) the functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a Quillen functor,

(ii) the model structure of $\mathcal{E}$ restricts to a model structure on the fiber $\mathcal{E}_A$, for every object A of the basis category $\mathcal{B}$.

This definition of Quillen bifibration deserves to be commented. The first requirement that $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a Quillen functor means that every cofibration, weak equivalence and fibration $f : X \longrightarrow Y$ of the total category $\mathcal{E}$ lies above a map $u : A \longrightarrow B$ of the same status in the model category $\mathcal{B}$. This condition makes sense, and we will see in section 3.3 that it is satisfied in a number of important examples. The second requirement means that the model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ combines into a single model structure on the total category $\mathcal{E}$ the family of all the model structures $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$ on the fiber categories $\mathcal{E}_A$.

A Grothendieck construction for Quillen bifibrations. The notion of Quillen bifibration is tightly connected to the notion of Quillen adjunction, thanks to the following observation that will be established in section 3.3.

Proposition. *In a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, the adjunction*

$$u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^* \tag{3.13}$$

is a Quillen adjunction, for every morphism $u : A \longrightarrow B$ of the basis category $\mathcal{B}$.

From this follows that a Quillen bifibration induces an *indexed model structure*

$$P : \mathcal{B} \longrightarrow \mathsf{Quil} \tag{3.14}$$

defined as a pseudofunctor from a model structure $\mathcal{B}$ to the 2-category Quil of model structures, Quillen adjunctions, and natural transformations. Our main contribution in this chapter is to formulate necessary and sufficient conditions for a Grothendieck construction to hold in this situation. More specifically, we resolve the following problem: suppose given an indexed Quillen category as we have just defined in (3.14) or equivalently, a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ where

- the basis category $\mathcal{B}$ is equipped with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$,
- every fiber $\mathcal{E}_A$ is equipped with a model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$,
- the adjunction $(u_!, u^*)$ is a Quillen adjunction, for every morphism $u : A \longrightarrow B$ of the basis category $\mathcal{B}$.

We find necessary and sufficient conditions to ensure that there exists a model structure $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ on the total category $\mathcal{E}$ such that

- the Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ defines a Quillen bifibration,
- for every object A of the basis category, the model structure of the total category $\mathcal{E}$ restricted to the fiber $\mathcal{E}_A$ coincide with the model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$ given in the hypothesis.

Alongside the resolution, it is shown (see section 3.3) that there exists at most one solution to the problem, which is obtained by defining the cofibrations and fibrations of the total category $\mathcal{E}$ in the following way:

- a morphism $f : X \longrightarrow Y$ of the total category $\mathcal{E}$ is a *total cofibration* when it factors as $X \longrightarrow Z \longrightarrow Y$ where $X \longrightarrow Z$ is a cocartesian map above a cofibration $u : A \longrightarrow B$ of $\mathcal{B}$, and $Z \longrightarrow Y$ is a cofibration in the fiber $\mathcal{E}_B$,
- a morphism $f : X \longrightarrow Y$ of the total category $\mathcal{E}$ is a *total fibration* when it factors as $X \longrightarrow Z \longrightarrow Y$ where $Z \longrightarrow Y$ is a cartesian map above a fibration $u : A \longrightarrow B$ of $\mathcal{B}$, and $X \longrightarrow Z$ is a fibration in the fiber $\mathcal{E}_A$.

Proposition (Uniqueness of the solution)**.** *When the solution $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ exists, it is uniquely determined by the fact that its fibrations and cofibrations are the total cofibrations and total fibrations of the total category $\mathcal{E}$, respectively.*

Besides the formulation of Quillen bifibrations, our main contribution is to devise two conditions called (hCon) for *homotopical conservativity* and (hBC) for *homotopical Beck-Chevalley*, and to show (see theorem 3.4.2) that they are sufficient and necessary for the solution to exist.

Related works. The interplay between bifibred categories and model structures was first explored by Roig in [Roi94], providing results in homological differentially graded algebra. Stanculescu then spotted a mistake in Roig's theorem and subsequently corrected it in [Sta12]. Finally, [HP15] tackles the problem of reflecting Lurie's Grothendieck construction for ∞-categories at the level of model categories, hence giving a model for lax colimits of diagrams of ∞-categories.

This work is directly in line with, and greatly inspired by, these papers. In our view, both Roig-Stanculescu's and Harpaz-Prasma's results suffer from flaws. The former introduces a very strong asymmetry, making natural expectations unmet. For example, for any Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, the opposite functor $p^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$ is also a Grothendieck bifibration. So we shall expect that when it is possible to apply Roig-Stanculescu's result to the functor p, providing this way a model structure on $\mathcal{E}$, it is also possible to apply it to p^{op}, yielding on $\mathcal{E}^{\mathrm{op}}$ the opposite model structure. This is not the case: for almost every such p for which the result applies, it does not for the functor p^{op}. The latter result by Harpaz and Prasma on the contrary forces the symmetry by imposing a rather strong assumption: the adjoint pair $(u_!, u^*)$ associated to a morphism u of the base $\mathcal{B}$, already required to be a Quillen adjunction in [Roi94] and [Sta12], needs in addition to be a Quillen equivalence whenever u is a weak equivalence. While it is a key property for their applications, it put aside real world examples that nevertheless satisfy the conclusion of the result. The goal of this chapter is to lay out a common framework fixing these flaws. This is achieved in theorem 3.4.2 by giving necessary and sufficient conditions for the resulting model structure on $\mathcal{E}$ to be the one described in both cited results.

Plan of the chapter. Section 3.2 recalls the basic facts we will need latter about Grothendieck bifibrations and model categories. It also introduces *intertwined weak factorization systems*, a notion that pops here and there on forums and the n-Category Café, but does not appear in the literature to the best of our knowledge. Its interest mostly resides in that it singles out the 2-out-of-3 property of weak equivalences in a model category from the other more *combinatorial* properties. Finally we recall and prove a result of [Sta12].

Section 3.4 contains the main theorem 3.4.2 that we previously announced. Its proof is cut into two parts: first we prove the necessity of conditions (hCon) and (hBC), and then we show that they are sufficient as well. The proof of necessity is the easy part and comes somehow as a bonus, while the proof of sufficiency is much harder and expose how conditions (hCon) and (hBC) play their role.

Section 3.5 illustrates theorem 3.4.2 with some applications in usual homotopical algebra. First, it gives an original view on Kan's theorem about Reedy model structures by stating it in a bifibrational setting. Here should it be said that this was our motivating example. We realized that neither Roig-Stanculescu's or Harpaz-Prasma's theorem could be apply to the Reedy construction, although the conclusion of these results was giving Kan's theorem back. As in any of those *too good no to be true* situations, we took that as an incentive to strip down the previous results in order to only keep what makes them *tick*, which eventually has led to the equivalence of theorem 3.4.2. Section 3.5.3 gives more details about Roig-Stanculescu's and Harpaz-Prasma's theorem, and explains how their analysis started the process of this work.

Convention. All written diagrams commute if not said otherwise. When objects are missing and replaced by a dot, they can be parsed from other informations on the diagram. Gray parts help to understand the diagram's context.

3.2 Liminaries

3.2.1 Grothendieck bifibrations

In this section, we recall a number of basic definitions and facts about Grothendieck bifibrations.

Given a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$, we shall use the following terminology. The categories $\mathcal{B}$ and $\mathcal{E}$ are called the *basis category* $\mathcal{B}$ and the *total category* $\mathcal{E}$ of the functor $p : \mathcal{E} \longrightarrow \mathcal{B}$. We say that an object X of the total category $\mathcal{E}$ is above an object A of the basis category $\mathcal{B}$ when $p(X) = A$ and, similarly, that a morphism $f : X \longrightarrow Y$ is above a morphism $u : A \longrightarrow B$ when $p(f) = u$. The fiber of an object A in the basis category $\mathcal{B}$ with respect to p is defined as the subcategory of $\mathcal{E}$ whose objects are the objects X such that $p(X) = A$ and whose morphisms are the morphisms f such that $p(f) = \mathrm{id}_A$. In other words, the fiber of A is the category of objects above A, and of morphisms above the identity id_A. The fiber is noted p_A or $\mathcal{E}_A$ when no confusion is possible.

A morphism $f : X \longrightarrow Y$ in a category $\mathcal{E}$ is called *cartesian* with respect to the functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ when the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(Z,X) \arrow[r, "f\circ-"] \arrow[d, "p"'] & \mathcal{E}(Z,Y) \arrow[d, "p"] \\
\mathcal{B}(C,A) \arrow[r, "u\circ-"] & \mathcal{B}(C,B)
\end{tikzcd}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. Here, we write $u : A \longrightarrow B$ and C for the images $u = p(f)$ and $C = p(Z)$ of the morphism f and of the object Z, respectively. Unfolding the definition, this means that for every pair of morphisms $v : C \longrightarrow A$ and $g : Z \longrightarrow Y$ above $u \circ v : C \longrightarrow B$, there exists a unique morphism $h : Z \longrightarrow X$ above v such that $h \circ f = g$. The situation may be depicted as follows:

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "h"] \arrow[dd, dotted, no head] & & \\
& X \arrow[r, "f"] \arrow[dd, dotted, no head] & Y \arrow[dd, dotted, no head] \\
C \arrow[rrd, bend left] \arrow[rd, "v"'] & & \\
& A \arrow[r, "u"'] & B
\end{tikzcd}$$

Dually, a morphism $f : X \longrightarrow Y$ in a category $\mathcal{E}$ is called *cocartesian* with respect to the functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ when the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(Y,Z) \arrow[r, "-\circ f"] \arrow[d, "p"'] & \mathcal{E}(X,Z) \arrow[d, "p"] \\
\mathcal{B}(B,C) \arrow[r, "-\circ u"] & \mathcal{B}(A,C)
\end{tikzcd}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. This means that for every pair of morphisms $v : A \longrightarrow C$ and $g : X \longrightarrow Z$ above $v \circ u : A \longrightarrow C$, there exists a unique morphism $h : Z \longrightarrow X$ above v such that $h \circ f = g$. Diagrammatically:

$$
\begin{tikzcd}
 & & Z \arrow[dd, dotted, no head] \\
X \arrow[r, "f"] \arrow[urr, bend left, "g"] \arrow[dd, dotted, no head] & Y \arrow[ur, dashed, "h"] \arrow[dd, dotted, no head] & \\
 & & C \\
A \arrow[r, "u"'] \arrow[urr, bend left] & B \arrow[ur, "v"'] &
\end{tikzcd}
$$

A functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is called a *Grothendieck fibration* when for every morphism $u : A \longrightarrow B$ and for every object Y above B, there exists a cartesian morphism $f : X \longrightarrow Y$ above u. Symmetrically, a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is called a *Grothendieck opfibration* when for every morphism $u : A \longrightarrow B$ and for every object X above A, there exists a cocartesian morphism $f : X \longrightarrow Y$ above u. Note that a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a Grothendieck opfibration precisely when the functor $p^{op} : \mathcal{E}^{op} \longrightarrow \mathcal{B}^{op}$ is a Grothendieck fibration. A *Grothendieck bifibration* is a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ which is at the same time a Grothendieck fibration and opfibration.

Definition 3.2.1. A *cloven Grothendieck bifibration* is a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ together with

- for any $Y \in \mathcal{E}$ and $u : A \longrightarrow pY$, an object $u^*Y \in \mathcal{E}$ and a cartesian morphism $\rho^p_{u,Y} : u^*Y \longrightarrow Y$ above u,
- for any $X \in \mathcal{E}$ and $u : pX \longrightarrow B$, an object $u_!X \in \mathcal{E}$ and a cocartesian morphism $\lambda^p_{u,X} : X \longrightarrow u_!X$ above u.

When the context is clear enough, we might omit the index p. The domain category $\mathcal{E}$ is often called the *total category* of p, and its codomain $\mathcal{B}$ the *base category*. We shall use this terminology when suited.

Remark 3.2.2. If $\mathcal{E}$ and $\mathcal{B}$ are small relatively to a universe $\mathbb{U}$ in which we suppose the axiom of choice, then a cloven Grothendieck bifibration is exactly the same as the original notion of Grothendieck bifibration. Hence, in this article, we treat the two names as synonym.

The data of such cartesian and cocartesian morphisms gives two factorizations of an arrow $f : X \longrightarrow Y$ above some arrow $u : A \longrightarrow B$, $f_{\triangleright}$ in the fiber $\mathcal{E}_B$ and $f^{\triangleleft}$ in the fiber $\mathcal{E}_A$: one goes through $\rho_{u,Y}$ and the other through $\lambda_{u,X}$. See the diagram below:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f^{\triangleleft}"'] \arrow[dr, "f"] & u_!X \arrow[d, "f_{\triangleright}"] \\
u^*Y \arrow[r] & Y
\end{tikzcd}
\tag{3.15}
$$

In turn, this allows $u_!$ and u^* to be extended as *adjoint functors*:

$$
u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^* \tag{3.16}
$$

where the action of $u_!$ on a morphism $k : X \longrightarrow X'$ of $\mathcal{E}_A$ is given by $(\lambda_{u,X'} \circ k)_{\triangleright}$ and the action of u^* on a morphism $\ell : Y' \longrightarrow Y$ is given by $(\ell \circ \rho_{u,Y'})^{\triangleleft}$:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "k"'] & u_!X \arrow[d, "u_!(k)"] \\
X' \arrow[r] & u_!X'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
u^*Y' \arrow[r] \arrow[d, "u^*\ell"'] & Y' \arrow[d, "\ell"] \\
u^*Y \arrow[r] & Y
\end{tikzcd}
\tag{3.17}
$$

This gives a mapping $\mathcal{B} \longrightarrow \mathsf{Adj}$ from the category $\mathcal{B}$ to the 2-category Adj of adjunctions: it maps an object A to the fiber $\mathcal{E}_A$, and a morphism u to the push-pull adjunction $(u_!, u^*)$. This mapping is even a pseudofunctor:

- For any $A \in \mathcal{B}$ and $X \in \mathcal{E}_A$, we can factor $\mathrm{id}_X : X \longrightarrow X$ through $\lambda_{\mathrm{id}_A,X}$ and $\rho_{\mathrm{id}_A,X}$:

$$
\begin{tikzcd}
X \arrow[r, "\lambda_{\mathrm{id}_A,X}"] \arrow[dr, equal] & (\mathrm{id}_A)_!X \arrow[d, "(\mathrm{id}_X)_{\triangleright}"] \\
& X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[d, "(\mathrm{id}_X)^{\triangleleft}"'] \arrow[dr, equal] & \\
(\mathrm{id}_A)^*X \arrow[r, "\rho_{\mathrm{id}_A,X}"'] & X
\end{tikzcd}
\tag{3.18}
$$

  In particular by looking at the diagram on the left, both $\lambda_{\mathrm{id}_A,X} \circ (\mathrm{id}_X)_{\triangleright}$ and the identity of $(\mathrm{id}_A)_!X$ are solution to the problem of finding an arrow f above id_A such that $f\lambda_{\mathrm{id}_A,X} = \lambda_{\mathrm{id}_A,X}$: by the uniqueness condition of the cocartesian morphisms, it means that they are equal, or otherwise said that $(\mathrm{id}_X)_{\triangleright}$ is an isomorphism with inverse $\lambda_{\mathrm{id}_A,X}$. Dually, looking at the diagram on the right, we deduce that $(\mathrm{id}_X)^{\triangleleft}$ is an isomorphism with inverse $\rho_{\mathrm{id}_A,X}$. All is natural in X, so we end up with

$$(\mathrm{id}_A)_! \simeq \mathrm{id}_{\mathcal{E}_A} \simeq (\mathrm{id}_A)^* \tag{3.19}$$

- For any $u : A \longrightarrow B$ and $v : B \longrightarrow C$ in $\mathcal{B}$, and for any $X \in \mathcal{E}_A$, the cocartesian morphism $\lambda_{vu,X} : X \longrightarrow (vu)_!X$ is above vu by definition hence should factorize as $h\lambda_{u,X}$ for some h above v, yielding $h_{\triangleright}$ as a morphism in $\mathcal{E}_C$ such that the following commutes:

$$
\begin{tikzcd}
X \arrow[r, "\lambda_{u,X}"] \arrow[drr, bend right, "\lambda_{vu,X}"'] & u_!X \arrow[r, "\lambda_{v,u_!X}"] \arrow[dr, "h"'] & v_!u_!X \arrow[d, "h_{\triangleright}"] \\
& & (vu)_!X
\end{tikzcd}
\tag{3.20}
$$

  Writing simply k for the composite $\lambda_{v,u_!X} \circ \lambda_{u,X}$, the following commutes:

$$
\begin{tikzcd}
X \arrow[rr, "\lambda_{vu,X}"] \arrow[dr, "\lambda_{u,X}"'] & & (vu)_!X \arrow[dd, "k_{\triangleright}"] \\
& u_!X \arrow[dr, "\lambda_{v,u_!X}"'] & \\
& & v_!u_!X
\end{tikzcd}
\tag{3.21}
$$

Clearly $h_\triangleright k_\triangleright$ and $\mathrm{id}_{v_! u_! X}$ both are solution to the problem of finding f above id_C such that $f\lambda_{vu,X} = \lambda_{vu,X}$: the uniqueness condition in the definition of cocartesian morphisms forces them to be equal. Conversely, we use the cocartesianness of $\lambda_{u,X}$ and $\lambda_{v,u_!X}$ in two steps: first $k_\triangleright h_\triangleright \lambda_{v,u_!X} = \lambda_{v,u_!X}$ because they both answer the problem of finding f above v such that $f\lambda_{u,X} = \lambda_{v,u_!X} \circ \lambda_{u,X}$; from which we deduce $k_\triangleright h_\triangleright = \mathrm{id}_{v_! u_! X}$ as they both answer the problem of finding a map f above id_C such that $f\lambda_{v,u_!X} = \lambda_{v,u_!X}$. In the end, $h_\triangleright$ and $k_\triangleright$ are isomorphisms, inverse to each other. All we did was natural in X, hence we have

$$(vu)_! \simeq v_! u_! \tag{3.22}$$

- The dual argument shows that $(vu)^* \simeq u^* v^*$.
- To prove rigorously the pseudo functoriality of $\mathcal{B} \longrightarrow \mathsf{Adj}$, we should show that the isomorphisms we have exhibited above are coherent. This is true, but irrelevant to this work, so we will skip it.

The pseudo functoriality relates through an isomorphism the chosen (co)cartesian morphism above a composite vu with the composite of the chosen (co)cartesian morphisms above u and v. The following lemma gives some kind of extension of this result.

Lemma 3.2.3. *Let $u : A \longrightarrow B$, $v : B \longrightarrow C$ and $w : C \longrightarrow D$ in $\mathcal{B}$. Suppose $f : X \longrightarrow Y$ in $\mathcal{E}$ is above the composite wvu. Then for the unique maps $h : v_! u_! X \longrightarrow Y$ and $k : (vu)_! X \longrightarrow Y$ above w that fill the commutative triangles*

$$\begin{tikzcd}
X \arrow[r, "\lambda_{u,X}"] \arrow[rrd, bend right, "f"'] & u_! X \arrow[r, "\lambda_{v,u_!X}"] & v_! u_! X \arrow[d, "h"] \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\lambda_{vu,X}"] \arrow[rd, bend right, "f"'] & (vu)_! X \arrow[d, "k"] \\
& Y
\end{tikzcd}
\tag{3.23}$$

there exists an isomorphism ϕ in the fiber $\mathcal{E}_C$ such that $h\phi = k$.

Proof. We know there is a isomorphism $\phi : (vu)_! X \longrightarrow u_! v_! X$ above id_C such that $\phi\lambda_{vu,X} = \lambda_{v,u_!X} \circ \lambda_{u,X}$. But then $h\phi : (vu)_! X \longrightarrow Y$ is above w and fills the same triangle k does in the statement: by uniqueness, $k = h\phi$.

$$\begin{tikzcd}
& & (vu)_! X \arrow[d, "\phi"] \arrow[dd, bend left, "k"] \\
X \arrow[rru, bend left, "\lambda_{vu,X}"] \arrow[r, "\lambda_{u,X}"] \arrow[rrd, bend right, "f"'] & u_! X \arrow[r, "\lambda_{v,u_!X}"] & v_! u_! X \arrow[d, "h"] \\
& & Y
\end{tikzcd}
\tag{3.24}$$

∎

Of course, we have the dual statement, that accepts a dual proof.

Lemma 3.2.4. *Let $u : A \longrightarrow B$, $v : B \longrightarrow C$ and $w : C \longrightarrow D$ in $\mathcal{B}$. Suppose $f : X \longrightarrow Y$ in $\mathcal{E}$ is above the composite wvu. Then for the unique maps $h : X \longrightarrow u^* v^* Y$ and $k : X \longrightarrow (vu)^* Y$ above w that fill the commutative triangles*

$$\begin{tikzcd}
X \arrow[dr, "h"'] \arrow[drr] \arrow[drrr, bend left, "f"] & & & \\
& u^* v^* Y \arrow[r, "\rho_{u,v^* Y}"'] & v^* Y \arrow[r, "\rho_{v,Y}"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "k"'] \arrow[drr, bend left, "f"] & & \\
& (vu)^* Y \arrow[r, "\rho_{vu,Y}"'] & Y
\end{tikzcd}
\tag{3.25}$$

there exists an isomorphism ϕ in the fiber $\mathcal{E}_C$ such that $\phi k = h$.

Suppose now that we have a chain of composable maps in $\mathcal{B}$:

$$A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} A_n \tag{3.26}$$

And let $f : X \longrightarrow Y$ be a map above the composite $u_n \dots u_1 u_0$. Choose $0 \leq i, j \leq n$ such that $i + j \leq n$. Then, using (co)cartesian choices above maps in $\mathcal{B}$, one can construct two canonical maps associated to f: these are the unique maps

$$\begin{aligned}
&h : (u_i)_! \cdots (u_0)_! X \longrightarrow (u_{n-j+1})^* \cdots (u_n)^* Y \\
&\text{and} \\
&k : (u_i \cdots u_0)_! X \longrightarrow (u_n \cdots u_{n-j+1})^* Y
\end{aligned} \tag{3.27}$$

above $u_{n-j} \cdots u_{i+1} : A_i \longrightarrow A_{n-j}$ (which is defined as id_{A_i} in case $i + j = n$) filling in the following commutative diagrams:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drrrrr, rounded corners, to path={-- ([yshift=-5ex]\tikztostart.south) -| (\tikztotarget)}, "f"'] & \dots \arrow[r, "\lambda"] & (u_i)_! \cdots (u_0)_! X \arrow[dr, "h"] & & & \\
& & & (u_{n-j+1})^* \cdots (u_n)^* Y \arrow[r, "\rho"] & \dots \arrow[r, "\rho"] & Y
\end{tikzcd}$$

$$\tag{3.28}$$

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drrr, rounded corners, to path={-- ([yshift=-5ex]\tikztostart.south) -| (\tikztotarget)}, "f"'] & (u_i \cdots u_0)_! X \arrow[dr, "k"] & & \\
& & (u_n \cdots u_{n-j+1})^* Y \arrow[r, "\rho"] & Y
\end{tikzcd}$$

By applying the previous lemmas multiples times, we get the following useful corollary.

Corollary 3.2.5. *There is fiber isomorphisms ϕ and ψ such that the following commutes:*

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[ddrr, "\lambda"'] \arrow[drrrrr, rounded corners, "f"] & \dots \arrow[r, "\lambda"] & (u_i)_! \cdots (u_0)_! X \arrow[dr, "h"] & & & \\
& & & (u_{n-j+1})^* \cdots (u_n)^* Y \arrow[r, "\rho"] & \dots \arrow[r, "\rho"] & Y \\
& & (u_i \cdots u_0)_! X \arrow[uu, "\phi"] \arrow[dr, "k"] & & & \\
& & & (u_n \cdots u_{n-j+1})^* Y \arrow[uu, "\psi"] \arrow[uurr, "\rho"'] & &
\end{tikzcd} \tag{3.29}$$

We will extensively use this corollary when $i + j = n$. Indeed, in that case h, k, ϕ, ψ all are in the same fiber $\mathcal{E}_{A_i}$ and then h and k are isomorphic as arrows in that fiber. Every property on h that is invariant by isomorphism of arrows will still hold on k, and conversely.

3.2.2 Weak factorization systems

Weak factorization systems have been defined in 2.2.1. Given a weak factorization system $(\mathfrak{L}, \mathfrak{R})$ on a category $\mathcal{M}$, call *left maps* the elements of $\mathfrak{L}$ and *right maps* the elements of $\mathfrak{R}$. So that left maps are precisely the maps having the left lifting property relatively to the right maps and vice versa, and every map of $\mathcal{M}$ is the composite of a left map followed by a right map.

Definition 3.2.6. Given categories $\mathcal{M}$ and $\mathcal{N}$, each with a weak factorization system, an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is said to be *wfs-preserving* if the left adjoint L preserves the left maps, or equivalently if the right adjoint R respects the right maps.

As a key ingredient in the proof of our main result, the following lemma deserves to be stated fully and independently. It explains how to construct a weak factorization system on the total category of a Grothendieck bifibration, given that the basis and fibers all have weak factorization systems in a way that the adjunctions arising from the bifibration are wfs-preserving.

Lemma 3.2.7 (Stanculescu)**.** *Let $\pi : \mathcal{F} \to \mathcal{C}$ be a Grothendieck bifibration with weak factorization systems $(\mathfrak{L}_C, \mathfrak{R}_C)$ on each fiber $\mathcal{F}_C$ and $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{C}$. If the adjoint pair $(u_!, u^*)$ is a wfs-adjunction for every morphism u of $\mathcal{C}$, then there is a weak factorization system $(\mathfrak{L}_{\mathcal{F}}, \mathfrak{R}_{\mathcal{F}})$ on $\mathcal{F}$ defined by*

$$\begin{aligned}
\mathfrak{L}_{\mathcal{F}} &= \{ f : X \to Y \in \mathcal{F} : \pi(f) \in \mathfrak{L}, f_{\triangleright} \in \mathfrak{L}_{\pi Y} \}, \\
\mathfrak{R}_{\mathcal{F}} &= \{ f : X \to Y \in \mathcal{F} : \pi(f) \in \mathfrak{R}, f^{\triangleleft} \in \mathfrak{R}_{\pi X} \}
\end{aligned} \tag{3.30}$$

For the proof in [Sta12, 2.2] is based on a different (yet equivalent) definition of weak factorization systems, here is a proof in our language for readers's convenience.

Proof. Let us begin with the easy part, which is the factorization property. For a map $f : X \longrightarrow Y$ of $\mathcal{F}$, one gets a factorization $\pi(f) = r\ell$ in $\mathcal{C}$ with $\ell : \pi X \longrightarrow C \in \mathfrak{L}$ and $r : C \longrightarrow \pi Y \in \mathfrak{R}$. It induces a fiber morphism $\ell_! X \longrightarrow r^* Y$ in $\mathcal{F}_C$ that we can in turn factor as $r_C \ell_C$ with $\ell_C \in \mathfrak{L}_C$ and $r_C \in \mathfrak{R}_C$.

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, dashed, "\tilde{\ell}"'] & \ell_! X \arrow[d, "\ell_C"] & \\
& \cdot \arrow[d, "r_C"'] \arrow[dr, dashed, "\tilde{r}"] & \\
& r^* Y \arrow[r] & Y
\end{tikzcd} \tag{3.31}$$

Then the wanted factorization of f is $\tilde{r}\tilde{\ell}$ where $\tilde{r}$ is the morphism of $\mathcal{F}$ such that $\pi(\tilde{r}) = r$ and $\tilde{r}^{\triangleleft} = r_C$, and $\tilde{\ell}$ the one such that $\pi(\tilde{\ell}) = \ell$ and $\tilde{\ell}_{\triangleright} = \ell_C$. This is summed up in the previous diagram.

Lifting properties follow the same kind of pattern: take the image by π and do the job in $\mathcal{C}$, then push and pull in $\mathcal{F}$ so that you end up in a fiber when everything goes smoothly. Take a map $j : X \longrightarrow Y \in \mathfrak{L}_{\mathcal{F}}$ and let us show that it lift against elements of $\mathfrak{R}_{\mathcal{F}}$. Consider in $\mathcal{F}$ a commutative square with the map q on the right in $\mathfrak{R}_{\mathcal{F}}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & V \arrow[d, "q"] \\
Y \arrow[r, "g"'] & W
\end{tikzcd} \tag{3.32}$$

By definition, $\pi(j) \in \mathfrak{L}$ has the left lifting property against $\pi(q)$, hence a lift h:

$$\begin{tikzcd}
\pi X \arrow[r, "\pi(f)"] \arrow[d, "\pi(j)"'] & \pi V \arrow[d, "\pi(q)"] \\
\pi Y \arrow[r, "\pi(g)"'] \arrow[ur, "h"] & \pi W
\end{tikzcd} \tag{3.33}$$

Now filling the original square with $\tilde{h} : Y \longrightarrow V$ above h is equivalent to fill the following induced solid square in $\mathcal{F}_{\pi Y}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "j"'] \arrow[rrr, bend left, "f"] & \cdot \arrow[r] \arrow[d, "j_{\triangleright}"'] & \cdot \arrow[r] \arrow[d, "h^*(q^{\triangleleft})"] & V \arrow[d, "q^{\triangleleft}"] \arrow[dr, "q"] & \\
& Y \arrow[r] \arrow[rrr, bend right, "g"'] & \cdot \arrow[r] & \cdot \arrow[r] & W
\end{tikzcd} \tag{3.34}$$

But $j_{\triangleright} \in \mathfrak{L}_{\pi Y}$, and h^* is the right adjoint of a wfs-preserving adjunction, hence maps the right map $q^{\triangleleft}$ of $\mathcal{F}_{\pi V}$ to a right map in $\mathcal{F}_{\pi Y}$: so there is such a filler.

Conversely, if $j : X \longrightarrow Y$ in $\mathcal{F}$ has the left lifting property relatively to all maps of $\mathfrak{R}_{\mathcal{F}}$, then one has to show that it is in $\mathfrak{L}_{\mathcal{F}}$. Consider in $\mathcal{F}_{\pi Y}$ a commutative square as

$$\begin{tikzcd}
X \arrow[r, "\lambda_{\pi(j),X}"] \arrow[dr, "j"'] & \cdot \arrow[r, "f"] \arrow[d, "j_{\triangleright}"] & Y' \arrow[d, "q"] \\
& Y \arrow[r, "g"'] & Y''
\end{tikzcd} \qquad q \in \mathfrak{R}_{\pi Y} \tag{3.35}$$

Then, because q also is in $\mathfrak{R}_{\mathcal{F}}$, there is an $h : Y \longrightarrow Y'$ such that $g = qh$ and $hj = f\lambda_{\pi(j),X}$. But then, $hj_{\triangleright}$ and f both are solution to the factorization problem of j through the cocartesian arrow $\lambda_{\pi(j),X}$, hence should be equal. Meaning h is a filler of the original square in the fiber $\mathcal{F}_{\pi Y}$. We conclude that $j_{\triangleright}$ is a left map in its fiber. Now consider a commutative square in $\mathcal{C}$:

$$\begin{tikzcd} \pi X \arrow[r, "f"] \arrow[d, "\pi(j)"'] & C \arrow[d, "q"] \\ \pi Y \arrow[r, "g"'] & D \end{tikzcd} \qquad q \in \mathfrak{R} \tag{3.36}$$

It induced a commutative square in $\mathcal{F}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & q^* g_! Y \arrow[d, "\kappa"] \\ Y \arrow[r] & g_! Y \end{tikzcd} \tag{3.37}$$

Now the arrow on the right is cartesian above a right map, hence is in $\mathfrak{R}_{\mathcal{F}}$ by definition. So j lift against it, giving us a filler $h : Y \longrightarrow q^* g_! Y$ whose image $\pi(h) : Y \longrightarrow C$ fills the square in $\mathcal{C}$. We conclude that $\pi(j)$ is a left map of $\mathcal{C}$. In the end, $j \in \mathfrak{L}_{\mathcal{F}}$ as we wanted to show.

Similarly, we can show that $\mathfrak{R}_{\mathcal{F}}$ is exactly the class of maps that have the right lifting property against all maps of $\mathfrak{L}_{\mathcal{F}}$. □

3.2.3 Intertwined weak factorization systems and model categories

Quillen introduced model categories in [Qui67] as categories with sufficient structural analogies with the category of topological spaces so that a sensible notion of *homotopy between maps* can be provided. Not necessarily obvious at first sight are the redundancies of Quillen's definition. Even though intentionally important in the conceptual understanding of a model category, the extra checkings required can make a simple proof into a painful process. To ease things a little bit, this part is dedicated to extract the minimal definition of a model category at the cost of trading topological intuition for combinatorial comfort.

Recall from definition 2.2.4 that a model structure on a category $\mathcal{M}$ consists in three classes of maps $\mathfrak{C}, \mathfrak{W}, \mathfrak{F}$ such that:

(i) $\mathfrak{W}$ has the 2-out-of-3 property, i.e. if two elements among $\{f, g, gf\}$ are in W for composable morphisms f and g, then so is the third,

(ii) $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ both are weak factorization systems.

The morphism in $\mathfrak{W}$ are called the *weak equivalences*, those in $\mathfrak{C}$ the *cofibrations* and those in $\mathfrak{F}$ the *fibrations*. Fibrations that are also weak equivalences are called *acyclic fibrations*, and cofibrations that are also weak equivalences are called*acyclic cofibrations*.

Remark also that in the wording of the previous section, Quillen adjunctions, as defined in 2.2.49, between two model structures $\mathcal{M}$ and $\mathcal{N}$ is an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ which is wfs-preserving for both the weak factorization systems (acyclic cofibrations, fibrations) and (cofibrations, acyclic fibrations).

Finally, to conclude those remainders about model structures, let us introduce some new vocabulary.

Definition 3.2.8. A functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ between model structures is said to be *homotopically conservative* if it preserves and reflects weak equivalences.

Remark 3.2.9. To get one's head around this terminology, let us make two observations:

(1) If $\mathcal{M}$ and $\mathcal{N}$ are endowed with the trivial model structure, in which weak equivalences are isomorphisms and cofibrations and fibrations are all morphisms, then the notion boils down to the usual conservative functors.

(2) Every functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ preserving weak equivalences induces a functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{M}) \longrightarrow \mathbf{Ho}(\mathcal{N})$. Given that weak equivalences are saturated in a model category, homotopically conservative functors are exactly those F such that $\mathbf{Ho}(F)$ is conservative as a usual functor.

Let us pursue with the following definition, apparently absent from literature.

Definition 3.2.10. A weak factorization system $(\mathfrak{L}_1, \mathfrak{R}_1)$ on a category $\mathcal{C}$ is *intertwined* with another $(\mathfrak{L}_2, \mathfrak{R}_2)$ on the same category when:

$$\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \qquad \text{and} \qquad \mathfrak{R}_2 \subseteq \mathfrak{R}_1. \tag{3.38}$$

The careful reader will notice that the properties $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ and $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$ are actually equivalent to each other, but the definition is more naturally stated in this way. A similar notion is formulated by Shulman for orthogonal factorization systems, in a blog post on the n-Category Café [Shu10] with a brief mention at the end of a version for weak factorization systems. This is the only appearance of such objects known to us.

The similarity with the weak factorization systems of a model category is immediately noticeable and in fact it goes further than a mere resemblance, as indicated in the following two results.

Proposition 3.2.11. *Let $(\mathfrak{L}_1, \mathfrak{R}_1)$ together with $(\mathfrak{L}_2, \mathfrak{R}_2)$ form intertwined weak factorization systems on a category $\mathcal{C}$. Denoting $\mathfrak{W} = \mathfrak{R}_2 \circ \mathfrak{L}_1$, the following class identities hold:*

$$\mathfrak{L}_1 = \mathfrak{W} \cap \mathfrak{L}_2, \qquad \mathfrak{R}_2 = \mathfrak{W} \cap \mathfrak{R}_1. \tag{3.39}$$

Proof. Let us prove the first identity only, as the second one is strictly dual. Suppose $f : A \longrightarrow B \in \mathfrak{L}_1$, then $f \in \mathfrak{L}_2$ by the very definition of intertwined weak factorization systems, and $f = \mathrm{id}_B f \in \mathfrak{W}$, hence the first inclusion: $\mathfrak{L}_1 \subseteq \mathfrak{W} \cap \mathfrak{L}_2$.

Conversely, take $f \in \mathfrak{W} \cap \mathfrak{L}_2$. Then in particular there exists $j \in \mathfrak{L}_1$ and $q \in \mathfrak{R}_2$ such that $f = qj$. Put otherwise, the following square commutes:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & C \arrow[d, "q"] \\ B \arrow[r, equal] & B. \end{tikzcd} \tag{3.40}$$

But f is in $\mathfrak{L}_2$ and q is in $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$, hence a lift $s : B \longrightarrow C$ such that $qs = \mathrm{id}_B$ and $sf = j$. Now for any $p \in \mathfrak{R}_1$ and any commutative square

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "j"] \arrow[dd, bend right, "f"'] & D \arrow[dd, "p"] \\ C \arrow[d, "q"] & \\ B \arrow[r, "y"] & E \end{tikzcd} \tag{3.41}$$

there is a lift $h : C \longrightarrow D$ taking advantage of j having the left lifting property against p. Then $hs : B \longrightarrow D$ provides a lift showing that f has the left lifting property against p: indeed $phs = yqs = y$ and $hsf = hsqj = hj = x$. Having the left lifting property against any morphism in $\mathfrak{R}_1$, the morphism f ought to be in $\mathfrak{L}_1$, hence providing the reverse inclusion: $\mathfrak{W} \cap \mathfrak{L}_2 \subseteq \mathfrak{L}_1$. □

Corollary 3.2.12. *Let $(\mathfrak{L}_1, \mathfrak{R}_1)$ and $(\mathfrak{L}_2, \mathfrak{R}_2)$ form intertwined weak factorization systems on a category $\mathcal{M}$, and denote again $\mathfrak{W} = \mathfrak{R}_2 \circ \mathfrak{L}_1$. The category $\mathcal{M}$ has a model structure with weak equivalences $\mathfrak{W}$, fibrations $\mathfrak{R}_1$ and cofibrations $\mathfrak{L}_2$ if and only if $\mathfrak{W}$ has the 2-out-of-3 property.*

Of course in that case, we also get the class of acyclic cofibrations as $\mathfrak{L}_1$ and the class of acyclic fibrations as $\mathfrak{R}_2$.

So there it is: we shredded apart the notion of a model structure to the point that what remains is the pretty tame notion of intertwined factorization systems $(\mathfrak{L}_1, \mathfrak{R}_1)$ and $(\mathfrak{L}_2, \mathfrak{R}_2)$ such that $\mathfrak{R}_2 \circ \mathfrak{L}_1$ has the 2-out-of-3 property. But it has the neat advantage to be easily checkable, especially in the context of formal constructions, as it is the case in this chapter. It also emphasizes the fact that Quillen adjunctions are really the right notion of morphisms for intertwined weak factorization systems and have *a priori* nothing to do with weak equivalences. We shall really put that on a stand because everything that follows in the main theorem can be restated with mere intertwined weak factorization systems in place of model structures and it still holds: in fact it represents the easy part of the theorem and all the hard core of the result resides in the 2-out-of-3 property, as usually encountered with model structures.

3.3 Quillen bifibrations

Recall from the introduction that a *Quillen bifibration* is a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ between categories with model structures such that:

(i) the functor p is both a left and right Quillen functor,

(ii) the model structure on $\mathcal{E}$ restricts to a model structure on the fiber $\mathcal{E}_A$, for every object A of the category $\mathcal{B}$.

In this section, we show that in a Quillen bifibration the model structure on the basis $\mathcal{B}$ and on every fiber $\mathcal{E}_A$ determines the original model structure on the total category $\mathcal{E}$. In the remainder of this section, we fix a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$.

Lemma 3.3.1. *For every morphism $u : A \longrightarrow B$ in $\mathcal{B}$, the adjunction $u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ is a Quillen adjunction.*

Proof. Let $f : X \longrightarrow Y$ be a cofibration in the fiber $\mathcal{E}_A$. We want to show that the morphism $u_!(f)$ of $\mathcal{E}_B$ is a cofibration. Take an arbitrary acyclic fibration $q : W \longrightarrow Z$ in $\mathcal{E}_B$ and a commutative square in that fiber:

$$\begin{tikzcd} u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\ u_!Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{3.42}$$

We need to find a lift $h : u_!Y \longrightarrow W$ making the diagram commutes, i.e. such that $qh = g'$ and $hu_!(f) = g$. Let us begin by precomposing with the square defining $u_!(f)$:

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\ Y \arrow[r, "\lambda"'] & u_!Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{3.43}$$

As a cofibration, f has the left lifting property against q, providing a map $k : Y \longrightarrow W$ that makes the following commute:

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\ Y \arrow[r, "\lambda"'] \arrow[urr, "k"] & u_!Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{3.44}$$

Now we use the cocartesian property of $\lambda_{u,Y} : Y \longrightarrow u_!Y$ on k, to find a map $h : u_!Y \longrightarrow W$ above the identity id_B such that $h\lambda_{u,Y} = k$. All it remains to show is that $qh = g'$ and $hu_!(f) = g$. Notice that both qh and g' answer to the problem of finding a map $x : u_!Y \longrightarrow Z$ above id_B such that $x\lambda_{u,Y} = qk$: hence, by the uniqueness condition in the cocartesian property of $\lambda_{u,Y}$, they must be equal. Similarly, $h \circ u_!(f)$ and g solve the problem of finding $x : u_!X \longrightarrow W$ above id_B such that $x\lambda_{u,X} = kf$: the cocartesian property of $\lambda_{u,X}$ allows us to conclude that they are equal. In the end, $u_!(f)$ has the left lifting property against every acyclic fibration of $\mathcal{E}_B$, so it is a cofibration. We prove dually that the image u^*f of a fibration f in $\mathcal{E}_B$ is a fibration of the fiber $\mathcal{E}_A$. □

Lemma 3.3.2. *A cocartesian morphism in $\mathcal{E}$ above a (acyclic) cofibration of $\mathcal{B}$ is a (acyclic) cofibration.*

Proof. Let $f : X \longrightarrow Y$ be cocartesian above a cofibration $u : A \longrightarrow B$ in $\mathcal{B}$. Given a commutative square of $\mathcal{E}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \arrow[d, "q"] \\ Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{3.45}$$

with q an acyclic fibration, we can take its image in $\mathcal{B}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & pW \arrow[d, "p(q)"] \\ B \arrow[r] & pZ \end{tikzcd} \tag{3.46}$$

Since u is a cofibration and $p(q)$ an acyclic fibration, there exists a morphism $h : B \longrightarrow pW$ making the expected diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & pW \arrow[d, "p(q)"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & pZ \end{tikzcd} \tag{3.47}$$

Because f is cocartesian, we know that there exists a (unique) map $\tilde{h} : Y \longrightarrow W$ above h making the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \\ Y \arrow[ur, "\tilde{h}"'] & \end{tikzcd} \tag{3.48}$$

For the morphism $\tilde{h}$ to be a lift in the first commutative square (3.45), there remains to show that $q\tilde{h} = g'$. Because $\tilde{h}$ is above h and $p(q)h = p(g')$, we have that the composite $q\tilde{h}$ is above g'. Moreover $q\tilde{h}f = qg = g'f$. Using the uniqueness property in the universal definition of cocartesian maps, we deduce $q\tilde{h} = g'$. We have just shown that the cocartesian morphism f is weakly orthogonal to every acyclic fibration, and we thus conclude that f is a cofibration. The case of cocartesian morphisms above acyclic cofibrations is treated in a similar way. □

The same argument establishes the dual statement:

Lemma 3.3.3. *A cartesian morphism in $\mathcal{E}$ above a (acyclic) fibration of $\mathcal{B}$ is a (acyclic) fibration.*

Proposition 3.3.4. *A map $f : X \longrightarrow Y$ in $\mathcal{E}$ is a (acyclic) cofibration if and only if $p(f)$ is a (acyclic) cofibration in $\mathcal{E}$ and $f_\triangleright$ is a (acyclic) cofibration in the fiber $\mathcal{E}_{pY}$.*

Proof. A direction of the equivalence is easy: if $p(f) = u : A \longrightarrow B$ is a cofibration, then so is the cocartesian morphism $\lambda_{u,X}$ above it by lemma 3.3.2; if moreover $f_\triangleright$ is a cofibration in the fiber $\mathcal{E}_B$, then $f = f_\triangleright \lambda_{u,X}$ is a composite of cofibration, hence it is a cofibration itself.

Conversely, suppose that $f : X \longrightarrow Y$ is a cofibration in $\mathcal{E}$. Then surely $p(f) = u : A \longrightarrow B$ also is a cofibration in $\mathcal{B}$, since p is a left Quillen functor. Now we want to show that $f_\triangleright : u_!X \longrightarrow Y$ is a cofibration in the fiber $\mathcal{E}_B$. Consider a commutative square in that fiber

$$\begin{tikzcd} u_!X \arrow[r, "g"] \arrow[d, "f_\triangleright"'] & W \arrow[d, "q"] \\ Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{3.49}$$

where q is an acyclic fibration of the fiber $\mathcal{E}_B$, and g, g' are arbitrary morphisms in that fiber. Since f itself is a cofibration in $\mathcal{E}$, we know that there exists a lift $h : Y \longrightarrow W$ for the outer square (with four sides f, q, $g\lambda_{u,X}$ and g') of the following diagram:

$$\begin{tikzcd}
\lambda_{u,X}X \arrow[r, "\lambda_{u,X}"] \arrow[dr, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "f_\triangleright"] & W \arrow[d, "q"] \\
& Y \arrow[r, "g'"'] \arrow[ur, "h"] & Z
\end{tikzcd} \tag{3.50}$$

Now, there remains to show that $hf_\triangleright = g$. We already know that $g\lambda_{u,X} = hf_\triangleright\lambda_{u,X}$, and taking advantage of the fact that the morphism $\lambda_{u,X}$ is cocartesian, we only need to show that $p(g) = p(hf_\triangleright)$. Since g and $f_\triangleright$ are fiber morphisms, it means we need to show that h also. This follows from the fact that $qh = g'$ and that q and g' are fiber morphisms. ☐

In the same way, we get the dual statement:

Proposition 3.3.5. *A map $f : X \longrightarrow Y$ in $\mathcal{E}$ is a (acyclic) cofibration if and only if $p(f)$ is a (acyclic) cofibration in $\mathcal{E}$ and $f_\triangleright$ is a (acyclic) cofibration in the fiber $\mathcal{E}_{pY}$.*

In particular, this means that the model structure on the total category $\mathcal{E}$ is entirely determined by the model structures on the basis $\mathcal{B}$ and on each fiber $\mathcal{E}_B$ of the bifibration. As these characterizations turn out to be important for what follows, we shall name them.

Definition 3.3.6. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a Grothendieck bifibration such that its basis $\mathcal{B}$ and each fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) have a model structure.

- a *total cofibration* is a morphism $f : X \longrightarrow Y$ of $\mathcal{E}$ above a cofibration $u : A \longrightarrow B$ of $\mathcal{B}$ such that $f_\triangleright$ is a cofibration in the fiber $\mathcal{E}_B$,
- a *total fibration* is a morphism $f : X \longrightarrow Y$ of $\mathcal{E}$ above a fibration $u : A \longrightarrow B$ of $\mathcal{B}$ such that $f^\triangleleft$ is a fibration in the fiber $\mathcal{E}_A$,
- a *total acyclic cofibration* is a morphism $f : X \longrightarrow Y$ of $\mathcal{E}$ above an acyclic cofibration $u : A \longrightarrow B$ of $\mathcal{B}$ such that $f_\triangleright$ is an acyclic cofibration in the fiber $\mathcal{E}_B$,
- a *total acyclic fibration* is a morphism $f : X \longrightarrow Y$ of $\mathcal{E}$ above an acyclic fibration $u : A \longrightarrow B$ of $\mathcal{B}$ such that $f^\triangleleft$ is an acyclic fibration in the fiber $\mathcal{E}_A$.

Using this terminology, the two propositions 3.3.4 and 3.3.5 just established come together as: the cofibrations, fibrations, acyclic cofibrations and acyclic fibrations of a Quillen bifibration are necessarily the total ones. Note also that the definitions of total cofibration and total fibration given in definition 3.3.6 coincides with the definition given in the introduction.

We end this section by giving simple examples of Quillen bifibrations. They should serve as both a motivation and a guide for the reader to navigate into the following definitions and proofs: it surely has worked that way for us authors.

Example(s) 3.3.7.

(1) One of the simplest instances of a Grothendieck bifibration other than the identity functor, is a projection from a product:

$$p : \mathcal{M} \times \mathcal{B} \longrightarrow \mathcal{B} \tag{3.51}$$

Cartesian and cocartesian morphisms coincide and are those of the form (id_M, u) for $M \in \mathcal{M}$ and u a morphism of $\mathcal{B}$. In particular, one have $(f, u)^{\triangleleft} = (f, \mathrm{id}_A)$ and $(f, u)_{\triangleright} = (f, \mathrm{id}_B)$ for any $u : A \longrightarrow B$ in $\mathcal{B}$ and any f in $\mathcal{M}$.

If $\mathcal{B}$ and $\mathcal{M}$ are model categories, each fiber $p_A \simeq \mathcal{M}$ inherits a model structure from $\mathcal{M}$ and the total fibrations and cofibrations coincide precisely with the one of the usual model structure on the product $\mathcal{M} \times \mathcal{B}$.

(2) For a category $\mathcal{B}$, one can consider the codomain functor:

$$\mathrm{cod} : \mathrm{Fun}(\mathbf{2}, \mathcal{B}) \longrightarrow \mathcal{B},\ (X \xrightarrow{f} A) \longmapsto A \tag{3.52}$$

Cocartesian morphisms above u relatively to cod are those commutative square of the form

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "uf"] \\ A \arrow[r, "u"] & B \end{tikzcd} \tag{3.53}$$

whereas cartesian morphisms above u are the pullback squares along u. Hence cod is a Grothendieck bifibration whenever $\mathcal{B}$ admits pullbacks.

If moreover $\mathcal{B}$ is a model category, then each fiber $\mathrm{cod}_A \simeq \mathcal{B}/A$ inherits a model structure (namely an arrow is a fibration or a cofibration if it is such as an arrow of $\mathcal{B}$), and the total fibrations and cofibrations coincide with the one in the injective model structure on $\mathrm{Fun}(\mathbf{2}, \mathcal{B})$: i.e. a cofibration is a commutative square with the top and bottom arrows being cofibrations in $\mathcal{B}$, whereas fibrations are those commutative squares

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, dashed, "h"] \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\ & A \times_{u,g} Y \arrow[ur] \arrow[dl] & \\ A \arrow[rr, "u"] & & B \end{tikzcd} \tag{3.54}$$

where both u and h are fibrations in $\mathcal{B}$.

(3) Similarly, the total fibrations and cofibrations of the Grothendieck bifibration $\mathrm{dom} : \mathrm{Fun}(\mathbf{2}, \mathcal{B}) \longrightarrow \mathcal{B}$ over a model category $\mathcal{B}$ are exactly those of the projective model structure on $\mathrm{Fun}(\mathbf{2}, \mathcal{B})$.

(4) In both [Sta12] and [HP15], the authors prove a theorem similar to our, putting a model structure on the total category of a Grothendieck bifibration under specific hypothesis. In both case, fibrations and cofibrations of this model structure end up being the total ones. The following theorem encompasses in particular this two results.

3.4 A Grothendieck construction for Quillen bifibrations

Now we have the tools to move on to the main goal of this chapter, which is to turn a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ into a Quillen bifibration whenever both the basis category $\mathcal{B}$ and every fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) admit model structures in such a way that all the pairs of adjoint push and pull functors between fibers are "homotopically well-behaved". To be more precise, we now suppose $\mathcal{B}$ to be equipped with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$, and each fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) to be equipped with a model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. We also make the following **fundamental assumption**:

$$\text{For all } u \text{ in } \mathcal{B}, \text{ the adjoint pair } (u_!, u^*) \text{ is a Quillen adjunction.} \tag{Q}$$

We defined in definition 3.3.6 notions of total cofibrations and total fibrations, as well as their acyclic counterparts. These are reminiscent of what happens with Quillen bifibrations, but they can be defined for any Grothendieck bifibration whose basis and fibers have model structures. We must insist that in that framework, *total cofibrations* and *total fibrations* are only names, and by no means are they giving the total category $\mathcal{E}$ a model structure. Indeed, the goal of this section, and to some extent even the goal of this entire chapter, is to provide a complete characterization, under hypothesis (Q), of the Grothendieck bifibrations $p : \mathcal{E} \longrightarrow \mathcal{B}$ for which the total cofibrations and total fibrations make p into a Quillen bifibration. For the rest of this section, we shall denote $\mathfrak{C}_{\mathcal{E}}$, $\mathfrak{F}_{\mathcal{E}}$, $\mathfrak{C}_{\tilde{\mathcal{E}}}$ and $\mathfrak{F}_{\tilde{\mathcal{E}}}$ for the respective classes of total cofibrations, total fibrations, total acyclic cofibrations, and total acyclic fibrations, that is:

$$\begin{aligned}
\mathfrak{C}_{\mathcal{E}} &= \{f : X \longrightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{C}, f_{\triangleright} \in \mathfrak{C}_{pY}\},\\
\mathfrak{F}_{\mathcal{E}} &= \{f : X \longrightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{F}, f^{\triangleleft} \in \mathfrak{F}_{pX}\},\\
\mathfrak{C}_{\tilde{\mathcal{E}}} &= \{f : X \longrightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{W} \cap \mathfrak{C}, f_{\triangleright} \in \mathfrak{W}_{pY} \cap \mathfrak{C}_{pY}\},\\
\mathfrak{F}_{\tilde{\mathcal{E}}} &= \{f : X \longrightarrow Y \in \mathcal{E} : p(f) \in \mathfrak{W} \cap \mathfrak{F}, f^{\triangleleft} \in \mathfrak{W}_{pX} \cap \mathfrak{F}_{pX}\}
\end{aligned} \tag{3.55}$$

3.4.1 Main theorem

In order to state the theorem correctly, we will need some vocabulary. Recall that the *mate* $\mu : u'_! v^* \longrightarrow v'^* u_!$ associated to a commutative square of $\mathcal{B}$

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\
C' \arrow[r, "v'"'] & B
\end{tikzcd} \tag{3.56}$$

is the natural transformation constructed at point $Z \in \mathcal{E}_C$ in two steps as follow: the composite

$$v^* Z \longrightarrow Z \longrightarrow u_! Z \tag{3.57}$$

which is above uv, factors through the cartesian arrow $\rho_{v', u_! Z} : v'^* u_! Z \longrightarrow u_! Z$ (because $v' u' = uv$) into a morphism $v^* Z \longrightarrow v'^* u_! Z$ above u', which in turn factors through the cocartesian arrow $\lambda_{u', v_! Z} : u'^* v_! Z \longrightarrow v'^* u_! Z$ giving rise to μ_Z, as summarized in the diagram below.

$$
\begin{tikzcd}
v^*Z \arrow[rrr, "\rho"] \arrow[d, "\lambda"] & & & Z \arrow[dd, "\lambda"] \\
u'_! v^* Z \arrow[drr, dashed, "\mu_Z"] & & & \\
& & v'^* u_! Z \arrow[r, "\rho"] & u_! Z
\end{tikzcd}
\tag{3.58}
$$

Definition 3.4.1. A commutative square of $\mathcal{B}$ is said to satisfy the *homotopical Beck-Chevalley condition* if its mate is pointwise a weak equivalence.

Consider then the following properties on the Grothendieck bifibration p:

Every commutative square of $\mathcal{B}$ of the form

$$
\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\
C' \arrow[r, "v'"'] & B
\end{tikzcd}
\qquad
\begin{array}{l} u, u' \in \mathfrak{C} \cap \mathfrak{W}, \\ v, v' \in \mathfrak{F} \cap \mathfrak{W} \end{array}
\tag{3.59}
$$
(hBC)

satisfies the homotopical Beck-Chevalley condition.

and

The functors $u_!$ and v^* are homotopically conservative whenever u is an acyclic cofibration and v an acyclic fibration. (hCon)

The theorem states that this is exactly what it takes to make the names "total cofibrations" and "total fibrations" legitimate, and to turn $p : \mathcal{E} \longrightarrow \mathcal{B}$ into a Quillen bifibration.

Theorem 3.4.2. *Under hypothesis (Q), the total category $\mathcal{E}$ admits a model structure with $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ as cofibrations and fibrations respectively if and only if properties (hBC) and (hCon) are satisfied.*

In that case, the functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a Quillen bifibration.

The proof begin with a very candid remark that we promote as a proposition because we shall use it several times in the rest of the proof.

Proposition 3.4.3. $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ *and* $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ *are intertwined weak factorization systems.*

Proof. Obviously $\mathfrak{C}_{\mathcal{E}}^{\sim} \subseteq \mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim} \subseteq \mathfrak{F}_{\mathcal{E}}$. Independently, a direct application of lemma 3.2.7 shows that $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ are both weak factorization systems on $\mathcal{E}$. □

The strategy to prove theorem 3.4.2 then goes as follow:

- first we will show the necessity of conditions (hBC) and (hCon): if $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ are the cofibrations and fibrations of a model structure on $\mathcal{E}$, then hypothesis (hBC) and (hCon) are met,
- next, the harder part is the sufficiency: because of proposition 3.4.3, it is enough to show that the induced class $\mathfrak{W}_{\mathcal{E}} = \mathfrak{F}_{\mathcal{E}}^{\sim} \circ \mathfrak{C}_{\mathcal{E}}^{\sim}$ of *total weak equivalences* has the 2-out-of-3 property to conclude through corollary 3.2.12.

3.4.2 Proof, part I: necessity

In all this section, we suppose that $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ provide respectively the cofibrations and fibrations of a model structure on the total category $\mathcal{E}$. We will denote $\mathfrak{W}_{\mathcal{E}}$ the corresponding class of weak equivalences.

First, we prove a technical lemma, directly following from proposition 3.4.3, that will be extensively used in the following. Informally, it states that the name given to the members of $\mathfrak{C}_{\tilde{\mathcal{E}}}$ and $\mathfrak{F}_{\tilde{\mathcal{E}}}$ are not foolish.

Lemma 3.4.4. *$\mathfrak{C}_{\tilde{\mathcal{E}}} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\tilde{\mathcal{E}}} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}}$.*

Proof. By proposition 3.4.3, we know that both $(\mathfrak{C}_{\tilde{\mathcal{E}}}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ are weak factorization systems with the same class of right maps, hence their class of left maps should coincide. Similarly the weak factorization systems $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\tilde{\mathcal{E}}})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}})$ have the same class of left maps, hence their class of right maps coincide.

□

Corollary 3.4.5. *For any object A of $\mathcal{B}$, the inclusion functor $\mathcal{E}_A \longrightarrow \mathcal{E}$ is homotopically conservative.*

Proof. The preservation of weak equivalences comes from the fact that acyclic cofibrations and acyclic fibrations of $\mathcal{E}_A$ are elements of $\mathfrak{C}_{\tilde{\mathcal{E}}}$ and $\mathfrak{F}_{\tilde{\mathcal{E}}}$ respectively. Thus, by lemma 3.4.4, they are elements of $\mathfrak{W}_{\mathcal{E}}$.

Conversely, suppose that f is a map of $\mathcal{E}_A$ which is a weak equivalence of $\mathcal{E}$. We want to show that f is a weak equivalence of the fiber $\mathcal{E}_A$. The map f factors in the fiber $\mathcal{E}_A$ as $f = qj$ where $j \in \mathfrak{C}_A \cap \mathfrak{W}_A$ and $q \in \mathfrak{F}_A$. We just need to show that $q \in \mathfrak{W}_A$. By lemma 3.4.4, j is also a weak equivalence of $\mathcal{E}$. By the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$, the map q is a weak equivalence of $\mathcal{E}$. As a fibration of $\mathcal{E}_A$, q is also a fibration of $\mathcal{E}$. This establishes that q is an acyclic fibration of $\mathcal{E}$. By lemma 3.4.4, q is thus an element of $\mathfrak{F}_{\tilde{\mathcal{E}}}$. This concludes the proof that $q = q^{\triangleleft}$ is an acyclic fibration, and thus a weak equivalence, in the fiber $\mathcal{E}_A$. □

Proposition 3.4.6 (Property (hCon))**.** *If $j : A \longrightarrow B$ in an acyclic cofibration in $\mathcal{B}$, then $j_! : \mathcal{E}_A \longrightarrow \mathcal{E}_B$ is homotopically conservative.*

If $q : A \longrightarrow B$ in an acyclic fibration in $\mathcal{B}$, then $q^* : \mathcal{E}_B \longrightarrow \mathcal{E}_A$ is homotopically conservative.

Proof. We only prove the first part of the proposition, as the second one is dual. Recall that the image $j_!(f)$ of a map $f : X \longrightarrow Y$ of $\mathcal{E}_A$ is computed as the unique morphism of $\mathcal{E}_B$ making the following square commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & j_! X \arrow[d, "j_!(f)"] \\ Y \arrow[r] & j_! Y \end{tikzcd} \tag{3.60}$$

The horizontal morphisms in the diagram are cocartesian above the acyclic cofibration j. As such they are elements $\mathfrak{C}_{\tilde{\mathcal{E}}}$, and thus weak equivalence in $\mathcal{E}$ by lemma 3.4.4. By the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$, f is a weak equivalence in $\mathcal{E}$ if and only if $j_!(f)$ is one also in $\mathcal{E}$. Corollary 3.4.5 allows then to conclude: f is a weak equivalence in the fiber $\mathcal{E}_A$ if and only if $j_!(f)$ is one in the fiber $\mathcal{E}_B$. □

Proposition 3.4.7 (Property (hBC)). *Commutative squares of $\mathcal{B}$ of the form*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"'] & B \end{tikzcd} \qquad u, u' \in \mathfrak{C} \cap \mathfrak{W} \quad v, v' \in \mathfrak{F} \cap \mathfrak{W} \tag{3.61}$$

satisfy the homotopical Beck-Chevalley condition.

Proof. Recall that for such a square in $\mathcal{B}$, the component of the mate $\mu : u'_! v^* \longrightarrow v'^* u_!$ at $Z \in \mathcal{E}_C$ is defined as the unique map of $\mathcal{E}_{Z'}$ making the following diagram commute:

$$\begin{tikzcd} v^*Z \arrow[rr, "\rho"] \arrow[d, "\lambda"] & & Z \arrow[dd, "\lambda"] \\ u'_! v^* Z \arrow[dr, "\mu_Z"] & & \\ & v'^* u_! Z \arrow[r, "\rho"] & u_! Z \end{tikzcd} \tag{3.62}$$

Arrows labeled ρ and λ are respectively cartesian above acyclic fibrations and cocartesian above acyclic cofibrations, hence weak equivalences of $\mathcal{E}$ by lemma 3.4.4. By applying the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$ three times in a row, we conclude that the fiber map μ_Z is a weak equivalence of $\mathcal{E}$, hence also of $\mathcal{E}_{C'}$ by corollary 3.4.5. □

3.4.3 Proof, part II: sufficiency

We have established the necessity of (hBC) and (hCon) in theorem 3.4.2. We now prove the sufficiency of these conditions. This is the hard part of the proof. Recall that every fiber $\mathcal{E}_A$ of the Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ is equipped with a model structure in such a way that (Q) is satisfied. From now on, we make the additional assumptions that (hBC) and (hCon) are satisfied.

We will use the notation $\mathfrak{W}_{\mathcal{E}} = \mathfrak{F}_{\tilde{\mathcal{E}}} \circ \mathfrak{C}_{\tilde{\mathcal{E}}}$ the class of maps that can be written as a total acyclic cofibration postcomposed with a total acyclic fibration. The overall goal of this section is to prove that

Claim. $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ *defines a model structure on the total category $\mathcal{E}$.*

By proposition 3.4.3, we already know that $(\mathfrak{C}_{\tilde{\mathcal{E}}}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\tilde{\mathcal{E}}})$ are intertwined weak factorization systems. From this follows that, by corollary 3.2.12, we only need to show that the class $\mathfrak{W}_{\mathcal{E}}$ of *total weak equivalences* satisfies the 2-out-of-3 property.

A first step is to get a better understanding of the total weak equivalences. For $f : X \longrightarrow Y$ in $\mathcal{E}$ such that $p(f) = vu$ for two composable morphisms $u : pX \longrightarrow C$ and $v : C \longrightarrow pY$ of $\mathcal{B}$, there is a unique morphism inside the fiber $\mathcal{E}_C$

$$_u f^v : u_! X \longrightarrow v^* Y \tag{3.63}$$

such that $f = \rho_{v,Y} \circ {}_u f^v \circ \lambda_{u,X}$. This morphism $_u f^v$ can be constructed as $k^{\triangleleft}$ where k is the unique morphism above v factorizing f through $\lambda_{u,X}$; or equivalently as $\ell_{\triangleright}$ where ℓ is the unique morphism above u factorizing f through $\rho_{v,Y}$. This is summed up in the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[dr, dashed, "\ell"'] & u_!X \arrow[d, "\ell_{\triangleright} = {}_u f^v = k^{\triangleleft}"] \arrow[dr, dashed, "k"] & \\
& v^*Y \arrow[r, "\rho"'] & Y
\end{tikzcd}
\tag{3.64}
$$

Notice that, in particular, a morphism f of $\mathfrak{W}_{\mathcal{E}}$ is exactly a morphism of $\mathcal{E}$ for which **there exists** a factorization $p(f) = qj$ with $j \in \mathfrak{W} \cap \mathfrak{C}$ and $q \in \mathfrak{W} \cap \mathfrak{F}$ such that ${}_j f^q$ is a weak equivalence in the corresponding fiber. We shall strive to show that, under our hypothesis (hCon) and (hBC), a morphism f of $\mathfrak{W}_{\mathcal{E}}$ satisfies the same property that ${}_j f^q$ is a weak equivalence **for all** such factorization $p(f) = qj$. This is the contain of proposition 3.4.10. We start by showing the property in the particular case where $p(f)$ is an acyclic cofibration (lemma 3.4.8) or an acyclic fibration (lemma 3.4.9).

Lemma 3.4.8. *Suppose that $f : X \longrightarrow Y$ is a morphism of $\mathcal{E}$ such that $p(f)$ is an acyclic cofibration in $\mathcal{B}$. If $p(f) = qj$ with $q \in \mathfrak{W} \cap \mathfrak{C}$ and $j \in \mathfrak{W} \cap \mathfrak{F}$, then $f_{\triangleright}$ is a weak equivalence if and only if ${}_j f^q$ is a weak equivalence.*

Proof. Since $p(f) = qj$, lemma 3.2.3 provides an isomorphism ϕ in the fiber $\mathcal{E}_{pY}$ such that $f_{\triangleright} = \tilde{f}_{\triangleright}\phi$, where $\tilde{f}_{\triangleright}$ is the morphism obtained by pushing in two steps:

$$
\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drr, "f"'] \arrow[rrr, bend left, "\lambda"] & j_!X \arrow[r, "\lambda"] & q_!j_!X \arrow[d, "\tilde{f}_{\triangleright}"] & (qj)_!X \arrow[l, "\phi"', "\simeq"] \arrow[dl, "f_{\triangleright}"] \\
& & Y &
\end{tikzcd}
\tag{3.65}
$$

By definition, ${}_j f^q$ is the image of $\tilde{f}_{\triangleright}$ under the natural bijection $\mathcal{E}_{pY}(q_!j_!X, Y) \xrightarrow{\simeq} \mathcal{E}_{pX}(j_!X, q^*Y)$. So it can be written ${}_j f^q = q^*(\tilde{f}_{\triangleright}) \circ \eta_{j_!X}$ using the unit η of the adjunction $(q_!, q^*)$. We can now complete the previous diagram as follow:

$$
\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drr, "f"] \arrow[rrr, bend left, "\lambda"] & j_!X \arrow[r, "\lambda"] \arrow[d, "\eta"] \arrow[dd, bend right=60, "{}_j f^q"'] & q_!j_!X \arrow[d, "\tilde{f}_{\triangleright}"] & (qj)_!X \arrow[l, "\phi"', "\simeq"] \arrow[dl, "f_{\triangleright}"] \\
& q^*q_!j_!X \arrow[ur, "\rho"] \arrow[d, "q^*(\tilde{f}_{\triangleright})"] & Y & \\
& q^*Y \arrow[ur, "\rho"'] & &
\end{tikzcd}
\tag{3.66}
$$

Proving that $\eta_{j_!X}$ is a weak equivalence is then enough to conclude: in that case ${}_j f^q$ is an weak equivalence if and only if $q^*(\tilde{f}_{\triangleright})$ is such by the two-of-three property ; $q^*(\tilde{f}_{\triangleright})$ is a weak equivalence if and only if $\tilde{f}_{\triangleright}$ is a weak equivalence in $\mathcal{E}_{pY}$ by (hCon) ; and finally $\tilde{f}_\triangleright$ is a weak equivalence if an only if $f_\triangleright$ is such because they are isomorphic as arrows in $\mathcal{E}_{pY}$.

So it remains to show that $\eta_{j_!X}$ is a weak equivalence in its fiber. Since $p(f) = qj$, the following square commutes in $\mathcal{B}$:

$$\begin{tikzcd} pX \arrow[r, equal, "\mathrm{id}_{pX}"] \arrow[d, "j"'] & pX \arrow[d, "qj"] \\ C \arrow[r, "q"'] & pY \end{tikzcd} \tag{3.67}$$

This is a square of the correct form to apply (hBC): hence the associated mate at component X

$$\mu_X : j_!(\mathrm{id}_{pX})^* X \longrightarrow q^*(qj)_! X \tag{3.68}$$

is a weak equivalence in the fiber $\mathcal{E}_C$. Corollary 3.2.5 ensures that μ_X is isomorphic as arrow of $\mathcal{E}_C$ to the unique fiber morphism that factors $\rho_{q,q_!j_!X}$ through $\lambda_{q,j_!X}$:

$$\begin{tikzcd} j_!X \arrow[r, "\lambda"] \arrow[d] & q_!j_!X \\ q^*q_!j_!X \arrow[ur, "\rho"'] & \end{tikzcd} \tag{3.69}$$

This is exactly the definition of the unit η at $j_!X$. Isomorphic morphisms being weak equivalences together, $\eta_{j_!X}$ is also acyclic in $\mathcal{E}_C$. □

Of course, one gets the dual lemma by dualizing the proof that we let for the reader to write down.

Lemma 3.4.9. *Let $f : X \longrightarrow Y$ a morphism of $\mathcal{E}$ such that $p(f)$ is an acyclic fibration in $\mathcal{B}$. If $p(f) = qj$ with $q \in \mathfrak{W} \cap \mathfrak{C}$ and $j \in \mathfrak{W} \cap \mathfrak{F}$, then $f^\triangleleft$ is a weak equivalence if and only if ${}_jf^q$ is a weak equivalence.*

We shall now prove the key proposition of this section.

Proposition 3.4.10. *Let $f : X \longrightarrow Y$ in $\mathcal{E}$. If $p(f) = qj = q'j'$ for some $j, j \in \mathfrak{W} \cap \mathfrak{C}$ and $q, q' \in \mathfrak{W} \cap \mathfrak{F}$, then ${}_jf^q$ is a weak equivalence if and only if ${}_{j'}f^{q'}$ is a weak equivalence.*

Proof. By hypothesis the following square commutes in $\mathcal{B}$:

$$\begin{tikzcd} pX \arrow[r, "j'"] \arrow[d, "j"'] & C' \arrow[d, "q'"] \\ C \arrow[r, "q"'] & pY \end{tikzcd} \tag{3.70}$$

Since j is an acyclic cofibration and q' a (acyclic) fibration, there is a filler $h : C \longrightarrow C'$ of the previous square, that is a weak equivalence by the 2-out-of-3 property. Hence it can be factored $h = h_f h_c$ as an acyclic cofibration followed by an acyclic fibration in $\mathcal{B}$. Write $j'' = h_c j$ and $q'' = q' h_f$ which are respectively an acyclic cofibration and an acyclic fibration as composite of such, and produce a new factorization of $p(f) = q'' j''$.

$$\begin{tikzcd}
pX \arrow[rr, "j'"] \arrow[dr, "j''"] \arrow[dd, "j"'] & & C' \arrow[dd, "q'"] \\
& C'' \arrow[ur, "h_f"] \arrow[dr, "q''"] & \\
C \arrow[ur, "h_c"] \arrow[rr, "q"'] & & pY
\end{tikzcd} \tag{3.71}$$

Write r for the composite ${}_{j'}f^{q'} \circ \lambda_{X,j'} : X \longrightarrow j'_! X \longrightarrow q'^* Y$. Then r is above the acyclic cofibration $j' = h_f j''$ and lemma 3.4.8 can be applied: $r_\triangleright$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f} : j''_! X \longrightarrow (h_f)^* q'^* Y$ is a weak equivalence in $\mathcal{E}_{C''}$. And by very definition $r_\triangleright = {}_f j'^{q'}$. So ${}_f j'^{q'}$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f}$ is such in $\mathcal{E}_{C''}$.

Similarly write s for the composite $\rho_{q,Y} \circ {}_j f^q : j_! X \longrightarrow q^* Y \longrightarrow Y$. Then s is above the acyclic fibration $q = q'' h_c$ and lemma 3.4.9 can be applied: $s^\triangleleft$ is a weak equivalence in $\mathcal{E}_C$ if and only if ${}_{h_c}s^{q''} : (h_c)_! j_! X \longrightarrow q''^* Y$ is a weak equivalence (in $\mathcal{E}_{C''}$). And by very definition $s^\triangleleft = {}_f j^q$. So ${}_f j^q$ is a weak equivalence in $\mathcal{E}_C$ if and only if ${}_{h_c}s^{q''}$ is such in $\mathcal{E}_{C''}$.

Now recall that $j'' = h_c j$ and $q'' = q' h_f$. By lemmas 3.2.3 and 3.2.4, there exists isomorphisms $j''_! X \simeq h_{c!} j_! X$ and $q''^* Y \simeq h_f{}^* q'^* Y$ in fiber $\mathcal{E}_{C''}$ making the following commute :

$$\begin{tikzcd}
h_{c!} j_! X \arrow[r, "\simeq"] \arrow[dr, "{}_{h_c}s^{q''}"'] & j''_! X \arrow[d, "{}_{j''}f^{q''}"] \arrow[dr, "{}_{j''}r^{h_f}"] & \\
& q''^* Y \arrow[r, "\simeq"] & h_f{}^* q'^* Y
\end{tikzcd} \tag{3.72}$$

In particular, the morphisms ${}_{j''}r^{h_f}$ and ${}_{h_c}s^{q''}$ are weak equivalences together. We conclude the argument: ${}_{j'}f^{q'}$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f}$ is such in $\mathcal{E}_{C''}$ if and only if ${}_{h_c}s^{q''}$ is such in $\mathcal{E}_{C''}$ if and only if ${}_j f^q$ is a weak equivalence in $\mathcal{E}_C$. □

The previous result allow the following "trick": to prove that a map f of $\mathcal{E}$ is in $\mathfrak{W}_\mathcal{E}$, you just need to find **some** factorization $p(f) = qj$ as an acyclic cofibration followed by an acyclic fibration such that ${}_j f^q$ is acyclic inside its fiber (this is just the definition of $\mathfrak{W}_\mathcal{E}$ after all); but if given that $f \in \mathfrak{W}_\mathcal{E}$, you can use that ${}_j f^q$ is a weak equivalence for **every** admissible factorization of $p(f)$!

We shall use that extensively in the proof of the two-out-of-three property for $\mathfrak{W}_\mathcal{E}$. This will conclude the proof of sufficiency in theorem 3.4.2.

Proposition 3.4.11. *The class $\mathfrak{W}_\mathcal{E}$ has the 2-out-of-3 property.*

Proof. We suppose given a commutative triangle $h = gf$ in the total category $\mathcal{E}$, and we proceed by case analysis.

First case: suppose that $f, g \in \mathfrak{W}_\mathcal{E}$, and we want to show that $h \in \mathfrak{W}_\mathcal{E}$. Since f and g are elements of $\mathfrak{W}_\mathcal{E}$, there exists a pair of factorizations $p(f) = qj$ and $p(g) = q'j'$ with j, j' acyclic cofibrations and q, q' acyclic fibrations of $\mathcal{B}$ such that both ${}_j f^q$ and

${}_{q'}g^{j'}$ are weak equivalences in their respective fibers. The weak equivalence $j'q$ of $\mathcal{B}$ can be factorized as $q''j''$ with j'' acyclic cofibration and q'' acyclic fibration. We write $i = j''j$ and $r = q'q''$ and we notice that $p(h) = ri$, as depicted below.

$$\begin{tikzcd} pX \arrow[r, "j"] \arrow[rr, bend left, "i"] \arrow[dr, "p(f)"'] \arrow[drr, "p(h)"] & A \arrow[r, "j''"] \arrow[d, "q"] & C \arrow[d, "q''"] \arrow[dd, bend left, "r"] \\ & pY \arrow[r, "j'"] \arrow[dr, "p(g)"'] & B \arrow[d, "q'"] \\ & & pZ \end{tikzcd} \tag{3.73}$$

Since i is an acyclic cofibration and r is an acyclic fibration, it is enough to show that ${}_ih^r : i_!X \longrightarrow r^*Y$ is a weak equivalence in $\mathcal{E}_C$ in order to conclude that $h \in \mathfrak{W}_{\mathcal{E}}$. Since $i = j''j$ and $r = q'q''$, corollary 3.2.5 states that it is equivalent to show that the isomorphic arrow $\tilde{h} : j''_!j_!X \longrightarrow q''^*q'^*$ is a weak equivalence, where $\tilde{h}$ is defined as the unique arrow in fiber $\mathcal{E}_C$ making the following commute:

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[drrrr, bend right, "h"] & j_!X \arrow[r, "\lambda"] & j''_!j_!X \arrow[d, "\tilde{h}"] & & \\ & & q''^*q'^*Z \arrow[r, "\rho"'] & q'^*Z \arrow[r, "\rho"'] & Z \end{tikzcd} \tag{3.74}$$

Since $h = gf$, such an arrow $\tilde{h}$ is given by the composite

$$j''_!j'_!X \xrightarrow{j''_!({}_jf^q)} j''_!q^*Y \xrightarrow{\mu_Y} q''^*j'_!Y \xrightarrow{q''^*({}_{j'}g^{q'})} q''^*q'^*Z \tag{3.75}$$

where μ_Y is the component at Y of the mate $\mu : j''_!q^* \longrightarrow q''^*j'_!$ of the commutative square $q''j'' = j'q$ of $\mathcal{B}$ (see diagram (3.73) above).

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[rrrr, bend left, "f"] & j_!X \arrow[r, "\lambda"] \arrow[d, "{}_jf^q"] & j''_!j_!X \arrow[d, "j''_!({}_jf^q)"] & & Y \arrow[ddl, "\lambda"] \arrow[dddd, bend left, "g"] \\ & q^*Y \arrow[r, "\lambda"] \arrow[urrr, "\rho"] & j''_!q^*Y \arrow[d, "\mu_Y"] & & \\ & & q''^*j'_!Y \arrow[r, "\rho"'] \arrow[d, "q''^*({}_{j'}g^{q'})"'] & j'_!Y \arrow[d, "{}_{j'}g^{q'}"'] & \\ & & q''^*q'^*Z \arrow[r, "\rho"'] & q'^*Z \arrow[r, "\rho"'] & Z \end{tikzcd} \tag{3.76}$$

We can conclude that $\tilde{h}$ is a weak equivalence in $\mathcal{E}_C$ because it is a composite of such. Indeed:

- hypothesis (hBC) can be applied to the square $q''j'' = j'q$, and so μ_Y is a weak equivalence in $\mathcal{E}_C$,

- and by hypothesis (hCon), the functors $j''_!$ and q''^* maps the weak equivalences ${}_jf^q$ and ${}_{j'}g^{q'}$ to weak equivalences in $\mathcal{E}_C$.

Suppose now that f and h are in $\mathfrak{W}_\mathcal{E}$ and we will show that g also is. Since $p(f)$ and $p(h)$ are weak equivalences in $\mathcal{B}$, we can use the two-out-of-three property of $\mathfrak{W}$ to deduce that also $p(g)$ is. By hypothesis, $p(f) = qj$ with $j \in \mathfrak{C} \cap \mathfrak{W}$ and $q \in \mathfrak{F} \cap \mathfrak{W}$ and ${}_jf^q$ a weak equivalence. Also write $p(g) = q'j'$ for some $j' \in \mathfrak{C} \cap \mathfrak{W}$ and $q' \in \mathfrak{F} \cap \mathfrak{W}$. We are done if we show that ${}_{j'}g^{q'}$ is a weak equivalence. But in that situation, one can define j'', q'', i, r, and $\tilde{h}$ as before. So we end up with the same big diagram, except that this time $j''_!({}_jf^q)$, μ_Y and the composite $\tilde{h}$ are weak equivalences of $\mathcal{E}_C$, yielding $q''^*({}_{j'}g^{q'})$ as a weak equivalence by the 2-out-of-3 property. But q''^* being homotopically conservative by (hCon), this shows that ${}_{j'}g^{q'}$ is a weak equivalence in $\mathcal{E}_B$.

The last case, where g and h are in $\mathfrak{W}_\mathcal{E}$ is strictly dual. □

3.5 Illustrations

Since the very start, our work is motivated by the idea that the Reedy model structure can be reconstructed by applying a series of Grothendieck constructions of model categories. The key observation is that the notion of latching and matching functors define a bifibration at each step of the construction of the model structure. We explain in 3.5.1 how the Reedy construction can be re-understood from our bifibrational point of view. In section 3.5.2, we describe how to adapt to express generalized Reedy constructions in a similar fashion. In section 3.5.3, we recall the previous notions of bifibration of model categories appearing in the literature and, although all of them are special cases of Quillen bifibrations, we indicate why they do not fit the purpose.

3.5.1 A bifibrational view on Reedy model structures

Recall that a *Reedy category* is a small category $\mathcal{R}$ together with two subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ and a degree function $d : \mathrm{Ob}\,\mathcal{R} \longrightarrow \lambda$ for some ordinal λ such that

- every morphism f admits a unique factorization $f = f^+f^-$ with $f^- \in \mathcal{R}^-$ and $f^+ \in \mathcal{R}^+$,
- non-identity morphisms of $\mathcal{R}^+$ strictly raise the degree and those of $\mathcal{R}^-$ strictly lower it.

For such a Reedy category, let $\mathcal{R}_\mu$ denote the full subcategory spanned by objects of degree strictly less than μ. In particular, $\mathcal{R} = \mathcal{R}_\lambda$. Remark also that every $\mathcal{R}_\mu$ inherits a structure of Reedy category from $\mathcal{R}$.

We are interested in the structure of the category of diagrams of shape $\mathcal{R}$ in a complete and cocomplete category $\mathcal{C}$. The category $\mathcal{C}$ is in particular tensored and cotensored over Set, those being respectively given by

$$S \odot C = \coprod_{s\in S} C, \qquad S \;\; C = \prod_{s\in S} C, \qquad S \in \mathsf{Set}, C \in \mathcal{C}. \tag{3.77}$$

For every $r \in \mathcal{R}$ of degree μ, a diagram $X : \mathcal{R}_\mu \longrightarrow \mathcal{C}$ induces two objects in $\mathcal{C}$, called the *latching* and *matching* objects of X at r, and respectively defined as:

$$\mathrm{L}_r\, X = \int^{s\in\mathcal{R}_\mu} \mathcal{R}\,(s,r) \odot X_s, \qquad \mathrm{M}_r\, X = \int_{s\in\mathcal{R}_\mu} \mathcal{R}\,(r,s)\ X_s \tag{3.78}$$

By abuse, we also denote $\mathrm{L}_r\, X$ and $\mathrm{M}_r\, X$ for the latching and matching objects of the restriction to $\mathcal{R}_\mu$ of some $X : \mathcal{R}_\kappa \longrightarrow \mathcal{C}$ with $\kappa \geq \mu$. In particular, when $\kappa = \lambda$, X is a diagram of shape the entire category $\mathcal{R}$ and we retrieve the textbook notion of latching and matching objects (see for instance [Hov99]). Universal properties of limits and colimits induce a family of canonical morphisms $\alpha_r : \mathrm{L}_r\, X \longrightarrow \mathrm{M}_r\, X$, which can also be understood in the following way. First, one notices that the two functors defined as $\mathcal{R}_{\mu+1} \longrightarrow \mathcal{C}$

$$r \mapsto \begin{cases} X_r \text{ if } d(r) < \mu \\ \mathrm{L}_r\, X \text{ if } d(r) = \mu \end{cases}, \qquad r \mapsto \begin{cases} X_r \text{ if } d(r) < \mu \\ \mathrm{M}_r\, X \text{ if } d(r) = \mu \end{cases} \tag{3.79}$$

are the skeleton and coskeleton X, which provide a left and a right Kan extensions X along the inclusion $i_\mu : \mathcal{R}_\mu \longrightarrow \mathcal{R}_{\mu+1}$. We will write these two functors $\mathrm{L}_\mu\, X$ and $\mathrm{M}_\mu\, X$ respectively. The family of morphisms α_r then describes the unique natural transformation $\alpha : \mathrm{L}_\mu\, X \longrightarrow \mathrm{M}_\mu\, X$ that restrict to the identity on $\mathcal{R}_\mu$.

The following property is, in our opinion, the key feature of Reedy categories.

Proposition 3.5.1. *Extensions of a diagram $X : \mathcal{R}_\mu \longrightarrow \mathcal{C}$ to $\mathcal{R}_{\mu+1}$ are in one-to-one correspondence with families of factorizations of the α_r's*

$$(\mathrm{L}_r\, X \longrightarrow \bullet \longrightarrow \mathrm{M}_r\, X)_{r\in\mathcal{R}, d(r)=\mu} \tag{3.80}$$

Proof. One direction is easy. Every extension $\hat{X} : \mathcal{R}_{\mu+1} \longrightarrow \mathcal{C}$ of X produces such a family of factorizations, but it has nothing to do with the structure of Reedy category: for every r of degree μ in $\mathcal{R}$, the functoriality of $\hat{X}$ ensures that there is a coherent family of morphisms $X_s = \hat{X}_s \longrightarrow \hat{X}_r$ for each arrow $s \longrightarrow r$, and symmetrically a coherent family of morphisms $\hat{X}_r \longrightarrow \hat{X}_{s'} = X_{s'}$ for each arrow $r \longrightarrow s'$. Hence the factorization of α_r given by the universal properties of limits and colimits

$$\mathrm{L}_r\, X \longrightarrow \hat{X}_r \longrightarrow \mathrm{M}_r\, X \tag{3.81}$$

The useful feature is the converse: when usually, to construct an extension of X, one should define images for arrows $r \longrightarrow r'$ between objects of degree μ in a functorial way, here every family automatically induces such arrows! This is a fortunate effect of the unique factorization property. Given factorizations $\mathrm{L}_r\, X \longrightarrow X_r \longrightarrow \mathrm{M}_r\, X$, one can define $X(f)$ for $f : r \longrightarrow r'$ as follow: factor $f = f^+ f^-$ with $f^- : r \longrightarrow s$ lowering the degree and $f^+ : s \longrightarrow r'$ raising it, so that in particular $s \in \mathcal{R}_\mu$; f^- then gives rise to a canonical projection $\mathrm{M}_r\, X \longrightarrow X_s$ and f^+ to a canonical injection $X_s \longrightarrow \mathrm{L}_{r'}\, X$; the wanted arrow $X(f)$ is given by the composite

$$X_r \longrightarrow \mathrm{M}_r\, X \longrightarrow X_s \longrightarrow \mathrm{L}_{r'}\, X \longrightarrow X_{r'} \tag{3.82}$$

Well-definition and functoriality of the said extension are following from uniqueness in the factorization property of the Reedy category $\mathcal{R}$. □

From now on, we fix a model category $\mathcal{M}$, that is a complete and cocomplete category $\mathcal{M}$ with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$. The motivation behind Kan's notion of Reedy categories is to gives sufficient conditions on $\mathcal{R}$ to equip $\mathrm{Fun}\,(\mathcal{R}, \mathcal{M})$ with a model structure where weak equivalences are pointwise.

Definition 3.5.2. Let $\mathcal{R}$ be Reedy. The *Reedy triple* on the functor category $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ is the data of the three following classes

- Reedy cofibrations : those $f : X \longrightarrow Y$ such that for all $r \in \mathcal{R}$, the map $\mathrm{L}_r\, Y \sqcup_{\mathrm{L}_r\, X} X_r \longrightarrow Y_r$ is a cofibration,
- Reedy weak equivalences : those $f : X \longrightarrow Y$ such that for $r \in \mathcal{R}$, $f_r : X_r \longrightarrow Y_r$ is a weak equivalence,
- Reedy fibrations : those $f : X \longrightarrow Y$ such that for all $r \in \mathcal{R}$, the map $X_r \longrightarrow \mathrm{M}_r\, X \times_{\mathrm{M}_r\, Y} Y_r$ is a fibration.

Kan's theorem about Reedy categories, whose our main result gives a slick proof, then states as follow: the Reedy triple makes $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ into a model category. A first reading of this definition/theorem is quite astonishing: the distinguished morphisms are defined through those latching and matching objects, and it is not clear, apart from being driven by the proof, why we should emphasize those construction that much. We shall say a word about that later.

Remark 3.5.3. Before going into proposition 3.5.4 below, we need to make a quick remark about extensions of diagrams up to isomorphism. Suppose given a injective-on-objects functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories and a category $\mathcal{C}$, then for every diagram $D : \mathcal{A} \longrightarrow \mathcal{C}$, every diagram $D' : \mathcal{B} \longrightarrow \mathcal{C}$ and every isomorphism $\alpha : D \longrightarrow D' i$, there exists a diagram $D'' : \mathcal{B} \longrightarrow \mathcal{C}$ isomorphic to D' such that $D'' i = D$ (and the isomorphism $\beta : D'' \longrightarrow D'$ can be chosen so that $\beta i = \alpha$). Informally it says that every "up to isomorphism" extension of D can be rectified into a strict extension of D.

Put formally, we are claiming that the restriction functor $i^* : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is an isofibration. Although it can be shown easily by hand, we would like to present an alternate proof based on homotopical algebra. Taking a universe $\mathbb{U}$ big enough for $\mathcal{C}$ to be small relatively to $\mathbb{U}$, we can consider the folk model structure on the category Cat of $\mathbb{U}$-small categories. With its usual cartesian product, Cat is a closed monoidal model category in which every object is fibrant. It follows that $\mathrm{Fun}(-, \mathcal{C})$ maps cofibrations to fibrations (see [Hov99, Remark 4.2.3]). Then, the injective-on-objects functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ is a cofibration, so it is mapped to a fibration $i^* : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$. Recall that fibrations in Cat are precisely the isofibrations and we obtain the result.

Proposition 3.5.4. *Let $\mathcal{R}$ be Reedy. The restriction functor $i_\mu^* : \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right) \longrightarrow \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$ is a Grothendieck bifibration.*

Proof. The claim is that a morphism $f : X \longrightarrow Y$ is cartesian precisely when the following diagram is a pullback square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathrm{M}_\mu\, pX \arrow[r, "\mathrm{M}_\mu\, p(f)"'] & \mathrm{M}_\mu\, pY \end{tikzcd} \tag{3.83}$$

where the vertical arrows are the component at X and Y of the unit η of the adjunction (p, M_μ). Indeed, such a diagram is a pullback square if and only if the following square is a pullback for all Z:

$$\begin{tikzcd}
\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)(Z, X) \arrow[r, "f\circ-"] \arrow[d, "\eta_X\circ-"] & \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)(Z, Y) \arrow[d, "\eta_Y\circ-"] \\
\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)\left(Z, \mathrm{M}_\mu\, pX\right) \arrow[r, "\mathrm{M}_\mu\, p(f)\circ-"'] & \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)\left(Z, \mathrm{M}_\mu\, pY\right)
\end{tikzcd} \tag{3.84}$$

We can take advantage of the adjunction (p, M_μ) and its natural isomorphism

$$\phi_{Z,A} : \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)\left(Z, \mathrm{M}_\mu\, A\right) \simeq \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)(pZ, A) \tag{3.85}$$

As in any adjunction, this isomorphism is related to the unit by the following identity: for any $g : Z \longrightarrow X$, $p(g) = \phi(\eta_X g)$. So in the end, the square in (3.83) is a pullback if and only if for every Z the outer square of the following diagram is a pullback:

$$\begin{tikzcd}
\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)(Z, X) \arrow[r, "f\circ-"] \arrow[d, "\eta_X\circ-"] \arrow[dd, bend right, "p"'] & \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)(Z, Y) \arrow[d, "\eta_Y\circ-"] \arrow[dd, bend left, "p"] \\
\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)\left(Z, \mathrm{M}_\mu\, pX\right) \arrow[r, "\mathrm{M}_\mu\, p(f)\circ-"'] \arrow[d, "\phi"'] & \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)\left(Z, \mathrm{M}_\mu\, pY\right) \arrow[d, "\phi"'] \\
\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)(pZ, pX) \arrow[r, "p(f)\circ-"'] & \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)(pZ, pY)
\end{tikzcd} \tag{3.86}$$

This is exactly the definition of a cartesian morphism. Dually, we can prove that cocartesian morphisms are those $f : X \longrightarrow Y$ such that the following is a pushout square:

$$\begin{tikzcd}
\mathrm{L}_\mu\, pX \arrow[r, "\mathrm{L}_\mu\, p(f)"] \arrow[d] & \mathrm{L}_\mu\, pY \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{3.87}$$

Now for $u : A \longrightarrow pY$ in $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$, one should construct a cartesian morphism $f : X \longrightarrow Y$ above u. First notice that we constructed M_μ in such a way that $p\,\mathrm{M}_\mu = \mathrm{id}$ (even more, the counit $p\,\mathrm{M}_\mu \longrightarrow \mathrm{id}$ is the identity natural transformation). So $\mathrm{M}_\mu\, A$ is above A and we could be tempted to take, for the wanted f, the morphism κ : $\mathrm{M}_\mu\, A \times_{\mathrm{M}_\mu\, pY} Y \longrightarrow Y$ appearing in the following pullback square:

$$\begin{tikzcd}
\bullet \arrow[r, "\kappa"] \arrow[d] & Y \arrow[d] \\
\mathrm{M}_\mu\, A \arrow[r, "\mathrm{M}_\mu\, u"'] & \mathrm{M}_\mu\, pY
\end{tikzcd} \tag{3.88}$$

But κ is not necessarily above u. Indeed, as a right adjoint, p preserves pullbacks. So we get that the following is a pullback in $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$:

$$\begin{tikzcd}
p(\bullet) \arrow[r, "p(\kappa)"] \arrow[d] & pY \arrow[d, "\mathrm{id}_Y"] \\
A \arrow[r, "u"'] & pY
\end{tikzcd} \tag{3.89}$$

We certainly know another pullback square of the same diagram, namely

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\mathrm{id}_A"'] & pY \arrow[d, "\mathrm{id}_Y"] \\ A \arrow[r, "u"'] & pY \end{tikzcd} \tag{3.90}$$

So, by universal property, we obtain an isomorphism $\alpha : A \longrightarrow p(\mathrm{M}_\mu A \times_{\mathrm{M}_\mu pY} Y)$. Now we summon remark 3.5.3 to get an extension X of A and an isomorphism $\beta : X \longrightarrow \mathrm{M}_\mu A \times_{\mathrm{M}_\mu pY} Y$ above α. The wanted $f : X \longrightarrow Y$ is then just the composite $\kappa\beta$, which is cartesian because the outer square in the following is a pullback (as we chose (3.88) to be one):

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\beta"] \arrow[dd] & & Y \arrow[dd] \\ & \bullet \arrow[ur, "\kappa"] \arrow[dl] & \\ \mathrm{M}_\mu A \arrow[rr, "\mathrm{M}_\mu u"'] & & \mathrm{M}_\mu pY \end{tikzcd} \tag{3.91}$$

The fact that the vertical map $X \longrightarrow \mathrm{M}_\mu A = \mathrm{M}_\mu pX$ is indeed the unit η of the adjunction at component X comes directly from the fact that its image by p is id_A. The existence of cocartesian morphism above any $u : pX \longrightarrow B$ is strictly dual, using this time the cocontinuity of p as a left adjoint. □

Remark 3.5.5. First, we should notice that proposition 3.5.1 make the following multi-evaluation functor an equivalence:

$$\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)_A \xrightarrow{\sim} \prod_{r\in\mathcal{R}, d(r)=\mu} \mathrm{L}_r A \backslash \mathcal{M} / \mathrm{M}_r A \tag{I}$$

The notation $\mathrm{L}_r A \backslash \mathcal{M} / \mathrm{M}_r A$ is slightly abusive and means the coslice category of $\mathcal{M}/\mathrm{M}_r A$ by α_r, or equivalently the slice category of $\mathrm{L}_r A \backslash \mathcal{M}$ by α_r.

Secondly, we can draw from the previous proof that for a morphism $f : X \longrightarrow Y$, the fiber morphisms $f^{\triangleleft}$ and $f_{\triangleright}$ are, modulo identification (I), the respective induced families defining the Reedy triple:

$$(X_r \longrightarrow \mathrm{M}_r X \times_{\mathrm{M}_r Y} Y_r)_{r, d(r)=\mu}, \qquad (X_r \sqcup_{\mathrm{L}_r X} \mathrm{L}_r Y \longrightarrow Y_r)_{r, d(r)=\mu} \tag{3.92}$$

So here it is: the reason behind those *a priori* mysterious morphisms, involving latching an matching, are nothing else but the witness of a hidden bifibrational structure. Putting this into light was a tremendous leap in our conceptual understanding of Reedy model structures and their generalizations.

The following proposition is the induction step for successor ordinals in the usual proof of the existence of Reedy model structures. Our main theorem 3.4.2 allows a very smooth argument.

Proposition 3.5.6. *If the Reedy triple on* $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$ *forms a model structure, then it is also the case on* $\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)$.

Proof. Our course, the goal is to use theorem 3.4.2 on the Grothendieck bifibration $i_\mu^* : \mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right) \longrightarrow \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$. By hypothesis, the base $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$ has a model structure given by the Reedy triple. Each fiber $(i_\mu{}^*)_A$ above a diagram A is endowed, via identification (I), with the product model structure: indeed, if $\mathcal{N}$ is a model category, so is its slices $\mathcal{N}/N$ and coslices $N\backslash\mathcal{N}$ categories, just defining a morphism to be a cofibration, a fibration or a weak equivalence if it is in $\mathcal{N}$; products of model categories are model categories by taking the pointwise defined structure. All in all, it means that the following makes the fiber $(i_\mu{}^*)_A$ into a model category: a fiber map $f : X \longrightarrow X'$ in $(i_\mu{}^*)_A$ is a cofibration, a fibration or a weak equivalence if and only if $f_r : X_r \longrightarrow X'_r$ is one for every $r \in \mathcal{R}$ of degree μ.

Now the proof amounts to show that hypothesis (Q), (hCon) and (hBC) are satisfied in this framework. Let us first tackle (Q). Suppose $u : A \longrightarrow B$ in $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$ and $f : Y \longrightarrow Y'$ a fiber morphism at B. Then by definition of the cartesian morphisms in $\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)$, $u^* f$ is the unique map above A making the following diagram commute for all r of degree μ:

$$\begin{tikzcd} (u^* Y)_r \arrow[r] \arrow[d, "(u^* f)_r"'] & Y_r \arrow[d, "f_r"] \\ (u^* Y')_r \arrow[r] \arrow[d] & Y'_r \arrow[d] \\ \mathrm{M}_r\, A \arrow[r, "\mathrm{M}_r\, u"'] & \mathrm{M}_r\, B \end{tikzcd} \tag{3.93}$$

where the lower square and outer square are pullback diagrams. By the pasting lemma, so is the upper square. Hence $(u^* f)_r$ is a pullback of f_r, and as such is a (acyclic) fibration whenever f_r is one. This proves that u^* is right Quillen for any u, that is (Q).

Goals (hCon) and (hBC) will be handle pretty much the same way one another and it lies on the following well know fact about Reedy model structures [Hir03, lemma 15.3.9]: for $r \in \mathcal{R}$ of degree μ, the functor $\mathrm{M}_r : \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right) \longrightarrow \mathcal{M}$ preserves acyclic fibrations[1]. This has a wonderful consequence: if u is an acyclic fibration of $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$, any pullback of $\mathrm{M}_r\, u$ is an acyclic fibration hence a weak equivalence. So the upper square of diagram (3.93) has acyclic horizontal arrows. By the 2-out-of-3 property, f_r on the right is a weak equivalence if an only if $(u^* f)_r$ is one. This being true for each $r \in \mathcal{R}$ of degree μ makes u^* homotopically conservative whenever u is an acyclic fibration. This validates half of the property (hCon). The other half is proven dually, resting on the dual lemma: for any $r \in \mathcal{R}$ of degree μ, the latching functor $\mathrm{L}_r : \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right) \longrightarrow \mathcal{M}$ preserves acyclic cofibrations; then deducing that pushouts of $\mathrm{L}_r\, u$ are weak equivalences whenever u is an acyclic cofibration.

It remains to show (hBC). Everything is already in place and it is just a matter of expressing it. For a commutative square of $\mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right)$

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"'] & B \end{tikzcd} \tag{3.94}$$

with u, u' Reedy acyclic cofibrations and v, v' Reedy acyclic fibrations, the mate at an extension Z of C is the unique fiber morphism $\nu_Z : (u'_! v^* Z) \longrightarrow (v'^* u_! Z)$ making the

[1] Actually it is right Quillen, but we will not need that much here.

following commute for every $r \in \mathcal{R}$ of degree μ:

$$\begin{tikzcd}
& \mathrm{M}_r\, A \arrow[rr, "\mathrm{M}_r\, v"] & & \mathrm{M}_r\, C & \\
\mathrm{L}_r\, A \arrow[r] \arrow[d, "\mathrm{L}_r\, u'"'] & (v^* Z)_r \arrow[u] \arrow[rr] \arrow[d] & & Z_r \arrow[u] \arrow[dd] & \mathrm{L}_r\, C \arrow[l] \arrow[dd, "\mathrm{L}_r\, v"] \\
\mathrm{L}_r\, C' \arrow[r] & (u'_! v^* Z)_r \arrow[dr, "(v_Z)_r"] & & & \\
& & (v'^* u_! Z)_r \arrow[r] \arrow[d] & (u_! Z)_r \arrow[d] & \mathrm{L}_r\, B \arrow[l] \\
& & \mathrm{M}_r\, C' \arrow[r, "\mathrm{M}_r\, v'"'] & \mathrm{M}_r\, B &
\end{tikzcd} \tag{3.95}$$

where gray-scaled square are either pullbacks (when involving matching objects) or pushouts (when involving latching objects). So by the same argument as above, the horizontal and vertical arrows of the pentagon are weak equivalences, making the r-component of the mate $(v_Z)_r$ a weak equivalence also by the 2-out-of-3 property.

Theorem 3.4.2 now applies, and yield a model structure on $\mathrm{Fun}\left(\mathcal{R}_{\mu+1}, \mathcal{M}\right)$ which is readily the Reedy triple. □

3.5.2 Notions of generalized Reedy categories

From time to time, people stumble across almost Reedy categories and build *ad hoc* workarounds to end up with a structure "à la Reedy". The most popular such generalizations are probably Cisinski's [Cis06] and Berger-Moerdijk's [BM11], allowing for non trivial automorphisms. In [Shu15], Shulman establishes a common framework for every such known generalization of Reedy categories (including enriched ones, which go behind the scope of this work). Roughly put, Shulman defines *almost-Reedy categories* to be those small categories $\mathcal{C}$ with a degree function on the objects that satisfy the following property: taking x of degree μ and denoting $\mathcal{C}_\mu$ the full subcategory of $\mathcal{C}$ of objects of degree strictly less than μ, and $\mathcal{C}_x$ the full subcategory of $\mathcal{C}$ spanned by $\mathcal{C}_\mu$ and x, then the diagram category $\mathrm{Fun}(\mathcal{C}_x, \mathcal{M})$ is obtained as the *bigluing* (to be defined below) of two nicely behaved functors $\mathrm{Fun}\left(\mathcal{C}_\mu, \mathcal{M}\right) \longrightarrow \mathcal{M}$, namely the weighted colimit and weighted limit functors, respectively weighted by $\mathcal{C}(-, x)$ and $\mathcal{C}(x, -)$. In particular, usual Reedy categories are recovered when realizing that the given formulas of latching and matching objects are precisely these weighted colimits and limits.

In order to understand completely the generalization proposed in [Shu15], we propose an alternative view on the Reedy construction that we exposed in detail in the previous section. For starter, here is a nice consequence of theorem 3.4.2:

Lemma 3.5.7. *Suppose there is a strict pullback square of categories*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "q"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{B}
\end{tikzcd} \tag{3.96}$$

in which $\mathcal{C}$ has a model structure and p is a Quillen bifibration. If

(i) $F(u)_!$ *and* $F(v)^*$ *are homotopically conservative whenever* u *is an acyclic cofibration and* v *an acyclic fibration in* $\mathcal{C}$*,*

(ii) F *maps squares of the form*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"] & B \end{tikzcd} \tag{3.97}$$

with u, u' *acyclic cofibrations and* v, v' *acyclic fibrations in* $\mathcal{C}$ *to squares in* $\mathcal{B}$ *that satisfy the homotopical Beck-Chevalley condition,*

then q *is also a Quillen bifibration.*

Proof. Denote $p' : \mathcal{B} \longrightarrow \mathsf{Adj}$ the pseudo functor $A \mapsto \mathcal{E}_A$ associated to p. Then it is widely known that the pullback q of p along F is the bifibration obtained by Grothendieck construction of the pseudo functor $p'F : \mathcal{C} \longrightarrow \mathsf{Adj}$. It has fiber $\mathcal{F}_C = \mathcal{E}_{FC}$ at $C \in \mathcal{C}$, which has a model structure; and for any $u : C \longrightarrow D$ in $\mathcal{C}$, the adjunction

$$u_! : \mathcal{F}_C \rightleftarrows \mathcal{F}_D : u^* \tag{3.98}$$

is given by the pair $(F(u)_!, F(u)^*)$ defined by p. Hence theorem 3.4.2 asserts that q is a Quillen bifibration as soon as (hBC) and (hCon) are satisfied. The conditions of the lemma are precisely there to ensure that this is the case. □

Now recall that $\Delta[1]$ and $\Delta[2]$ are the posetal categories associated to $\{0 < 1\}$ and $\{0 < 1 < 2\}$ respectively, and write $c : \Delta[1] \longrightarrow \Delta[2]$ for the functor associated with the mapping $0 \mapsto 0, 1 \mapsto 2$. Given a Reedy category $\mathcal{R}$ and an object r of degree μ, denote $i_r : \mathcal{R}_\mu \longrightarrow \mathcal{R}_r$ the inclusion of the full subcategory of $\mathcal{R}$ spanned by the object of degree strictly less than μ into the one spanned by the same objects plus r. Then proposition 3.5.1 asserts that the following is a strict pullback square of categories:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{R}_r, \mathcal{M}) \arrow[r] \arrow[d, "i_r^*"'] & \mathrm{Fun}(\Delta[2], \mathcal{M}) \arrow[d, "c^*"] \\ \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right) \arrow[r, "\alpha_r"'] & \mathrm{Fun}(\Delta[1], \mathcal{M}) \end{tikzcd} \tag{3.99}$$

where the bottom functor maps every diagram $X : \mathcal{R}_\mu \longrightarrow \mathcal{M}$ to the canonical arrow $\alpha_r : \mathrm{L}_r X \longrightarrow \mathrm{M}_r X$. Moreover the functor c^* is a Grothendieck bifibration: one can easily verify that an arrow in $\mathrm{Fun}(\Delta[2], \mathcal{M})$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "g"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "h"] & \bullet \end{tikzcd} \tag{3.100}$$

is cartesian if and only if the bottom square is a pullback, and is cocartesian if and only if the top square is a pushout. In particular, for each object $k : A \longrightarrow B$ of $\mathrm{Fun}(\Delta[1], \mathcal{M})$ we have a model structure on its fiber $(c^*)_k \simeq A \backslash \mathcal{M} / B$. Stability of cofibrations by pushout and of fibrations by pullback in the model category $\mathcal{M}$ translates to say that hypothesis Q is satisfied by c^*. In other word, by equipping the basis category $\mathrm{Fun}(\Delta[1], \mathcal{M})$ with the trivial model structure, theorem 3.4.2 applies ((hBC) and (hCon) are vacuously met) and makes c^* a Quillen bifibration. The content of the proof of proposition 3.5.6 is precisely showing conditions (i) and (ii) of lemma 3.5.7. We can then conclude that $i^*r : \mathrm{Fun}(\mathcal{R}_r, \mathcal{M}) \longrightarrow \mathrm{Fun}(\mathcal{R}_r, \mathcal{M})$ is a Quillen bifibration as in proposition 3.5.6.

The result of [Shu15, Theorem 3.11] fall within this view. Shulman defines the *bigluing* of a natural transformation $\alpha : F \longrightarrow G$ between two functors $F, G : \mathcal{M} \longrightarrow \mathcal{N}$ as the category $\mathcal{G}\ell(\alpha)$ whose:

- objects are factorizations

$$\alpha_M : FM \xrightarrow{f} N \xrightarrow{g} GM \tag{3.101}$$

- morphisms $(f, g) \xrightarrow{(h,k)} (f', g')$ are commutative diagrams of the form

$$\begin{tikzcd} FM \arrow[r, "f"] \arrow[d, "F(h)"'] & N \arrow[r, "g"] \arrow[d, "k"] & GM \arrow[d, "G(h)"] \\ FM' \arrow[r, "f'"'] & N' \arrow[r, "g'"'] & GM' \end{tikzcd} \tag{3.102}$$

Otherwise put, the category $\mathcal{G}\ell(\alpha)$ is a pullback as in:

$$\begin{tikzcd} \mathcal{G}\ell(\alpha) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta[2], \mathcal{N}) \arrow[d, "c^*"] \\ \mathcal{M} \arrow[r, "\alpha"'] & \mathrm{Fun}(\Delta[1], \mathcal{N}) \end{tikzcd} \tag{3.103}$$

In the same fashion as in the proof of proposition 3.5.6, we can show that conditions (i) and (ii) are satisfied for the bottom functor (that we named abusively α) when F maps acyclic cofibrations to *couniversal weak equivalences* and G maps acyclic fibrations to *universal weak equivalences*. By a couniversal weak equivalence is meant a map every pushout of which is a weak equivalence; and by a universal weak equivalence is meant a map every pullback of which is a weak equivalence. Now lemma 3.5.7 directly proves Shulman's theorem.

Theorem 3.5.8 (Shulman)**.** *Suppose $\mathcal{N}$ and $\mathcal{M}$ are both model categories. Let $\alpha : F \longrightarrow G$ between $F, G : \mathcal{M} \longrightarrow \mathcal{N}$ satisfying that:*

- *F is cocontinuous and maps acyclic cofibrations to couniversal weak equivalences,*
- *G is continuous and maps acyclic fibrations to universal weak equivalence.*

Then $\mathcal{G}\ell(\alpha)$ is a model category whose:

- *cofibrations are the maps (h, k) such that both h and the map $FM' \sqcup_{FM} N \longrightarrow N'$ induced by k are cofibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*
- *fibrations are the maps (h, k) such that both h and the map $N \longrightarrow GM \times_{GM'} N'$ induced by k are fibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*

- *weak equivalences are the maps (h, k) where both h and k are weak equivalences in $\mathcal{M}$ and $\mathcal{N}$ respectively.*

Maybe the best way to understand this theorem is to see it at play. Recall that a generalized Reedy category in the sense of Berger and Moerdijk is a kind of Reedy category with degree preserving isomorphism: precisely it is a category $\mathcal{R}$ with a degree function $d : \mathrm{Ob}\,\mathcal{R} \longrightarrow \lambda$ and wide subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ such that:

- **non-invertible** morphisms of $\mathcal{R}^+$ strictly raise the degree while those of $\mathcal{R}^-$ strictly lower it,
- isomorphisms all preserve the degree,
- $\mathcal{R}^+ \cap \mathcal{R}^-$ contains exactly the isomorphisms as morphisms,
- every morphism f can be factorized as $f = f^+ f^-$ with $f^+ \in \mathcal{R}^+$ and $f^- \in \mathcal{R}^-$, and such a factorization is unique up to isomorphism,
- if θ is an isomorphism and $\theta f = f$ for some $f \in \mathcal{R}^-$, then θ is an identity.

The central result in [BM11] goes as follow:

(1) the latching and matching objects at $r \in \mathcal{R}$ of some $X : \mathcal{R} \longrightarrow \mathcal{M}$ are defined as in the classical case, but now the automorphism group $\mathrm{Aut}\,(r)$ acts on them, so that $\mathrm{L}_r X$ and $\mathrm{M}_r X$ are objects of $\mathrm{Fun}\,(\mathrm{Aut}\,(r), \mathcal{M})$ rather than mere objects of $\mathcal{M}$.

(2) suppose $\mathcal{M}$ such that every $\mathrm{Fun}\,(\mathrm{Aut}\,(r), \mathcal{M})$ bears the projective model structure, and define Reedy cofibrations, Reedy fibrations and Reedy weak equivalences as usual but considering the usual induced maps $X_r \sqcup_{\mathrm{L}_r X} \mathrm{L}_r Y \longrightarrow Y_r$ and $X_r \longrightarrow Y_r \times_{\mathrm{M}_r Y} \mathrm{M}_r X$ in $\mathrm{Fun}\,(\mathrm{Aut}\,(r), \mathcal{M})$, not in $\mathcal{M}$.

(3) then Reedy cofibrations, Reedy fibrations and Reedy weak equivalences give $\mathrm{Fun}\,(\mathcal{R}, \mathcal{M})$ a model structure.

In that framework, theorem 3.5.8 is applied repeatedly with α being the canonical natural transformation between $\mathrm{L}_r, \mathrm{M}_r : \mathrm{Fun}\left(\mathcal{R}_\mu, \mathcal{M}\right) \longrightarrow \mathrm{Fun}\,(\mathrm{Aut}\,(r), \mathcal{M})$ whenever r is of degree μ. In particular, here we see the importance to be able to vary the codomain category $\mathcal{N}$ of Shulman's result in each successor step, and not to work with an homogeneous $\mathcal{N}$ all along.

3.5.3 Related works on Quillen bifibrations

Our work builds on the papers [Roi94], [Sta12] on the one hand, and [HP15] on the other hand, whose results can be seen as special instances of our main theorem 3.4.2. In these two lines of work, a number of sufficient conditions are given in order to construct a Quillen bifibration. The fact that their conditions and constructions are special cases of ours follows from the equivalence established in theorem 3.4.2. As a matter of fact, it is quite instructive to review and to point out the divergences between the two approaches and ours, since it also provides a way to appreciate the subtle aspects of our construction.

Let us state the two results and comment them.

Theorem 3.5.9 (Roig, Stanculescu). *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a Grothendieck bifibration. Suppose that $\mathcal{B}$ is a model category with structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ and that each fiber $\mathcal{E}_A$ also with structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. Suppose also assumption (Q). Then $\mathcal{E}$ is a model category with*

- *cofibrations the total ones,*
- *weak equivalences those $f : X \longrightarrow Y$ such that $p(f) \in \mathfrak{W}$ and $f^{\triangleleft} \in \mathfrak{W}_{pX}$,*
- *fibrations the total ones,*

provided that

(i) u^* is homotopically conservative for all $u \in \mathfrak{W}$,

(ii) for $u : A \longrightarrow B$ an acyclic cofibration in $\mathcal{B}$, the unit of the adjoint pair $(u_!, u^*)$ is pointwise a weak equivalence in $\mathcal{E}_A$.

The formulation of the theorem is not symmetric, since it emphasizes the cartesian morphisms over the cocartesian ones in the definition of weak equivalences. This lack of symmetry in the definition of the weak equivalences has the unfortunate effect of giving a similar bias to the sufficient conditions: in order to obtain the weak factorization systems, cocartesian morphisms above acyclic cofibrations should be acyclic, which is the meaning of this apparently weird condition (ii); at the same time, cartesian morphisms above acyclic fibrations should also be acyclic but this is vacuously true with the definition of weak equivalences in theorem 3.5.9. Condition (i) is only here for the 2-out-of-3 property, which boils down to it.

Theorem 3.5.10 (Harpaz, Prasma). *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a Grothendieck bifibration. Suppose that $\mathcal{B}$ is a model category with structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ and that each fiber $\mathcal{E}_A$ also with structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. Suppose also assumption (Q). Then $\mathcal{E}$ is a model category with*

- *cofibrations the total ones,*
- *weak equivalences those $f : X \longrightarrow Y$ such that $u = p(f) \in \mathfrak{W}$ and $u^*(r) \circ f^{\triangleleft} \in \mathfrak{W}_{pX}$, where $r : Y \longrightarrow Y^{\mathrm{fib}}$ is a fibrant replacement of Y in $\mathcal{E}_{pY}$,*
- *fibrations the total ones,*

provided that

(i') the adjoint pair $(u_!, u^*)$ is a Quillen equivalence for all $u \in \mathfrak{W}$,

(ii') $u_!$ and v^* preserves weak equivalences whenever u is an acyclic cofibration and v an acyclic fibration.

At first glance, Harpaz and Prasma introduces the same asymmetry that Roig and Stanculescu in the definition of weak equivalences. They show however that, under condition (i'), weak equivalences can be equivalently described as those $f : X \longrightarrow Y$ such that $u = p(f) \in \mathfrak{W}$ and

$$u_! X^{\mathrm{cof}} \longrightarrow u_! X \longrightarrow Y \in \mathfrak{W}_{pY} \tag{3.104}$$

where the first arrow is the image by $u_!$ of a cofibrant replacement $X^{\mathrm{cof}} \longrightarrow X$. Hence, they manage to adapt Roig-Stanculescu's result and to make it self dual. There is a cost however, namely condition (i'). Informally, it says that weakly equivalent objects of $\mathcal{B}$ should have fibers with the same homotopy theory. Harpaz and Prasma observe moreover that under (i'), (i) and (ii) implies (ii'). The condition is quite strong: in particular for the simple Grothendieck bifibration $\mathrm{cod} : \mathrm{Fun}(\mathbf{2}, \mathcal{B}) \longrightarrow \mathcal{B}$ of example 3.3.7, it is equivalent to the fact that the model category $\mathcal{B}$ is right proper. This explains why condition (i') has to be weakened in order to recover the Reedy construction, as we did in this chapter.

It is possible to understand our work as a reflection on these results, in the following way. A common pattern in the train of thoughts developed in the three papers [Roi94, Sta12, HP15] is their strong focus on cartesian and cocartesian morphisms above *weak equivalences*. Looking at what it takes to construct weak factorization systems using Stanculescu's lemma (cf. lemma 3.2.7), it is quite unavoidable to *push* along (acyclic) cofibrations and *pull* along (acyclic) fibrations in order to put everything in a common fiber, and then to use the fiberwise model structure. On the other hand, *nothing* compels us apparently to push or to pull along weak equivalences of $\mathcal{B}$ in order to define a model structure on $\mathcal{E}$. This is precisely the Ariadne's thread which we followed in this work: organize everything so that cocartesian morphisms above (acyclic) cofibrations are (acyclic) cofibrations, and cartesian morphisms above (acyclic) fibrations are (acyclic) fibrations. This line of thought requires in particular to see every weak equivalences of the basis category $\mathcal{B}$ as the *composite* of an acyclic cofibration followed by an acyclic fibration. All the rest, and in particular hypothesis (hCon) and (hBC), follows from that perspective, together with the idea of applying the framework to re-understand the Reedy construction from a bifibrational point of view.

Chapter 4

Homotopy categories of Quillen bifibrations

This chapter is devoted to show that the functor $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \longrightarrow \mathbf{Ho}(\mathcal{B})$ associated with a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ (as introduced in definition 3.1.2) can be constructed in two stages: a fiberwise localization which does not alter the basis category $\mathcal{B}$ nor its model structure, followed by a basis localization. As such, the construction in two stages provides an instructive and first example of iterated homotopy localization.

Recall that a Quillen bifibration is a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ that have a model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ on $\mathcal{E}$ and a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ on $\mathcal{B}$ and satisfying the two following properties:

- p is left Quillen (i.e. preserves cofibrations and trivial cofibrations) and right Quillen (i.e. preserves fibrations and trivial fibrations),
- for every $A \in \mathcal{B}$, the classes

$$\mathfrak{C}_{\mathcal{E}} \cap \mathrm{Mor}(\mathcal{E}_A), \mathfrak{W}_{\mathcal{E}} \cap \mathrm{Mor}(\mathcal{E}_A), \mathfrak{F}_{\mathcal{E}} \cap \mathrm{Mor}(\mathcal{E}_A),$$

  defines a model structure on $\mathcal{E}_A$, called the *restricted model structure* on $\mathcal{E}_A$.

Theorem 3.4.2 gives a characterization of the Quillen bifibrations among the Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ that have a model structure on $\mathcal{B}$ and on each fiber $\mathcal{E}_A$ in such a way that every adjunction $(u_!, u^*)$ over a morphism u of $\mathcal{B}$ is a Quillen adjunction. One can interpret this result as a Grothendieck construction for a special class of pseudo functors $\mathcal{B} \longrightarrow \mathsf{Quil}$ with values in the 2-category of model categories, Quillen adjunctions and natural transformations. In particular, a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ is entirely determined by its Grothendieck "deconstruction", i.e. the pseudo functor $\tilde{p} : \mathcal{B} \longrightarrow \mathsf{Quil}$ it induces.

One predominant and leading example is the codomain bifibration

$$\mathrm{cod} : \mathrm{Fun}(\mathbf{2}, \mathcal{C}) \longrightarrow \mathcal{C} \tag{4.1}$$

when $\mathcal{C}$ is a model category with pullbacks and $\mathrm{Fun}\,(\mathbf{2}, \mathcal{C})$ is equipped with the injective model structure. Recall that the cartesian morphisms for cod are precisely the cartesian squares and that the cocartesian morphisms are the squares of the form

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "x"'] & X \arrow[d, "fx"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{4.2}$$

Consider the pullback in Cat of cod along itself:

$$\begin{tikzcd} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \arrow[d, "\mathrm{cod}"] \\ \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \arrow[r, "\mathrm{cod}"'] & \mathcal{C} \end{tikzcd} \tag{4.3}$$

Choosing pullbacks in $\mathscr{C}$ permits to craft a functor, hereafter called *substitution*:

$$\begin{gathered} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \xrightarrow{\sigma} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \\ \begin{tikzcd} & Y \arrow[d, "y"] & & Y \times_B A \arrow[d, "u^* y"] \\ A \arrow[r, "u"] & B & \longmapsto & A \end{tikzcd} \end{gathered} \tag{4.4}$$

Remark that the domain category identifies with the diagram category $\mathrm{Fun}\,(\lrcorner, \mathcal{C})$ where $\lrcorner$ is the walking cospan category. As such is recovered the pullback functor $\lim_{\lrcorner} : \mathrm{Fun}\,(\lrcorner, \mathcal{C}) \to \mathcal{C}$ as the composite:

$$\mathrm{Fun}\,(\lrcorner, \mathcal{C}) \xrightarrow{\ \sigma\ } \mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \xrightarrow{\mathrm{dom}} \mathcal{C} \tag{4.5}$$

Moreover, the constant diagram functor $c : \mathcal{C} \to \mathrm{Fun}\,(\lrcorner, \mathcal{C})$, which is left adjoint to the pullback functor, decomposes as a left adjoint c_{dom} to dom followed by a left adjoint c_σ to σ. These left adjoints are given by:

$$c_{\mathrm{dom}} : X \mapsto \begin{tikzcd} X \arrow[d, equal, "\mathrm{id}_X"] \\ X \end{tikzcd} \quad \text{and} \quad c_\sigma : \begin{tikzcd} X \arrow[d, "x"] \\ A \end{tikzcd} \mapsto \begin{tikzcd} & X \arrow[d, "x"'] \\ A \arrow[r, equal] & A \end{tikzcd} \tag{4.6}$$

The category $\mathrm{Fun}\,(\lrcorner, \mathcal{C})$ can be equipped with the injective model structure, which incidentally correspond to the "pointwise" model structure when looking at $\mathrm{Fun}\,(\lrcorner, \mathcal{C})$ as the pullback $\mathrm{Fun}\,(\mathbf{2}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}\,(\mathbf{2}, \mathcal{C})$. Then both c_{dom} and c_σ are left Quillen. It follows that the homotopy pullback, expressed as the right derived functor of $\lim_{\lrcorner}$ can be written as the composite functor:

$$\mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C})) \xrightarrow{\mathbf{R}\,\sigma} \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C})) \xrightarrow{\mathbf{R}\,\mathrm{dom}} \mathbf{Ho}\,(\mathcal{C}) \tag{4.7}$$

The functor of interest here is the *homotopy substitution* $\mathbf{R}\,\sigma$. The vertical projection $\pi_1 : \mathrm{Fun}\,(\lrcorner, \mathcal{C}) \to \mathrm{Fun}\,(\mathbf{2}, \mathcal{C})$ from the pullback square (4.3) respects weak equivalences (it is even a Quillen bifibration), hence we end up with a functor

$$\mathbf{Ho}\,(\pi_1) : \mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C})) \to \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C})) \tag{4.8}$$

for which we can take a fiber at an object u (that is a map $u : A \longrightarrow B$ of $\mathcal{C}$):

$$\begin{tikzcd}
\mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C}))_u \arrow[r, hook] \arrow[d] & \mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C})) \arrow[d] \\
1 \arrow[r, "u"'] & \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C}))
\end{tikzcd} \tag{4.9}$$

Then the homotopy substitution along u, by which is meant the functor $(\mathbf{R}\,\sigma)_u$ obtained as the composition

$$\mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C}))_u \hookrightarrow \mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C})) \xrightarrow{\mathbf{R}\,\sigma} \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C})) \tag{4.10}$$

is to be compared with the right derived functor of substitution along u

$$\mathbf{Ho}\,(\mathcal{C}_B) \xrightarrow{\mathbf{R}\,u^*} \mathbf{Ho}\,(\mathcal{C}_A) \tag{4.11}$$

More precisely, there is obvious functors $\mathbf{Ho}\,(\mathcal{C}_B) \longrightarrow \mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C}))_u$ and $\mathbf{Ho}\,(\mathcal{C}_A) \longrightarrow \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C}))$ and one could wonder if the following square of functors is commutative:

$$\begin{tikzcd}
\mathbf{Ho}\,(\mathcal{C}_B) \arrow[r, "\mathbf{R}\,u^*"] \arrow[d] & \mathbf{Ho}\,(\mathcal{C}_A) \arrow[d] \\
\mathbf{Ho}\,(\mathrm{Fun}\,(\lrcorner, \mathcal{C}))_u \arrow[r, "(\mathbf{R}\,\sigma)_u"'] & \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathcal{C}))
\end{tikzcd} \tag{4.12}$$

The answer is generally **no** except if u is a fibration and either B is fibrant or $\mathcal{C}$ is right proper. Explicitly, the top right path in (4.12) is computed as follow: given a map $y : Y \longrightarrow B$, choose a factorization $y = y'i'$ with $i' : Y \longrightarrow Y'$ an acyclic cofibration and $y' : Y \longrightarrow B$ a fibration, then the result is the substitution $u^*y' : Y' \times_B A$. On the other hand, the bottom left path is computed by first taking a fibrant replacement $j : B \longrightarrow B'$ of B, and factoring ju as $u'j'$ with $u' : A' \longrightarrow B'$ a fibration and $j' : A \longrightarrow A'$ an acyclic cofibration: then given $y : Y \longrightarrow B$, factor jy as $y''i''$ with $i'' : Y \longrightarrow Y''$ acyclic cofibration ans $y'' : Y'' \longrightarrow B'$ fibration; the result is the substitution $(u')^*y'' : Y'' \times_{B'} A'$. The two constructions are summarized in the commutative diagram of $\mathcal{C}$ below:

$$\begin{tikzcd}
 & & Y \arrow[dr, "i'"] \arrow[ddr, "i''"] \arrow[ddd, "y"] & \\
 & Y' \times_B A \arrow[rr] \arrow[d, dashed] \arrow[dl, "u^*y'"'] & & Y' \arrow[ddl, "y'"] \\
 & Y'' \times_{B'} A' \arrow[rr] \arrow[dd, "(u')^*y''"'] & & Y'' \arrow[dd, "y''"] \\
A \arrow[rr, "u"'] \arrow[dr, "j'"'] & & B \arrow[dr, "j"] & \\
 & A' \arrow[rr, "u'"'] & & B'
\end{tikzcd} \tag{4.13}$$

The dotted arrow exists by mean of the universal property of the pullback $Y'' \times_{B'} A'$ but there is no reason for it to be a weak equivalence, which would correspond to (4.12) being commutative up to isomorphism, except in the already mentioned cases.

In the setting of a general Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, the functor $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \longrightarrow \mathbf{Ho}(\mathcal{B})$ nurture the same kind of relationship with the Grothendieck construction of the functor $\mathbf{D}\,\tilde{p} : \mathcal{B} \longrightarrow \mathsf{Adj}$ constructed as

$$\mathcal{B} \xrightarrow{\tilde{p}} \mathsf{Mod} \xrightarrow{\mathbf{D}} \mathsf{Adj} \tag{4.14}$$

where Mod denotes the 2-category of model categories, Quillen adjunctions and natural transformations, and where $\mathbf{D}$ is the pseudo 2-functor that maps a model category to its localization and that maps a Quillen adjunction to the corresponding derived adjunction (see section 4.1). Let us insist on the fact that object of Mod are *model categories* and not just mere *model structures*, so that localizations can be given by the $\mathbf{Ho}(-)$ construction (see proposition 2.2.42) and so that Quillen adjunctions give rise to derived adjunctions (see proposition 2.2.52). Recall that we work with definition 2.2.10 of a model category that only requires the existence of finite products and finite coproducts. It is crucial to use this less restrictive version of model categories in this chapter because the setting is such that we shall encounter localizations of model categories in which equalizers and coequalizers do not necessarily exists.

Let us give a quick overview of what follows. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a given Quillen bifibration, so that in particular the fibers $\mathcal{E}_A$ are model categories satisfying (Q). Consider $\mathcal{B}$ with the trivial model structure; then conditions (hCon) and (hBC) are vacuously met, so theorem 3.4.2 gives a new model structure on $\mathcal{E}$ that we call the *fiberwise* model structure. Let us denote $p_{\mathrm{fw}} : \mathcal{E}_{\mathrm{fw}} \longrightarrow \mathcal{B}_{\mathrm{fw}}$ the newly created Quillen bifibration to distinguish it clearly from $p : \mathcal{E} \longrightarrow \mathcal{B}$, although p and p_{fw} are the same functors and only differ by their homotopical content. The induced functor $\mathbf{Ho}(p_{\mathrm{fw}}) : \mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}) \longrightarrow \mathbf{Ho}(\mathcal{B}_{\mathrm{fw}})$ will be showed to be isomorphic to the Grothendieck construction of $\mathbf{D}\,\tilde{p} : \mathcal{B} \longrightarrow \mathsf{Adj}$ (cf. proposition 4.3.3). In particular, $\mathbf{Ho}(p_{\mathrm{fw}})$ is a Grothendieck bifibration and its base category $\mathbf{Ho}(\mathcal{B}_{\mathrm{fw}})$ is isomorphic to $\mathcal{B}$, hence it is a model category. Suppose each of the fiber of $\mathbf{Ho}(p_{\mathrm{fw}})$ is equipped with the trivial model structure. Then the main result of our analysis can be stated as follow:

Theorem. *The functor $\mathbf{Ho}(p_{\mathrm{fw}})$ is a Quillen bifibration and the induced homotopy functor*

$$\mathbf{Ho}(\mathbf{Ho}(p_{\mathrm{fw}})) : \mathbf{Ho}(\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})) \longrightarrow \mathbf{Ho}(\mathcal{B})$$

is isomorphic to the homotopy functor induced by p:

$$\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \longrightarrow \mathbf{Ho}(\mathcal{B})$$

Morally speaking, it means that the homotopy category $\mathbf{Ho}(\mathcal{E})$ of the total category $\mathcal{E}$ in a Quillen bifibration can be obtained in two steps: first by quotienting up to *vertical* homotopies, and next quotienting up to *horizontal* ones. There should be something annoying to read in the previous paragraph for the eye of a trained homotopy theorist: the iterated localization $\mathbf{Ho}(\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}))$ usually makes no sense because it is fairly rare for an homotopy category such as $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$ to be again a model category. The main obstruction is that homotopy categories usually do not have all finite limits and finite colimits. However they do have finite products and small coproducts, directly computed in the original model category. This is precisely why we took on Egger's definition of model categories in this work.

4.1 Localization is a pseudo 2-functor

In this section, we will show that the mapping $\mathsf{Mod} \longrightarrow \mathsf{Adj}$ that associates a model category $\mathcal{M}$ with its localization $\mathbf{Ho}(\mathcal{M})$ and a Quillen adjunction (F, G) to its derived version $(\mathbf{L}\,F, \mathbf{R}\,G)$ can be turned into a pseudo 2-functor. To make that absolutely rigorous, we shall now fix a choice of left derived functor $\mathbf{L}\,F$ for every left Quillen adjoint F and a choice of right derived functor $\mathbf{R}\,G$ for every right Quillen adjoint G. In particular, each left Quillen adjoint $F : \mathcal{M} \longrightarrow \mathcal{N}$ comes equipped with its canonical 2-cell

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{N} \arrow[d] \\ \mathbf{Ho}(\mathcal{M}) \arrow[r, "\mathbf{L}\,F"'] \arrow[ur, phantom, "\overset{\varepsilon_F}{\Nearrow}"] & \mathbf{Ho}(\mathcal{N}) \end{tikzcd} \tag{4.15}$$

That being done, the rest actually has little to do with model categories per se and is solely based on the properties of left and right derived functors as Kan extensions.

Lemma 4.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. The mapping*

$$\mathsf{Mod}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathsf{Adj}(\mathbf{Ho}(\mathcal{M}), \mathbf{Ho}(\mathcal{N})), \quad (F, G) \mapsto (\mathbf{L}\,F, \mathbf{L}\,G) \tag{4.16}$$

can be extended as a functor.

Proof. Given Quillen adjunctions (F, G) and (F', G') with a natural transformation $\alpha : F \longrightarrow F'$, we can consider the natural transformation

$$\mathbf{L}\,F\gamma_M \xrightarrow{\varepsilon_F} \gamma_N F \xrightarrow{\mathrm{id}_{\gamma_N}\cdot\alpha} \gamma_N F' \tag{4.17}$$

Using the fact that $\mathbf{L}\,F'$ is a right Kan extension of $\gamma_N F'$ along γ_M, there exists a cell $\mathbf{L}\,F \longrightarrow \mathbf{L}\,F'$ factorizing the one above and we should denote it $\mathbf{L}\,\alpha$.

Now if $\alpha : F \longrightarrow F'$ and $\beta : F' \longrightarrow F''$ are 2-cells between Quillen adjunctions (F, G), (F', G') and (F'', G''), then the 2-cells $\mathbf{L}\,\alpha'$ and $\mathbf{L}\,\alpha$ are constructed as to make the following diagram commutes

$$\begin{tikzcd} \mathbf{L}\,F\gamma_M \arrow[r, "\mathbf{L}\,\alpha\cdot\mathrm{id}_{\gamma M}"] \arrow[d] & \mathbf{L}\,F'\gamma_M \arrow[r, "\mathbf{L}\,\alpha'\cdot\mathrm{id}_{\gamma M}"] \arrow[d] & \mathbf{L}\,F''\gamma M \arrow[d] \\ \gamma_N F \arrow[r, "\mathrm{id}_{\gamma_N}\cdot\alpha"] & \gamma_N F' \arrow[r, "\mathrm{id}_{\gamma_N}\cdot\alpha'"] & \gamma_N F'' \end{tikzcd} \tag{4.18}$$

where the vertical arrows are the canonical 2-cells ε_F, $\varepsilon_{F'}$ and $\varepsilon_{F''}$. Remark that the composite bottom arrow is just $\mathrm{id}_{\gamma_N} \cdot (\alpha' \circ \alpha)$. So we can use uniqueness of factorization through right Kan extensions to conclude that the top arrow is nothing less that $\mathbf{L}(\alpha' \circ \alpha)$. Similarly, the identity cell clearly fills in the following commutative square:

$$\begin{tikzcd} \mathbf{L}\,F\gamma_M \arrow[r, "\mathrm{id}_{\mathbf{L}\,F}\cdot\mathrm{id}_{\gamma M}"] \arrow[d] & \mathbf{L}\,F\gamma_M \arrow[d] \\ \gamma_N F \arrow[r, "\mathrm{id}_{\gamma_N}\cdot\mathrm{id}_F"] & \gamma_N F \end{tikzcd} \tag{4.19}$$

Hence, by uniqueness in the factorization through Kan extension, it follows that $\mathbf{L}\,\mathrm{id}_F = \mathrm{id}_{\mathbf{L}\,F}$. This concludes the proof of functoriality. ☐

We are now going to describe the structural 2-cells of the pseudo functoriality. We shall use again the properties of left derived functors as right Kan extensions. Recall that the left derived functor of $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ is defined as a right Kan extension:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{M}}"] \arrow[d, "\gamma_M"'] \arrow[dr, phantom, "\overset{\varepsilon}{\Nearrow}"] & \mathcal{M} \arrow[d, "\gamma_M"] \\ \mathbf{Ho}\,(\mathcal{M}) \arrow[r, "\mathbf{L}\,\mathrm{id}_{\mathcal{M}}"'] & \mathbf{Ho}\,(\mathcal{M}) \end{tikzcd} \tag{4.20}$$

So the identity 2-cell $\mathrm{id}_{\mathbf{Ho}(\mathcal{M})} \circ \gamma_M \longrightarrow \gamma_M \circ \mathrm{id}_{\mathcal{M}}$ factorizes through it, yielding

$$u_{\mathcal{M}} : \mathrm{id}_{\mathbf{Ho}(\mathcal{M})} \longrightarrow \mathbf{L}\,\mathrm{id}_{\mathcal{M}}. \tag{4.21}$$

Similarly we can define a 2-cell

$$m_{F,F'} : \mathbf{L}\,F' \circ \mathbf{L}\,F \longrightarrow \mathbf{L}(F'F) \tag{4.22}$$

for composable functors $F : \mathcal{M} \longrightarrow \mathcal{N}$ and $F' : \mathcal{N} \longrightarrow \mathcal{P}$ as the factorization through the right Kan extension $\mathbf{L}(F'F)$ of the pasting composite 2-cell:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\overset{\varepsilon_F}{\Nearrow}"] & \mathcal{N} \arrow[r, "F'"] \arrow[d] \arrow[dr, phantom, "\overset{\varepsilon_{F'}}{\Nearrow}"] & \mathcal{P} \arrow[d] \\ \mathbf{Ho}\,(\mathcal{M}) \arrow[r, "\mathbf{L}\,F"'] & \mathbf{Ho}\,(\mathcal{N}) \arrow[r, "\mathbf{L}\,F'"'] & \mathbf{Ho}\,(\mathcal{P}) \end{tikzcd} \tag{4.23}$$

From the remark below proposition 2.2.46, we get that all the $u_{\mathcal{M}}$ and $m_{F,F'}$ are invertible.

Proposition 4.1.2. *Given a left Quillen adjoint $F : \mathcal{M} \longrightarrow \mathcal{N}$ between model categories, the following relation holds:*

$$\begin{aligned} m_{\mathrm{id}_{\mathcal{M}},F} \circ (\mathrm{id}_{\mathbf{L}\,F} \cdot u_{\mathcal{M}}) &= \mathrm{id}_{\mathbf{L}\,F} \\ m_{F,\mathrm{id}_{\mathcal{N}}} \circ (u_{\mathcal{N}} \cdot \mathrm{id}_{\mathbf{L}\,F}) &= \mathrm{id}_{\mathbf{L}\,F} \end{aligned} \tag{4.24}$$

Proof. We only prove the first relation, the other one is strictly similar. By definition, the composition $\varepsilon_F \circ (m_{\mathrm{id}_{\mathcal{M}},F} \cdot \mathrm{id}_{\gamma_M})$ is equal to the pasting composite

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{M}}"] \arrow[d] \arrow[dr, phantom, "\overset{\varepsilon_{\mathrm{id}_{\mathcal{M}}}}{\Nearrow}"] & \mathcal{M} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\overset{\varepsilon_F}{\Nearrow}"] & \mathcal{N} \arrow[d] \\ \mathbf{Ho}\,(\mathcal{M}) \arrow[r, "\mathbf{L}\,\mathrm{id}_{\mathcal{M}}"'] & \mathbf{Ho}\,(\mathcal{M}) \arrow[r, "\mathbf{L}\,F"'] & \mathbf{Ho}\,(\mathcal{N}) \end{tikzcd} \tag{4.25}$$

Now $u_{\mathcal{M}}$ is precisely the 2-cell such that $\varepsilon_{\mathrm{id}_{\mathcal{M}}} \circ (u_{\mathcal{M}} \cdot \mathrm{id}_{\gamma_M}) = \mathrm{id}_{\gamma_M}$. So in the end,

$$\begin{aligned} \varepsilon_F \circ \left(\left(m_{\mathrm{id}_{\mathcal{M}},F} \circ (u_{\mathcal{M}} \cdot \mathrm{id}_{\mathbf{L}\,F})\right) \cdot \mathrm{id}_{\gamma_M}\right) &= \varepsilon_F \circ \left(m_{\mathrm{id}_{\mathcal{M}},F} \cdot \mathrm{id}_{\gamma_M}\right) \circ \left((\mathrm{id}_{\mathbf{L}\,F} \cdot u_{\mathcal{M}}) \cdot \mathrm{id}_{\gamma_M}\right) \\ &= (\varepsilon_F \cdot \mathrm{id}_{\mathrm{id}_{\mathcal{M}}}) \circ (\mathrm{id}_{\mathbf{L}\,F} \cdot \varepsilon_{\mathrm{id}_{\mathcal{M}}}) \circ \left(\mathrm{id}_{\mathbf{L}\,F} \cdot u_{\mathcal{M}} \cdot \mathrm{id}_{\gamma_M}\right) \\ &= \varepsilon_F \circ (\mathrm{id}_{\mathbf{L}\,F} \cdot \mathrm{id}_{\gamma_M}) \end{aligned} \tag{4.26}$$

By uniqueness of factorization through right Kan extensions, it follows that

$$m_{\mathrm{id}_{\mathcal{M}},F} \circ (\mathrm{id}_{\mathbf{L}\,F} \cdot u_{\mathcal{M}}) = \mathrm{id}_{\mathbf{L}\,F} \tag{4.27}$$

□

Proposition 4.1.3. *Given left Quillen adjoints $F : \mathcal{M} \longrightarrow \mathcal{N}$, $F' : \mathcal{N} \longrightarrow \mathcal{P}$ and $F'' : \mathcal{P} \longrightarrow \mathcal{Q}$, the following diagram of 2-cells commutes:*

$$\begin{tikzcd} \mathbf{L}\,F'' \circ \mathbf{L}\,F' \circ \mathbf{L}\,F \arrow[r, "m_{F'',F'} \cdot \mathrm{id}_{\mathbf{L}\,F}"] \arrow[d, "\mathrm{id}_{\mathbf{L}\,F''} \cdot m_{F',F}"'] & \mathbf{L}(F'' \circ F') \circ \mathbf{L}\,F \arrow[d, "m_{F'' \circ F', F}"] \\ \mathbf{L}\,F'' \circ \mathbf{L}(F' \circ F) \arrow[r, "m_{F'',F' \circ F}"'] & \mathbf{L}(F'' \circ F' \circ F) \end{tikzcd} \tag{4.28}$$

Proof. By definition of the 2-cells $m_{F'',F}$, $m_{F',F}$, $m_{F'',F'\circ F}$, $m_{F''\circ F',F}$, both

$$\varepsilon_{F''\circ F'\circ F} \circ \left(\left(m_{F''\circ F',F} \circ \left(m_{F'',F'} \cdot \mathrm{id}_{\mathbf{L}\,F} \right) \right) \cdot \mathrm{id}_{\gamma_M} \right) \tag{4.29}$$

and

$$\varepsilon_{F''\circ F'\circ F} \circ \left(\left(m_{F'',F'\circ F} \circ \left(\mathrm{id}_{\mathbf{L}\,F''} \cdot m_{F',F} \right) \right) \cdot \mathrm{id}_{\gamma_M} \right) \tag{4.30}$$

are equal to the pasting composite

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \arrow[r, "F'"] \arrow[d, "\gamma_{\mathcal{N}}"'] \arrow[dl, phantom, "\Swarrow \varepsilon_F"] & \mathcal{P} \arrow[r, "F''"] \arrow[d, "\gamma_{\mathcal{P}}"'] \arrow[dl, phantom, "\Swarrow \varepsilon_{F'}"] & \mathcal{Q} \arrow[d, "\gamma_{\mathcal{Q}}"] \arrow[dl, phantom, "\Swarrow \varepsilon_{F''}"] \\ \mathbf{Ho}\,(\mathcal{M}) \arrow[r, "\mathbf{L}\,F"'] & \mathbf{Ho}\,(\mathcal{N}) \arrow[r, "\mathbf{L}\,F'"'] & \mathbf{Ho}\,(\mathcal{P}) \arrow[r, "\mathbf{L}\,F''"'] & \mathbf{Ho}\,(\mathcal{Q}) \end{tikzcd} \tag{4.31}$$

The uniqueness in the factorization trough a right Kan extension allows us to conclude that the diagram of the proposition does in fact commute. □

We can summarize propositions 4.1.2 and 4.1.3 as follow.

Theorem 4.1.4. *The morphisms $m_{F,F'}$ and $u_{\mathcal{M}}$ gives the mapping*

$$\mathsf{Mod} \longrightarrow \mathsf{Adj}, \quad \mathcal{M} \mapsto \mathbf{Ho}\,(\mathcal{M}), \quad (F, G) \mapsto (\mathbf{L}\,F, \mathbf{R}\,G) \tag{4.32}$$

the structure of a pseudo 2-functor.

4.2 Fiberwise homotopies versus total homotopies

In this section we formulate a notion of *fiberwise localization* of a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ and compare it to the *total localization* $\mathbf{Ho}\,(p) : \mathbf{Ho}\,(\mathcal{E}) \longrightarrow \mathbf{Ho}\,(\mathcal{B})$. Recall that we are working with definition 2.2.10 of model categories, in which finite products and finite coproducts exist but equalizers and coequalizers are no longer required. In this framework, $\mathbf{Ho}\,(\mathcal{M})$ admits finite products and finite coproducts, directly create by $\gamma : \mathcal{M} \longrightarrow \mathbf{Ho}\,(\mathcal{M})$. This opens the possibility to perform iterated localizations, as will be illustrated in section 4.4.

Section 4.2.1 start by describing the cylinder and path objects, hence also the homotopies, in the total category of a Quillen bifibration. Section 4.2.1 gives an example of a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ where the fiber of $\mathbf{Ho}\,(p)$ over on object A of the basis does not coincide with the homotopy category $\mathbf{Ho}\,(\mathcal{E}_A)$. This occurs precisely because the basis category $\mathcal{B}$ has non trivial homotopies. It serves as a starting point for section 4.3, which is devoted to the study of Quillen bifibrations where the basis category is equipped with the trivial model structure.

4.2.1 Cylinder and path objects in the total category

The purpose of this section is to establish basic properties of cylinder and path objects in the total category $\mathcal{E}$. Here might be a good place to talk about limits and colimits in $\mathcal{E}$ from the point of view of the Grothendieck construction. We thus recall the following well-known fact:

Lemma 4.2.1. *Suppose given a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ whose basis category $\mathcal{B}$ admits all limits of shape $\mathcal{J}$. Then the following assertions are equivalent:*

(i) the total category $\mathcal{E}$ admits limits of shape $\mathcal{J}$ and p preserves them,

(ii) all the fibers $\mathcal{E}_A$ $(A \in \mathcal{B})$ admit limits of shape $\mathcal{J}$.

Proof. We want to emphasize the *computation* of those (co)limits. We only treat the case of limits, as the case of colimits is dual. A diagram $d : \mathcal{J} \longrightarrow \mathcal{E}$ induces a diagram $pd : \mathcal{J} \longrightarrow \mathcal{E} \longrightarrow \mathcal{B}$ in $\mathcal{B}$. By hypothesis, pd has a limit $L \in \mathcal{B}$ with canonical projections $\pi_J : L \longrightarrow pdJ, J \in \mathcal{J}$. Pulling every dJ over π_J, we end up with a diagram $\mathcal{J} \longrightarrow \mathcal{E}_L$ mapping $J \mapsto {\pi_J}^*(dJ)$, which admits a limit K with canonical projections $\rho_J : K \longrightarrow {\pi_J}^*(dJ)$.

The claim is that K, together with the maps $\varpi_J = \kappa\rho_J : K \longrightarrow {\pi_J}^*(dJ) \longrightarrow dJ$, is actually a limit for the diagram d. Indeed, any other cone over d with apex X induces a cone in $\mathcal{B}$ over pd with apex pX: hence a morphism $u : pX \longrightarrow L$ given by the universal property of the limit L. Now, using the cartesianity of all those ${\pi_J}^*(dJ) \longrightarrow dJ$ and the cocartesianity of $X \longrightarrow u_!X$, a morphism of cone in $\mathcal{E}$ between X and K amount to a morphism of cone $\in \mathcal{E}_L$ between $u_!X$ and K, which uniquely exists by universal property of K. □

From now on and for this entire section, $p : \mathcal{E} \longrightarrow \mathcal{B}$ is supposed to be a Quillen bifibration between model categories, where one requires moreover that p preserves finite products and finite coproducts *on the nose*. By the lemma just established, this is equivalent to the existence of finite products and finite coproducts in both the basis $\mathcal{B}$ and in each fiber $\mathcal{E}_A$.

It is worth noticing that a diagram $d : \mathcal{J} \longrightarrow \mathcal{E}_A$ in a fiber has not necessarily the same (co)limit in $\mathcal{E}_A$ than (its composite with $i_A : \mathcal{E}_A \hookrightarrow \mathcal{E}$) in $\mathcal{E}$. There is of course canonical maps

$$\lim(i_A \circ d) \longrightarrow i_A(\lim(d)), \qquad i_A(\operatorname{colim}(d)) \longrightarrow \operatorname{colim}(i_A \circ d) \tag{4.33}$$

but they are not a priori isomorphisms.

In particular, the previous lemma gives a construction for the initial and terminal objects of $\mathcal{E}$: they are respectively the initial object of $\mathcal{E}_0$ and terminal object of $\mathcal{E}_1$ where 0 and 1 denotes the initial and terminal objects of $\mathcal{B}$.

Lemma 4.2.2. *An object of $\mathcal{E}$ is cofibrant precisely when it is cofibrant in its fiber and it lies above a cofibrant object of $\mathcal{B}$. Dually, an object of $\mathcal{E}$ is fibrant precisely when it is fibrant in its fiber and it lies above a fibrant object of $\mathcal{B}$.*

Proof. A map $f : 0_{\mathcal{E}} \longrightarrow X$ from the initial object of $\mathcal{E}$ is a cofibration precisely when $p(f)$ is a cofibration in $\mathcal{B}$ and $f_\triangleright$ is a cofibration in $\mathcal{E}_{pX}$. Since the initial object $0_{\mathcal{E}}$ is above the initial object 0 of the basis category, $p(f)$ is a cofibration precisely when pX is cofibrant; and since $0_{\mathcal{E}}$ is initial in $\mathcal{E}_0$ and left adjoints preserve initial objects, the map $f_\triangleright : p(f)_!0 \longrightarrow X$ is a cofibration in $\mathcal{E}_{pX}$ precisely when X is cofibrant in this fiber. The characterization of fibrant objects is dual. □

Recall that a *cylinder* for an object X in a model category $\mathcal{M}$ is an object C together with a weak equivalence $q : C \longrightarrow X$ and two sections $j_0, j_1 : X \longrightarrow C$. A cylinder is *good* when the induced map $\langle j_0, j_1 \rangle : X + X \longrightarrow C$ is a cofibration. A cylinder is *very good* if it is good and moreover q is an acyclic fibration. Notice that there is always a very good cylinder for X since the fold map $\nabla : X + X \longrightarrow X$ can be factored as a cofibration followed by an acyclic fibration. The notion of path object, good path object and very good path object is defined dually.

Lemma 4.2.3. *The functor p transports every cylinder (respectively good cylinder, very good cylinder) in $\mathcal{E}$ to a cylinder (respectively good cylinder, very good cylinder) in $\mathcal{B}$.*

The functor p transports every path object (respectively good path object, very good path object) in $\mathcal{E}$ to a path object (respectively good path object, very good path object) in $\mathcal{B}$.

Proof. By the computation of coproducts in $\mathcal{E}$, the fold map $\nabla_X : X + X \longrightarrow X$ in $\mathcal{E}$ is above the fold map $\nabla_{pX} : pX + pX \longrightarrow pX$ in $\mathcal{B}$. So any factorization $\nabla_X = qj$ with q a weak equivalence (respectively q a weak equivalence and j a cofibration, q an acyclic fibration and j a cofibration) yields a factorization of ∇_{pX} of the same type in $\mathcal{B}$. The case of path objects is treated dually. □

In particular it shows that any left homotopy is mapped to a left homotopy and any right homotopy to a right homotopy.

Lemma 4.2.4. *For every very good cylinder C_A of A in $\mathcal{B}$, and for any $X \in \mathcal{E}_A$, there is a very good cylinder C_X in $\mathcal{E}$ such that $pC_X = C_A$. For any very good path object P_A of A in $\mathcal{B}$, and for any $X \in \mathcal{E}_A$, there is a very good path object P_X in $\mathcal{E}$ such that $pP_X = P_A$.*

Proof. Suppose a factorization in $\mathcal{B}$ of the fold map

$$A + A \xrightarrow{u} C_A \xrightarrow{v} A, \quad u \in \mathfrak{C}, v \in \mathfrak{F} \cap \mathfrak{W} \tag{4.34}$$

The functors p maps the fold map $X + X \longrightarrow X$ to the fold map of A. This induces a factorization of the fold map below:

$$\begin{tikzcd} & v^*X \arrow[r, "\rho"] & X \\ X + X \arrow[r, "\lambda"'] \arrow[urr, gray] & u_!(X + X) \arrow[u, "f"] & \end{tikzcd} \tag{4.35}$$

where f lives in the fiber $\mathcal{E}_{C_A}$. By definition, λ is a cofibration and ρ an acyclic fibration, so any factorization of f as qj with $j \in \mathfrak{C}_{C_A}$ and $q \in \mathfrak{F}_{C_A} \cap \mathfrak{W}_{C_A}$ yields in $\mathcal{E}_{C_A}$ a very good cylinder C_X for the object X in $\mathcal{E}$. □

4.2.2 The homotopy lifting property for Quillen bifibrations

One of the nicest properties of fibrations in a model category is the so-called *homotopy lifting property* that roughly states that given a shape in the domain X of a fibration $q : X \longrightarrow Y$, and a deformation of its image in the codomain Y, the deformation can be performed in before in X prior to taking the image through q.

Proposition 4.2.5. *Let A be cofibrant and $q : X \longrightarrow Y$ a fibration in model category $\mathcal{M}$. For every map $f : A \longrightarrow X$ and $g : A \longrightarrow Y$, if $qf \sim g$ then there exists $f' : A \longrightarrow X$ such that $f \sim f'$ and $qf' = g$.*

Proof. Let us quickly recall the proof for pedagogical reasons. Denote $h : C \longrightarrow Y$ a left homotopy from qf to g though a good cylinder C for X. Denote $j_0, j_1 : X \longrightarrow C$ and $r : C \longrightarrow X$ the structural maps of the cylinder C. In particular, j_0 is an acyclic cofibration because X is cofibrant and C is a good cylinder. Hence, the following commutative square admits a filler $k : C \longrightarrow X$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_0"'] & X \arrow[d, "q"] \\ C \arrow[r, "h"'] & Y \end{tikzcd} \tag{4.36}$$

Define f' as kj_1, then in one hand $k : C \longrightarrow X$ is a left homotopy from $kj_0 = f$ to $kj_1 = f'$ and in the other hand $qf = qkj_1 = hj_1 = g$ as wanted. ☐

Remark 4.2.6. We shall insist that the proof might be more important than the result as stated in 4.2.5. Indeed, the proof is more precise in that the left homotopy k from f to f' can be chosen to be such that qk equals any given good left homotopy h from qf to g.

In this section, we advocate that the homotopy lifting property is a blissful fallout of the fact that the codomain bifibration cod : Fun$(\mathbf{2}, \mathcal{M}) \longrightarrow \mathcal{M}$ is a Quillen bifibration. For the rest of this section, let $p : \mathcal{E} \longrightarrow \mathcal{B}$ denote a Quillen bifibration, where as before the finite (co)products of $\mathcal{E}$ are chosen above the chosen finite (co)products of $\mathcal{B}$.

Lemma 4.2.7. *Let $\iota : X \longrightarrow Y$ and $\varphi : X \longrightarrow Z$ be maps of $\mathcal{E}$ with ι an acyclic cofibration and Z a fibrant object. Denote $j : A \longrightarrow B$ for $p(\iota)$ and $f : A \longrightarrow C$ for $p(f)$. Then for any $v : B \longrightarrow C$ such that commutes the triangle on the left of (4.37), there exists υ above v such that commutes the triangle on the right of (4.37).*

$$\begin{tikzcd} & & C \\ A \arrow[r, "j"'] \arrow[urr, bend left, "f"] & B \arrow[ur, "v"'] & \end{tikzcd} \quad \overset{p}{\longleftarrow\!\shortmid} \quad \begin{tikzcd} & & Z \\ X \arrow[r, "\iota"'] \arrow[urr, bend left, "\varphi"] & Y \arrow[ur, dashed, "\upsilon"'] & \end{tikzcd} \tag{4.37}$$

Proof. Factor ι as $\iota_{\triangleright}\lambda_{j,X}$. The map $\lambda_{j,X}$ is cocartesian above j and $vj = f = p(\varphi)$, so that there exist $\upsilon' : j_!X \longrightarrow C$ above v such that $\upsilon'\lambda_{j,X} = \varphi$. By definition of the total acyclic cofibration, $\iota_{\triangleright}$ is an acyclic cofibration in $\mathcal{E}$. Hence, because Z is fibrant, there is an extension $\upsilon : Y \longrightarrow Z$ of υ' along $\iota_{\triangleright}$.

$$\begin{tikzcd} & & Z \\ X \arrow[r, "\iota"'] \arrow[urr, bend left, "\varphi"] \arrow[dr, "\lambda"'] & Y \arrow[ur, dashed, "\upsilon"'] & \\ & j_!X \arrow[u, "\iota_{\triangleright}"'] \arrow[uur, bend right, "\upsilon'"'] & \end{tikzcd} \tag{4.38}$$

Moreover $v = p(\upsilon') = p(\upsilon\iota_{\triangleright}) = p(\upsilon)$ as $\iota_{\triangleright}$ is a fiber morphism. ☐

Combining lemma 4.2.7 and lemma 2.2.27 applied in $\mathcal{E}$, acyclic cofibration in $\mathcal{E}$ are some kind of *up-to-homotopy cocartesian* morphism relatively to fibrant objects.

Proposition 4.2.8. *Let $\varphi : X \longrightarrow Y$ be a map of $\mathcal{E}$ between a cofibrant object X and a fibrant object Y. Denote $f : A \longrightarrow B$ for $p(\varphi)$ and suppose f is homotopic to some map g. Then there exists γ over g such that $\varphi \sim \gamma$.*

Proof. Denote $h : C \longrightarrow B$ an homotopy from f to g through a very good cylinder C for A, with structural inclusions $j_0, j_1 : A \longrightarrow C$. From lemma 4.2.4, one obtains a very good cylinder Z for X with structural inclusions $\iota_0, \iota_1 : X \longrightarrow Z$ such that $p(\iota_0) = j_0$ and $p(\iota_1) = j_1$. Now use lemma 4.2.7 to obtain $\eta : Z \longrightarrow Y$ above h as in the right triangle:

$$\begin{tikzcd} & & B \\ A \arrow[urr, bend left, "f"] \arrow[r, "j_0"'] & C \arrow[ur, "h"'] & \end{tikzcd} \quad \overset{p}{\longmapsfrom} \quad \begin{tikzcd} & & Y \\ X \arrow[urr, bend left, "\varphi"] \arrow[r, "\iota_0"'] & Z \arrow[ur, dashed, "\eta"'] & \end{tikzcd} \tag{4.39}$$

Let $\gamma = \eta\iota_1$. By definition γ is left homotopic to $\eta\iota_0 = \varphi$, and $p(\gamma) = hj_1 = g$. □

Remark 4.2.9. Here again the proof is more precise that the bare statement of proposition 4.2.8. Indeed, the left homotopy η from φ to γ can be chosen to be above any given very good left homotopy h from f to g.

If one was to play this proof for the special case of $p = \mathrm{cod}$, one would recover proposition 4.2.5 and its proof. The homotopy lifting property is crucial in the demonstration of the following fact in any model category $\mathcal{M}$: given a span $f : A \longrightarrow B \longleftarrow Y : q$ where q is a fibration and all three objects are fibrant, then any ordinary pullback

$$\begin{tikzcd} P \arrow[r, "u'"] \arrow[d, "q'"'] & Y \arrow[d, "q"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{4.40}$$

is an homotopy pullback diagram, in the sense that: for all cofibrant object X together with maps $f : X \longrightarrow Y$ and $g : X \longrightarrow A$ and for all good left homotopy h from qf to ug, there exists a map $k : X \longrightarrow P$, unique up to homotopy, together with homotopies h_1 from $u'h$ to f and h_2 from $q'h$ to g satisfying that $qh_i = h$ for $i = 1, 2$.

The bifibrational version of the homotopy lifting property yields the same kind of result.

Proposition 4.2.10. *Let $\rho : X \longrightarrow Y$ be a cartesian map between fibrant objects in $\mathcal{E}$. Let $\varphi : Z \longrightarrow Y$ be another map with Z cofibrant. Denote $u : A \longrightarrow B$ for $p(\rho)$ and $f : C \longrightarrow B$ for $p(\varphi)$. Then for any $v : C \longrightarrow A$ and any very good left homotopy h from f to uv in $\mathcal{B}$, there exists a map $\upsilon : Z \longrightarrow X$ above v, unique up-to-homotopy, together with a (very good) left homotopy above h from φ to $\rho\upsilon$ in $\mathcal{E}$.*

$$\begin{tikzcd} C \arrow[drr, bend left, "f"] \arrow[dr, "v"'] & & \\ & A \arrow[r, "u"'] \arrow[u, phantom, "\sim"] & B \end{tikzcd} \quad \overset{p}{\longmapsfrom} \quad \begin{tikzcd} Z \arrow[drr, bend left, "\varphi"] \arrow[dr, dashed, "\upsilon"'] & & \\ & X \arrow[r, "\rho"'] \arrow[u, phantom, "\sim"] & Y \end{tikzcd} \tag{4.41}$$

Proof. The existence follows from the bifibrational version of the homotopy lifting property and the following remark. Indeed, because $f \sim uv$ and φ is above f between a cofibrant domain and a fibrant codomain, proposition 4.2.8 applies and gives a map

$\gamma \sim \varphi$ such that $p(\gamma) = uv$. Moreover, the good left homotopy h being given, we can choose a left good homotopy η above h to witness $\gamma \sim \varphi$. Then the cartesian property of ρ yields a map υ above v such that $\rho\upsilon = \gamma$.

Suppose there are υ and υ' above v together with η and η' above h such that η and η' are left homotopies from φ to respectively $\rho\upsilon$ and $\rho\upsilon'$. In particular, η and η' share their domain C_Z which is a very good cylinder for Z with structural inclusion maps $\iota_0, \iota_1 : Z \longrightarrow C_Z$. Using (a variant of) Ken Brown's lemma 2.2.35, we can factor u between fibrant objects as qj with q a fibration and j an acyclic cofibration such that there is an acyclic fibration q' satisfying $q'j = \mathrm{id}_A$. Then ρ can be rewritten as $\tilde{\rho}\chi$ where $\tilde{\rho}$ is cartesian above q and χ is cartesian above j. Because $q'j = \mathrm{id}_A$, I can find a retraction ρ' of χ such that ρ' is cartesian above q'. Being a cartesian map above an acyclic fibration ρ' is an acyclic fibration in $\mathcal{E}$, so it is a weak equivalence. Hence so is its section χ. Now choose a filler k for the commutative square of (4.42) in which $p(\iota_1)$ is an acyclic cofibration and q a fibration.

$$\begin{tikzcd} C \arrow[r, "jv"] \arrow[d, "p(\iota_1)"'] & A' \arrow[d, "q"] \\ pC_Z \arrow[r, "h"'] \arrow[ur, dashed, "k"] & B \end{tikzcd} \tag{4.42}$$

Recall now that fillers in $\mathcal{E}$ can be chosen above given fillers in $\mathcal{B}$, so that there is $\kappa, \kappa' : C_Z \longrightarrow X'$ where X' is the domain of $\tilde{\rho}$ such that commute both diagrams of (4.43):

$$\begin{tikzcd} Z \arrow[r, "\chi\upsilon"] \arrow[d, "\iota_1"'] & X' \arrow[d, "\tilde{\rho}"] \\ C_Z \arrow[r, "\eta"'] \arrow[ur, dashed, "\kappa"] & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "\chi\upsilon'"] \arrow[d, "\iota_1"'] & X' \arrow[d, "\tilde{\rho}"] \\ C_Z \arrow[r, "\eta'"'] \arrow[ur, dashed, "\kappa'"] & Y \end{tikzcd} \tag{4.43}$$

Now both $\kappa\iota_0$ and $\kappa'\iota_0$ are solution to the universal problem of finding ϑ above $kp(\iota_0)$ such that $\tilde{\rho}\vartheta = \varphi$. So by the cartesian essence of $\tilde{\rho}$, they must be equal. As ι_0 is an acyclic fibration, we deduce that κ and κ' are (right) homotopic. Composing with ι_1 yields that $\kappa\iota_1 = \chi\upsilon$ and $\kappa'\iota_1 = \chi\upsilon'$ are (right) homotopic. But χ is a weak equivalence and Z is cofibrant while X and X' are fibrant, so that it implies υ is (left) homotopic to υ'. □

4.2.3 Homotopy categories of fibers and fibers of the homotopy category

A Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ always respects weak equivalences. So, by proposition 2.2.42, it induces a functor $\mathbf{Ho}\,(p)$ that fills the following commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] & \mathbf{Ho}\,(\mathcal{E}) \arrow[d, "\mathbf{Ho}(p)"] \\ \mathcal{B} \arrow[r] & \mathbf{Ho}\,(\mathcal{B}) \end{tikzcd} \tag{4.44}$$

We might sometimes refer to $\mathbf{Ho}\,(p)$ as the *homotopy quotient* of p. In the same way we were able to reconstruct the model structure on $\mathcal{E}$ from the sole informations of the model structures on $\mathcal{B}$ and the fibers $\mathcal{E}_A$, it seems quite natural to investigate the possibility to reconstruct $\mathbf{Ho}\,(\mathcal{E})$ solely from the homotopical information of $\mathcal{B}$ and the fibers $\mathcal{E}_A$.

More formally, consider the pseudo functor $\mathbf{D}\,\tilde{p} : \mathcal{B} \longrightarrow \mathsf{Adj}$ defined by mapping each object A to $\mathbf{Ho}\,(\mathcal{E}_A)$ and each morphism u to the derived adjunction $(\mathbf{L}\,u_!, \mathbf{R}\,u^*)$. This is indeed a pseudo functor, because it is the composition of the pseudo functor $\tilde{p} : \mathcal{B} \longrightarrow \mathsf{Mod}$ whose p is the Grothendieck construction with the pseudo 2-functor $\mathbf{D} : \mathsf{Mod} \longrightarrow \mathsf{Adj}$ of theorem 4.1.4. What is the link between the two constructions $\mathbf{Ho}\,(p)$ and $\mathbf{D}\,p$? A first guess would be that $\mathbf{Ho}\,(p)$ can be recovered as the Grothendieck construction of a pseudo functor $P : \mathbf{Ho}\,(\mathcal{B}) \longrightarrow \mathsf{Adj}$ fitting in a (pseudo) commutative square as follow:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\tilde{p}"] \arrow[d, "\gamma"'] & \mathsf{Mod} \arrow[d, "\mathbf{D}"] \\ \mathbf{Ho}\,(\mathcal{B}) \arrow[r, "P"'] & \mathsf{Adj} \end{tikzcd} \tag{4.45}$$

This is hopeless for two reasons: first, this would require that $(u_!, u^*)$ is a Quillen equivalence whenever u is a weak equivalence of $\mathcal{B}$, which can fail as shown in counterexample 4.2.11; secondly, even adding such an assumption (called *relativeness* in [HP15]) on p do not fix the issue, as it is shown in counterexample 4.2.12 that fiber maps in some $\mathcal{E}_A$ can be homotopic in the total category $\mathcal{E}$ without being homotopic in $\mathcal{E}_A$, preventing $\mathbf{Ho}\,(\mathcal{E}_A)$ to be the fiber of $\mathbf{Ho}\,(p)$ over A. However it holds that $\mathbf{Ho}\,(p)_A \hookrightarrow \mathbf{Ho}\,(\mathcal{E}_A)$ as we shall see later, so one could expect a 2-cell $P\gamma \longrightarrow \mathbf{D}\,\tilde{p}$ in the square (4.45) instead of mere commutativity. But in fact $\mathbf{Ho}\,(p)$ can not be a Grothendieck bifibration in full generality as shown by counterexample 4.2.13.

Counterexample(s) 4.2.11. Consider the codomain bifibration $\mathrm{cod} : \mathrm{Fun}\,(\mathbf{2}, \mathcal{M}) \longrightarrow \mathcal{M}$ on a model category $\mathcal{M}$. Recall that $u_!$ is obtained by postcomposing with u, whereas u^* is computed as a pullback. Endow $\mathrm{Fun}\,(\mathbf{2}, \mathcal{M})$ with the injective model structure makes cod a Quillen bifibration. But asking for

$$u_! : \mathcal{M}/A \rightleftarrows \mathcal{M}/B : u^* \tag{4.46}$$

to be a Quillen equivalence whenever $u : A \longrightarrow B$ is a weak equivalence of $\mathcal{M}$ is exactly asking $\mathcal{M}$ to be right proper (see [Rez02, Proposition 2.7]).

Counterexample(s) 4.2.12. Consider Top with its classical model structures where the weak equivalences are the weak homotopy equivalences, the fibrations are Serre's fibrations and the cofibrations are generated by the boundary inclusions $S^{n-1} \longrightarrow D^n$, $n \geq 1$. It is right proper as every object is fibrant. Still, we show that there exists continuous maps $f, g : X \longrightarrow Y$ in Top/A between bifibrant objects that can be homotopic in $\mathrm{Fun}\,(\mathbf{2}, \mathsf{Top})$ but not in the fiber Top/A. In particular, it prevent the canonical map $\mathbf{Ho}\,(\mathsf{Top}/A) \longrightarrow \mathbf{Ho}\,(\mathrm{Fun}\,(\mathbf{2}, \mathsf{Top}))_A$ to be an isomorphism.

Take A to be the circle $S^1 \subseteq \mathbb{C}$. Specify the following object of Top/S^1

$$\exp : \mathbb{R} \longrightarrow S^1,\ x \longmapsto e^{ix} \tag{4.47}$$

As $\mathbb{R}$ is a CW-complex, it is a cofibrant topological space. Hence exp is cofibrant in Top/A. Moreover, exp is a covering map hence a Serre fibration, so exp is also fibrant in Top/A. Now both $f : x \longmapsto x$ and $g : x \longmapsto x + 2\pi$ are fiber maps from exp to exp. And there is an obvious (topological) homotopy from f to g

$$h : I \times \mathbb{R} \longrightarrow \mathbb{R},\ (t, x) \longmapsto x + 2t\pi \tag{4.48}$$

The spaces $I \times \mathbb{R}$ and $I \times \mathbb{S}^1$ being cylinder objects for $\mathbb{R}$ and $\mathbb{S}^1$ in the model category Top, the map $I \times \exp : I \times \mathbb{R} \longrightarrow I \times S^1$ is a cylinder for exp in $\mathrm{Fun}(\mathbf{2}, \mathsf{Top})$. Then h gives rise to a (left) homotopy in $\mathrm{Fun}(\mathbf{2}, \mathsf{Top})$ from (f, id_{S^1}) to (g, id_{S^1}):

$$\begin{tikzcd}
\mathbb{R} \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[d, "\exp"'] & I \times \mathbb{R} \arrow[r, "h"] \arrow[d, "I\times\exp"] & \mathbb{R} \arrow[d, "\exp"] \\
S^1 \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"'] & I \times S^1 \arrow[r, "k"] & S^1
\end{tikzcd} \qquad \text{where } k : (t, z) \mapsto z e^{2it\pi} \tag{4.49}$$

Now, for f and g to be homotopic in the fiber Top/A, and because exp is bifibrant in this fiber, there should be a left homotopy through any good cylinder, in particular $q \circ (I \times \exp) : I \times \mathbb{R} \longrightarrow S^1$ where $q : I \times S^1 \longrightarrow S^1$ is the canonical projection. Suppose there is such an homotopy:

$$\begin{tikzcd}
I \times \mathbb{R} \arrow[r, "\ell"] \arrow[d, "I\times\exp"'] & \mathbb{R} \arrow[d, "\exp"] \\
I \times S^1 \arrow[r, "q"'] & S^1
\end{tikzcd} \tag{4.50}$$

Then $\tilde{\ell} = \ell(-, 0) : I \longrightarrow \mathbb{R}$ still is continuous. As $\tilde{\ell}(0) = f(0) = 0$ and $\tilde{\ell}(1) = g(0) = 2\pi$, one can invoke the intermediate value theorem to get $t_0 \in I$ such that $\tilde{\ell} = \pi$. Here comes the contradiction as in one hand $\exp(\tilde{\ell}(t_0)) = e^{i\pi} = -1$, and in the other $\exp(\tilde{\ell}(t_0)) = (\exp \circ \ell)(t_0, 0) = q(t_0, e^{i0}) = 1$.

Counterexample(s) 4.2.13. Let $L : \mathcal{E}_0 \rightleftarrows \mathcal{E}_1 : R$ be an adjunction with R conservative. This is typically the case when $\mathcal{E}_1$ is a reflexive subcategory of $\mathcal{E}_0$ and we shall suppose that this is the case here. Suppose also that both $\mathcal{E}_0$ and $\mathcal{E}_1$ have finite products and coproducts and equip them the trivial model structure. Then the associated Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathbf{2}$ is a Quillen bifibration when $\mathbf{2}$ has the model structure where $\varepsilon : 0 \longrightarrow 1$ is both a weak equivalence and a fibration but not a cofibration. So $\mathbf{Ho}(\mathbf{2})$ is the walking isomorphism. Hence, if $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \longrightarrow \mathbf{Ho}(\mathbf{2})$ were a Grothendieck bifibration, it would give an equivalence of categories between $\mathbf{Ho}(\mathcal{E})_0$ and $\mathbf{Ho}(\mathcal{E})_1$. However, $\mathbf{Ho}(\mathcal{E})_0 \cong \mathcal{E}_0$ and $\mathbf{Ho}(\mathcal{E})_1 \cong \mathcal{E}_1$. Indeed:

- First remark that every object of $\mathcal{E}_0$ is bifibrant in $\mathcal{E}$, and every object of $\mathcal{E}_1$ is fibrant in $\mathcal{E}$. Hence the bifibrant replacement of $X \in \mathcal{E}_0$ is X itself and the bifibrant replacement of $X \in \mathcal{E}_1$ is RX.

- Next note that maps $f, g : X \longrightarrow Y$ between objects of $\mathcal{E}_0$ are homotopic in $\mathcal{E}$ only when they are equal. This follows from the fact there is no maps from objects of $\mathcal{E}_1$ to objects of $\mathcal{E}_0$, and so a cylinder in $\mathcal{E}$ for an object $X \in \mathcal{E}_0$ is necessarily a cylinder for X in the trivial model category $\mathcal{E}_0$.

- Then for $X, Y \in \mathcal{E}_0$, it holds that

$$\mathbf{Ho}(\mathcal{E})(X, Y) = \pi(X, Y) \cong \mathcal{E}_0(X, Y) \tag{4.51}$$

- And for $X, Y \in \mathcal{E}_1$, it holds that

$$\mathbf{Ho}(\mathcal{E})(X, Y) = \pi(RX, RY) \cong \mathcal{E}_0(RX, RY) \cong \mathcal{E}_1(X, Y) \tag{4.52}$$

In fact, $\mathbf{Ho}(\mathcal{E})$ only differs from $\mathcal{E}$ in that $\mathbf{Ho}(\mathcal{E})(X, Y)$ can be non empty for $X \in \mathcal{E}_1$ and $Y \in \mathcal{E}_0$. This shows that $\mathbf{Ho}(p)$ is a bifibration only if there is an equivalence of categories between $\mathcal{E}_0$ and $\mathcal{E}_1$. However, not every reflexive inclusion is an equivalence: for example, take $\mathcal{E}_0$ to be Set and $\mathcal{E}_1$ to be the terminal category , and $R : 1 \longrightarrow \mathsf{Set}$ selects a terminal object of Set.

The relationship between $\mathbf{D}\,p$ and $\mathbf{Ho}(p)$ is subtler and is explained in every details in section 4.4. The study of this relationship in a formal setting requires a careful account of the situation where the basis $\mathcal{B}$ of the bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ comes with the trivial model structure, whose weak equivalences are reduced to isomorphisms. The next section is devoted to these particular cases.

4.3 Quillen bifibrations over a trivial basis

Counterexample 4.2.12 relies on the fact that there are non trivial homotopies in the basis category. This leads us to the study of Quillen bifibrations whose basis has a trivial model structure. By *trivial* model structure is meant the only model structure in which the weak equivalences are nothing but the isomorphisms. It follows from the lifting properties that all maps are both fibrations and cofibrations.

Consider a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ respecting (Q) such that $\mathcal{B}$ has a trivial model structure. Then (hCon) and (hBC) are vacuously true, hence p is a Quillen bifibration. Detailing the total cofibrations, fibrations and weak equivalences in that case, we find that in $\mathcal{E}$: the fibrations are those f such that $f^{\triangleleft}$ is a fibration; the cofibrations are those f such that $f_{\triangleright}$ is a cofibration; the weak equivalences are those f such that $p(f)$ is an isomorphism and $f^{\triangleleft}$ is a weak equivalence (equivalently $f_{\triangleright}$ is a weak equivalence).

Lemma 4.3.1. *Two maps $f, g : X \longrightarrow Y$ in $\mathcal{E}$ are left homotopic if and only if they have the same image $u : A \longrightarrow B$ in $\mathcal{B}$ by p and $f^{\triangleleft}, g^{\triangleleft}$ are left homotopic in the fiber $\mathcal{E}_A$.*

Proof. One direction is easy and does not rely on the hypothesis that $\mathcal{B}$ is trivial. Suppose $f^{\triangleleft}$ and $g^{\triangleleft}$ are left homotopic in the fiber $\mathcal{E}_A$. Then there is an homotopy $h : C_X \longrightarrow u^*Y$ through a cylinder C_X for X in $\mathcal{E}_A$: it comes with the weak equivalence $q : C_X \longrightarrow X$ and its two sections $j_0, j_1 : X \longrightarrow C_X$, such that $hj_0 = f^{\triangleleft}$ and $hj_1 = g^{\triangleleft}$. Observe that C_X is still a cylinder for X in $\mathcal{E}$: indeed $q : C_X \longrightarrow X$ is still a weak equivalence in $\mathcal{E}$ with two sections j_0, j_1. Then $\rho_{u,Y}h$ is an homotopy from $f = \rho_{u,Y}f^{\triangleleft}$ to $g = \rho_{u,Y}g^{\triangleleft}$.

Conversely, suppose f and g are left homotopic in $\mathcal{E}$. There is a cylinder C for X in $\mathcal{E}$, that comes with a weak equivalence $q : C \longrightarrow X$ and two sections $j_0, j_1 : X \longrightarrow C$; and there is $h : C \longrightarrow Y$ such that $hj_0 = f$ and $hj_1 = g$. In particular, $\phi = p(q)$ is a weak equivalence in $\mathcal{B}$, thus a isomorphism with inverse $p(j_0) = \phi^{-1} = p(j_1)$. If we write $\eta = p(h)$, then the images by p of the equations $hj_0 = f$ and $hj_1 = g$ give

$$p(f) = \eta\phi^{-1} = p(g). \tag{4.53}$$

Let us write u for this arrow $p(f) = p(g)$. We end up with the following commutative diagram in $\mathcal{E}_A$:

$$
\begin{tikzcd}
X \arrow[d, shift right, "{j_0}^{\triangleleft}"'] \arrow[d, shift left, "{j_1}^{\triangleleft}"] \arrow[dr, shift left, "j_1"] \arrow[dr, shift right, "j_0"'] \arrow[ddd, bend right=60, "f^{\triangleleft}"'] \arrow[ddd, bend left=60, "g^{\triangleleft}"] & & \\
(\phi^{-1})^* C \arrow[r, "\rho"] \arrow[d, "(\phi^{-1})^*(h^{\triangleleft})"] & C \arrow[d, "h^{\triangleleft}"] \arrow[ddr, "h"] & \\
(\phi^{-1})^* \eta^* Y \arrow[r, "\rho"] \arrow[d] & \eta^* Y \arrow[dr, "\rho"] & \\
u^* Y \arrow[rr, "\rho"] & & Y
\end{tikzcd}
\tag{4.54}
$$

We are left to prove that $(\phi^{-1})^* C$ is a cylinder for X in $\mathcal{E}_A$. We already have good candidate for the canonical injections, namely ${j_0}^{\triangleleft}$ and ${j_1}^{\triangleleft}$. Thus we only need to find a shared retraction $\tilde{q} : (\phi^{-1})^* C \longrightarrow X$ in $\mathcal{E}_A$ which is a weak equivalence. Because ϕ is an isomorphism in $\mathcal{B}$, there is an isomorphism $\alpha : (\phi^{-1})^* C \longrightarrow \phi_! C$ such that the following triangle commutes:

$$
\begin{tikzcd}
(\phi^{-1})^* C \arrow[r, "\rho"] \arrow[d, "\alpha"'] & C \arrow[dl, "\lambda"] \\
\phi_! C &
\end{tikzcd}
\tag{4.55}
$$

Then define $\tilde{q}$ as the composite $q^{\triangleleft}\alpha$. First, this is a morphism of the fiber $\mathcal{E}_A$. Next, it is a weak equivalence: q is a weak equivalence and so is $q^{\triangleleft}$ by the description above of the model structure on $\mathcal{E}$; and α is an isomorphism so in particular a weak equivalence. Finally, $\tilde{q}$ is readily a retraction for ${j_0}^{\triangleleft}$ and ${j_1}^{\triangleleft}$. It makes $(\phi^{-1})^* C$ into a cylinder for X in $\mathcal{E}_A$ and exhibits the composite vertical map $(\phi^{-1})^* C \longrightarrow u^* Y$ in diagram (4.54) as a left homotopy between $f^{\triangleleft}$ and $g^{\triangleleft}$. □

We get a dual statement by dualizing the previous proof.

Lemma 4.3.2. *Two maps $f, g : X \longrightarrow Y$ in $\mathcal{E}$ are right homotopic if and only if they have the same image $u : A \longrightarrow B$ in $\mathcal{B}$ by p and $f_{\triangleright}, g_{\triangleright}$ are right homotopic in the fiber $\mathcal{E}_B$.*

The archetypal example to keep in mind of a Quillen bifibration with trivial model structure on the basis is that of a Quillen adjunction $\ell : \mathcal{M}_0 \rightleftarrows \mathcal{M}_1 : r$. From such a adjunction is crafted a Grothendieck bifibration $\mathcal{M} \longrightarrow \mathbf{2}$, usually called the *collage*, where $\mathbf{2}$ is the category with two objects, say 0 and 1, and a unique non-identity morphism $0 \longrightarrow 1$, and where the push-pull adjunction over this morphism is precisely (ℓ, r). Considering $\mathbf{2}$ as a trivial model category makes this functor $\mathcal{M} \longrightarrow \mathbf{2}$ a Quillen bifibration, giving the Grothendieck construction $\mathcal{M}$ a model structure in which the weak equivalences are those of $\mathcal{M}_0$ and $\mathcal{M}_1$. What is striking there is that the quotient functor $\mathbf{Ho}(\mathcal{M}) \longrightarrow \mathbf{Ho}(\mathbf{2}) \simeq \mathbf{2}$ is exactly the collage of the derived adjunction $\mathbf{L}\ell : \mathbf{Ho}(\mathcal{M}_0) \rightleftarrows \mathbf{Ho}(\mathcal{M}_1) : \mathbf{R}r$. The goal of the next proposition is to show that this is not specific to the category $\mathbf{2}$, but actually happens for any Quillen bifibration with trivial basis.

Proposition 4.3.3. *Let $p : \mathcal{E} \to \mathcal{B}$ be a Quillen bifibration where $\mathcal{B}$ bears the trivial model structure. Then* $\mathbf{Ho}(p)$ *is isomorphic to the Grothendieck construction of* $\mathbf{D}\, p$.

Proof. Because $\mathcal{B}$ has trivial model structure, every object in $\mathcal{B}$ is bifibrant and every morphism f is the only element of its homotopy class $[f]$: this gives an identity-on-objects isomorphism $\mathbf{Ho}(\mathcal{B}) \simeq \mathcal{B}$ so we shall just use $\mathcal{B}$ in place of $\mathbf{Ho}(\mathcal{B})$. Let us now describe $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \to \mathcal{B}$. Recall in this regard that we chose the bifibrant replacement X^{cf} of an object X in $\mathcal{E}$ above A to be as follow: consider the chosen replacement of A in $\mathcal{B}$

$$A \xleftarrow{q_A} A^{\mathrm{c}} \xrightarrow{j_A} A^{\mathrm{cf}} \tag{4.56}$$

the replacement X^{cf} is chosen to be the fibrant replacement in $\mathcal{E}_{A^{\mathrm{cf}}}$ of $j_{A!}((q_A{}^*X)^{\mathrm{c}})$ where $(q_A{}^*X)^{\mathrm{c}}$ is the cofibrant replacement in $\mathcal{E}_{A^{\mathrm{c}}}$ of $q_A{}^*X$. Here A is bifibrant in $\mathcal{B}$, so that $A = A^{\mathrm{c}} = A^{\mathrm{cf}}$ and $q_A = j_A = \mathrm{id}_A$. So in particular, the replacement X^{cf} in $\mathcal{E}$ coincides with the chosen bifibrant replacement of X in $\mathcal{E}_A$. More precisely, in our situation, the bifibrant replacement of X in $\mathcal{E}$ and the associated morphisms

$$X \longleftarrow X^{\mathrm{c}} \longrightarrow X^{\mathrm{cf}} \tag{4.57}$$

actually lives in the fiber $\mathcal{E}_A$. In particular, we can safely use X^{cf} to means either the replacement of X inside $\mathcal{E}$ or inside $\mathcal{E}_A$. The functor $\mathbf{Ho}(p)$ maps an object X of $\mathcal{E}$ to pX in $\mathcal{B}$, and a map $[f] : X \to Y$ of $\mathbf{Ho}(\mathcal{E})$, represented by $f : X^{\mathrm{cf}} \to Y^{\mathrm{cf}}$, is mapped to $p(f)$ in $\mathcal{B}$ (this is correctly defined by what precedes). In particular, the fiber of $\mathbf{Ho}(p)$ at an object A is the category whose objects are those X with $pX = A$ and whose morphisms $X \to Y$ are the homotopy classes, relatively to $\mathcal{E}$, of those $f : X^{\mathrm{cf}} \to Y^{\mathrm{cf}}$ such that $p(f) = \mathrm{id}_A$. To conclude that

$$\mathbf{Ho}(\mathcal{E})_A \cong \mathbf{Ho}(\mathcal{E}_A) \tag{4.58}$$

all it remains to show is that two maps $f, g : X^{\mathrm{cf}} \to Y^{\mathrm{cf}}$ in the fiber $\mathcal{E}_A$ are homotopic in $\mathcal{E}$ precisely when they are in $\mathcal{E}_A$: lemma 4.3.1 ensures that two such maps are left homotopic if and only if $f^{\lhd}$ and $g^{\lhd}$ are homotopic in $\mathcal{E}_A$; as f and g are fiber-maps, we have that $f^{\lhd} = f$ and $g^{\lhd} = g$, so we can conclude.

As for now, we have just showed that the fiber of $\mathbf{Ho}(p)$ at A is indeed $\mathbf{D}\, p(A)$. It remains to show that for any $u : A \to B$, $X \in \mathcal{E}_A$ and $Y \in \mathcal{E}_B$, there is respectively cocartesian and cartesian morphisms in $\mathbf{Ho}(\mathcal{E})$

$$\lambda : X \to \mathbf{L}\, u_! X \qquad \rho : \mathbf{R}\, u^* Y \to Y \tag{4.59}$$

such that for all fiber maps $[f] : X \to X' \in \mathbf{Ho}(\mathcal{E}_A)$ and $[g] : Y \to Y' \in \mathbf{Ho}(\mathcal{E}_B)$ the following commute in $\mathbf{Ho}(\mathcal{E})$:

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "{[f]}"] & \mathbf{L}\, u_! X \arrow[d, "{\mathbf{L}\, u_!([f])}"] \\ X' \arrow[r, "\lambda"] & \mathbf{L}\, u_! X' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{R}\, u^* Y \arrow[r, "\rho"] \arrow[d, "{\mathbf{R}\, u^*([g])}"] & Y \arrow[d, "{[g]}"] \\ \mathbf{R}\, u^* Y' \arrow[r, "\rho"] & Y' \end{tikzcd} \tag{4.60}$$

Recall that $\mathbf{L}\, u_! X = u_!(X^{\mathrm{c}})$ and $\mathbf{R}\, u^* Y = u^*(Y^{\mathrm{f}})$. Moreover X^{c} being cofibrant, so is $u_!(X^{\mathrm{c}})$; in the same way, Y^{f} being fibrant, so is $u^*(Y^{\mathrm{f}})$. We now claim that λ can be chosen as follow: because $u_!(X^c)^f$ is fibrant, there is a filler $\tilde{\lambda} : X^{cf} \longrightarrow u_!(X^c)^f$ in the following diagram

$$\begin{tikzcd} X^{cf} \arrow[r, "\tilde{\lambda}"] & u_!(X^c)^f \\ X^c \arrow[u, "j_X"] \arrow[r, "\lambda"] & u_!(X^c) \arrow[u, "j_{u_!(X^c)}"'] \end{tikzcd} \tag{4.61}$$

where vertical morphisms are fibrant replacements, hence in particular acyclic cofibrations. Remark that any other map filling the square is homotopic to $\tilde{\lambda}$, so $\lambda = [\lambda]$ in $\mathbf{Ho}(\mathcal{E})$ is well-defined. Let us now show that the map λ is cocartesian for $\mathbf{Ho}(p)$: suppose given a map $[f] : X \longrightarrow Z$ in $\mathbf{Ho}(\mathcal{E})$ together with a map $v : B \longrightarrow pZ$ in $\mathcal{B}$ such that $vu = p(f)$. We shall find a unique $[g] : u_!(X^c) \longrightarrow Z$ above v such that $[g]\,\lambda = [f]$. Because $\lambda : X^c \longrightarrow u_!(X^c)$ is cocartesian in $\mathcal{E}$ above u, there is $\tilde{g} : u_!(X^c) \longrightarrow Z^{cf}$ that make the following commute:

$$\begin{tikzcd} & & Z^{cf} \\ X^{cf} \arrow[urr, bend left, "f"] & & \\ X^c \arrow[u] \arrow[r, "\lambda"] & u_!(X^c) \arrow[uur, "\tilde{g}"'] & \end{tikzcd} \tag{4.62}$$

Because Z is fibrant, $\tilde{g}$ can be lifted along the fibrant replacement of $u_!(X^c)$ to give $g : u_!(X^c)^f \longrightarrow Z^{cf}$ as in the diagram:

$$\begin{tikzcd} & & Z^{cf} \\ X^{cf} \arrow[urr, bend left, "f"] & u_!(X^c)^f \arrow[ur, "g"] & \\ X^c \arrow[u, "j"] \arrow[r, "\lambda"] & u_!(X^c) \arrow[u, "j"] \arrow[uur, "\tilde{g}"'] & \end{tikzcd} \tag{4.63}$$

This homotopy class of g is the one we are looking for: indeed

$$g\tilde{\lambda}\mathrm{j}_{X^c} = g\mathrm{j}_{u_!(X^c)}\lambda = \tilde{g}\lambda = f\mathrm{j}_{X^c} \tag{4.64}$$

Hence $g\tilde{\lambda}$ and f are equal when precomposed by the acyclic cofibration j_{X^c}, so they are (right) homotopic.

It remains to show that such a $[g]$ is unique. Suppose $g_1, g_2 : u_!(X^c)^f \longrightarrow Z^{cf}$ both satisfy $p(g_1) = v = p(g_2)$ and $\left[g_1\tilde{\lambda}\right] = [f] = \left[g_2\tilde{\lambda}\right]$. Because the diagonal $B \longrightarrow B \times B = B \times B$ is a good path object in $\mathcal{B}$ for B, lemma 4.2.4 ensures there is a good path object $Z^{cf} \longrightarrow P_Z \longrightarrow Z^{cf} \times Z^{cf}$ in $\mathcal{E}$ for Z^{cf} such that the fibration $q : P_Z \longrightarrow Z^{cf} \times Z^{cf}$ is a fiber-map. Because $g_1\tilde{\lambda} \sim g_2\tilde{\lambda}$, also $g_1\tilde{\lambda}\mathrm{j}_{X^c} \sim g_2\tilde{\lambda}\mathrm{j}_{X^c}$, so there is a right homotopy h as in:

$$\begin{tikzcd} & & & P_Z \arrow[d, "q"] \\ X^c \arrow[urrr, bend left, "h"] \arrow[r, "\lambda"'] & u_!(X^c) \arrow[r, "\mathrm{j}"'] & u_!(X^c)^f \arrow[r, "\langle g_1, g_2 \rangle"'] & Z^{cf} \times Z^{cf} \end{tikzcd} \tag{4.65}$$

Recall that λ is cocartesian, so that there is $h' : u_!(X^c) \longrightarrow P_Z$ above $\langle v, v\rangle$ such that $h'\lambda h$. Remark now that qh' and $\langle g_1, g_2\rangle \mathrm{j}_{u_!(X^c)}$ are solution to the problem of finding ϕ above $\langle v, v\rangle$ such that $\phi\lambda = h$, so that they are equal. In particular h' is a right homotopy from $g_1 \mathrm{j}_{u_!(X^c)}$ to $g_2 \mathrm{j}_{u_!(X^c)}$, and $\mathrm{j}_{u_!(X^c)}$ being an acyclic cofibration it means that g_1 and g_2 are (right) homotopic. So we get $[g_1] = [g_2]$ as wanted.

The construction of ρ is almost dual. Build $\tilde{\rho}$ as a filler in the diagram:

$$\begin{tikzcd} u^*(Y^f)^c \arrow[r, "\tilde{\rho}"] \arrow[d, "\mathrm{q}"] & (Y^f)^c \arrow[d, "\mathrm{q}"] \\ u^*(Y^f) \arrow[r, "\rho"] & Y^f \end{tikzcd} \tag{4.66}$$

Such a filler exists because $u^*(Y^f)^c$ is cofibrant and q_{Y^f} is an acyclic fibration. The cartesian nature of $[\tilde{\rho}]$ is proven in the same fashion as the cocartesian nature of λ. Hence, the composite arrow $\rho = [w]^{-1}[\tilde{\rho}]$ is also cartesian where w is any weak equivalence w in $\mathcal{E}_B$ making the following diagram commute:

$$\begin{tikzcd} Y^{cf} \arrow[rr, "w"] & & (Y^f)^c \arrow[d, "\mathrm{q}"] \\ Y^c \arrow[u, "\mathrm{j}"] \arrow[r, "\mathrm{q}"] & Y \arrow[r, "\mathrm{j}"] & Y^f \end{tikzcd} \tag{4.67}$$

□

This closes our study of the Quillen bifibrations with trivial basis. It will be used in the following sections.

4.4 Iterated localizations

We carry on the discussion started in section 4.2.3, and study the general case of a Quillen bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$. We show below that we can compute $\mathbf{Ho}(p)$ from $\mathbf{D}\,p$ in two steps. Here is the plan of what comes next:

(1) First, write $\mathcal{B}_{\mathrm{triv}}$ for the category $\mathcal{B}$ equipped with the trivial model structure. The Grothendieck bifibration p still satisfies the hypothesis of theorem 3.4.2 when $\mathcal{B}$ is replaced by $\mathcal{B}_{\mathrm{triv}}$. Hence the total category $\mathcal{E}$ inherits a new model structure $\mathcal{E}_{\mathrm{fw}}$, called the *fiberwise model structure*. We write $p_{\mathrm{fw}} : \mathcal{E}_{\mathrm{fw}} \longrightarrow \mathcal{B}_{\mathrm{triv}}$ to distinguish it from the original $p : \mathcal{E} \longrightarrow \mathcal{B}$, although both Quillen bifibrations have the same underlying categories and functor.

(2) Then we show that the quotient $\mathbf{Ho}(p_{\mathrm{fw}})$ coincide with the Grothendieck construction of $\mathbf{D}\,p : \mathcal{B} \longrightarrow \mathsf{Adj}$. In particular it defines a Grothendieck bifibration

$$\mathbf{Ho}(p_{\mathrm{fw}}) : \mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}) \longrightarrow \mathbf{Ho}(\mathcal{B}_{\mathrm{triv}}) \simeq \mathcal{B} \tag{4.68}$$

(3) Lastly, by using the main theorem 3.4.2, we show that the resulting bifibration $\mathbf{Ho}(p_{\mathrm{fw}})$ is a Quillen bifibration on the model category $\mathcal{B}$ where all fibers $(\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}))_A$ are equipped with the trivial model structure. Taking the quotient $\mathbf{Ho}(\mathbf{Ho}(p_{\mathrm{fw}}))$ relatively to the induced model category $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$, we get back the original quotient $\mathbf{Ho}(p)$.

Informally speaking, we decompose the quotient $\mathbf{Ho}(p)$ into a *vertical* (or *fiberwise*) quotient followed by an *horizontal* (or *basis driven*) quotient. The vertical quotient $\mathbf{Ho}(p_{\mathrm{fw}})$ localizes each fiber independently while the horizontal quotient acknowledges the homotopies in the basis.

From now on, we suppose that $p : \mathcal{E} \to \mathcal{B}$ is a Quillen bifibration. In particular, hypothesis (Q) is verified: that is the adjunctions $(u_!, u^*)$ are Quillen adjunctions. But this fact is independent of the model structure on the basis: so we can take the trivial model structure $\mathcal{B}_{\mathrm{triv}}$ instead of $\mathcal{B}$ and hypothesis (Q) is still satisfied. By the previous section, we can apply theorem 3.4.2 and we get a new model structure on $\mathcal{E}$, which we will call the *fiberwise model structure* and write $\mathcal{E}_{\mathrm{fw}}$. To emphasize the distinction from the initial Quillen bifibration p, we shall write $p_{\mathrm{fw}} : \mathcal{E}_{\mathrm{fw}} \to \mathcal{B}_{\mathrm{triv}}$ for the newly obtained Quillen bifibration. Of course, p and p_{fw} are the same as functors, but they are very different in regards to the homotopy informations they contain.

By proposition 4.3.3, $\mathbf{Ho}(p_{\mathrm{fw}}) : \mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}) \to \mathbf{Ho}(\mathcal{B}_{\mathrm{triv}})$ is the Grothendieck construction of $\mathbf{D}(p_{\mathrm{fw}}) = \mathbf{D}\,p$. We know that the category $\mathbf{Ho}(\mathcal{B}_{\mathrm{triv}})$ is isomorphic to $\mathcal{B}$. So $\mathbf{Ho}(p_{\mathrm{fw}})$ can be consider as a Grothendieck bifibration $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}) \to \mathcal{B}$. Now recall that $\mathcal{B}$ is a model category. Moreover one can equip each fiber $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})_A$, which is isomorphic to $\mathbf{Ho}(\mathcal{E}_A)$, with the trivial model structure.

Proposition 4.4.1. *The Grothendieck bifibration* $\mathbf{Ho}(p_{\mathrm{fw}}) : \mathbf{Ho}(\mathcal{E}_{\mathrm{fw}}) \to \mathcal{B}$ *is a Quillen bifibration.*

Proof. The fibers being trivial model categories, the adjunctions $(u_!, u^*)$ are vacuously Quillen adjunctions. Hence, by theorem 3.4.2, we only need to show that (hBC) and (hCon) are satisfied.

Begin with (hBC). Take a commutative square of $\mathcal{B}$

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"'] & B \end{tikzcd} \tag{4.69}$$

Recall from (the proof of) proposition 4.3.3 that cocartesian and cartesian morphisms of $\mathbf{Ho}(p_{\mathrm{fw}})$ above $u : A \to B$ are constructed from those of p_{fw} as the homotopy classes of the top morphisms in the following commutative squares of $\mathcal{E}_{\mathrm{fw}}$:

$$\begin{tikzcd} X^{\mathrm{cf}} \arrow[r, "\tilde{\lambda}"] & u_!(X^{\mathrm{c}})^{\mathrm{f}} \\ X^{\mathrm{c}} \arrow[u, "\mathrm{j}"] \arrow[r, "\lambda"] & u_!(X^{\mathrm{c}}) \arrow[u, "\mathrm{j}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} u^*(Y^{\mathrm{f}})^{\mathrm{c}} \arrow[r, "\tilde{\rho}"] \arrow[d, "\mathrm{q}"] & (Y^{\mathrm{f}})^{\mathrm{c}} \arrow[d, "\mathrm{q}"] \\ u^*(Y^{\mathrm{f}}) \arrow[r, "\rho"] & Y^{\mathrm{f}} \end{tikzcd} \tag{4.70}$$

The mate associated to (4.69) at component $Z \in \mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})_C$ is the morphism $[v_Z] : \mathbf{L}\,u'_!\,\mathbf{R}\,v^*Z \to \mathbf{R}\,v'^*\,\mathbf{L}\,u_!Z$ in $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})_{C'}$ computed in two steps:

- factor the composite arrow $\mathbf{R}\,v^*Z \to Z \to \mathbf{L}\,u_!Z$ through the cocartesian arrow $\mathbf{R}\,v^*Z \to \mathbf{L}\,u'_!\,\mathbf{R}\,v^*Z$, resulting in an arrow $\mathbf{L}\,u'_!\,\mathbf{R}\,v^*Z \to \mathbf{L}\,u_!Z$ above v',
- factor this resulting arrow through the cartesian arrow $\mathbf{R}\,v'^*\,\mathbf{L}\,u_!Z \to \mathbf{L}\,u_!Z$.

We appeal to the proof of proposition 4.3.3 concerning the detail of such factorizations, that end up giving a representative v_Z of the mate as any morphism in $\mathcal{E}_{C'}$ making the following black diagram commute:

$$
\begin{tikzcd}
 & & & v^*(Z^{\mathrm{f}})^{\mathrm{c}} \arrow[rr, "\tilde{\rho}"] \arrow[d, "\mathrm{q}"] \arrow[dddll, bend right, "\lambda"'] & & (Z^{\mathrm{f}})^{\mathrm{c}} \arrow[d, "\mathrm{q}"] \arrow[ddrr, bend left, "w"] & & \\
 & & & v^*(Z^{\mathrm{f}}) \arrow[rr, "\rho"] \arrow[ddl, bend right, "\lambda"'] & & Z^{\mathrm{f}} & & \\
 & & & v^*Z \arrow[u, "v^*(\mathrm{j})"] \arrow[rr, "\rho"] \arrow[d, "\lambda"] & & Z \arrow[u, "\mathrm{j}"] \arrow[dd, "\lambda"] & Z^{\mathrm{c}} \arrow[l, "\mathrm{q}"] \arrow[r, "\mathrm{j}"] \arrow[dd, "\lambda"] & Z^{\mathrm{cf}} \arrow[dd, "\tilde{\lambda}"] \\
u'_!(v^*(Z^{\mathrm{f}})^{\mathrm{c}})^{\mathrm{f}} \arrow[ddddrrrr, dashed, "v_Z"'] & u'_!(v^*(Z^{\mathrm{f}})^{\mathrm{c}}) \arrow[l, "\mathrm{j}"] \arrow[r] \arrow[dddrrr] & u'_!(v^*(Z^{\mathrm{f}})) & u'_!v^*Z \arrow[l] \arrow[dr, "\mu_Z"] & & & & \\
 & & & & v'^*u_!Z \arrow[r, "\rho"] & u_!Z & u_!(Z^{\mathrm{c}}) \arrow[l, "u_!(\mathrm{q})"] \arrow[r, "\mathrm{j}"] & u_!(Z^{\mathrm{c}})^{\mathrm{f}} \\
 & & & & v'^*u_!(Z^{\mathrm{c}}) \arrow[u] \arrow[urr, bend right, "\rho"] \arrow[d] & & & \\
 & & & & v'^*(u_!(Z^{\mathrm{c}})^{\mathrm{f}}) \arrow[uurrr, bend right, "\rho"] & & & \\
 & & & & v'^*(u_!(Z^{\mathrm{c}})^{\mathrm{f}})^{\mathrm{c}} \arrow[u, "\mathrm{q}"] & & &
\end{tikzcd}
\tag{4.71}
$$

where w is a representative for the isomorphism in $\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})$ induced by the zigzag

$$(Z^{\mathrm{f}})^{\mathrm{c}} \xrightarrow{\mathrm{q}} Z^{\mathrm{f}} \xleftarrow{\mathrm{j}} Z \xleftarrow{\mathrm{q}} Z^{\mathrm{c}} \xrightarrow{\mathrm{j}} Z^{\mathrm{cf}} \tag{4.72}$$

In particular w is a weak equivalence. Suppose now that the commutative square (4.69) has $u, u' \in \mathfrak{C} \cap \mathfrak{W}$ and $v, v' \in \mathfrak{F} \cap \mathfrak{W}$. Then the cartesian morphism $\rho_{v', u_!(Z^{\mathrm{c}})^{\mathrm{f}}}$ above the acyclic fibration v' is a weak equivalence, as is the cocartesian morphism $\lambda_{u', v^*(Z^{\mathrm{f}})^{\mathrm{c}}}$ above the acyclic cofibration u'. In the same way the morphism $\tilde{\rho}$ is a weak equivalence because it is the cofibrant replacement of the cartesian morphism $\rho_{v, Z^{\mathrm{f}}}$ over the acyclic fibration v; and the morphism $\tilde{\lambda}$ is a weak equivalence because it is the fibrant replacement of the cocartesian morphism $\lambda_{u, Z^{\mathrm{c}}}$ over the acyclic cofibration u. The cofibrant and fibrant replacement morphisms q and j is the left bottom square are weak equivalences by definition. By the two-out-of-three property, we can conclude that v_Z is a weak equivalence, and so that $[v_Z]$ is an isomorphism. (The gray part of the diagram is not strictly needed, but it shows the construction bits from which the outer diagram is constructed. In particular, we recognize in the middle the pentagon defining the mate μ_Z for the Quillen bifibration p_{fw}, and the bottom part of the diagram show how its image in $\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})$ is isomorphic to $[v_Z]$).

Continue with (hCon). Given an acyclic cofibration $u : A \longrightarrow B$ in $\mathcal{B}$, and a map $[f] : X \longrightarrow Y$ in the fiber $\mathbf{Ho}\,(\mathcal{E}_A)$, its image $\mathbf{L}(u_!)([f])$ is given by the unique arrow making the following commute in $\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "{[f]}"] & \mathbf{L}\,u_!X \arrow[d, "{\mathbf{L}\,u_!([f])}"] \\
Y \arrow[r] & \mathbf{L}\,u_!Y
\end{tikzcd}
\tag{4.73}
$$

where the horizontal morphisms are the chosen cocartesian morphism described in the proof of theorem 4.3.3. Unfolding the definitions, $\mathbf{L}\, u_!([f])$ is the homotopy class of the dotted arrow in the following commutative diagram of $\mathcal{E}$:

$$
\begin{tikzcd}
& Y^{\mathrm{cf}} \arrow[r, "\tilde{\lambda}"] & u_!(Y^{\mathrm{c}})^{\mathrm{f}} \\
X^{\mathrm{cf}} \arrow[ur, "f"] & u_!(X^{\mathrm{c}})^{\mathrm{f}} \arrow[ur, dashed] & \\
X^{\mathrm{c}} \arrow[u, "\mathrm{j}"] \arrow[r, "\lambda"] & u_!(X^{\mathrm{c}}) \arrow[u, "\mathrm{j}"] \arrow[uur, bend right] &
\end{tikzcd}
\tag{4.74}
$$

The cocartesian morphism $\lambda_{u,X^{\mathrm{c}}}$ is above the acyclic cofibration u, hence it is a weak equivalence. For the same reason, $\lambda_{u,Y^{\mathrm{c}}}$ also is, and so its fibrant replacement $\tilde{\lambda}$ is a weak equivalence. Next, the fibrant replacement morphisms j are weak equivalences by definition. By the two-out-of-three property, f is a weak equivalence exactly when the dotted arrow is one.

Otherwise put, $[f]$ is an isomorphism in $\mathbf{Ho}\,(\mathcal{E}_A)$ if and only if $[g] = \mathbf{L}(u_!)([f])$ is an isomorphism in $\mathbf{Ho}\,(\mathcal{E}_B)$. Meaning that $\mathbf{L}\, u_!$ is conservative. Dually for an acyclic fibration v, the functor $\mathbf{R}\, v^*$ is conservative. Because the model structures considered in the fibers of $\mathbf{Ho}\,(p_{\mathrm{fw}})$ are trivial, this is exactly (hCon) for $\mathbf{Ho}\,(p_{\mathrm{fw}})$. ☐

We shall call $\mathbf{Ho}\,(p_{\mathrm{fw}})$ the *vertical quotient* (or *fiberwise quotient*) of the bifibration p. It somehow only takes into account the homotopies of the fibers of p, by rendering the homotopies of the basis $\mathcal{B}$ trivial. But we just showed that, giving back its whole structure to $\mathcal{B}$ after the vertical quotient, $\mathbf{Ho}\,(p_{\mathrm{fw}}) : \mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}}) \to \mathcal{B}$ is again a Quillen bifibration. This gives us the opportunity to now consider the homotopies of $\mathcal{B}$ and to perform the *horizontal quotient* (or *basis quotient*): this is just $\mathbf{Ho}\,(\mathbf{Ho}\,(p_{\mathrm{fw}}))$.

Proposition 4.4.2. *The functor*

$$\mathbf{Ho}\,(\mathbf{Ho}\,(p_{\mathrm{fw}})) : \mathbf{Ho}\,(\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})) \longrightarrow \mathbf{Ho}\,(\mathcal{B}) \tag{4.75}$$

is isomorphic (as a functor) to $\mathbf{Ho}\,(p)$.

Proof. Recall that $\mathbf{Ho}\,(\mathbf{Ho}\,(p_{\mathrm{fw}}))$ is defined as the unique functor that makes the following square commutative:

$$
\begin{tikzcd}
\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}}) \arrow[r] \arrow[d, "\mathbf{Ho}(p_{\mathrm{fw}})"'] & \mathbf{Ho}\,(\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})) \arrow[d, "\mathbf{Ho}(\mathbf{Ho}(p_{\mathrm{fw}}))"] \\
\mathcal{B} \arrow[r] & \mathbf{Ho}\,(\mathcal{B})
\end{tikzcd}
\tag{4.76}
$$

where the horizontal functors are localizations. Let us denote $\gamma_{\mathrm{fw}} : \mathcal{E} \to \mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}})$ for the localization of $\mathcal{E}$ with the fiberwise model structure. In order to prove the proposition, we shall show that γ_{fw} pass to the homotopy categories: that is there is $\tilde{\gamma_{\mathrm{fw}}}$ such that

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "\gamma_{\mathrm{fw}}"] \arrow[d] & \mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}}) \arrow[d] \\
\mathbf{Ho}\,(\mathcal{E}) \arrow[r, dashed, "\tilde{\gamma_{\mathrm{fw}}}"'] & \mathbf{Ho}\,(\mathbf{Ho}\,(\mathcal{E}_{\mathrm{fw}}))
\end{tikzcd}
\tag{4.77}
$$

and that $\tilde{\gamma_{\mathrm{fw}}}$ is an isomorphism of categories. Translated in terms of homotopies in our various model categories, we need to show that maps $f, g : X \longrightarrow Y$ in $\mathcal{E}$ between bifibrant objects are homotopic in the model category $\mathcal{E}$ if and only if the classes $[f], [g] : X \longrightarrow Y$ are homotopic in the model category $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$. Suppose that the representative f and g are homotopic in $\mathcal{E}$, which homotopy $h : C \longrightarrow Y$ starting C is a very good cylinder for X in $\mathcal{E}$:

$$\begin{tikzcd} X \arrow[r, shift left, "j_0"] \arrow[r, shift right, "j_1"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & C \arrow[r, "h"] & Y \end{tikzcd} \tag{4.78}$$

By definition of a very good cylinder for X, the retraction $q : C \longrightarrow X$ of j_0 and j_1 is an acyclic fibration of $\mathcal{E}$. From this follows that the cylinder C is bifibrant in $\mathcal{E}$. As bifibrant objects of $\mathcal{E}$, X, Y, C are also bifibrant in $\mathcal{E}_{\mathrm{fw}}$. Transporting the map of that diagram to their homotopy classes relatively to the structure $\mathcal{E}_{\mathrm{fw}}$, we obtain the following commutative diagram in $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$ (it makes sense to do so because C, being a very good cylinder is bifibrant in $\mathcal{E}$, so in particular in its fiber):

$$\begin{tikzcd} X \arrow[r, shift left, "{[j_0]}"] \arrow[r, shift right, "{[j_1]}"'] \arrow[rr, bend left, "{[f]}"] \arrow[rr, bend right, "{[g]}"'] & C \arrow[r, "{[h]}"] & Y \end{tikzcd} \tag{4.79}$$

In order to show $[f]$ homotopic to $[g]$, it is enough to prove that the object C defines a cylinder for X with injections $[j_0], [j_1]$ in $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$. The map $[q] : C \longrightarrow X$ is a retraction of $[j_0]$ and $[j_1]$ because $[-]$ is functorial. At this point, it simply remains to show is that $[q]$ is a weak equivalence in $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$. By definition of an acyclic fibration in $\mathcal{E}$, the fiber map $q^{\triangleleft} : C \longrightarrow v^* X$ (where $v = p(q)$) is an acyclic fibration. Since C is cofibrant in its fiber $\mathcal{E}_{pC}$, it is a cofibrant replacement of $v^* X$, hence $[q] = \left[\rho_{v,X} \circ q^{\triangleleft}\right]$ has precisely the form of the cartesian morphisms of $\mathbf{Ho}(p_{\mathrm{fw}})$ (see proposition 4.3.3). So in the end, $[q]$ is a cartesian morphism of $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$ above the acyclic fibration v of $\mathcal{B}$. As such $[q]$ is a total acyclic fibration of the Quillen bifibration $\mathbf{Ho}(p_{\mathrm{fw}})$ and in particular $[q]$ is a weak equivalence.

Conversely, if $[f]$ and $[g]$ are homotopic in the sense of the model category $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$, we want to show that f and g are homotopic in the sense of the model category $\mathcal{E}$. Write $[h] : C \longrightarrow Y$ for such an homotopy from $[f]$ to $[g]$ through a very good cylinder C for X in $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$:

$$\begin{tikzcd} X \arrow[r, shift left, "{[j_0]}"] \arrow[r, shift right, "{[j_1]}"'] \arrow[rr, bend left, "{[f]}"] \arrow[rr, bend right, "{[g]}"'] & C \arrow[r, "{[h]}"] & Y \end{tikzcd} \tag{4.80}$$

The commutativity of this diagram says that, in the model category $\mathcal{E}_{\mathrm{fw}}$, the maps f and hj_0 on one side and g and hj_1 on the other are homotopic. Now recall that the very good cylinder C comes with an acyclic fibration $[q] : C \longrightarrow X$ that is a retraction for both $[j]_0$ and $[j]_1$: in particular $[qj_0] = [qj_1]$, hence qj_0 is homotopic to qj_1 in $\mathcal{E}_{\mathrm{fw}}$. However, homotopies in $\mathcal{E}_{\mathrm{fw}}$ being homotopies in $\mathcal{E}$, the maps f and hj_0 on one side and g and hj_1 on the other are homotopic in $\mathcal{E}$. Similarly qj_0 and qj_1 are homotopic in $\mathcal{E}$. Let us prove that q is a weak equivalence of $\mathcal{E}$, so that then j_0 and j_1 are homotopic in $\mathcal{E}$. Because $[q]$ is an acyclic fibration in the total category $\mathbf{Ho}(\mathcal{E}_{\mathrm{fw}})$, it can be written as $k \circ [w]$ where k is the chosen cartesian morphism above $u = \mathbf{Ho}(p_{\mathrm{fw}})([q]) = p(q)$ and $[w]$ is an acyclic fibration in its fiber. Since fibers are trivial model categories, $[w]$ is an isomorphism: otherwise said, w is a weak equivalence in its fibers, so also in $\mathcal{E}$. Recall from proposition 4.3.3 that k is chosen to be the homotopy class (relatively to $\mathcal{E}_{\mathrm{fw}}$) of

$$(u^*X)^{\mathrm{c}} \longrightarrow u^*X \xrightarrow{\rho} X \tag{4.81}$$

where the first arrow is a cofibrant replacement in the fiber and the second arrow is a cartesian map for p above u. In particular this composite is a weak equivalence of $\mathcal{E}$. It follows that $[q]$ is the homotopy class relatively to $\mathcal{E}_{\mathrm{fw}}$ to a weak equivalence of $\mathcal{E}$. Once again, homotopies in $\mathcal{E}_{\mathrm{fw}}$ are also homotopies of $\mathcal{E}$, so we obtain that q is homotopic in $\mathcal{E}$ to a weak equivalence of $\mathcal{E}$: it means that q itself is a weak equivalence of $\mathcal{E}$. We conclude by using the fact that the homotopy relation between bifibrant objects in $\mathcal{E}$ is transitive and closed under composition: j_0 is homotopic to j_1, so hj_0 is homotopic to hj_1; we already know that f is homotopic to hj_0 and g to hj_1, so we obtain that f is homotopic to g. $\square$

Chapter 5

Dependent type theory and hypertribes

In this chapter, we design a categorical structure that is meant to reconcile Lawvere's approach of logic and the interpretation of dependent types as fibrations as advocated by Voevodsky, Awodey and Warren. It starts by a careful study of the notion of *tribe*, developed by Joyal in the past few years. In a tribe $\mathcal{C}$, it is possible to interpret the Id-type of a type A as a kind of path object P_A together with an "acyclic cofibration" $u : A \longrightarrow P_A$. This tribe $\mathcal{C}$ induces a Grothendieck fibration $\mathcal{F} \longrightarrow \mathcal{C}$ and the map u induces a map in $\mathcal{F}$ that closely resembles a cocartesian map above the diagonal $\Delta_A : A \longrightarrow A \times A$. Cocartesian maps satisfies a universal property that, provided some specific data, asks for the existence of a unique morphism satisfying some property. The map in $\mathcal{F}$ induced by u satisfies the same kind of universal property: provided the same kind of data, it asks for the *mere existence* of a morphism satisfying some property; two answers to the universal problem are not necessarily equal, but only homotopic in this case. It displays u as a kind of "weak cocartesian morphism" above Δ_A, and P_A as the "weak push" over Δ_A of the terminal object of the fiber at A. This connects to Lawvere's presentation of the equality predicate $x =_A y$ in first-order logic as the push $(\Delta_A)_!(\star A)$ over Δ_A of the total predicate $x =_A x$. The name of "hypertribe" appearing in the chapter's title is an incentive towards this goal. However we shall not use "hypertribe" as a name for our structures because it is not self-explanatory on its own.

The chapter is organized as follows. The first section is a detailed overview of Martin-Löf type theory, with an emphasis on the homotopy interpretation of the rules. The presentation we make of the syntax in section 5.1.1 is inspired by Garner's paper [Gar15]. The rest of the presentation is more or less classical, except we emphasize algebraic theories view through the scope of MLTT over the use of MLTT as a foundation.

Section 5.2 is a review of the theory of tribes. It recalls some definitions and results of [Joy17]. Our contribution in that part is limited to a thorough explanation of the link between tribes and dependent type theories. The only two results that are note already in [Joy17] are propositions 5.2.5 and 5.2.14. Proposition 5.2.5 is a recast of a result in Paige North's thesis [Nor17], with a more detailed proof than the one in her memoir. Proposition 5.2.14 however is new to our knowledge.

Section 5.3 introduces a new structure inspired by orthogonal and weak factorization systems. It lifts the principle behind these notions to a relative setting. In this new framework, it became possible to put in the same language the cocartesian maps of a Grothendieck opfibration and the "acyclic cofibrations" of a tribe. This is the structure that will allow to make precise the connection with Lawvere's hyperdoctrines which we gave the outline of above. To our knowledge, all the definitions and results of this section are new in the literature.

Section 5.4 gives a natural generalization of tribes through the new framework designed in section 5.3. We call this generalization *relative tribes*. These structures are inspired both by Joyal's tribes and the D-categories of Ehrhard (see [Ehr88]). The section ends with an opening on a new kind of semantics, designed to correct some defect of relative tribes and to take into account the "many level" structure of type theory. This last part is still a work in progress and much remains to be done.

5.1 Dependent type theories

The dawn of type theory can be traced back to Russell and its *Principia Mathematica*. Russell's initial motivation was to get rid of its now famous paradox that can take place in a naive set theory, and that goes as follows: if we can construct a set X whose elements are all those sets s such that $s \notin s$, then both statements $X \in X$ and $X \notin X$ lead to contradictions, making the theory inconsistent. The solution advertised by Russell was to give *structure* to the mathematical constructions. Mathematical objects are naturally *typed* (natural numbers are not the same as groups, which are in turn very different from smooth functions, etc.) and keeping track of these types through the constructions of mathematical objects should prevent mathematicians to construct inconsistent beasts as Russell's "set" X. Indeed, if we restrict the statement $x \in s$ to be meaningful only when the type of x matches the type allowed by the construction of the set s, then $s \notin s$ is not a correct statement, preventing the set $\{s \mid s \notin s\}$ to even make sense. Broadly speaking, the purpose of Russell is to provide a formal framework for the following kind of thoughts: if we define the set E of even numbers as

> "the set of these natural numbers that are divisible by 2"

then the statement $x \in E$ can only be tested for an x that meets the prerequisite of being a natural number. In particular, $E \notin E$ is not a meaningful statement. In spite of its attractive features, Russell's type theory did not overtake foundational issues at the time, mostly because of its verbose notations and lack of precise definition of the syntax. In comparison, ZF(C) eliminates Russell's paradox not by stripping the statement $s \notin s$ from its meaning, but by giving an axiomatic of sets that does not allow the construction of the "set" $\{s \mid s \in s\}$. It does not say much about the very nature of mathematical objects, but it discards enough so that paradoxical beasts can't be made into sets anymore and it gives sufficient power to encode all modern mathematics. But even in such a nice and powerful framework as ZFC, every object is just a set, making a natural number of the same kind than a function or a group… For example, once everything encoded in ZFC, it is meaningful to ask question such as:

"Is π a cyclic group of order $x \mapsto \exp(x)$?"

Of course, the encodings of notions such as π, cyclic groups and exp into ZFC are sufficiently robust so that the answer is "no" in every model whatsoever. But it is already quite wrong to be able to ask such a question. That is the second point that tackles type theory: to give sufficient meaning to mathematical objects so that only sensitive statement are expressible in the syntax. This is through this view that type theory has made a come back in the late 20$^{\text{th}}$ century through computer science. The ability to check for errors at compilation time in a program calls for something that ZFC-like encoding can not grasp, precisely because one wants the flexibility to discard a program when applied to the wrong kind of argument (which could eventually lead to a runtime error). This remark could be consider off-topic by the mathematical community dealing with foundations if it were not for the Curry-Howard equivalence that establishes a one-to-one correspondence between the formal proofs of propositional calculus in intuitionistic logic and the simply typed λ-terms (understand a very small functional programming language). Work about type theory since has been back-and-forth trying to extend this correspondence to more general calculi, resorting to change what have been known to be a proof until then if needed.

This is the context in which emerged Per Martin-Löf's dependent type theory. This is a variation on the simply typed λ-calculus that tries to extend the Curry-Howard correspondence to encompass predicate intuitionistic logic. The rest of this section is devoted to introduce the vocabulary and rules of Martin-Löf's type theory, afterwards denoted MLTT, and variations of it.

5.1.1 Basic definitions

For the rest of the chapter is given an infinite countable set[1] Var. Informally it is a bag in which fresh variables can be picked from whenever needed.

Definition 5.1.1. Given an alphabet $\mathscr{A}$, the *term-expressions* over $\mathscr{A}$ are inductively defined as:

- any variable $x \in \mathsf{Var}$ is an expression,
- any string $f(e_1, \dots, e_n)$ is an expression whenever $f \in \mathscr{A}$ and each e_i is an expression.

Denote $\mathscr{A}^t$ for the set of term-expressions over $\mathscr{A}$.

Given a second alphabet $\mathscr{B}$, the *type-expression* over $(\mathscr{A}, \mathscr{B})$ are defined to be strings $B(e_1, \dots, e_n)$ where each $e_i \in \mathscr{A}^t$. Denote $(\mathscr{A}, \mathscr{B})^T$, or simply $\mathscr{B}^T$ when context permits it, for the set of type-expressions over $(\mathscr{A}, \mathscr{B})$.

For term- or type-expressions of the second form with $n = 0$, we write f and B instead of $f()$ and $B()$.

Definition 5.1.2. For any $e \in \mathscr{A}^t$, the *free variables* of e are defined inductively as:

- the free variables of $x \in \mathsf{Var}$ are just x,
- the free variables of $f(e_1, \dots, e_n)$ are all free variables of each of the e_i's.

[1] This is a "set" in the most naive sense and needs not any kind of set theory to support it.

Denote $\mathsf{fv}(e)$ for the set of free variables of e. This is always a finite subset of Var.

The free variables of a type-expression $\mathrm{B} = B(e_1, \dots, e_n)$ are defined to be all the free variables of each e_i as just defined. We denote also $\mathsf{fv}(\mathrm{B})$ for the set of free variables of B.

Definition 5.1.3. For any $t, e \in \mathscr{A}^t$ and $x \in \mathsf{Var}$, the *substitution* $t\,[x \leftarrow e]$ is defined by induction as:

- $x\,[x \leftarrow e] = e$ and $y\,[x \leftarrow e] = y$ for any variable $y \neq x$,
- $f(e_1, \dots, e_n)\,[x \leftarrow e] = f(e_1\,[x \leftarrow e], \dots, e_n\,[x \leftarrow e])$ for $f \in \mathscr{A}$ and each $e_i \in \mathscr{A}^*$.

The substitution of x by e in a type-expression $\mathrm{B} = B(e_1, \dots, e_n)$, also denoted $\mathrm{B}\,[x \leftarrow e]$ is defined as $B(e_1\,[x \leftarrow e], \dots, e_n\,[x \leftarrow e])$.

Definition 5.1.4. A *context* over $(\mathscr{A}, \mathscr{B})$ is a (possibly empty) string

$$x_1 : A_1, \dots, x_n : A_n \tag{5.1}$$

where the x_i's are pairwise distinct in Var, and each $A_i \in \mathscr{B}^T$ has free variables among $x_1, \dots, x_{i-1}$.

Definition 5.1.5. *Judgments* over $(\mathscr{A}, \mathscr{B})$ are of five forms:

(i) *type judgments* are strings of the form $\Gamma \vdash A\ \mathsf{type}$,

(ii) *term judgments* are strings of the form $\Gamma \vdash t : A$

(iii) *type equality judgments* are strings of the form $\Gamma \vdash A \equiv B\ \mathsf{type}$,

(iv) *term equality judgments* are strings of the form $\Gamma \vdash t \equiv u : A$

where $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is a context over $(\mathscr{A}, \mathscr{B})$, A, B are type expressions with free variables among the x_i's, and t, u are term expressions with free variables among the x_i's.

For now, judgments are just syntactical constructs, but it provides better understanding by starting to put some meanings behind the symbols. The judgment $\Gamma\ \mathsf{context}$ read "Γ is a well-formed context". The symbol "$\vdash$" should be thought as a relation symbol between the context on the left and the judgment stated on the right. A judgment $\Gamma \vdash \mathscr{J}$ should be read "$\mathscr{J}$ holds in context Γ" or "Γ entails $\mathscr{J}$". In that reading, the type judgment $\Gamma \vdash A\ \mathsf{type}$ is declaring that A is a type in context Γ. The term judgment $\Gamma \vdash t : A$ states that t is a term of type A in context Γ. The symbol "$\equiv$" stands for "definitional equality", which is not to be confused with identity types that will be introduced latter (hence the particular symbol $\equiv$ instead of the usual $=$). The judgment $\Gamma \vdash A \equiv B\ \mathsf{type}$ states that A and B are definitionally equal types, meaning they are the same types: a term of type A *is* a term of type B. Put this in contrast with intentionally equal types (see section 5.1.4) where a term of one type *is transported* (along a proof) to a term of the other type. The same goes for term equality judgments $\Gamma \vdash t \equiv u : A$ which read as "t equals u as terms of type A in context Γ".

We now focus on ground rules that control the judgments in MLTT. There will be more rules as we add type-constructors latter, but these rules are the basic one that will always be assumed for each variations of MLTT. Notations for rules are

$$\frac{\Gamma_1 \vdash \mathscr{J}_1 \quad \dots \quad \Gamma_n \vdash \mathscr{J}_n}{\Gamma \vdash \mathscr{J}} \tag{5.2}$$

where $\mathcal{J}$ and the $\mathcal{J}_i$ stand for the right hand side of either of the four forms of judgments. For now this is just a syntactical rewriting of judgments, but this is intended to be read as "the judgment $\Gamma \vdash \mathcal{J}$ follow from the judgments $\Gamma_1 \vdash \mathcal{J}_1 \ldots \Gamma_n \vdash \mathcal{J}_n$ together". In order to render the rules a little less cumbersome, we use curvy letters as just above when the rule apply to either of the four forms of judgments.

Structural rules. This rule simply says that changing the name of the variables all together does not change the validity of a judgment. Denote $\mathfrak{S}(\mathsf{Var})$ for the group of automorphisms of Var, and denote $\sigma \cdot$ for the obvious action of such an automorphism σ on contexts and on the right hand side of judgments.

$$\frac{\Gamma \vdash \mathcal{J}}{\sigma \cdot \Gamma \vdash \sigma \cdot \mathcal{J}} \quad \forall \sigma \in \mathfrak{S}(\mathsf{Var}) \tag{5.3}$$

Equality rules. These are the rules that govern definitional equalities of types and terms. They are stating that definitional equality is an equivalence relation and that definitional equal types are interchangeable.

$$\begin{gathered}
\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}} \quad \frac{\Gamma \vdash A_1 \equiv A_2 \text{ type}}{\Gamma \vdash A_2 \equiv A_1 \text{ type}} \quad \frac{\Gamma \vdash A_1 \equiv A_2 \text{ type} \quad \Gamma \vdash A_2 \equiv A_3 \text{ type}}{\Gamma \vdash A_1 \equiv A_3 \text{ type}} \\
\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv t : A} \quad \frac{\Gamma \vdash t_1 \equiv t_2 : A}{\Gamma \vdash t_2 \equiv t_1 : A} \quad \frac{\Gamma \vdash t_1 \equiv t_2 : A \quad \Gamma \vdash t_2 \equiv t_3 : A}{\Gamma \vdash t_1 \equiv t_3 : A} \\
\frac{\Gamma \vdash A_1 \equiv A_2 \text{ type} \quad \Gamma \vdash t : A_1}{\Gamma \vdash t : A_2} \quad \frac{\Gamma \vdash A_1 \equiv A_2 \text{ type} \quad \Gamma \vdash t_1 \equiv t_2 : A_1}{\Gamma \vdash t_1 \equiv t_2 : A_2} \\
\frac{\Gamma \vdash A_1 \equiv A_2 \text{ type} \quad \Gamma, x : A_1, \Delta \vdash J}{\Gamma, x : A_2, \Delta \vdash J}
\end{gathered} \tag{5.4}$$

Substitution rules. These rules manage substitutions in both types and terms. They state that substituting a variable by a term in a valid judgment yields a new valid judgment, and that substituting a variable by definitionally equal terms in a given type/term yields definitionally equal types/terms in the end.

$$\begin{gathered}
\frac{\Gamma \vdash t : A \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta\,[x \leftarrow t] \vdash \mathcal{J}\,[x \leftarrow t]} \\
\frac{\Gamma \vdash t_1 \equiv t_2 : A \quad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta\,[x \leftarrow t_2] \vdash B\,[x \leftarrow t_1] \equiv B\,[x \leftarrow t_2] \text{ type}} \\
\frac{\Gamma \vdash t_1 \equiv t_2 : A \quad \Gamma, x : A, \Delta \vdash t : B}{\Gamma, \Delta\,[x \leftarrow t_2] \vdash t\,[x \leftarrow t_1] \equiv t\,[x \leftarrow t_2] : B\,[x \leftarrow t_2]}
\end{gathered} \tag{5.5}$$

The choice of substitution by t_2 over t_1 in Δ in the second rule does not break symmetry. Using symmetry of definitional equality of types shows this choice is equivalent to the choice of substituting by t_1. The same goes for the last rule, except that showing the equivalence requires the use of the second rule and of the invariance rule of definitional equality between terms under definitional equality of types.

Weakening rules. This rule is saying that whenever a judgment holds, we can add a variable in its context without changing its validity.

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}} \quad x \notin \mathsf{fv}(\Gamma) \cup \mathsf{fv}(\Delta) \tag{5.6}$$

This is the rule that could be skipped if we were to study linear variations of MLTT.

Identity rules. The last rule states that a variable that appears in a context is a valid term.

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A} \quad x \notin \mathsf{fv}(\Gamma) \tag{5.7}$$

Definition 5.1.6. Let S be a set of judgments over $(\mathscr{A}, \mathscr{B})$. A *derivable* judgment $\Gamma \vdash \mathcal{J}$ is a judgment over the same alphabets that is the root of a **finite** labeled rooted tree

- whose leaves are elements of S,
- and whose nodes are instances of the previous rules.

Here, a *labeled rooted tree* over any set S is defined inductively as either an element $s \in S$ (called a leaf), or a pair (s, L) with $s \in S$ and L is a finite sequence of labeled rooted trees (called a node). In both cases, s is called the root.

Example(s) 5.1.7. Let $\mathscr{A} = \emptyset$ and $\mathscr{B} = \{A, B\}$. Then the judgment $x : A \vdash B(x, x)$ type is derivable from $\{\vdash A \text{ type}; x : A, y : A \vdash B(x, y) \text{ type}\}$. Indeed, we can construct the following tree

$$\text{subs}\,\frac{\text{id}\,\dfrac{\vdash A \text{ type}}{x : A \vdash x : A} \quad x : A, y : A \vdash B(x, y) \text{ type}}{x : A \vdash B(x, x) \text{ type}} \tag{5.8}$$

The following lemma allows one to use a derivable rule as any of the previous rule to create new derivable rules.

Lemma 5.1.8. *Given a set S of judgments over $(\mathscr{A}, \mathscr{B})$ such that a judgment $\Gamma \vdash \mathcal{J}$ is the root of a finite labeled rooted tree whose leaves are elements of S and whose nodes are of the form:*

$$\frac{\Gamma'_1 \vdash \mathcal{J}'_1 \quad \dots \quad \Gamma'_n \vdash \mathcal{J}'_n}{\Gamma' \vdash \mathcal{J}'} \tag{5.9}$$

with the bottom judgment being derivable from the top ones, then $\Gamma \vdash \mathcal{J}$ is derivable from S.

Proof. This is just a matter of grafting the derivation trees into the nodes of the given tree. This has little to do with type theory per se and is true for any formal system. □

Lemma 5.1.9. *The following judgment*

$$\Gamma, y : A, x : A, \Delta \vdash \mathcal{J} \tag{5.10}$$

is derivable from $\{\Gamma \vdash A \text{ type}; \Gamma, x : A, y : A, \Delta \vdash \mathcal{J}\}$

Proof. The finite labeled rooted tree is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \text{ type}}{\Gamma, z : A \vdash z : A}\text{id} \qquad \dfrac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A, y : A, \Delta \vdash \mathcal{J}}{\Gamma, z : A, x : A, y : A, \Delta \vdash \mathcal{J}}\text{weak}}{\Gamma, z : A, x : A, \Delta[y \leftarrow z] \vdash \mathcal{J}[y \leftarrow z]}\text{subs}}{\Gamma, y : A, x : A, \Delta \vdash \mathcal{J}}\sigma \tag{5.11}$$

The last rule is an instance of the structural rule with σ being the transposition $(y\ z)$. Indeed, on an expression ε with $z \notin \mathsf{fv}(\varepsilon)$, $\varepsilon[y \leftarrow z] = (y\ z) \cdot \varepsilon$, and $(y\ z)$ being an involution we get that $(y\ z) \cdot (\varepsilon[y \leftarrow z]) = \varepsilon$. □

This lemma says that when subsequent variables in a context are not independent, then the order in which they appear does not matter. In virtue of lemma 5.1.8, this will be used implicitly as a rule whenever needed.

Definition 5.1.10. A *algebraic* MLTT over $(\mathscr{A}, \mathscr{B})$ is a collection $\mathbb{T}$ of judgments such that

(i) for each type symbol $B \in \mathscr{B}$, there is exactly one type judgment in $\mathbb{T}$ of the form

$$x_1 : A_1, \dots, x_n : A_n \vdash B(x_1, \dots, x_n) \text{ type} \tag{5.12}$$

(ii) for each term symbol $f \in \mathscr{A}$, there is exactly one term judgment in $\mathbb{T}$ of the form

$$x_1 : A_1, \dots, x_n : A_n \vdash f(x_1, \dots, x_n) : A \tag{5.13}$$

(iii) all other judgments are type and terms equality judgments,

(iv) for each judgment $x_1 : A_1, \dots, x_n : A_n \vdash \mathcal{J}$ in $\mathbb{T}$, and for each $i \in \{1, \dots, n\}$, the judgment

$$x_1 : A_1, \dots, x_{i-1} : A_{i-1} \vdash A_i \text{ type} \tag{5.14}$$

is derivable from $\mathbb{T}$,

(v) for each term judgment $\Gamma \vdash t : A$ in $\mathbb{T}$, the type judgment $\Gamma \vdash A$ type is derivable from $\mathbb{T}$.

(vi) for any type equality judgment $\Gamma \vdash A_1 \equiv A_2$ type, both $\Gamma \vdash A_1$ type and $\Gamma \vdash A_2$ type are derivable from $\mathbb{T}$,

(vii) for any term equality judgment $\Gamma \vdash t_1 \equiv t_2 : A$, both $\Gamma \vdash t_1 : A$ and $\Gamma \vdash t_2 : A$ are derivable from $\mathbb{T}$.

EXAMPLE(S) 5.1.11. Any algebraic theory can be made into a simple MLTT with only one type symbol. For example the theory of monoids can be encoded over the alphabets $\mathscr{A} = \{m, e\}$ and $\mathscr{B} = \{\star\}$ by the simple MLTT with judgments:

$$\begin{gathered} \vdash \star \text{ type} \qquad \vdash e : \star \qquad x : \star, y : \star \vdash m(x,y) : \star \\ x : \star \vdash m(x,e) \equiv x : \star \qquad x : \star \vdash m(e,x) \equiv x : \star \\ x : \star, y : \star, z : \star \vdash m(m(x,y),z) \equiv m(x, m(y,z)) : \star \end{gathered} \tag{5.15}$$

The first three conditions of the definition are clearly met. The fourth is just a matter to prove that $x_1 : \star, \dots, x_n : \star \vdash \star$ type is derivable for all $n \in \mathbb{N}$ (actually here $n = 0, 1, 2$

is enough). Case $n > 0$ is obtained from case $n - 1$ by constructing the following tree:

$$\text{weak}\;\dfrac{\dfrac{\vdots}{x_1 : \star, \dots, x_{n-1} : \star \vdash \star\ \mathsf{type}} \qquad \dfrac{\vdots}{x_1 : \star, \dots, x_{n-1} : \star \vdash \star\ \mathsf{type}}}{x_1 : \star, \dots, x_n : \star \vdash \star\ \mathsf{type}} \tag{5.16}$$

This also takes care of the fifth point. The sixth one is vacuously true. The seventh one is about proving the derivability of these judgments:

$$\begin{gathered} x : \star \vdash x : \star \qquad x : \star \vdash m(x, e) : \star \qquad x : \star \vdash m(e, x) : \star \\ x : \star, y : \star, z : \star \vdash m(m(x, y), z) : \star \qquad x : \star, y : \star, z : \star \vdash m(x, m(y, z)) : \star \end{gathered} \tag{5.17}$$

The first judgment is proved by applying the identity rule on the judgment $\vdash \star$ type. The second one is derivable through:

$$\text{subs}\;\dfrac{\text{weak}\;\dfrac{\vdash \star\ \mathsf{type} \qquad \vdash e : \star}{x : \star \vdash e : \star} \qquad x : \star, y : \star \vdash m(x, y) : \star}{x : \star \vdash m(x, e) : \star} \tag{5.18}$$

The judgment $x : \star \vdash m(e, x) : \star$ is treated in a similar fashion. The two remaining judgments are derived the same way and we only detail the derivation tree for one of them:

$$\dfrac{x : \star, y : \star \vdash m(x, y) : \star \qquad \dfrac{\dfrac{\vdash \star\ \mathsf{type} \qquad \vdash \star\ \mathsf{type}}{x : \star \vdash \star\ \mathsf{type}} \qquad \dfrac{\vdash \star\ \mathsf{type} \qquad z' : \star, z : \star \vdash m(z', z) : \star}{x : \star, z' : \star, z : \star \vdash m(z', z) : \star}}{x : \star, y : \star, z' : \star, z : \star \vdash m(z', z) : \star}}{x : \star, y : \star, z : \star \vdash m(m(x, y), z) : \star} \tag{5.19}$$

The previous example illustrates how cumbersome it can be to check carefully the condition for a set of judgment to be a theory. But it illustrates also that there is nothing difficult about it: it just a matter to formally construct the tree of rules that are obvious under reading by a mathematician. Reading $m(e, x)$ for example is automatically translated by our brain as "substitute the first argument of the multiplication by the unit of the monoid". The tree constructed to justify that this term exists in the type $\star$ is just a formal translation of this implicit thinking. For the next examples, we shall leave these checks up to the reader when there are no difficulties.

Of course, simple MLTT theories are not limited to recast universal algebra as in the previous example. They reveal their expressive power when we have multiple types, and especially dependent ones.

EXAMPLE(S) 5.1.12. There is a simple MLTT of categories, constituted by the following judgments over the alphabets $\mathscr{A} = \{\mathrm{id}, \circ\}$ and $\mathscr{B} = \{\mathrm{Ob}, \mathrm{Hom}\}$:

$$\begin{gathered} \vdash \mathrm{Ob}\ \mathsf{type} \qquad x : \mathrm{Ob}, y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \mathsf{type} \\ x : \mathrm{Ob} \vdash \mathrm{id}_x : \mathrm{Hom}(x, x) \\ x, y, z : \mathrm{Ob}, f : \mathrm{Hom}(x, y), g : \mathrm{Hom}(y, z) \vdash g \circ f : \mathrm{Hom}(x, z) \\ x, y : \mathrm{Ob}, f : \mathrm{Hom}(x, y) \vdash f \circ \mathrm{id}_x \equiv f : \mathrm{Hom}(x, y) \\ x, y : \mathrm{Ob}, f : \mathrm{Hom}(x, y) \vdash \mathrm{id}_y \circ f \equiv f : \mathrm{Hom}(x, y) \\ x, y, z, w : \mathrm{Ob}, f : \mathrm{Hom}(x, y), g : \mathrm{Hom}(y, z), h : \mathrm{Hom}(z, w) \vdash \\ \hfill h \circ (g \circ f) \equiv (h \circ g) \circ f : \mathrm{Hom}(x, w) \end{gathered} \tag{5.20}$$

Here are used some abuse of notation: we wrote $x_1, \dots, x_k : A$ as a shorthand version of $x_1 : A, \dots, x_k : A$; also we wrote $g \circ f$ instead of $\circ(f, g)$ and id_x instead of $\mathrm{id}(x)$ to meet the usual writing of category theory.

As said before, this would be very cumbersome and not that enlightening for the reader to prove thoroughly that it is a simple MLTT as defined above. So we only give one example of the derivability to be checked, namely that $f \circ \mathrm{id}_x$ is correctly typed in the fifth judgment, in figure 5.1.

5.1.2 Dependent pairs

As for now, we have presented a properly behaved syntax that encompasses the usual universal algebra together with more general algebraic theories taking benefit of the dependent nature of Martin-Löf type theory. This additional expressive power provided by dependent types enables us in particular to describe operations whose domains are defined by equations. Let us illustrate that with the example of the theory of categories that has been presented as an algebraic MLTT in example 5.1.12. In the usual first-order multisorted setting, one typically could define a category with two sorts: one for the objects O, and one for the morphisms M. One would have to define operations source s and target t of kind $M \longrightarrow O$, a identity operation i of kind $O \longrightarrow M$ and a "composition" operation c of kind $M \times M \longrightarrow M$. Then one would have to add axioms saying that $c(f, g)$ is only meaningful when $s(g) = t(f)$ is derivable. Moreover one would have to add axioms saying that the source of $c(f, g)$ is $s(f)$ and that its target is $t(g)$. In particular, this is not doable without an implication symbol in the syntax of formulas. The use of an algebraic MLTT allows to avoid such a process because each morphism comes directly with its source and target encoded in its type, so that defining the composition term in this theory coerces us to give the correct domain of appliance and to specify the source and target of the output. This is the power of dependent types: to embed in the *terms* the necessary conditions that are usually reflected through *axioms*.

This expressiveness comes at a cost: given a type $B(x)$ depending on $x : A$, one can not talk about a term of type $B(x)$ without explicitly giving $x : A$. In the example of the theory of categories, it means that the word "morphism" is not meaningful anymore, only "morphism from x to y" for previously defined objects x and y makes sense. This flaw is fixed by the introduction of *dependent pair types*, also called Σ-*types*. For each type $B(x)$ depending on $x : A$, it provides a new type, denoted $\Sigma_{x:A}B(x)$ whose inhabitants are to be understood as "inhabitants of $B(x)$ for some possibly irrelevant $x : A$". More precisely, the rules about Σ-types ensure that every inhabitant of $\Sigma_{x:A}B(x)$ is a pair (a, b) with $a : A$ and $b : B(a)$ (hence the name *dependent pair*). Coming back to the example of the theory of categories, the bare notion "morphism" now makes sense as inhabitants of $\Sigma_{x:\mathrm{Ob}}\Sigma_{y:\mathrm{Ob}}\,\mathrm{Hom}(x, y)$.

Formally speaking we shall extend the inductive definition of term-expressions over $\mathscr{A}$ with

- For every term-expressions e_1 and e_2, the string (e_1, e_2) is again a term-expression.
- For every term-expression e, the strings $\pi_1(e)$ and $\pi_2(e)$ is again a term-expression.

Similarly we extend the inductive definition of type-expressions over $(\mathscr{A}, \mathscr{B})$ with

- For every type-expressions A and B and any variable x, the string $\Sigma_{x:A}B$ is a type-expression.

Finally, substitution should also be extended in the following manner:

$$
\begin{aligned}
(\Sigma_{x:A}B)\,[y \leftarrow t] &= \Sigma_{x:A[y\leftarrow t]}B\,[y \leftarrow t] \\
(e_1, e_2)\,[y \leftarrow t] &= (e_1\,[y \leftarrow t]\,, e_2\,[y \leftarrow t]) \\
\pi_i(e)\,[y \leftarrow t] &= \pi_i(e\,[y \leftarrow t]) \quad i \in \{1, 2\}
\end{aligned}
\tag{5.21}
$$

The rules also have to be completed accordingly, as we do below.

Σ-introduction rules. These rules control the creation of dependent pairs, and encode into the formal system the intuitions given above.

$$
\frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \Sigma_{x:A}B \text{ type}} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B\,[x \leftarrow t]}{\Gamma \vdash (t, u) : \Sigma_{x:A}B}
\tag{5.22}
$$

Σ-elimination rules. These rules explain out to use a dependent pair. It is formally saying that given a dependent pair, one can either use its first or second component.

$$
\frac{\Gamma \vdash t : \Sigma_{x:A}B}{\Gamma \vdash \pi_1(t) : A} \qquad \frac{\Gamma \vdash t : \Sigma_{x:A}B}{\Gamma \vdash \pi_2(t) : B\,[x \leftarrow \pi_1(t)]}
\tag{5.23}
$$

Σ-computation rules.

$$
\begin{gathered}
\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B\,[x \leftarrow t]}{\Gamma \vdash \pi_1(t, u) \equiv t : A} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B\,[x \leftarrow t]}{\Gamma \vdash \pi_2(t, u) \equiv u : B\,[x \leftarrow t]} \\
\frac{\Gamma \vdash t : \Sigma_{x:A}B}{\Gamma \vdash (\pi_1(t), \pi_2(t)) \equiv t : \Sigma_{x:A}B}
\end{gathered}
\tag{5.24}
$$

Definition 5.1.13. A MLTT *with Σ-types* is a simple MLTT where judgments can contain extended type- and term-expressions and where derivability contains the new rules concerning Σ-types.

REMARK 5.1.14. From any pair of judgments $\Gamma \vdash A$ type and $\Gamma \vdash B$ type, the following Σ-derivation tree can be constructed:

$$
\frac{\dfrac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x : A \vdash B \text{ type}}}{\Gamma \vdash \Sigma_{x:A}B \text{ type}}
\tag{5.25}
$$

It is the formal version of saying: although B does not depend on A, make it so artificially by weakening, and consider the type of "dependent" pairs. The application of the weakening in the previous derivation tree ensures that x is not a free variable of B, so that $B\,[x \leftarrow t] = B$ for any substitution. Hence, the introduction rule becomes

$$
\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash (t, u) : \Sigma_{x:A}B}
\tag{5.26}
$$

The rule says that any ordinary pair consisting of an inhabitant t in A and an inhabitant u in B is a inhabitant in $\Sigma_{x:A}B$. The elimination and computations rules ensures that any inhabitant is of this form:

$$
\frac{\Gamma \vdash t : \Sigma_{x:A}B}{\Gamma \vdash \pi_1(t) : A} \qquad \frac{\Gamma \vdash t : \Sigma_{x:A}B}{\Gamma \vdash \pi_2(t) : B}
\tag{5.27}
$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \pi_1(t,u) \equiv t : A} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \pi_2(t,u) \equiv u : B}$$

$$\frac{\Gamma \vdash t : \Sigma_{x:A} B}{\Gamma \vdash (\pi_1(t), \pi_2(t)) \equiv t : \Sigma_{x:A} B} \tag{5.28}$$

Hence $\Sigma_{x:A} B$ acts exactly as the usual product type, and for that reason we denote it $A \times B$ in this case.

Example(s) 5.1.15. Recall from example 5.1.12 the simple MLTT of categories. Make it a theory of monoidal category by extending the term alphabet with

$$\{\otimes^0, \otimes^1, \alpha, \alpha^{-1}, \mathbb{I}, \rho, \rho^{-1}, \lambda, \lambda^{-1}\}$$

and adding the following judgment yields the theory of monoidal categories:

$$\begin{array}{l}
x : \mathrm{Ob}, y : \mathrm{Ob} \vdash x \otimes^0 y : \mathrm{Ob} \qquad \vdash \mathbb{I} : \mathrm{Ob} \\
x, x', y, y' : \mathrm{Ob}, f : \mathrm{Hom}(x,y), f' : \mathrm{Hom}(x',y') \vdash f \otimes^1 f' : \mathrm{Hom}(x \otimes^0 x', y \otimes^0 y') \\
x, y : \mathrm{Ob} \vdash \mathrm{id}_x \otimes^1 \mathrm{id}_y \equiv \mathrm{id}_{x \otimes^0 y} : \mathrm{Hom}(x \otimes^0 y, x \otimes^0 y) \\
x, x', y, y', z, z' : \mathrm{Ob}, \\
\qquad f : \mathrm{Hom}(x,y), f' : \mathrm{Hom}(x',y'), g : \mathrm{Hom}(y,z), g' : \mathrm{Hom}(y',z') \vdash \\
\qquad\qquad (g \circ f) \otimes^1 (g' \circ f') \equiv (g \otimes^1 g') \circ (f \otimes^1 f') : \mathrm{Hom}(x \otimes^0 x', z \otimes^0 z') \\
x : \mathrm{Ob} \vdash \lambda(x) : \mathrm{Hom}(\mathbb{I} \otimes^0 x, x) \qquad x : \mathrm{Ob} \vdash \lambda^{-1}(x) : \mathrm{Hom}(x, \mathbb{I} \otimes^0 x) \\
x : \mathrm{Ob} \vdash \rho(x) : \mathrm{Hom}(x \otimes^0 \mathbb{I}, x) \qquad x : \mathrm{Ob} \vdash \rho^{-1}(x) : \mathrm{Hom}(x, x \otimes^0 \mathbb{I}) \\
x, y, z : \mathrm{Ob} \vdash \alpha(x,y,z) : \mathrm{Hom}((x \otimes^0 y) \otimes^0 z, x \otimes^0 (y \otimes^0 z)) \\
x, y, z : \mathrm{Ob} \vdash \alpha^{-1}(x,y,z) : \mathrm{Hom}(x \otimes^0 (y \otimes^0 z), (x \otimes^0 y) \otimes^0 z) \\
x : \mathrm{Ob} \vdash \lambda(x) \circ \lambda^{-1}(x) \equiv \mathrm{id}_x : \mathrm{Hom}(x,x) \\
x : \mathrm{Ob} \vdash \lambda(x)^{-1} \circ \lambda(x) \equiv \mathrm{id}_{\mathbb{I} \otimes^0 x} : \mathrm{Hom}(\mathbb{I} \otimes^0 x, \mathbb{I} \otimes^0 x) \\
x : \mathrm{Ob} \vdash \rho(x) \circ \rho^{-1}(x) \equiv \mathrm{id}_x : \mathrm{Hom}(x,x) \\
x : \mathrm{Ob} \vdash \rho(x)^{-1} \circ \rho(x) \equiv \mathrm{id}_{x \otimes^0 \mathbb{I}} : \mathrm{Hom}(x \otimes^0 \mathbb{I}, x \otimes^0 \mathbb{I}) \\
x, y, z : \mathrm{Ob} \vdash \alpha(x,y,z) \circ \alpha^{-1}(x,y,z) \equiv \mathrm{id}_{x \otimes^0 (y \otimes^0 z)} : \mathrm{Hom}(x \otimes^0 (y \otimes^0 z), x \otimes^0 (y \otimes^0 z)) \\
x, y, z : \mathrm{Ob} \vdash \alpha^{-1}(x,y,z) \circ \alpha(x,y,z) \equiv \mathrm{id}_{(x \otimes^0 y) \otimes^0 z} : \mathrm{Hom}((x \otimes^0 y) \otimes^0 z, (x \otimes^0 y) \otimes^0 z) \\
x, y : \mathrm{Ob}, f : \mathrm{Hom}(x,y) \vdash f \circ \lambda(x) \equiv \lambda(y) \circ (\mathrm{id}_{\mathbb{I}} \otimes^1 f) : \mathrm{Hom}(\mathbb{I} \otimes^0 x, y) \\
x, y : \mathrm{Ob}, f : \mathrm{Hom}(x,y) \vdash f \circ \rho(x) \equiv \rho(y) \circ (f \otimes^1 \mathrm{id}_{\mathbb{I}}) : \mathrm{Hom}(x \otimes^0 \mathbb{I}, y) \\
x, y, z, x', y', z' : \mathrm{Ob}, f : \mathrm{Hom}(x,x'), g : \mathrm{Hom}(y,y'), h : \mathrm{Hom}(z,z') \vdash \\
\qquad (f \otimes^1 (g \otimes^1 h)) \circ \alpha(x,y,z) \equiv \alpha(x',y',z') \circ ((f \otimes^1 g) \otimes^1 h) \\
\qquad\qquad : \mathrm{Hom}((x \otimes^0 y) \otimes^0 z, x' \otimes^0 (y' \otimes^0 z')) \\
x, y : \mathrm{Ob} \vdash (\mathrm{id}_x \otimes^1 \lambda(y)) \circ \alpha(x, \mathbb{I}, y) \equiv \rho(x) \otimes^1 \mathrm{id}_y : \mathrm{Hom}((x \otimes^0 \mathbb{I}) \otimes^0 y, x \otimes^0 y) \\
x, y, z, w : \mathrm{Ob} \vdash \\
\qquad (\mathrm{id}_x \otimes^1 \alpha(y,z,w)) \circ \alpha(x, (y \otimes^0 z), w) \circ (\alpha(x,y,z) \otimes^1 \mathrm{id}_w) \\
\qquad\qquad \equiv \alpha(x, y, z \otimes^0 w) \circ \alpha(x \otimes^0 y, z, w) \\
\qquad\qquad\qquad : \mathrm{Hom}(((x \otimes^0 y) \otimes^0 z) \otimes^0 w, (x \otimes^0 (y \otimes^0 (z \otimes^0 w))))
\end{array} \tag{5.29}$$

Instead of having two symbols λ and λ^{-1} and to define both

$$x : \mathrm{Ob} \vdash \lambda(x) : \mathrm{Hom}(\mathbb{I} \otimes x, x) \qquad x : \mathrm{Ob} \vdash \lambda^{-1}(x) : \mathrm{Hom}(x, \mathbb{I} \otimes x)$$

We could have been more concise with a unique symbol Λ and a unique declaration

$$x : \mathrm{Ob} \vdash \Lambda(x) : \mathrm{Hom}(\mathbb{I} \otimes x, x) \times \mathrm{Hom}(x, \mathbb{I} \otimes x)$$

And then replace all later occurrences of $\lambda(-)$ by $\pi_1(\Lambda(-))$ and all occurrences of $\lambda^{-1}(-)$ by $\pi_2(\Lambda(-))$. Not only it makes the definition of the theory slightly more compact, but it also reflects the fact that a monoidal structure contains an invertible left unitor rather than a left unitor in both direction that happen to be inverse of each other. Indeed, usually λ^{-1} is not considered to be part of the monoidal *structure* because it is definable from the *property* of λ being invertible. The same goes for ρ and α.

Let us continue to add symbols and axioms to define get the theory of dualizable monoidal category (meaning every object has a dual):

$$\begin{array}{l} x : \mathrm{Ob} \vdash x^* : \mathrm{Ob} \\ x : \mathrm{Ob} \vdash \varepsilon(x) : \mathrm{Hom}(x^* \otimes^0 x, x) \qquad x : \mathrm{Ob} \vdash \eta(x) : \mathrm{Hom}(x, x \otimes^0 x^*) \\ x : \mathrm{Ob} \vdash \lambda(x) \equiv \rho(x) \circ (\mathrm{id}_x \otimes^1 \varepsilon(x)) \circ \alpha(x, x^*, x) \circ (\eta(x) \otimes^1 \mathrm{id}_x) : \mathrm{Hom}(\mathbb{I} \otimes^0 x, x) \\ x : \mathrm{Ob} \vdash \rho(x) \equiv \lambda(x) \circ (\varepsilon(x) \otimes^1 \mathrm{id}_{x^*}) \circ \alpha^{-1}(x^*, x, x^*) \circ (\mathrm{id}_{x^*} \otimes^1 \eta(x)) : \mathrm{Hom}(x \otimes^0 \mathbb{I}, x) \end{array} \tag{5.30}$$

Here one could have taken advantage of the Σ-types to avoid three new symbols and three declarations by only writing:

$$x : \mathrm{Ob} \vdash x^\star : \Sigma_{y : \mathrm{Ob}} \mathrm{Hom}(y \otimes^0 x, x) \times \mathrm{Hom}(x, x \otimes^0 y)$$

And then replace all occurrences of x^* by $\pi_1(x^\star)$, those of $\varepsilon(x)$ by $\pi_1(\pi_2(x^\star))$ and those of $\eta(x)$ by $\pi_2(\pi_2(x^\star))$. Once again, it matches more faithfully the intention behind the definition of a dual object of x: it is an object x^* *together* with evaluation and coevaluation maps.

Remark 5.1.16. Note that a context $x : A, y : B$ is the same thing that the context $z : \Sigma_{x:A} B$. Iterating the process, any context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ is equivalent to a context with a single non-dependent type, namely

$$z : \Sigma_{x_1 : A_1} \Sigma_{x_2 : A_2} \ldots \Sigma_{x_{n-1} : A_{n-1}} A_n \tag{5.31}$$

In that regard, in a MLTT with Σ-types, every rules could be stated with a context of length 1 every time that a Γ or a Δ was written.

5.1.3 Dependent function type

We continue to extend the language of type theories by adding a new type constructor, called *dependent function type*, much in the same fashion than for Σ-types. For a type $B(x)$ that depends on $x : A$, it constructs the type $\Pi_{x:A} B(x)$ that is to be thought as the type of "functions" taking an argument x of type A and resulting in a inhabitant of $B(x)$. Dependent function types bring a huge jump in expressiveness by making term-judgment into first-class citizen of the type theory itself. More precisely, dependent function types allow to make into an axiom of a theory a property of the following form:

> For a (non necessarily derivable) judgment $\Gamma, x : A, y : B \vdash \mathcal{J}$ and for any derivable term-judgment $\Gamma, x : A \vdash t : B$, then the judgment $\Gamma, x : A \vdash \mathcal{J}\,[y \leftarrow t]$ is derivable.

In a type theory with Π-types, this can be expressed by just adding the following judgment to the theory at play:

$$\Gamma, f : \Pi_{x:A} B, x : A \vdash \mathcal{J}\,[y \leftarrow f(x)] \tag{5.32}$$

As with Σ-types, let us start by extending the inductive definition of term-expressions over $\mathscr{A}$ with

- For every term-expression e, and any variable x, the string $\lambda x.e$ is a term-expression.

- For every term-expressions e_1, e_2, the string $e_1(e_2)$ is a term-expression.

We now extend the inductive definition of type-expression over $(\mathscr{A}, \mathscr{B})$ with:

- For type-expressions A and B, the string $\Pi_{x:A}B$ is a type-expression.

Finally, substitution is extended accordingly by:

$$\begin{aligned} (\Pi_{x:A}B)[y \leftarrow t] &= \Pi_{x:A[y\leftarrow t]}B[y \leftarrow t] \\ (\lambda x.e)[y \leftarrow t] &= \lambda x.e[y \leftarrow t] \\ e_1(e_2)[y \leftarrow t] &= e_1[y \leftarrow t](e_2[y \leftarrow t]) \end{aligned} \tag{5.33}$$

The rules are then completed by the following ones.

Π-introduction rules. These rules control the creation of dependent functions.

$$\frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \Pi_{x:A}B \text{ type}} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi_{x:A}B} \tag{5.34}$$

Π-elimination rules. This rule explain out to apply a dependent function. It is formally saying that given a dependent function, applying it to some argument x yields an inhabitant of a type dependent on x.

$$\frac{\Gamma \vdash f : \Pi_{x:A}B \quad \Gamma \vdash t : A}{\Gamma \vdash f(t) : B[x \leftarrow t]} \tag{5.35}$$

Π-computation rules.

$$\begin{gathered} \frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x.t)(u) \equiv t[x \leftarrow u] : B[x \leftarrow u]} \\ \\ \frac{\Gamma \vdash f : \Pi_{x:A}B}{\Gamma \vdash (\lambda x.f(x)) \equiv f : \Pi_{x:A}B} \end{gathered} \tag{5.36}$$

Definition 5.1.17. A MLTT *with* Σ- *and* Π-*types* is a MLTT with Σ-types where judgments can contain extended type- and term-expressions and where derivability is defined by adding the new set of rules concerning Π-types.

Remark 5.1.18. As for the case of Σ-types, there is a special kind of dependent function type that should be elaborated on. Indeed, from the judgments $\Gamma \vdash A$ type and $\Gamma \vdash B$ type, one can construct the following derivation tree

$$\frac{\Gamma \vdash A \text{ type} \qquad \dfrac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x : A \vdash B \text{ type}}}{\Gamma \vdash \Pi_{x:A}B \text{ type}} \tag{5.37}$$

It is the formal construction of seeing B as trivially depending on A and then forming the corresponding Π-type. The introduction rule describes the term of $\Pi_{x:A}B$ as these

$\lambda x.t$ where $\Gamma, x : A \vdash t : B$ is derivable. Although B does not depend on the variable x, the term t can very much rely on x. That is to say that substitution realizes a function

$$\begin{aligned} \{u \mid \Gamma \vdash u : A \text{ derivable}\} &\longrightarrow \{v \mid \Gamma \vdash v : B \text{ derivable}\} \\ u &\mapsto t\,[x \leftarrow u] \end{aligned} \tag{5.38}$$

For that reason, inhabitants of $\Pi_{x:A}B$ can be thought as functions from A to B and we denote $A \longrightarrow B$ instead of $\Pi_{x:A}B$. One recovers the usual rules of the simply typed λ-calculus when restricting Π-types to non-dependent types

Remark 5.1.19. The curryfication process of Π-types is a way to "empty" the context of a term judgment at the expense of making the type of the term more complex. Given a term judgment

$$x_1 : A_1, \dots, x_n : A_n \vdash t : B \tag{5.39}$$

the rules of Π-types give an equivalent form with empty context, namely

$$\vdash \lambda.x_1(\lambda x_2. \dots (\lambda x_n.t) \dots) : \Pi_{x_1:A_1}\Pi_{x_2:A_2} \dots \Pi_{x_n:A_n}B \tag{5.40}$$

5.1.4 Identity type

Much like Π-types make term-judgments first-class citizens of a type theory, identity types are introduced to make equalities first-class citizens. As for now indeed, there is no way in a MLTT with Σ- and Π-types to assume that inhabitants are equal in the premises of a judgment. For example, to state the uniqueness of identities in the theory of small categories, one would have to say:

From the judgments

$$\begin{aligned} \Gamma &\vdash f : \Pi_{x:\mathrm{Ob}} \operatorname{Hom}(x,x) \\ \Gamma, x, y : \mathrm{Ob}, g : \operatorname{Hom}(x,y) &\vdash g \circ f(x) \equiv g : \operatorname{Hom}(x,y) \\ \Gamma, x, y : \mathrm{Ob}, g : \operatorname{Hom}(x,y) &\vdash f(y) \circ g \equiv g : \operatorname{Hom}(x,y) \end{aligned}$$

one can derive in the MLTT of categories that:

$$\Gamma \vdash f \equiv \mathrm{id} : \Pi_{x:\mathrm{Ob}} \operatorname{Hom}(x,x)$$

Given a type A and two inhabitants $x : A$ and $y : A$, the identity type $\mathrm{Id}_A(x,y)$ is inhabited by the "witnesses of equalities between x and y". This terminology sometimes is confusing, especially in an intentional framework as we shall discuss later: this is why we will prefer to refer to $p : \mathrm{Id}_A(x,y)$ as an "identification of x and y" or even, in the style of HoTT, as a "path between x and y". Now that identifications are inhabitants of some type, they can appear in contexts, and we can express the previous property of uniqueness of identities in a new way: in the type theory with Id-types of small categories, there is a derivable judgment of the form

$$\begin{aligned} &f : \Pi_{x:\mathrm{Ob}} \operatorname{Hom}(x,x), \\ &p : \Pi_{x,y:\mathrm{Ob}}\Pi_{g:\operatorname{Hom}(x,y)} \mathrm{Id}_{\operatorname{Hom}(x,y)}(g, g \circ f(x)) \times \mathrm{Id}_{\operatorname{Hom}(x,y)}(g, f(y) \circ g) \\ &\quad \vdash t : \mathrm{Id}_{\Pi_{x:\mathrm{Ob}} \operatorname{Hom}(x,x)}(f, \mathrm{id}) \end{aligned} \tag{5.41}$$

The explicit description of the term t does not really matter for now. This judgment says that given a dependent function f such that for any inhabitant $g : \operatorname{Hom}(x,y)$

there is an identification of $g \circ f(x)$ with g and an identification of $f(y) \circ g$ with g, then one can craft an identification of f with id. The explicit form of t is a formal version of the following argument: because $f(x) \equiv f(x) \circ \mathrm{id}_x$ there is an identification between them; by hypothesis, by putting $y = x$ and $g = \mathrm{id}_x$ in p, there is also an identification between id_x and $f(x) \circ \mathrm{id}_x$ (formally it is given by $\pi_2(p(x,x)(\mathrm{id}_x)))$; as we shall see identifications respect transitivity, which yields the wanted identification between $f(x)$ and id_x.

The syntax is now extended so that the inductive definition of term-expressions over $\mathscr{A}$ contains the rule

- For every term-expression e, the string refl_e is a term-expression.
- For every term-expressions e_1, e_2, e_3, e_4, the string $\mathsf{j}(e_1, e_2, e_3, e_4)$ is a term-expression

The inductive definition of type-expressions over $(\mathscr{A}, \mathscr{B})$ is also modified so that the following rule becomes available:

- For term-expressions e_1, e_2 and a type-expression B, the string $\mathrm{Id}_B(e_1, e_2)$ is a type-expression.

The substitution is extended as well with the rules:

$$\begin{aligned} \mathrm{Id}_B(e_1, e_2)[x \leftarrow t] &= \mathrm{Id}_{B[x \leftarrow t]}(e_1[x \leftarrow t], e_2[x \leftarrow t]) \\ \mathsf{refl}_e[x \leftarrow t] &= \mathsf{refl}_{e[x \leftarrow t]} \\ \mathsf{j}(e_1, e_2, e_3, e_4)[x \leftarrow t] &= \mathsf{j}(e_1[x \leftarrow t], e_2[x \leftarrow t], e_3[x \leftarrow t], e_4[x \leftarrow t]) \end{aligned} \tag{5.42}$$

We complete the set of rules with the following ones.

Id-introduction rules. Any type has an associated identity type, and there is only one way to get out of nowhere an identification of an inhabitant with itself: the reflexivity of a given inhabitant, which can be thought as the constant path.

$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma, x : A, y : A \vdash \mathrm{Id}_A(x, y) \ \mathsf{type}} \qquad \frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma, x : A \vdash \mathsf{refl}_x : \mathrm{Id}_A(x, x)} \tag{5.43}$$

NOTATION 5.1.20. To ease a little the reading of the next rules, we shall write $E(x_1, \dots, x_n)$ when we want to emphasize that the free variables of the type- or term-expression E are among $x_1, \dots, x_n$. Then given terms $t_1, \dots, t_n$, the string $E(t_1, \dots, t_n)$ denotes the simultaneous substitution of the x_i's by the t_i's in E.

Id-elimination rules. The elimination rule for Id-types is kind of subtle. It roughly says that whatever can be constructed for an inhabitant $x : A$ by means of the constant path refl_x can also be transported to any other inhabitant $y : A$ through an identification from x to y. This is sometimes called *path induction*. This has many consequences that we shall explore further afterwards.

$$\frac{\Gamma, x : A, y : A, p : \mathrm{Id}_A(x, y) \vdash C(x, y, p) \ \mathsf{type} \quad \Gamma, x : A \vdash c(x) : C(x, x, \mathsf{refl}_x)}{\Gamma, x : A, y : A, p : \mathrm{Id}_A(x, y) \vdash \mathsf{j}(c, x, y, p) : C(x, y, p)}$$
(5.44)

Id-**computation rules.** The computation rule states that transporting along the constant path is just doing nothing.

$$\frac{\Gamma, x : A, y : A, p : \mathrm{Id}_A(x,y) \vdash C(x,y,p) \text{ type} \quad \Gamma, x : A \vdash c(x) : C(x,x,\mathsf{refl}_x)}{\Gamma, x : A \vdash \mathsf{j}(c,x,x,\mathsf{refl}_x) \equiv c : C(x,x,\mathsf{refl}_x)}$$

(5.45)

Definition 5.1.21. A MLTT *with* Σ-,Π- *and* Id-*types* is a MLTT with Σ- and Π-types where judgments can contain extended type- and term-expressions and where derivability is defined by adding the new set of rules concerning Id-types.

When the type A of terms t and u is inferable or irrelevant, we write $\mathrm{Id}(t,u)$ instead of $\mathrm{Id}_A(t,u)$. It is usual in the literature to find the vocabulary *propositionally equal*[2] for terms t and u such that $\mathrm{Id}(t,u)$ is inhabited. This is to be compared with *definitionally equal* terms that we defined earlier and denoted $t \equiv u$. Remark that definitional equality implies propositional equality: indeed from $\Gamma \vdash A$ type, $\Gamma \vdash t : A$ and $\Gamma \vdash t \equiv u : A$, one can construct the following derivation tree

$$\frac{\Gamma \vdash \mathsf{refl}_t : \mathrm{Id}_A(t,t) \qquad \dfrac{\Gamma, y : A \vdash \mathrm{Id}_A(t,y) \text{ type} \quad \Gamma \vdash t \equiv u : A}{\Gamma \vdash \mathrm{Id}_A(t,t) \equiv \mathrm{Id}_A(t,u) : A}}{\Gamma \vdash \mathsf{refl}_t : \mathrm{Id}_A(t,u)} \tag{5.46}$$

The leaves $\Gamma \vdash \mathsf{refl}_t : \mathrm{Id}_A(t,t)$ and $\Gamma, y : A \vdash \mathrm{Id}_A(t,y)$ type directly follow from the introduction rules of Id-types in which t is substituted. So that in the end the following rule is derivable in any MLTT with Id-types:

$$\frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash \mathsf{refl}_t : \mathrm{Id}_A(t,u)} \tag{5.47}$$

Extensional vs intentional. This is where intentional and extensional Martin-Löf type theories diverge. Extensional type theory require the following extra rule, called *equality reflection*:

$$\frac{\Gamma \vdash p : \mathrm{Id}_A(t,u)}{\Gamma \vdash t \equiv u : A} \tag{5.48}$$

This is morally a converse for the previous derivable rule. It says that propositionally equal terms are definitionally equal. In regards to the motivation we gave about Id-types, it is a very reasonable rule: identity-types give access to definitional equality as hypothesis in the context and are no longer relegated to the sole conclusion of judgments. But it has a nasty effect, that is better understood from a computational point of view: it makes definitional equality and type checking undecidable! A proof is given by Hofmann in his thesis [Hof95]. The argument is basically a diagonal argument much like Gödel's encoding of the liar paradox. The details of Hofmann's proof are not really relevant for this work and we will not retranscript it here. However it is quite easy to understand the informal idea hidden behind the proof: if an algorithm were to decide if two given terms t and u are definitionally equal in an extensional type theory, it would have to decide when the equality reflection rule is needed; in particular, from $t \equiv u$ and the fact that the rule have been used in order to obtained that, it would have to "guess" the inhabitant $p : \mathrm{Id}_A(t,u)$ that the rule has been using. The rest of the proof by Hofmann concentrates on showing that there is a case in which such a

[2] Beside in this section, we try to avoid this terminology, because of the confusion that it can bring.

p cannot be computed. From there, because the derivability of $\Gamma \vdash \mathsf{refl}_t : \mathrm{Id}_A(t,u)$ is now equivalent to the derivability of $\Gamma \vdash t \equiv u : A$, type-checking of a term also becomes undecidable.

So although the equality reflection rule seems natural, it is not practical if the goal is to use MLTT to implement a proof assistant by means of a type-checker. Incidentally, discarding the rule opens a new world of possibilities. If we are to think of inhabitants of $\mathrm{Id}_A(x,y)$ as the "witnesses that x and y are equal", then intentional type theory favor proofs of equality over the mere existence of such a proof: this is usually referred to as *proof-relevance*. This is crucial for the development of *Homotopy Type Theory*. By allowing inhabitants $x : A$ and $y : A$ to be definitionally distinct but yet related by multiple identifications, the type A is given the structure of a graph instead of just a set (in the extensional version, the type would have been identified with the connected component of this graph). But recall that every type comes with a identity type, in particular for the type $\mathrm{Id}_A(x,y)$, there is a type $\mathrm{Id}_{\mathrm{Id}(x,y)}(p,q)$ for any identification p and q between x and y: so these paths p and q can be identified in many ways even if not definitionally equal; and so on. It gives to the type A not only the structure of a graph but the structure of an ω-graph. The rules of Id-types equip this ω-graph with "composition operations" that actually makes it an ∞-groupoid. If the reflection rule was required, the ∞-groupoid structure on a type A would collapse onto the set of connected component, making all this higher structure unavailable.

At this point, the reader could be surprise that there is only one computation rule for Id-types, where there where systematically two such rules for other type formers. The expected second rule would have been the following:

$$\frac{\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash C(x,y,p)\ \mathsf{type} \qquad \Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash c(x,y,p) : C(x,y,p)}{\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash \mathsf{j}(c(x,x,\mathsf{refl}_x), x, y, p) \equiv c(x,y,p) : C(x,y,p)} \tag{5.49}$$

Informally this rule states that transporting a general term that have been restricted to the constant path recover the general term itself. Again, it seems quite natural to require such a computation rule. However, applied with the type $C(x,y,p) = A$ and $c(x,y,p) = x$ (after weakening), it gives the definitional equality

$$\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash \mathsf{j}(x,x,y,p) \equiv x : A \tag{5.50}$$

Now applied with the same type A but with $c(x,y,p) = y$ to find

$$\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash \mathsf{j}(x,x,y,p) \equiv y : A \tag{5.51}$$

By transitivity of definitional equality, it follows that:

$$\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash x \equiv y : A \tag{5.52}$$

Given terms t and u of type A, substitution gives back the equality reflection rule:

$$\dfrac{\dfrac{\dfrac{\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash x \equiv y : A \qquad \Gamma \vdash t : A}{\Gamma, y : A, p : \mathrm{Id}(t,y) \vdash t \equiv y : A} \qquad \Gamma \vdash u : A}{\Gamma, p : \mathrm{Id}(t,u) \vdash t \equiv u : A} \qquad \Gamma \vdash p' : \mathrm{Id}(t,u)}{\Gamma \vdash t \equiv u : A} \tag{5.53}$$

That is to say that from computation rule (5.49) is recovered the reflection rule one want to avoid. So we should restrain from adopting this computation rule.

Consequences of path induction. Even without the reflection rule (hence preserving the decidability of definitional equality), the elimination rule for Id-types has great implications.

First the so-called *indiscernibility of identicals*. It states that the following rule is derivable:

$$\frac{\Gamma, x : A \vdash C(x) \text{ type}}{\Gamma, x, y : A, p : \mathrm{Id}(x, y) \vdash \mathsf{trans}_p : C(x) \longrightarrow C(y)} \tag{5.54}$$

where trans_p stands for "transport along p" and is defined as the term $\mathsf{j}(\lambda c.c, x, y, p)$ obtained by introduction as follows:

$$\frac{\Gamma, x, y : A, p : \mathrm{Id}(x, y) \vdash C(x) \longrightarrow C(y) \text{ type} \quad \Gamma, x : A \vdash \lambda c.c : C(x) \longrightarrow C(x)}{\Gamma, x, y : A, p : \mathrm{Id}(x, y) \vdash \mathsf{j}(\lambda c.c, x, y, p) : C(x) \longrightarrow C(y)} \tag{5.55}$$

Through the computation rule is then derived the following equality judgment:

$$\Gamma, x : A, c : C(x) \vdash \mathsf{trans}_{\mathsf{refl}_x}(c) \equiv c : C(x) \tag{5.56}$$

Another way to view transport is to curryfy on the path p, to obtain a function $\mathsf{trans} : \mathrm{Id}(x, y) \times C(x) \longrightarrow C(y)$. If one think of the type $C(x)$ as a type of some kind of *structure* on x, then this form trans convey the idea that taking a structure on x and a way to rewrite x into y yields a structure of the same kind on y. In particular, carefully chosen transports allow to construct inverses and compositions for paths, giving a first glance at the structure of ∞-groupoids of types. Indeed, taking $C(x)$ to be $\mathrm{Id}(x, z)$, the transport rule yields $\mathsf{trans}_p : \mathrm{Id}(x, z) \longrightarrow \mathrm{Id}(y, z)$ for any path $p : \mathrm{Id}(x, y)$. By substituting z for x and applying the resulting function to refl_x on gets the following judgment:

$$\Gamma, x, y : A, p : \mathrm{Id}(x, y) \vdash \mathsf{trans}_p[z \leftarrow x](\mathsf{refl}_x) : \mathrm{Id}(y, x) \tag{5.57}$$

The term $\mathsf{trans}_p[z \leftarrow x](\mathsf{refl}_x)$ will be more simply denoted p^{-1}. The notation will be justified afterwards. Remark also that $\mathsf{refl}_x{}^{-1} \equiv \mathsf{refl}_x$. Hence to any path from x to y is associated a path from y to x, which we called its (pseudo)inverse. The next application of transport is composition. This time take $C(x)$ to be $\mathrm{Id}(z, x)$ in order to obtain the following judgment:

$$\Gamma, x, y : A, p : \mathrm{Id}(x, y) \vdash \mathsf{trans}_p : \mathrm{Id}(z, x) \longrightarrow \mathrm{Id}(z, y) \tag{5.58}$$

From any path $q : \mathrm{Id}(z, x)$ is obtained a path $\mathsf{trans}_p(q) : \mathrm{Id}(z, y)$ that is more simply denoted $p \circ q$ or even pq. The computation rule gives in this case that $\mathsf{refl}_x \circ q \equiv q$ for any path $q : \mathrm{Id}(z, x)$. So refl_x acts as definitional left neutral for the composition. It is not realistic to expect it to be also a definitional right neutral, but it is possible to give it the structure of a propositional right neutral. Indeed, because in particular $\mathsf{refl}_x \circ \mathsf{refl}_x \equiv \mathsf{refl}_x$, rule (5.47) derives the judgment:

$$\Gamma, x : A \vdash \mathsf{refl}_{\mathsf{refl}_x} : \mathrm{Id}_{\mathrm{Id}(x,x)}(\mathsf{refl}_x, \mathsf{refl}_x \circ \mathsf{refl}_x) \tag{5.59}$$

From there the elimination rule can be applied on the type $C(x, y, p) = \mathrm{Id}_{\mathrm{Id}(x,y)}(p, p \circ \mathsf{refl}_x)$ to obtain the judgment:

$$\Gamma, x : A, y : A, p : \mathrm{Id}(x, y) \vdash \mathsf{j}(\mathsf{refl}_{\mathsf{refl}_x}, x, y, p) : \mathrm{Id}_{\mathrm{Id}(x,y)}(p, p \circ \mathsf{refl}_x) \tag{5.60}$$

In the same manner, the elimination rule can be used to prove that the composition operation $\circ$ is propositionally associative. Indeed, by what is preceding, there is a definitional equality $\mathsf{refl}_x \circ (q \circ r) \equiv q \circ r \equiv (\mathsf{refl}_x \circ q) \circ r$, yielding through (5.47) the judgment

$$\Gamma, x, z, w : A, r : \mathrm{Id}\,(w,z)\,, q : \mathrm{Id}\,(z,x) \vdash \mathsf{refl}_{qr} : \mathrm{Id}_{\mathrm{Id}(w,x)}\,(\mathsf{refl}_x(qr), (\mathsf{refl}_x q)r) \tag{5.61}$$

To this term can be applied the elimination rule with $C(x,y,p) = \mathrm{Id}\,(p(qr),(pq)r)$ to get in the end a term judgment:

$$\begin{aligned}\Gamma, x, y, z, w : A, r : \mathrm{Id}\,(w,z)\,, q : \mathrm{Id}\,(z,x)\,,\\ p : \mathrm{Id}\,(x,y) \vdash \mathsf{j}(\mathsf{refl}_{qr}, x, y, p) : \mathrm{Id}\,(p(qr),(pq)r)\end{aligned} \tag{5.62}$$

Now the name of "pseudo inverse" for p^{-1} can make sense. Using elimination from the term-judgment (5.59) and for the type $C(x,y,p) = \mathrm{Id}\left(\mathsf{refl}_x, p \circ p^{-1}\right)$ (this make sense as $\mathsf{refl}_x{}^{-1} \equiv \mathsf{refl}_x$ as already pointed out) yields a term-judgment:

$$\Gamma, x, y : A, p : \mathrm{Id}\,(x,y) \vdash \mathsf{j}(\mathsf{refl}_{\mathsf{refl}_x}, x, y, p) : \mathrm{Id}\left(\mathsf{refl}_x, p \circ p^{-1}\right) \tag{5.63}$$

That is p^{-1} is a propositional right inverse for p (relatively to the propositional neutral refl_x). A similar argument shows that it is also a propositional left inverse.

To sum things up, we did show that: propositional equality is an equivalence relation on paths. But we did much more, because we did not just prove that some Id-types were merely inhabited, we gave explicit constructions of inhabitants, which we can compute with. This is to be taken into account, otherwise our type theory would just be a fancy presentation of Bishop's setoids.

Uniqueness principle for identity types. Although there can be non definitionally equal paths between given inhabitants $x : A$ and $y : A$, it is reasonable to ask whether two paths always are propositionally equal. Meaning precisely, is the following type always inhabited?

$$\Gamma, x, y : A, p, q : \mathrm{Id}\,(x,y) \vdash \mathrm{Id}_{\mathrm{Id}(x,y)}\,(p,q)\ \mathsf{type} \tag{5.64}$$

This statement is known as UIP (Uniqueness of Identity Proofs). Hofmann and Streicher showed in their seminal paper [HS96] that UIP can not be derived for every MLTT. However, there is still a uniqueness principle involving identity types. Given the type

$$\Gamma, x, y : A, p : \mathrm{Id}\,(x,y) \vdash \mathrm{Id}_{\Sigma_{z:A}\mathrm{Id}(x,z)}\,((x,\mathsf{refl}_x),(y,p))\ \mathsf{type} \tag{5.65}$$

the elimination rule can be applied on the term-judgment

$$\Gamma, x : A \vdash \mathsf{refl}_{(x,\mathsf{refl}_x)} : \mathrm{Id}\,((x,\mathsf{refl}_x),(x,\mathsf{refl}_x)) \tag{5.66}$$

to obtain the following term-judgment

$$\Gamma, x, y : A, p : \mathrm{Id}\,(x,y) \vdash \mathsf{j}(\mathsf{refl}_{(x,\mathsf{refl}_x)}, x, y, p) : \mathrm{Id}\,((x,\mathsf{refl}_x),(y,p)) \tag{5.67}$$

This says that every inhabitant of $\Sigma_{z:A}\mathrm{Id}\,(x,z)$ is propositionally equal to (x,refl_x). Otherwise put this type $\Sigma_{z:A}\mathrm{Id}\,(x,z)$ is propositionally a singleton. In the interpretation of propositional equalities as paths, it says that the type is contractible. So even if two paths $p : \mathrm{Id}\,(x,y)$ and $q : \mathrm{Id}\,(x,y)$ are not necessarily propositionally equal, they become such when "liberating" the y extremity: formally (y,p) ans (y,q) are propositionally equal in $\Sigma_{z:A}\mathrm{Id}\,(x,z)$. The situation is comparable to situations encountered in topology: e.g. on the circle $S^1 \subseteq \mathbb{C}$, the "upper" arc $t \mapsto \mathrm{e}^{i\pi t}$ and the "lower" arc $t \mapsto \mathrm{e}^{-i\pi t}$ from 1 to -1 are not homotopic relatively to their boundary, but they are relatively to their starting point; and for the simplest reason, each of them being homotopic relatively to the starting point to the constant path at 1.

5.1.5 Universes and univalence

The last extension we wish to make to the syntax concerns *universe types*. Like Π-types and Id-types bring respectively term and term equality judgments as first-class citizen, universe types translate type judgment (hence type equality judgment also) into the syntax. A universe $\mathbb{U}$ in a MLTT is roughly a type whose inhabitants are the other types. In such a MLTT type judgments $\Gamma, x : A \vdash B$ type become term judgments $\Gamma \vdash \ulcorner B \urcorner : A \longrightarrow \mathbb{U}$. Universe types make it possible to quantify over dependent types, so that rules of the form

$$\frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma, \Delta \vdash \mathcal{J}} \tag{5.68}$$

can be rewritten as simple judgments

$$\Gamma, B : A \longrightarrow \mathbb{U}, \Delta \vdash \mathcal{J} \tag{5.69}$$

As an extreme example, the presence of a universe $\mathbb{U}$ allows to state the elimination rule for Id-types as the mere existence of the following term

$$\vdash \mathsf{j} : \Pi_{\Gamma : \mathbb{U}} \Pi_{A : \Gamma \to \mathbb{U}} \Pi_{C : \Pi_{x,y:A} \mathrm{Id}_A(x,y)} \Big((\Pi_{x:A} C(x, x, \mathsf{refl}_x)) \longrightarrow \Pi_{x,y:A} \Pi_{p : \mathrm{Id}_A(x,y)} C(x, y, p) \Big) \tag{5.70}$$

To formalize this idea properly, we need to modify slightly our syntax one last time, by allowing the following rule in the inductive definition of type-expressions:

- Each term-expression is a type-expression.[3]

Definition 5.1.22. A *universe* in a MLTT with Σ-,Π-,Id-types is a type-expression U such that are derivable the following rules

$$\frac{}{\vdash U \text{ type}} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash A \text{ type}}$$

$$\frac{\Gamma \vdash A : U \quad \Gamma, x : A \vdash B : U}{\Gamma \vdash \Sigma_{x:A} B : U} \qquad \frac{\Gamma \vdash A : U \quad \Gamma, x : A \vdash B : U}{\Gamma \vdash \Pi_{x:A} B : U} \tag{5.71}$$

$$\frac{\Gamma \vdash A : U}{\Gamma, x : A, y : A \vdash \mathrm{Id}_A(x, y) : U}$$

Definition 5.1.23. A MLTT *with Σ-types, Π-types,* Id*-types and a cumulative hierarchy of universes*, or *full* MLTT for short, is a MLTT with with Σ-,Π-,Id-types together with the data $\mathbb{U}$ of universes $(U_i)_{i \geq 0}$ such that are derivable the following rules for all $i \geq 0$

$$\frac{}{\vdash U_i : U_{i+1}} \qquad \frac{\Gamma \vdash A : U_i}{\Gamma \vdash A : U_{i+1}} \tag{5.72}$$

and such that for all type judgment $\Gamma \vdash A$ type there exists $i \geq 0$ such that $\Gamma \vdash A : U_i$ is derivable.

[3]This way, types **are** terms of some type that will be axiomatized to be a universe. The resulting universe types are called *universes à la Russel*. The other way to do it is to keep distinct the syntax of types and terms as before, and add a constructor El(A) giving a type-expression for each term-expression A. The universe types of this method are called *universes à la Tarski*. It is much more cumbersome to work with and offers mainly formal advantages that we are not concerned about in this work.

Where the height on the hierarchy is not relevant, we might write $\mathbb{U}$ in place of the U_i's. So that the last part of the definition becomes

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A : \mathbb{U}} \tag{5.73}$$

And in particular, the cumulative hierarchy allows to write $\vdash \mathbb{U} : \mathbb{U}$ without worries.

Remark 5.1.24. Until now we have tried to give example of MLTTs modeling algebraic objects. The presence of universes is not usual for such examples, and the rules of such universes are crafted to present MLTTs whose purpose is to model a foundation for mathematics. This is what is implicitly understood in the literature when referring to **the** (intentional) Martin-Löf's type theory. In the language developed here, it corresponds to the full MLTT on the alphabet $\mathscr{A} = \emptyset$ and $\mathscr{B} = \emptyset$ with no axiomatic judgment, or maybe more realistically to the full MLTT on the alphabet $\mathscr{A} = \{0, s, \mathsf{rec}\}$ and $\mathscr{B} = \{\mathbb{N}\}$ with axiomatic judgments:

$$\begin{aligned}
&\vdash \mathbb{N} \text{ type}\\
&\vdash 0 : \mathbb{N}\\
&\vdash s : \mathbb{N} \longrightarrow \mathbb{N}\\
&\vdash \mathsf{rec} : \Pi_{C : \mathbb{N} \to \mathbb{U}} (C(0) \longrightarrow (\Pi_{n:\mathbb{N}} (C(n) \longrightarrow C(n+1))) \longrightarrow \Pi_{n:\mathbb{N}} C(n))
\end{aligned} \tag{5.74}$$

where $\mathbb{N}$ model the natural numbers with 0 being its initial element, s the successor function and rec the induction principle for natural numbers.

Identity and equivalence types of types. One of the most impressive gain in expressiveness through universes is the availability of identity types between two types (seen as inhabitants of the universe). More precisely, in the presence of a cumulative hierarchy $\mathbb{U}$, for any two types $A, B : U_i$ one can craft $\mathrm{Id}_{U_i}(A, B) : U_{i+1}$ by the last rule of (5.71) applied as $\vdash U_i : U_{i+1}$. Ignoring heights in the hierarchy, it is simply stated as:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathrm{Id}_{\mathbb{U}}(A, B) \text{ type}} \tag{5.75}$$

Informally speaking, the inhabitants of this type $\mathrm{Id}(A, B)$ are the possible identifications of the type A as a whole with the type B as a whole. But there is another type that can be constructed in a full MLTT, which can be understood as the pointwise identifications of the type A with the type B, namely:

$$\mathrm{Eq}(A, B) = \Sigma_{f : A \to B} \Pi_{y : B} \mathsf{isContr}(\Sigma_{x:A} \mathrm{Id}(f(x), y)) \tag{5.76}$$

where $\mathsf{isContr}(A)$ is syntactic sugar for $\Sigma_{a:A} \Pi_{z:A} \mathrm{Id}(a, z)$. Let us unfold the construction of this type:

- $\mathsf{isContr}(A)$ is the type of pairs (a, p) with $a : A$ and p a dependent function associated to each $z : A$ a path $p(x)$ from a to z; in other words it is the type of those $a : A$ on which A can contract. As soon as $\mathsf{isContr}(A)$ is inhabited, A is contractible (i.e. a propositional singleton).
- The type $\Sigma_{x:A} \mathrm{Id}(f(x), y)$ for a given $f : A \longrightarrow B$ and $y : B$ is the type of pairs (x, p) with $x : A$ and p a path from $f(x)$ to y in B. Otherwise put it is the homotopy fiber $f^{-1}(y)$.

- $\mathrm{Eq}(A, B)$ is then the type of pairs (f, c) with $f : A \longrightarrow B$ and c a dependent function associating to $y : B$ a witness $c(y)$ of the contractibility of the homotopy fiber $f^{-1}(y)$. Said otherwise, $\mathrm{Eq}(A, B)$ is the type of homotopy equivalence $f : A \longrightarrow B$.

For convenience, let us denote $\mathsf{isEquiv}(f)$ for the type $\Pi_{y:B}\mathsf{isContr}(\Sigma_{x:A}\,\mathrm{Id}\,(f(x), y))$ of witnesses that f is an homotopy equivalence, so that $\mathrm{Eq}(A, B)$ just becomes $\Sigma_{f:A\to B}\mathsf{isEquiv}(f)$.

Univalence. The *univalence axiom* refers to a revolutionary idea of Voevodsky. This axiom is the formal counterpart to a genuine activity that mathematicians experiment in a daily basis: identifying isomorphic structures. For a group theorist, there is not doubt that the subgroup $\{-1, 1\}$ of the circle S^1 *is* the cyclic group of order 2, as *is* the quotient of $\mathbb{Z}$ by its ideal $2\mathbb{Z}$, or as *is* the set $\{0, 1\}$ when defining 0 as neutral and $1 + 1$ as 0. The "is" of the previous sentence certainly behaves in a bizarre way, because each of the three **distinct** presented objects *is* the cyclic group of order 2. The usual way for the mathematician to get around this issue[4] is to arbitrarily choose the cyclic group of order 2 among all its representatives, say the quotient $\mathbb{Z}/2\mathbb{Z}$, and then consider the other representatives only as isomorphic to the chosen one. It is not an issue as long as only group-theoretic properties are considered, in the sense that if another representative were to be chosen as the canonical one, precisely the same group-theoretic properties would hold. Otherwise put, isomorphic groups are the same from the point of view of propositions on them. The univalence axiom takes advantage of the built-in Id-types of MLTTs, whose purpose are precisely to designated *propositionally* equal objects even when definitionally distinct.

Formally, for any type $\Gamma \vdash A$ type, the identity function $\lambda x.x : A \longrightarrow A$, also denoted id_A, is a homotopy equivalence. Indeed, the homotopy fiber of id_A at $a : A$ is given by $\Sigma_{x:A}\,\mathrm{Id}\,(a, x)$, which has already been proved contractible when discussing the *uniqueness principle for identity types* in section 5.1.4. So that there is a term:

$$\Gamma \vdash \mathsf{idIsEquiv}_A : \mathsf{isEquiv}(\mathrm{id}_A) \tag{5.77}$$

To be completely precise, the term $\mathsf{idIsEquiv}_A$ is actually given by

$$\lambda a.((a, \mathsf{refl}_a), \lambda z.\mathsf{j}(\mathsf{refl}_{(a,\mathsf{refl}_a)}, a, \pi_1(z), \pi_2(z)))$$

However, the explicit form does not matter much and knowing the type $\mathsf{isEquiv}(\mathrm{id}_A)$ is inhabited is enough. The elimination rule for $\mathrm{Id}_{\mathbb{U}}(-,-)$ can then be applied:

$$\frac{\Gamma, A, B : \mathbb{U}, p : \mathrm{Id}\,(A, B) \vdash \mathrm{Eq}\,(A, B)\ \mathsf{type} \qquad \Gamma, A : \mathbb{U} \vdash (\mathrm{id}_A, \mathsf{idIsEquiv}_A) : \mathrm{Eq}\,(A, A)}{\Gamma, A, B : \mathbb{U}, p : \mathrm{Id}\,(A, B) \vdash \mathsf{j}((\mathrm{id}_A, \mathsf{idIsEquiv}_A), A, B, p) : \mathrm{Eq}\,(A, B)}$$
(5.78)

Curryfying on p then A and B and the variable of type Γ (seen as an iterated Σ-type), one obtains a dependent function

$$\vdash \mathsf{IdtoEq} : \Pi_{\gamma:\Gamma}\Pi_{A,B:\mathbb{U}}\,\mathrm{Id}\,(A, B) \longrightarrow \mathrm{Eq}\,(A, B) \tag{5.79}$$

where IdtoEq stands for $\lambda\gamma.\lambda A.\lambda B.\lambda p.\mathsf{j}((\mathrm{id}_A, \mathsf{idIsEquiv}_A), A, B, p)$. The univalence axiom states that the following type is inhabited:

$$\vdash \Pi_{\gamma:\Gamma}\Pi_{A,B:\mathbb{U}}\mathsf{isEquiv}(\mathsf{IdtoEq})\ \mathsf{type} \tag{5.80}$$

[4] Actually most mathematicians do not even consider this an issue. All of the things spelled out here occur at a very informal level for the working mathematician, and only through his/her mathematical maturity is he/she able to get past these considerations with confidence in the consistency of the objects manipulated.

5.2 A primer on tribes

Tribes have been introduced by André Joyal through several talks before releasing a first set of notes [Joy17] very recently. Tribes are an attempt to support the types-as-fibrations philosophy, initiated by Awodey and Warren in [AW09], and design a minimal framework to interpret MLTT's. They are reminiscent of Brown's fibration categories.

Joyal's process in designing tribes is very incremental, starting from the notion of *clan* that interprets Σ-MLTT's, then moving on to Π-*clans* to add interpretation for Π-types, and finally culminating in the notion of *tribe* to implement Id-types. Actually, in the same fashion that Id-types does not require the presence of Π-types in MLTT's, a tribe can be define on top of a mere clan. The structure interpreting MLTT's with Σ-,Π- and Id-types is then called a Π-tribe, namely a tribe with underlying clan being a Π-clan and a good interaction with the tribe structure. Adding universes and stating univalence in these structures is certainly possible and was presented by Joyal in various workshops and conferences, but is not part of the notes [Joy17]. We will review all these notions and results about them here, referring directly to [Joy17] when possible, and trying to fill the gaps when needed. We claim no originality in this presentation, except for the propositions already mentioned in the very beginning of chapter 5, and we just follow Joyal's treatment of his theory. It is worth noting that most results on clans and Π-clans were already worked out earlier under the name *display maps* by Hyland and Pitts in [HP89] and by Streicher in [Str92], themselves building on work by Taylor in his thesis [Tay86].

5.2.1 Clans

Recall that a map $y : Y \longrightarrow B$ in a category $\mathfrak{C}$ is carrable if for any $f : A \longrightarrow B$ there exists a pullback square as follows

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "y"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{5.81}$$

The vertical map $X \longrightarrow A$ is called a base change of y. A class of maps $\mathfrak{F}$ is said to be closed under base changes if every elements of $\mathfrak{F}$ is carrable and any base change of such an element is again in $\mathfrak{F}$.

Definition 5.2.1. A *clan structure* on a category $\mathfrak{C}$ with terminal object 1 is a class of maps $\mathfrak{F}$ of $\mathfrak{C}$ such that

(i) for every object A, the unique map $A \longrightarrow 1$ is in $\mathfrak{F}$,

(ii) every isomorphism is in $\mathfrak{F}$,

(iii) $\mathfrak{F}$ is closed under base change,

(iv) $\mathfrak{F}$ is closed under composition.

A *clan* is a category with a clan structure.

Every clan $\mathcal{C}$ has an associated MLTT with Σ-types $\mathbb{T}_{\mathcal{C}}$. This theory contains:

- a type judgment $\vdash A$ type for each object A of $\mathcal{C}$,
- a judgment $x : A \vdash p(x)$ type for each fibration $p : B \longrightarrow A$,
- a judgment $x : A \vdash t(x) : p(x)$ for each section $t : A \longrightarrow B$ of the fibration $p : B \longrightarrow A$.

There is also a type equality judgment

$$\vdash \Sigma_{x:A} p(x) \equiv B \text{ type} \tag{5.82}$$

for each fibration $p : B \longrightarrow A$. Composition of fibrations reflects Σ-types, by which is meant that $\mathbb{T}_{\mathcal{C}}$ contains type equality judgments of the from (5.83) for every fibrations $p : B \longrightarrow A$ and $q : C \longrightarrow B$.

$$x : A \vdash (p \circ q)(x) \equiv \Sigma_{y:p(x)} q(x, y) \text{ type} \tag{5.83}$$

Weakening is obtained by pullback, meaning that if diagram (5.84) below is a (chosen) pullback diagram with p, q, r fibrations then one add the following type equality judgment: $x : C \vdash p(\pi_1(x)) \equiv r(x)$ type (notice that $\pi_1(x)$ makes sense for $x : C$ as C is definitionally equal to $\Sigma_{y:A} q(y)$).

$$\begin{tikzcd} D \arrow[r] \arrow[d, "r"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & A \end{tikzcd} \tag{5.84}$$

Because sections of the fibrations of the form $\varpi_1 : A \times A' \longrightarrow A$ are necessarily given by (id_A, u) for $u : A \longrightarrow A'$, we can make the abuse to write $x : A \vdash u(x) : \varpi_1(x)$ instead of $x : A \vdash (\mathrm{id}_A, u)(x) : \varpi_1(x)$. And because ϖ_1 is the pullback of the fibration $A : A \longrightarrow 1$ along the fibration $A' \longrightarrow 1$, the rules of definitional equality makes it meaningful to write $x : A \vdash u(x) : A'$ for a map $u : A \longrightarrow A'$. As fibrations are also just maps in $\mathcal{C}$, we shall now add to $\mathbb{T}_{\mathcal{C}}$ a term equality judgment

$$x : B \vdash p(x) \equiv \pi_1(x) : A \tag{5.85}$$

for any fibration $p : B \longrightarrow A$. Generic pullbacks of fibration can now be encoded in the theory $\mathbb{T}_{\mathcal{C}}$ as substitutions, meaning that whenever the diagram (5.84) is a (chosen) pullback diagram with p and r fibrations (but not necessarily q anymore), then the theory $\mathbb{T}_{\mathcal{C}}$ should contain a type equality judgment as in (5.86).

$$x : C \vdash r(x) \equiv (p(y)\,[y \leftarrow q(x)] \text{ type}) \tag{5.86}$$

If moreover $t : A \longrightarrow B$ is a section of p and $s : C \longrightarrow D$ is the section of r deduced from t, then the following term equality judgment is added to $\mathbb{T}_{\mathcal{C}}$:

$$x : C \vdash s(x) \equiv (t(y)\,[y \leftarrow q(x)]) \tag{5.87}$$

In particular, if a in an inhabitant of the *plain* type A (that is $a : 1 \longrightarrow A$ in $\mathcal{C}$), then the fiber of a dependent type $p : B \longrightarrow A$ at a is the type $p(a)$:

$$\begin{tikzcd} p(a) \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd} \tag{5.88}$$

Given fibrations $p : A \longrightarrow B$ and $q : C \longrightarrow B$, a section t of p and a section u of the pullback of the pullback of q along t as in diagram (5.89), one can construct a section of the fibration pq as $q^* t \circ u$.

$$\begin{tikzcd} t^*C \arrow[r, "q^*t"] \arrow[d, "t^*q"'] & C \arrow[d, "q"] \\ A \arrow[r, "t"'] \arrow[dr, equal] & B \arrow[d, "p"] \\ & A \end{tikzcd} \tag{5.89}$$

One need to add a judgment to $\mathbb{T}_{\mathcal{C}}$ to identify this section with the dependent pair of t and u:

$$x : A \vdash (q^* t \circ u)(x) \equiv (t(x), u(x)) : \Sigma_{y : p(x)} q(x, y) \tag{5.90}$$

Conversely every section s of pq can be decomposed into a section qs of p and a section s' of $(qs)^* q$ such that $q^*(qs) \circ s' = s$. One should then add judgments in $\mathbb{T}_{\mathcal{C}}$ identifying these two sections with the dependent projection of s:

$$\begin{aligned} x : A &\vdash \pi_1(s(x)) \equiv qs(x) : p(x) \\ x : A &\vdash \pi_2(s(x)) \equiv s'(x) : q(x, y)\,[y \leftarrow qs(x)] \end{aligned} \tag{5.91}$$

Remark 5.2.2. What is the utility of $\mathbb{T}_{\mathcal{C}}$? This is the *internal type theory* of the clan $\mathcal{C}$, and it gives the possibility to "prove" statement about $\mathcal{C}$ in the syntax of the type theory: any derivable judgment in $\mathbb{T}_{\mathcal{C}}$ can be reinterpreted as a result in $\mathcal{C}$. The presentation above is a mapping between fibrations in $\mathcal{C}$ and types in $\mathbb{T}_{\mathcal{C}}$, and between morphisms of $\mathcal{C}$ and terms in $\mathbb{T}_{\mathcal{C}}$. We like to emphasize that these mapping are **not** one-to-one correspondence because pullbacks are only defined up to isomorphisms in $\mathcal{C}$. Even if we choose pullbacks in $\mathcal{C}$, the chosen pullback over a composite morphism will only be isomorphic to the composition of the pullbacks. But in $\mathbb{T}$ the types and terms found by substituting twice are definitionally with the ones found by substituting by the composite. So for example, if one can derive a judgment like (5.92) in the theory $\mathbb{T}_{\mathcal{C}}$,

$$x : A \vdash t(x) \equiv t'(x) : p(x) \tag{5.92}$$

it should not be taken as a result that the maps corresponding to t and t' in $\mathcal{C}$ are equal. For all we know the domain and codomain of t and t' can even be distinct! However they will be isomorphic objects in $\mathcal{C}$, and t ad t' will be isomorphic as arrows of $\mathcal{C}$. This kind of subtleties about the coherence of those isomorphisms relating definitionally equal entities of the internal type theory of $\mathcal{C}$ has been studied (in slightly different frameworks) by several authors, among which Curien [Cur93] that solves the issue by incorporating explicit substitutions in the syntax, and Hofmann [Hof94] that shows how to "strictify" the pseudo functorial substitution. We shall not bother with these subtleties here and just take the derivability in $\mathbb{T}_{\mathcal{C}}$ has an incentive to reprove the statement in $\mathcal{C}$ from an external point of view. In this sense, a derivable judgment in $\mathbb{T}_{\mathcal{C}}$ is a "hint" that some property might be true in $\mathcal{C}$, and the derivation tree that goes with it is a "road map" for a proof of the said property. This kind of thinking is illustrated in this chapter at proposition 5.2.14 that draws on the type-theoretic observation of remark 5.2.9.

Remark 5.2.3. One could be tempted to take axiom (iv) out from the definition of clan to interpret only algebraic MLTT's (which is to say Σ-types). However, as demonstrated by the description above, Σ-types are necessary to the correct interpretation of weakening and substitution in a types-as-fibrations philosophy. It can be circumvent if objects interpret contexts instead of mere types, and morphisms interpret dependent tuples of terms instead of mere terms, but this solution is just a way to talk about Σ-types without naming them.

We will end up with the same kind of behavior when generalizing clans and tribes in section 5.4. In a sense, categorical semantics of dependent type theories can not do without Σ-types. This is to relate with example 5.1.15 in which is illustrated how Σ-types are merely syntactic sugar on top of algebraic MLTT's, and does not add expressiveness per se.

5.2.2 Π-clans

Recall that given a carrable morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$, one can define a substitution functor

$$f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A \tag{5.93}$$

that takes a map $q : Y \longrightarrow B$ to a map $f^*q : X \longrightarrow A$ such that there is a pullback square of the form (5.94).

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f^*q"'] & Y \arrow[d, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{5.94}$$

The *internal product along* f of a morphism $p : X \longrightarrow A$ is defined, if exists, as a morphism $\Pi_f p : Y \longrightarrow B$ such that exists $\varepsilon : f^*\Pi_f p \longrightarrow p$ making $(\Pi_f p, \varepsilon)$ a terminal object in the category $\left(f^* \downarrow p\right)$. Whenever internal products along f exist for all objects of $\mathcal{C}/A$, it defines a functor

$$\Pi_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B \tag{5.95}$$

which is right adjoint to the substitution functor f^*. We shall make the abuse of writing $\Pi_f X$ for the domain of $\Pi_f p$ when not harmful.

Definition 5.2.4. A Π-*clan* is a clan in which every fibration $p : X \longrightarrow A$ admits an internal product along any fibration $f : A \longrightarrow B$ such that $\Pi_f p : \Pi_f X \longrightarrow B$ is a fibration.

Given a clan $\mathcal{C}$, follow Joyal's notation and denote $\mathcal{C}(A)$ for the *local clan* above $A \in \mathcal{C}$, that is the full subcategory of $\mathcal{C}/A$ spanned by the objects $X \longrightarrow A$ that are fibrations in $\mathcal{C}$. Then $\mathcal{C}(A)$ is again a clan where fibrations are those maps h such that the image of h through the forgetful functor $\mathcal{C}(A)$ is a fibration in $\mathcal{E}$. Definition 5.2.4 seems a little odd: a Π-clan $\mathcal{C}$ is a clan in which for each fibration $f : A \longrightarrow B$, and for each $p \in \mathcal{C}(A)$ there is an object $\Pi_f p$ in $\mathcal{C}/B$ which is cofree for the substitution functor $f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A$; moreover this cofree object is an object of the local tribe $\mathcal{C}(B)$, producing *de facto* a functor

$$\Pi_f : \mathcal{C}(A) \longrightarrow \mathcal{C}(B) \tag{5.96}$$

which acts as a kind of "partial right adjoint" to the substitution functor f^*. The next proposition aims to state the definition of a Π-clan in a more conceptually clear form. It is due to Paige North (cf. [Nor17, Proposition 2.4.3]), rewritten here in full details to make this section self-contained.

Proposition 5.2.5. *A clan $\mathcal{C}$ is a Π-clan if and only if it satisfies both*

(i) the (restricted) substitution functor $f^* : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$ admits a right adjoint, denoted Π_f, for every fibration $f : A \longrightarrow B$ of $\mathcal{C}$,

(ii) every pullback square as in (5.97) with f (hence f') a fibration meets the Beck-Chevalley condition, meaning that the mate $g^*\Pi_f \longrightarrow \Pi_{f'}(g')^*$ is a natural isomorphism.

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{5.97}$$

Proof. Suppose first that both conditions of the proposition are met. Then exploit the fact that for a carrable morphism $q : Y' \longrightarrow B$, the morphisms $g \longrightarrow q$ in $\mathcal{C}/B$ are in natural bijection with the section of $g^*q : Z \longrightarrow Y$, which in turn identifies with the morphism $\mathrm{id}_Y \longrightarrow g^*q$ in $\mathcal{C}/Y$. Applied to $q = \Pi_f p$, for some fibrations f and p with same codomain, it gives:

$$\mathcal{C}/B\left(g, \Pi_f p\right) \simeq \mathcal{C}/Y\left(\mathrm{id}_Y, g^*\Pi_f p\right) \simeq \mathcal{C}(Y)\left(\mathrm{id}_Y, g^*\Pi_f p\right) \tag{5.98}$$

where the last isomorphism comes from the fact that both id_Y and $g^*\Pi_f p$ are fibrations and that $\mathcal{C}(Y)$ is a full subcategory of $\mathcal{C}/Y$. Form the pullback of f and g in $\mathcal{C}$ (f is carrable) and obtain a pullback square as in (5.97). Using the Beck-condition, the last hom-set is in natural bijection with $\mathcal{C}(Y)\left(\mathrm{id}_Y, \Pi_{f'}(g')^* p\right)$. Now apply the adjunction property of Π_f to get:

$$\mathcal{C}(Y)\left(\mathrm{id}_Y, \Pi_{f'}(g')^* p\right) \simeq \mathcal{C}(Y)\left((f')^*\mathrm{id}_Y, (g')^* p\right) \tag{5.99}$$

Finally, remark that $(f')^*\mathrm{id}_Y$ is isomorphic in $\mathcal{C}(Y)$ to id_X, and reuse the first trick to obtain:

$$\mathcal{C}(Y)\left(\mathrm{id}_X, (g')^* p\right) \simeq \mathcal{C}/Y\left(\mathrm{id}_X, (g')^* p\right) \simeq \mathcal{C}/Y\left(g', p\right) \tag{5.100}$$

It remains to realize that g' has been defined as the pullback of g along f so that $g' = f^*g$, and by putting (5.98), (5.99) and (5.100) together, one gets: $\mathcal{C}/B\left(g, \Pi_f p\right) \simeq \mathcal{C}/Y\left(f^*g, p\right)$. This expresses exactly that $\Pi_f p$ is cofree on p relatively to the functor f^*.

Conversely, suppose $\mathcal{C}$ is a Π-clan and let us prove both conditions of the statement. The first one is easily taken care of: given fibrations f, p, g

$$\mathcal{C}(A)\left(f^*g, p\right) \simeq \mathcal{C}/A\left(f^*g, p\right) \simeq \mathcal{C}/B\left(g, \Pi_f p\right) \simeq \mathcal{C}(B)\left(g, \Pi_f p\right) \tag{5.101}$$

where the first and last isomorphisms are following from $\mathcal{C}(X)$ being a full subcategory of $\mathcal{C}/X$ for any X, and the middle one is the cofree property of $\Pi_f p$ on p relatively to $f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A$. To prove the Beck-Chevalley condition given a pullback square as in (5.97), first write for any object $p \in \mathcal{C}(A)$ the following chain of isomorphisms, natural in q:

$$\begin{aligned} \mathcal{C}(Y)\left(q, g^*\Pi_f p\right) &\simeq \mathcal{C}/B\left(gq, \Pi_f p\right) \\ &\simeq \mathcal{C}/A\left(f^*(gq), p\right) \\ &\simeq \mathcal{C}/A\left(g' \circ (f')^* q, p\right) \\ &\simeq \mathcal{C}/X\left((f')^* q, (g')^* p\right) \\ &\simeq \mathcal{C}(Y)\left(q, \Pi_{f'}(g')^* p\right) \end{aligned} \tag{5.102}$$

All the isomorphisms are the adjunctions at play except the third one which is a direct use of the pullback pasting lemma. This natural transformation maps a morphism $h : q \longrightarrow g^*\Pi_f p$ to first the composite

$$gq \xrightarrow{g_! h} g \circ (g^*\Pi_f p) \xrightarrow{\varepsilon_{\Pi_f p}} \Pi_f p \tag{5.103}$$

where the first morphism is the image of h through the functor $g \circ -$, and where the second morphism is the counit of the adjunction $g \circ - \dashv g^*$. It then maps it to

$$f^*(gq) \xrightarrow{f^*(g_! h)} f^* g \circ (g^*\Pi_f p) \xrightarrow{f^*(\varepsilon_{f\,(p)})} f^*(\Pi_f p) \xrightarrow{\ \epsilon\ } p \tag{5.104}$$

where the last morphism is the counit morphism associated with the cofree object $\Pi_f p$ on p. Then comes the pullback pasting lemma, that replace the first morphism by an isomorphic one

$$\begin{tikzcd}
g' \circ (f')^* q \arrow[r, "g'_!(f')^*(h)"] \arrow[d, "\simeq"] & g' \circ (f')^*(g^*\Pi_f p) \arrow[d, "\simeq"] & & \\
f^*(gq) \arrow[r, "f^*(g_! h)"] & f^* g \circ (g^*\Pi_f p) \arrow[r, "f^*(\varepsilon_{f\,(p)})"] & f^*(\Pi_f p) \arrow[r, "\epsilon"] & p
\end{tikzcd} \tag{5.105}$$

The next isomorphism applies $(g')^*$ to that morphism and precompose it with the unit of the adjunction $g' \circ - \dashv (g')^*$, yielding

$$\begin{tikzcd}
(f')^* q \arrow[r, "(f')^*(h)"] & (f')^*(g^*\Pi_f p) \arrow[d, "\simeq"] & & \\
& (g')^* f^* g \circ (g^*\Pi_f p) \arrow[r, "(g')^* f^*(\varepsilon_{f\,(p)})"] & (g')^* f^*(\Pi_f p) \arrow[r, "(g')^*(\epsilon)"] & (g')^* p
\end{tikzcd} \tag{5.106}$$

The last isomorphism in (5.102) is given by the application of $\Pi_{f'}$ and the precomposition of the result with the unit of the corresponding cofree object of the form $\Pi_{f'} x$. In our case it gives the morphism:

$$\begin{tikzcd}
q \arrow[r, "h"] & g^*\Pi_f p \arrow[d, "\eta"] & & \\
& \Pi_{f'}(f')^*(g^*\Pi_f p) \arrow[d, "\simeq"] & & \\
& \Pi_{f'}(g')^* f^* g \circ (g^*\Pi_f p) \arrow[r, "\Pi_{f'}(g')^* f^*(\varepsilon_{f\,(p)})"] & \Pi_{f'}(g')^* f^*(\Pi_f p) \arrow[r, "\Pi_{f'}(g')^*(\epsilon)"] & \Pi_{f'}(g')^* p
\end{tikzcd}$$

(5.107)

But the composite morphism following h in (5.107) is precisely the mate $\mu_p : g^*\Pi_f p \longrightarrow \Pi_{f'}(g')^* p$. So the chain of natural bijections in (5.102) actually show that the image of μ_p through the Yoneda embedding is an isomorphism. However the Yoneda embedding is fully faithful, hence conservative, so that μ_p is an isomorphism in $\mathcal{C}(Y)$. This is true for any p, so that μ is a natural isomorphism. □

Π-clans are modeling MLTT's with Σ- and Π-types. To the MLTT $\mathbb{T}_{\mathcal{C}}$ associated with the clan $\mathcal{C}$, one add the judgment of (5.108) for each fibrations $p : X \longrightarrow A$ and $q : A \longrightarrow B$.

$$x : B \vdash (\Pi_q p)(x) \equiv \Pi_{y:q(x)} p(x,y) \text{ type} \tag{5.108}$$

Recall that $(\Pi_q p, \varepsilon : q^*\Pi_q p \longrightarrow p)$ is terminal in $(q^* \downarrow p)$, so that for a section t of p there exists a morphism

$$\Lambda_t : (\mathrm{id}_B, q^*\mathrm{id}_B \cong \mathrm{id}_A \xrightarrow{t} p) \longrightarrow (\Pi_q p, \varepsilon) \tag{5.109}$$

In particular Λ_t is a morphism $\mathrm{id}_B \longrightarrow \Pi_q p$ in $\mathcal{C}(B)$, hence it is a section of $\Pi_q p$ in $\mathcal{C}$. Then given a term $x : B, y : q(x) \vdash t(x,y) : p(x,y)$ defining a section t of p, the following axiom is added to $\mathbb{T}_{\mathcal{C}}$:

$$x : B \vdash \Lambda_t(x) \equiv \lambda y.t(x,y) : \Pi_{y:q(x)} p(x,y) \tag{5.110}$$

Conversely, given a section t of $\Pi_q p$, one gets a section q^*t of $q^*\Pi_q p$. Then $\varepsilon \circ q^*t$ is a section of p: indeed by definition of ε as a morphism in $\mathcal{C}(A)$, it holds that $p\varepsilon = q^*\Pi_q p$. We then add to $\mathbb{T}_{\mathcal{C}}$ the following equality judgment:

$$x : B, y : q(x) \vdash (\varepsilon \circ q^*t)(x,y) \equiv t(x)(y) : p(x,y) \tag{5.111}$$

5.2.3 Tribes and Id-types

This section introduces the structure needed on a clan to interprets Id-types.

Definition 5.2.6. Let $\mathcal{C}$ be a clan. An *anodyne* morphism in $\mathcal{C}$ is a morphism having the left lifting property against all fibrations.

In notation, the class of anodyne maps is ${}^{\boxslash}\mathfrak{F}$ when $\mathfrak{F}$ denotes the class of fibrations of the clan $\mathcal{C}$.

Lemma 5.2.7. *A map $u : A \longrightarrow B$ in a clan is anodyne if and only if for any commutative triangle as in (5.112) with p a fibration, there exists a section s of p such that $t = su$.*

$$\begin{tikzcd} & X \arrow[d, "p"] \\ A \arrow[ur, "t"] \arrow[r, "u"'] & B \end{tikzcd} \tag{5.112}$$

Proof. Suppose that u is anodyne and find a filler s for the outer commutative square of (5.113).

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[d, "u"'] & X \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed, "s"] & B \end{tikzcd} \tag{5.113}$$

Such a filler s satisfies both requirements: the upper triangle is exactly the second condition of the statement and the lower triangle expresses that s is a section of p.

Conversely if u satisfies the property of the statement, then given any commutative square as in (5.114) with p a fibration, then one can form the pullback y^*p of p along y.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "u"'] & X \arrow[d, "p"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \tag{5.114}$$

The commutative square of (5.114) gives a morphism $t : A \longrightarrow y^*X$ that makes the triangle in (5.114) commute.

$$\begin{tikzcd} & y^*X \arrow[d, "y^*p"] \\ A \arrow[ur, "t"] \arrow[r, "u"'] & B \end{tikzcd} \tag{5.115}$$

Moreover t is such that $x = p^*y \circ t$. By hypothesis, it exists then a section s of y^*p such that $su = t$. Now $h = p^*y \circ s$ is a filler for (5.114): indeed $hu = p^*y \circ t = x$ and $ph = pp^*y \circ s = yy^*ps = y$. □

This lemma makes a nice interpretation of anodyne maps in type theoretical terms. A map $u : A \longrightarrow B$ is anodyne if for any type p dependent on B the function of (5.116), that substitutes along u the terms of type p in the context A, is surjective.

$$\begin{gathered} \{t \mid x : B \vdash t : p(x)\} \longrightarrow \{t' \mid y : A \vdash t' : p(u(y))\} \\ t \longmapsto t\,[x \leftarrow u(y)] \end{gathered} \tag{5.116}$$

Definition 5.2.8. A *tribe* is a clan in which

(i) every morphism f factors as pu with p a fibration and u anodyne,

(ii) the pullback of an anodyne map along a fibration is anodyne.

The first axiom of a tribe is reminiscent of a model category and will prove crucial in modeling Id-types. The second axiom is less usual in from an homotopical point of view and is reminiscent of model categories in which the trivial cofibrations are stable under pullbacks (as in a Cisinski model structure for example). However it is clearly justified from the type-theoretical point of view: given a pullback square as in (5.117) with p a fibration, the theory $\mathbb{T}_{\mathcal{C}}$ derives that $u'(x, y) \equiv (u(x), y)$ in the type $\Sigma_{z:B}p(z)$ in the context $x : A, y : p(u(x))$.

$$\begin{tikzcd} u^*X \arrow[r, "u'"] \arrow[d, "u^*p"'] & X \arrow[d, "p"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{5.117}$$

Axiom (ii) of tribes, together with the interpretation of lemma 5.2.7 we made just before, states that if the substitution of terms along $u(x)$ is surjective then so is the term substitution along the dependent pair $(u(x), y)$, which seems quite reasonable.

Remark 5.2.9. Even better, in the presence of Π-types, this is automatically true. Indeed, from a judgment

$$x : A, y : p(u(x)) \vdash t(x, y) : q(u(x), y) \tag{5.118}$$

one can derive the term $\lambda y.t(x, y)$ of type $\Pi_{y:p(u(x))}q(u(x), y)$ which is precisely $\Pi_{y:p(z)}q(z, y)\,[z \leftarrow u(x)]$, so the surjectivity of the substitution along $u(x)$ applies and gives a term

$$z : B \vdash f(z) : \Pi_{y:p(z)}q(z, y) \tag{5.119}$$

such that $f(u(x)) \equiv \lambda y.t(x, y)$. It then follows that

$$x : A, y : p(u(x)) \vdash f(u(x))(y) \equiv t(x, y) : q(u(x), y) \tag{5.120}$$

Notice then that $f(u(x))(y)$ is just another notation for $f(\pi_1(w))(\pi_2(w))\,[w \leftarrow (u(x), y)]$ to conclude that the substitution along $(u(x), y)$ is surjective. In other words, u' ought to be anodyne.

Tribes have enough structure to interpret Id-types. The idea is to follow the usual construction of a very good path object in a model category. Given an object A, consider its diagonal $\Delta_A : A \longrightarrow A \times A$ and factor it as

$$A \xrightarrow{u} \mathrm{Id}_A \xrightarrow{p} A \times A \tag{5.121}$$

with u anodyne and p a fibration. Then add to $\mathbb{T}_{\mathcal{C}}$ the following judgments

$$\begin{aligned} x : A, y : A &\vdash p(x,y) \equiv \mathrm{Id}_A(x,y) \ \mathsf{type} \\ x : A &\vdash \mathsf{refl}_x \equiv u(x) : \mathrm{Id}_A(x,x) \end{aligned} \tag{5.122}$$

In particular Id_A appears as $\Sigma_{x,y:A} \mathrm{Id}_A(x,y)$. Given a fibration $q : C \longrightarrow \mathrm{Id}_A$ and a section $c : A \longrightarrow u^*C$ of the fibration $u^*q : u^*C \longrightarrow A$, one gets a commutative triangle as in (5.123) where $\bar{c}$ is the composite of c with the projection $u^*C \longrightarrow C$ from the pullback.

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "\bar{c}"] \arrow[r, "u"'] & \mathrm{Id}_A \end{tikzcd} \tag{5.123}$$

Because u is anodyne, there exists a section j of the fibration q such that $ju = \bar{c}$. This construction emulates the elimination rule of the identity type. More precisely, we now add to the theory $\mathbb{T}_{\mathcal{C}}$ the following judgment

$$x : A, y : A, \alpha : \mathrm{Id}_A(x,y) \vdash j \equiv \mathsf{j}(c,x,y,\alpha) : q(x,y,\alpha) \tag{5.124}$$

The discussion above describes how a tribe can model the identity type of a *plain* type A, meaning that it interprets $\mathrm{Id}_A(x,y)$ whenever $\vdash A$ type is derivable. However, Id-types are defined for dependent types with non empty contexts and those can not be emulated by Id-types of plain types: e.g. for a derivable type judgment $\gamma : \Gamma \vdash A(\gamma)$ type, there is a drastic difference between $\mathrm{Id}_{\Sigma_{z:\Gamma}A(z)}((\gamma,x),(\gamma,y))$ and $\mathrm{Id}_A(x,y)$ in the context $\gamma : \Gamma, x, y : A$; the inhabitants of the former allow for a non trivial identification from γ to itself, while the one in the latter forbid such a thing. Proposition 5.2.10 states that tribes are equipped with sufficient structure to also interpret Id-types in contexts and proposition 5.2.12 explains that these interpretations behave coherently under substitution.

Proposition 5.2.10. *Given an object A in a tribe $\mathcal{C}$, the local clan $\mathcal{C}(A)$ is a tribe.*

Proof. Lemma 5.2.7 has the immediate consequence that anodyne maps of $\mathcal{C}(A)$ are exactly the maps $u : p \longrightarrow q$ in $\mathcal{C}(A)$ such that $u : \mathrm{dom} p \longrightarrow \mathrm{dom} q$ is anodyne in $\mathcal{C}$. So given a map $f : p \longrightarrow q$ as in the left of (5.125), we can factor it in $\mathcal{C}$ as $f = p'u'$ with p' fibration of $\mathcal{C}$ and u' anodyne in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "p"'] & B \arrow[d, "q"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u'"] \arrow[dr, "p"'] & C' \arrow[r, "p'"] \arrow[d, "qp'"] & B \arrow[dl, "q"] \\ & C & \end{tikzcd} \tag{5.125}$$

We end up with a factorization of f in $\mathcal{C}(A)$ as a local anodyne map followed by a local fibration as depicted on the right of (5.125). Hence axiom (i) is satisfied by $\mathcal{C}(A)$. Axiom (ii) is immediate because pullbacks in $\mathcal{C}(A)$ coincide with pullbacks taken in $\mathcal{C}$. $\square$

Now one can do the same kind of construction as before but in the local tribe $\mathcal{C}(A)$, and add the same kind of equality judgment to $\mathbb{T}_{\mathcal{C}}$. More specifically, for any fibration $p : X \longrightarrow A$ in the tribe $\mathcal{C}$, the object p of $\mathcal{C}(A)$ admits a path objects Id_p together with a factorization

$$p \xrightarrow{u} \mathrm{Id}_p \xrightarrow{q} p \times p \tag{5.126}$$

with u and q respectively anodyne and fibration in $\mathcal{C}(A)$. Then is added in $\mathbb{T}_{\mathcal{C}}$ the following judgments:

$$\begin{aligned} &x : A, y, y' : p(x) \vdash \mathrm{Id}_{p(x)}(y, y') \equiv q(x, y, y') \ \mathsf{type} \\ &x : A, y : p(x) \vdash u(x, y) \equiv \mathsf{refl}_y : \mathrm{Id}_{p(x)}(y, y) \\ &x : A, y, y' : p(x), \alpha : \mathrm{Id}_{p(x)}(y, y') \vdash j \equiv \mathsf{j}(c, y, y', \alpha) : p'(x, y, y', \alpha) \end{aligned} \tag{5.127}$$

where in the last judgment, j refers to the section of $p' : C \longrightarrow \mathrm{dom}(\mathrm{Id}_p)$ defined by a section c of $u^* p'$ and the anodyne property of u in the following diagram:

$$\begin{tikzcd} u^*C \arrow[r] & C \arrow[d, "q"] \\ X \arrow[u, "c"] \arrow[r, "u"] & \mathrm{dom}(\mathrm{Id}_p) \end{tikzcd} \tag{5.128}$$

In particular the type $\mathrm{Id}_p(x)$ is definitionally equal to $\Sigma_{y,y' : p(x)} \mathrm{Id}_{p(x)}(y, y')$ in the context $x : A$.

Lemma 5.2.11. *Given a commutative triangle as in (5.129) in a clan $\mathcal{C}$, if v and w are anodyne, then so is u.*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "w"'] & B \arrow[d, "v"] \\ & C \end{tikzcd} \tag{5.129}$$

Proof. Anodyne morphisms are split monomorphisms: indeed given $v : B \longrightarrow C$ anodyne, it should lift against all fibration and especially against the fibration $B \longrightarrow 1$, so that the outer commutative square of (5.130) admits a filler r.

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, "v"'] & B \arrow[d] \\ C \arrow[r] \arrow[ur, dashed, "r"] & 1 \end{tikzcd} \tag{5.130}$$

Now given a commutative square as on the left in (5.131) with p a fibration, using v and its retraction r one obtains a diagram as the solid one on the right of (5.131).

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "u"'] & X \arrow[d, "p"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "u"] \arrow[dd, bend right, "w"'] & X \arrow[d, "p"] \\ B \arrow[d, "v"] & Y \\ C \arrow[uur, dashed, "h"] \arrow[ur, "yr"'] & \end{tikzcd} \tag{5.131}$$

Because w is anodyne and p a fibration, there is a filler $h : C \longrightarrow X$ that satisfies $hw = x$ and $ph = yr$. Denote then $h' = hv$, so that $h'u = hw = x$ and $ph' = yrv = y$. In other words, h' is a filler of the square on the left in (5.131), which proves that u is anodyne. □

Proposition 5.2.12. *Given a map $f : A \longrightarrow B$ in a tribe $\mathcal{C}$, the substitution functor*

$$f^* : \mathcal{C}(B) \longrightarrow \mathcal{C}(A) \tag{5.132}$$

maps anodyne morphisms of $\mathcal{C}(B)$ to anodyne morphisms of $\mathcal{C}(A)$.

Proof. First factor f as pu for some fibration p and some anodyne map u, and note that $f^* \cong u^* \circ p^*$. Hence we only need to prove the statement for f a fibration and for f anodyne to have prove it in full generality.

Suppose f is a fibration, and take an anodyne map $v : p \longrightarrow q$ in $\mathcal{C}(B)$. By the pasting lemma of pullbacks, the image $f^*(v)$ is the pullback of v along the morphism f' obtained as the pullback of f along q (see diagram (5.133)). Being the pullback of a fibration, f' is a fibration, so that axiom (ii) of tribes applies to show that $f^*(v)$ is anodyne.

$$\begin{tikzcd}
 & f^*X \arrow[rr, "f''"] \arrow[dl, "f^*(v)"'] \arrow[ddl, two heads, "f^*p" near end] & & X \arrow[dl, "v"'] \arrow[ddl, two heads, "p"] \\
f^*Y \arrow[rr, "f'" near start] \arrow[d, two heads, "f^*q"'] & & Y \arrow[d, two heads, "q"] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd} \tag{5.133}$$

Suppose now that f is anodyne, so that in diagram (5.133), f' and f'' are also anodyne as the respective pullbacks of f along the fibrations q and p. If v is anodyne, then so is the composite vf''. Lemma 5.2.11 allows to conclude that $f^*(v)$ is anodyne. □

This last proposition is giving a kind of stability of identity types under substitution. More precisely, let us say that a type of the form

$$\Gamma, x : A, y : A \vdash I(x, y) \ \mathsf{type} \tag{5.134}$$

derivable in a given MLTT is *identity-like* for A if there is a derivable term

$$\Gamma, x : A \vdash r_x : I(x, y) \tag{5.135}$$

such that there is a derivable rule

$$\frac{\Gamma, x : A, y : A, p : I(x, y) \vdash C(x, y, p) \ \mathsf{type} \quad \Gamma, x : A \vdash c(x) : C(x, x, r_x)}{\Gamma, x : A, y : A, p : I(x, y) \vdash j(c, x, y, p) : C(x, y, p)} \tag{5.136}$$

such that the following computation rule is derivable:

$$\frac{\Gamma, x : A, y : A, p : I(x, y) \vdash C(x, y, p) \ \mathsf{type} \quad \Gamma, x : A \vdash c(x) : C(x, x, r_x)}{\Gamma, x : A \vdash j(c, x, x, r_x) \equiv c : C(x, x, r_x)} \tag{5.137}$$

Then of course every identity type $\mathrm{Id}_A(x, y)$ is identity-like for A, but nothing prevent an MLTT to have other identity-like types for such an A. Proposition 5.2.12 states in particular that in $\mathbb{T}_{\mathcal{C}}$, if $q(x, y, y')$ is identity-like for $x : B \vdash p(x)$ type, then also $q(x, y, y')[x \leftarrow f(x')]$ is identity-like for $x' : A \vdash p(x)[x \leftarrow f(x')]$ type.

Remark 5.2.13. The reader has to realize that the judgment of the internal type theory $\mathbb{T}_{\mathcal{C}}$ and the properties and constructions that hold in $\mathcal{C}$ grow further apart. We already mentioned in remark 5.2.2 that the judgment of $\mathbb{T}_{\mathcal{C}}$ only reflects equalities and constructions in $\mathcal{C}$ up to isomorphisms, and that we should not rely completely on the theory $\mathbb{T}_{\mathcal{C}}$ to prove statement in $\mathcal{C}$. This is even worst now that we have introduced Id-types: any derivable judgment of $\mathbb{T}_{\mathcal{C}}$ that involves Id-types reflects equalities and constructions that hold in the tribe $\mathcal{C}$ only up to homotopy/weak equivalences.

An homotopy in the tribe $\mathcal{C}$ between parallel morphisms $f, g : A \longrightarrow B$ is the data of a map $A \longrightarrow P_B$ where P_B is a path object of B as presented in (5.138) such that $(f, g) = p_B h$

$$B \xrightarrow{u} P_B \xrightarrow{p_B} B \times B \tag{5.138}$$

A weak equivalence is then defined as an homotopy equivalence, namely a map $f : A \longrightarrow B$ such that there exists a map $g : B \longrightarrow A$ satisfying that gf and fg both are homotopic to the identities id_A and id_B respectively.

One can show that two path objects for the same object B in a tribe $\mathcal{C}$ are weakly equivalent (cf. [Joy17]). So the remark above on the stability of *identity-like* types by substitution can be restated as follows: the usual definitional equality

$$x : A \vdash \Sigma_{y,z : p(f(x))} \mathrm{Id}_{p(f(x))}(y, z) \equiv \left(\Sigma_{y,z : p(w)} \mathrm{Id}_{p(w)}(y, z)\right) [w \leftarrow f(x)] \text{ type} \tag{5.139}$$

is only to be interpreted in $\mathcal{C}$ as a weak equivalence, and not an isomorphism. So we emphasize again that, in this work,t the type theory $\mathbb{T}_{\mathcal{C}}$ and the statement it derives are used as a guide to prove statements in the tribe $\mathcal{C}$.

5.2.4 Π-tribes

Interpretation of Id-types provided by tribes does not draw on the presence of internal products for fibrations along fibrations as demanded by Π-clans. That is to say that the underlying clan of a tribe need not be a Π-clan. When this is the case, more structure is available, and this structure should behave correctly with the anodyne maps.

The next proposition shows how the presence of internal products in a clan helps to determine if it has the properties of a tribe. This result does not appear in Joyal's notes [Joy17] on tribes, so we give a detailed proof.

Proposition 5.2.14. *Given a Π-clan $\mathcal{C}$, if every map factors as an anodyne morphism followed by a fibration then $\mathcal{C}$ is a tribe.*

Proof. In order to show that $\mathcal{C}$ is a tribe, we only need to show axiom (ii) of tribes as axiom (i) is already part of the hypothesis. Suppose we have a pullback square as in (5.140) with p a fibration and u anodyne.

$$\begin{tikzcd} X \arrow[r, "u'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{5.140}$$

We need to show that u' is anodyne and to do so we shall use the characterization of lemma 5.2.7. Consider the following map between hom-sets for a fibration $q : Z \longrightarrow Y$:

$$\begin{aligned}\mathcal{C}(Y)(\mathrm{id}_Y, q) \cong \mathcal{E}(B)\left(\mathrm{id}_B, \Pi_p q\right) \longrightarrow \mathcal{C}(A)\left(\mathrm{id}_A, u^*\Pi_p q\right) \\ \cong \mathcal{C}(A)\left(\mathrm{id}_A, \Pi_{p'}(u')^* q\right) \\ \cong \mathcal{C}(X)\left((u')^*\mathrm{id}_Y, (u')^* q\right)\end{aligned} \tag{5.141}$$

where:

- the first isomorphism takes advantage of $\mathrm{id}_Y \cong p^*\mathrm{id}_B$ before using the adjunction $p^* \dashv \Pi_p$,
- the second map is the application of u^*, which is surjective by lemma 5.2.7 because u is anodyne, immediately followed by the fact that $\mathrm{id}_A \cong u^*\mathrm{id}_B$,
- the next isomorphism is postcomposing by the mate

$$\mu_q : u^*\Pi_p q \longrightarrow \Pi_{p'}(u')^* q \tag{5.142}$$

which is invertible by hypothesis on the clan $\mathcal{C}$,
- and the last isomorphism is given by the adjunction $(p')^* \dashv \Pi_{p'}$ immediately followed by the fact that $(p')^*\mathrm{id}_A \cong \mathrm{id}_X \cong (u')^*\mathrm{id}_Y$.

In particular this map is surjective. All it remains to show is that it actually is the maps of hom-set $\mathcal{C}(Y)(\mathrm{id}_Y, q) \longrightarrow \mathcal{C}(X)\left((u')^*\mathrm{id}_Y, (u')^* q\right)$ induced by the functor $(u')^*$. This is mostly a matter of writing in full the image $\varphi(t)$ of a section $t : Y \longrightarrow Z$ of q through the application described in (5.141), and reducing the morphism obtained in this way by ways of the definition of the mate μ_q, the natural isomorphism $(u')^* p^* \cong (p')^* u^*$ and the triangular identities of the adjunctions at play. This is summed up in the large diagram of figure 5.2. ☐

However a Π-tribe is not just a tribe with an underlying Π-clan. The internal products should behave properly relatively to the anodyne maps. This is made precise by Joyal's definition of a Π-tribe.

Definition 5.2.15. A Π-tribe is a tribe whose underlying clan is a Π-clan and such that the internal product functor

$$\Pi_f : \mathcal{C}(A) \longrightarrow \mathcal{C}(B) \tag{5.143}$$

along a fibration $f : A \longrightarrow B$ preserves anodyne maps.

Corollary 5.2.16. *A Π-clan for which Π_f preserves anodyne maps for each fibration f is a Π-tribe if and only if every maps factors as qj for some anodyne j and some fibration q.*

5.3 Relative factorization systems

To be able to present identity types in the same kind of settings than the usual equality predicate in Lawvere's hyperdoctrines, we start by putting together a framework where the anodyne maps in a tribe have similar properties as the cocartesian morphisms of an Grothendieck opfibration. Beyond the application in mind, this framework we called *relative factorization systems* is a structure of interest of its own that is probably hidden in other constructions of categorical logic. This part can be read without any knowledge of type theory and is merely a combinatorial study of specific structures of interest.

5.3.1 Definitions

Definition 5.3.1. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor and let φ, ψ, χ, v form a commutative square in $\mathcal{E}$ as follows:

$$\begin{tikzcd} X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & X' \arrow[d, "\psi"] \\ Y \arrow[r, "v"'] & Y' \end{tikzcd} \tag{5.144}$$

The morphism φ has the *left lifting property against ψ relatively to p* (equivalently, ψ has the *right lifting property against φ relatively to p*) when for all morphism $h : pY \longrightarrow pY'$ making the diagram on the right of (5.145) commutative, there is map η above h such that the diagram on the left is commutative. We denote $\varphi \boxslash_p \psi$.

$$\begin{tikzcd} X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & X' \arrow[d, "\psi"] \\ Y \arrow[r, "v"'] \arrow[ur, dashed, "\eta"] & Y' \end{tikzcd} \quad \overset{p}{\longmapsto} \quad \begin{tikzcd} pX \arrow[r, "p(\chi)"] \arrow[d, "p(\varphi)"'] & pX' \arrow[d, "p(\psi)"] \\ pY \arrow[r, "p(v)"'] \arrow[ur, "h"] & pY' \end{tikzcd} \tag{5.145}$$

If moreover such an η is unique, φ is said to has the *strict* left lifting property against ψ relatively to p (equivalently ψ has the *strict* right lifting property against φ relatively to p). We denote $\varphi \perp_p \psi$.

Remark 5.3.2. Clearly when $\mathcal{B} = 1$ is the terminal category, the usual notions of strict and weak left and right lifting properties are recovered.

The following definition is a slight variation on the notion of weak factorization system. No standard name is available for this notion in the literature.

Definition 5.3.3. A *right weak factorization system*, or right wfs for short, on a category $\mathcal{C}$ is the data of two classes $(\mathfrak{L}, \mathfrak{R})$ such that:

(i) $\mathfrak{L} = {}^{\boxslash}\mathfrak{R}$,

(ii) any morphism of $\mathcal{C}$ factors as qj for some $q \in \mathfrak{R}$ and $j \in \mathfrak{L}$.

So the main difference with a weak factorization system is that $\mathfrak{R}$ is not necessarily of the form $\mathfrak{L}^{\boxslash}$. As such it might not be closed under existing pullbacks and retracts. Of course, for any such right wfs $(\mathfrak{L}, \mathfrak{R})$, one have a honest weak factorization system $(\mathfrak{L}, ({}^{\boxslash}\mathfrak{R})^{\boxslash})$ on $\mathcal{C}$. However, a right wfs is *a priori* a finer notion than just a class $\mathfrak{R}$ such that $({}^{\boxslash}\mathfrak{R}, ({}^{\boxslash}\mathfrak{R})^{\boxslash})$ is a weak factorization system. Indeed, in the former every morphism factors as qj with $q \in \mathfrak{R}$ and $j \in {}^{\boxslash}\mathfrak{R}$, while in the latter q is merely an element of $({}^{\boxslash}\mathfrak{R})^{\boxslash}$. A prominent example of a right wfs is the couple (anodynes, fibrations) of a tribe.

Accordingly we have a version relative to a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$.

Definition 5.3.4. A *right weak factorization system relative to* $p : \mathcal{E} \longrightarrow \mathcal{B}$ consists of classes of morphisms $\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}$ of $\mathcal{E}$ and classes of morphisms $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of $\mathcal{B}$ such that:

(i) $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$,

(ii) the elements of $\mathfrak{L}_{\mathcal{E}}$ are exactly those φ such that $\varphi \boxslash_p \psi$ for every $\psi \in \mathfrak{R}$,

(iii) for every morphism φ of $\mathcal{E}$ and every factorization $p(\varphi) = qj$ with $q \in \mathfrak{R}_{\mathcal{B}}$ and $j \in \mathfrak{L}_{\mathcal{B}}$, there exists $\chi \in \mathfrak{R}_{\mathcal{E}}$ above q and $\iota \in \mathfrak{L}_{\mathcal{E}}$ above j such that $\varphi = \chi\iota$.

If moreover $\mathfrak{R}_{\mathcal{E}}$ happens to coincide with ${\mathfrak{L}_{\mathcal{E}}}^{\boxslash_p}$, then the system is just called a *weak factorization system relative to p*.

As much as weak factorization systems admit orthogonal factorization systems as strict counterpart, we can define *(right) orthogonal factorization system relative to p* by changing each $\boxslash_p$ by $\perp_p$ in the previous definition.

Remark 5.3.5. The two classes on the basis $\mathcal{B}$ need not have any kind of lifting property one against the other. In particular, whenever $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}})$ is a right weak factorization system relative to p, then so is $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}, \mathfrak{R})$ for any $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L} \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R} \subseteq \mathfrak{R}_{\mathcal{B}}$.

5.3.2 Grothendieck fibrations and relative factorization systems

Relative factorization systems are relevant to the study of Grothendieck fibrations because of the following propositions.

Proposition 5.3.6. *Let $\mathcal{E}$ and $\mathcal{B}$ have terminal objects, and let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor that preserves terminal objects. A morphism φ in $\mathcal{E}$ is cocartesian relatively to p if and only if $\varphi \perp_p \psi$ for every ψ in $\mathcal{E}$.*

Dually, let $\mathcal{E}$ and $\mathcal{B}$ have initial objects and let $p : \mathcal{E} \longrightarrow \mathcal{B}$ preserve them. Then φ is cartesian relatively to p if and only if $\psi \perp_p \varphi$ for every ψ in $\mathcal{E}$.

Proof. Suppose φ is cocartesian relatively to p. Then for any commutative square as in (5.144) and any lift h as in the right hand side of (5.145), one can use the cocartesian nature of φ to find η above h such that $\eta\varphi = \chi$. Now both $\psi\eta$ and v are answers to the universal problem of finding a map θ above $p(v) = p(\psi)h$ such that $\theta\varphi = \psi\chi$. Hence they are equal: $\psi\eta = v$.

Conversely, suppose $\varphi : X \longrightarrow Y$ verifies that $\varphi \perp_p \psi$ for any ψ of $\mathcal{E}$. Given a map $\chi : X \longrightarrow X'$, and a map $h : pY \longrightarrow pX'$ in $\mathcal{B}$ such that $hp(\varphi) = p(\chi)$, both diagram commute in (5.146).

$$\begin{tikzcd} X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & X' \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd} \quad \overset{p}{\longmapsto} \quad \begin{tikzcd} pX \arrow[r, "p(\chi)"] \arrow[d, "p(\varphi)"'] & pX' \arrow[d] \\ pY \arrow[r] \arrow[ur, dashed, "h"] & 1 \end{tikzcd} \tag{5.146}$$

Since φ has the left lifting property relatively to p against the map $X' \longrightarrow 1$, there exists a unique η above h such that $\eta\varphi = \chi$. Hence φ is cocartesian. ☐

Proposition 5.3.7. *A terminal object preserving functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ between categories with terminal objects is a Grothendieck opfibration if and only if there is a strict right factorization system $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}})$ relative to p with:*

$$\mathfrak{R}_{\mathcal{E}} = \mathrm{Mor}(\mathcal{E}), \qquad \mathfrak{L}_{\mathcal{B}} = \mathrm{Mor}(\mathcal{B}) \tag{5.147}$$

Proof. Suppose p is a Grothendieck opfibration. Then $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}})$ is a weak factorization system relative to p when:

- $\mathfrak{L}_{\mathcal{E}}$ is the class of cocartesian maps of $\mathcal{E}$,
- $\mathfrak{R}_{\mathcal{E}} = \mathrm{Mor}(\mathcal{E})$,
- $\mathfrak{R}_{\mathcal{B}} = \mathrm{Mor}(\mathcal{B})$,
- and $\mathfrak{L}_{\mathcal{B}} = \mathrm{Mor}(\mathcal{B})$

Indeed, surely $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$. Proposition 5.3.6 proves that $\mathfrak{L}_{\mathcal{E}}$ is exactly the class of maps having the left lifting property against those in $\mathfrak{R}_{\mathcal{E}}$ relatively to p. So it only remains to show the relative factorization property. Take $\chi : X \longrightarrow X'$ such that $p(\chi) = gf$ for some $f : pX \longrightarrow B$ and $g : B \longrightarrow pX'$. By hypothesis, there exists a cocartesian morphism $\varphi : X \longrightarrow Y$ above f. Because $gf = gp(\varphi) = p(\chi)$, there exists a morphism γ above g such that $\gamma\varphi = \chi$. Now φ is in $\mathfrak{L}_{\mathcal{E}}$ by definition and γ is in $\mathfrak{R}_{\mathcal{E}}$ vacuously.

Conversely, suppose that there is a strict right factorization system $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}})$ relative to p as in the statement. The condition $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$ forces $\mathfrak{R}_{\mathcal{B}}$ to contains any map of $\mathcal{B}$ and proposition 5.3.6 ensures that $\mathfrak{L}_{\mathcal{E}} = {}^{\boxslash_p}\mathfrak{R}_{\mathcal{E}}$ contains exactly the cocartesian maps. Now given an object $X \in \mathcal{E}$ and a morphisms $f : pX \longrightarrow B$ in $\mathcal{B}$, one can find a cocartesian map φ above f as follows: consider the unique map $X \longrightarrow 1$ from X to the terminal object of $\mathcal{E}$; it gets mapped though p to the unique map $pX \longrightarrow 1$ from pX to the terminal 1 of $\mathcal{B}$, which obviously factors through $f : pX \longrightarrow B$; as $f \in \mathfrak{L}_{\mathcal{B}}$ and $(B \longrightarrow 1) \in \mathfrak{R}_{\mathcal{B}}$ vacuously, it follows that there exists $\varphi : X \longrightarrow Y$ in $\mathfrak{L}_{\mathcal{E}}$ above f and $\psi : Y \longrightarrow 1$ in $\mathfrak{R}_{\mathcal{E}}$ above $B \longrightarrow 1$. Of course the latter gives no information, but the former exactly says that there exists a cocartesian morphism with domain X above f. ☐

5.3.3 Tribes and relative factorization systems

Given a clan $\mathcal{C}$ whose class of fibrations is $\mathfrak{F}_{\mathcal{C}}$, we shall make the abuse of also denoting by $\mathfrak{F}_{\mathcal{C}}$ the full subcategory of $\mathrm{Fun}(\mathbf{2}, \mathcal{C})$ spanned by the elements of $\mathfrak{F}_{\mathcal{C}}$, and we shall call it the *category of fibrations* of $\mathcal{C}$. The codomain functor restricted to the category of fibrations will be denoted $\mathfrak{p}_{\mathcal{C}} : \mathfrak{F}_{\mathcal{C}} \longrightarrow \mathcal{C}$. Cartesian morphism for $\mathfrak{p}_{\mathcal{C}}$ are the pullback square from a fibration to another. Because every fibration is carrable and that every pullback of it is still a fibration, $\mathfrak{p}_{\mathcal{C}}$ admits cartesian morphisms with codomain a given fibration $p : Y \longrightarrow B$ above any morphism $u : A \longrightarrow B$. In other words, $\mathfrak{p}_{\mathcal{C}}$ is a Grothendieck fibration. Its fiber at A is precisely the local clan denoted $\mathcal{C}(A)$ in section 5.2. In particular, the morphisms $p \longrightarrow q$ of the fiber $\mathcal{C}(A)$, for fibrations $p : X \longrightarrow A$ and $q : Y \longrightarrow A$, may be described as the morphisms $X \longrightarrow Y$ in $\mathcal{C}$ such that the triangle of (5.148) commutes.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & A \end{tikzcd} \tag{5.148}$$

Hence we shall not restrain from seeing such a morphism $p \longrightarrow q$ as in $\mathcal{C}$ when needed.

Recall from section 3.2.1 the notation $\varphi^{\triangleleft}$ for a morphism $\varphi : X \longrightarrow Y$ in the total category of a (cloven) Grothendieck fibration $\mathfrak{p}$: it is the unique morphism such that

$\mathfrak{p}(\varphi^{\triangleleft}) = \mathrm{id}_{\mathfrak{p}X}$ and $\rho \circ \varphi^{\triangleleft} = \varphi$ where ρ if the chosen cartesian morphism above $\mathfrak{p}(\varphi)$ with codomain Y.

Definition 5.3.8. In a clan $\mathcal{C}$, a *total fibration* is a morphism φ of $\mathfrak{F}_{\mathcal{C}}$ such that both $\mathfrak{p}_{\mathcal{C}}(\varphi)$ and $\varphi^{\triangleleft}$ are fibrations of $\mathcal{C}$.

The definition is reminiscent and inspired by the notion of total fibration of Quillen bifibrations (see chapter 3). Let us unfold it a little: a total fibration is a commutative square as the outer one of (5.149) such that both u and the *cartesian gap* g are fibrations.

$$\begin{tikzcd} X \arrow[dr, dashed, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "p"'] & & \\ & A \times_B Y \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\ & A \arrow[r, "u"'] & B \end{tikzcd} \tag{5.149}$$

Proposition 5.3.9. *Given a clan $\mathcal{C}$, a morphism $p \longrightarrow q$ of $\mathcal{F}_{\mathcal{C}}$ as depicted in (5.150) has the left lifting property against all total fibrations relatively to $\mathfrak{p}_{\mathcal{C}}$ if and only if f is anodyne.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{5.150}$$

Proof. Suppose that f is anodyne and that a commutative diagram of the form (5.151) is given such that the map $(g, v) : r \longrightarrow s$ on the right is a total fibration in $\mathfrak{F}_{\mathcal{C}}$.

$$\begin{tikzcd} X \arrow[rr, "z"] \arrow[dr, "f"] \arrow[dd, "p"'] & & Z \arrow[dr, "g"] \arrow[dd, "r", near end] & \\ & Y \arrow[rr, crossing over, "t", near start] & & T \arrow[dd, "s"] \\ A \arrow[rr, "c", near start] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\ & B \arrow[rr, "d"'] \arrow[ur, "h"] \arrow[from=uu, crossing over, "q", near start] & & D \end{tikzcd} \tag{5.151}$$

Because $st = vhq$, there is a morphism $h' : Y \longrightarrow T \times_D C$ into the pullback that makes (h', h) a morphism $q \longrightarrow v^* s$ in $\mathfrak{F}_{\mathcal{C}}$. Denote abusively $g^{\triangleleft} : Z \longrightarrow T \times_D C$ for the cartesian gap of the square on the right. Then the square of (5.152) commutes.

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "f"'] & Z \arrow[d, "g^{\triangleleft}"] \\ Y \arrow[r, "h'"'] & T \times_D C \end{tikzcd} \tag{5.152}$$

Indeed both $g^{\triangleleft} z$ and $h' f$ have projection $rz = hqf$ on C and projection $gz = tf$ on T. Note then that g is a fibration by hypothesis, and f is given to be anodyne, so there is a filler $k : Y \longrightarrow Z$. If we write π_C for the projection of $T \times_D C \longrightarrow C$, then

$rk = \pi_1 g^{\triangleleft} k = \pi_1 h' = hq$; it shows that (k, h) is indeed a map $q \longrightarrow r$ in $\mathfrak{F}_{\mathfrak{C}}$, which is a filler for the outer cube of (5.151). In other words, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"] \arrow[dd, "p"'] & & Z \arrow[r, "g^{\triangleleft}"] \arrow[drr, "g"] \arrow[dd, "r"] & T \times_D C \arrow[dr] \arrow[dl, "\pi_C"] & \\
& Y \arrow[ur, "k"] \arrow[urr, "h'"] \arrow[rrr, "t"] \arrow[dd, "q"] & & & T \arrow[dd, "s"] \\
A \arrow[rr, "c"] \arrow[dr, "u"] & & C \arrow[drr, "v"] & & \\
& B \arrow[ur, "h"] \arrow[rrr, "d"] & & & D
\end{tikzcd} \tag{5.153}$$

Conversely, if the map $p \longrightarrow q$ of (5.150) has the left lifting property against all total fibrations relatively to $\mathfrak{p}_{\mathfrak{C}}$, then in particular it lifts again the maps of the form of the square in the left of (5.154) with g a fibration. So that there is filler as in the cube on the right of (5.154).

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "!"'] & T \arrow[d, "!"] \\
1 \arrow[r, equals] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"] \arrow[dd, "p"'] & & Z \arrow[dr, "g"] \arrow[dd, "!"] & \\
& Y \arrow[ur, "k"] \arrow[rr, "t"] \arrow[dd, "q"] & & T \arrow[dd, "!"] \\
A \arrow[rr, "!"] \arrow[dr, "u"] & & 1 \arrow[dr, equals] & \\
& B \arrow[ur, "!"] \arrow[rr, "!"] & & 1
\end{tikzcd} \tag{5.154}$$

The top commutative square hence admits a filler $k : Y \longrightarrow Z$ for any z and t, meaning that f is anodyne. $\square$

Such a map $p \longrightarrow q$ having the left lifting property against all total fibrations relatively to $\mathfrak{p}_{\mathfrak{C}}$ will be called *totally anodyne*, or a *total anodyne map*.

Proposition 5.3.10. *A clan $\mathfrak{C}$ is a tribe if and only if*

(i) there is a right weak factorization system $(\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}}, \mathfrak{R}_{\mathfrak{F}_{\mathfrak{C}}}, \mathfrak{L}_{\mathfrak{C}}, \mathfrak{R}_{\mathfrak{C}})$ relative to $\mathfrak{p}_{\mathfrak{C}}$ where $\mathfrak{L}_{\mathfrak{C}}$ contains all morphisms of $\mathfrak{C}$ and $\mathfrak{R}_{\mathfrak{F}_{\mathfrak{C}}}$ contains exactly the total fibrations,

(ii) the cartesian maps above an anodyne morphism are totally anodyne.

Proof. Suppose the clan $\mathfrak{C}$ is a tribe. Then surely (ii) is satisfied: a cartesian map above an anodyne morphism u is a pullback square as in (5.155).

$$\begin{tikzcd}
X \arrow[r, "u'"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
A \arrow[r, "u"'] & B
\end{tikzcd} \tag{5.155}$$

In this situation axiom (ii) of tribes states that u' is anodyne, meaning precisely that (u', u) is totally anodyne in $\mathfrak{F}_{\mathfrak{C}}$. This takes care of the second condition of the statement. Let us focus on the first one: we claim that taking for $\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}}$ the class of totally anodyne maps, and for $\mathfrak{R}_{\mathfrak{C}}$ the fibration of $\mathfrak{C}$ completes the system $(\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}}, \mathfrak{R}_{\mathfrak{F}_{\mathfrak{C}}}, \mathfrak{L}_{\mathfrak{C}}, \mathfrak{R}_{\mathfrak{C}})$ into a right weak factorization system relative to $\mathfrak{p}_{\mathfrak{C}}$. Vacuously one gets $\mathfrak{p}_{\mathfrak{C}}(\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}}) \subset \mathfrak{L}_{\mathfrak{C}}$, and $\mathfrak{p}_{\mathfrak{C}}(\mathfrak{R}_{\mathfrak{F}_{\mathfrak{C}}}) \subset \mathfrak{R}_{\mathfrak{C}}$ is given by definition of the total fibrations. Total anodyne maps are defined to be the one having the left lifting property against total fibrations relatively to $\mathfrak{p}_{\mathfrak{C}}$ so that $\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}} = {}^{\boxslash_{\mathfrak{p}_{\mathfrak{C}}}}\mathfrak{R}_{\mathfrak{F}_{\mathfrak{C}}}$. It only remains to check the relative factorization property: for any map $(f, u) : p \longrightarrow q$, and given a factorization of u as wv in $\mathfrak{C}$ with w a fibration, start by factorizing vp as a anodyne map v' followed by a fibration p', as exhibited in the diagram (5.156).

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "v'"] \arrow[dd, "p"'] & & Y \arrow[dd, "q"] \\
& Z \arrow[ur, dashed, "h"] \arrow[dd, "p'"] & \\
A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \\
& C \arrow[ur, "w"'] &
\end{tikzcd} \tag{5.156}$$

Because v' is anodyne, q is a fibration and $qf = wp'v'$, there is a filler $h : Z \longrightarrow Y$ as in the diagram above. Then consider the cartesian gap $h^{\triangleleft} : Z \longrightarrow Y \times_B C$ and factor it as an anodyne map t' followed by a fibration r'. Denote $r = w^* q \circ r'$, $t = t'v'$ and $k = q^* w \circ r'$. The map $(t, v) : p \longrightarrow r$ is totally anodyne because t is anodyne in $\mathfrak{C}$ (as a composition of anodyne maps). The map $(k, w) : r \longrightarrow q$ is a total fibration because w is a fibration and its cartesian gap is the fibration r'. And finally $(f, u) = (k, w) \circ (t, v)$ as summed up in the diagram (5.157).

$$\begin{tikzcd}
X \arrow[rrrr, "f"] \arrow[drr, "t"] \arrow[dr, "v'"'] \arrow[dd, "p"'] & & & & Y \arrow[dd, "q"] \\
& Z \arrow[r, "t'"'] \arrow[urrr, bend left, "h"] \arrow[dd, "p'"] & Z' \arrow[r, "r'"'] \arrow[urr, "k"] \arrow[ddl, "r"] & Y \times_B C \arrow[ur] \arrow[ddll] & \\
A \arrow[rrrr, "u"] \arrow[dr, "v"'] & & & & B \\
& C \arrow[urrr, "w"'] & & &
\end{tikzcd} \tag{5.157}$$

Conversely, suppose that $\mathfrak{p}_{\mathfrak{C}}$ satisfies both conditions of the statement. By definition of a right weak factorization system, the class $\mathfrak{L}_{\mathfrak{F}_{\mathfrak{C}}}$ contains exactly the total anodyne maps, and $\mathfrak{R}_{\mathfrak{C}}$ contains at least the fibration of $\mathfrak{C}$ because for each fibration $p : X \longrightarrow A$ there is a total fibration in $\mathfrak{F}_{\mathfrak{C}}$, namely $(p, p) : \mathrm{id}_X \longrightarrow \mathrm{id}_A$, such that $\mathfrak{p}_{\mathfrak{C}}$ maps it to p. Now take $f : A \longrightarrow B$ in $\mathfrak{C}$ and consider the unique morphism $\mathrm{id}_A \longrightarrow \mathrm{id}_1$ in $\mathfrak{F}_{\mathfrak{C}}$ whose image $! : A \longrightarrow 1$ through $\mathfrak{p}_{\mathfrak{C}}$ certainly factors through f; by assumption then there exists a fibration $p : Y \longrightarrow B$ and two maps $u : A \longrightarrow Y$ and $q : Y \longrightarrow 1$ such that $(q, !) \circ (u, f) = (!, !)$ with (u, f) totally anodyne and $(q, !)$ a total fibration. Of course q must be the unique map $! : Y \longrightarrow 1$ and does not bring information, but the total anodyne map $(u, f) : \mathrm{id}_A \longrightarrow p$ precisely gives a factorization of f as pu with

p a fibration and u anodyne, as wanted.

$$
\begin{tikzcd}
A \arrow[rr, "!"] \arrow[dr, "u"'] \arrow[dd, equal] & & 1 \arrow[dd, equal] \\
& Y \arrow[ur, "q"'] \arrow[dd, "p"] & \\
A \arrow[rr, "!", crossing over] \arrow[dr, "f"'] & & 1 \\
& B \arrow[ur, "!"'] &
\end{tikzcd}
\tag{5.158}
$$

It only remains to show that pullback of anodyne along fibrations are anodyne: such a pullback as in (5.155) of u along a fibration q produces a cartesian map $(u', u) : p \longrightarrow q$ which is totally anodyne when u is anodyne, which mean by proposition 5.3.9 that u' is anodyne when u is. □

Corollary 5.3.11. *A Π-clan $\mathcal{C}$ is a tribe if and only if it satisfies condition (i) of proposition 5.3.10.*

Proof. This is a direct use of proposition 5.2.14 and the fact, already displayed in the previous proof, that axiom (ii) of tribes are equivalent to condition (ii) of proposition 5.3.10. □

5.4 Relative tribes

In this section, we take the presentation of tribes given in section 5.3.3 in the language of relative factorization systems as an incentive to design a new notion of model of intentional type theory: moving away from categories with display maps (or clans), we build a framework based on Grothendieck fibration equipped with a good relative factorization system. That is we try to make a conceptual leap similar to the one given by Lawvere's hyperdoctrines compared to the subobject interpretation of a (first-order) theory. In doing so, we obtain the interpretation of identity types as an *up-to-homotopy* version of the usual equality predicate of a Lawvere hyperdoctrine.

5.4.1 Comprehension systems

Definition 5.4.1. A *comprehension system* is the data of a Grothendieck fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ together with

(i) a section $\star : \mathcal{B} \longrightarrow \mathcal{E}$ such that $\star(f)$ is cartesian for every morphism f of $\mathcal{B}$,

(ii) and a functor $\{-\} : \mathcal{E} \longrightarrow \mathcal{B}$, right adjoint to $\star$.

Given an object X of $\mathcal{E}$ in such a system, the counit $\varepsilon_X : \star\{X\} \longrightarrow X$ of the adjunction at X furnishes a morphism $p(\varepsilon_X) : \{X\} \longrightarrow pX$. Let us denote this morphism by pr_X. This construction extend to a full functor

$$
\mathsf{pr} : \mathcal{E} \longrightarrow \mathrm{Fun}(\mathbf{2}, \mathcal{B}) \tag{5.159}
$$

as follows: given a map $f : X \longrightarrow Y$ in $\mathcal{E}$, the naturality of the counit ε makes the square on the left of (5.160) commute; take its image through p and it still commutes as on the right of (5.160); then define pr_f as this commutative square.

$$\begin{tikzcd} \star\{X\} \arrow[r, "\star\{f\}"] \arrow[d, "\varepsilon_X"'] & \star\{Y\} \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \overset{p}{\longmapsto} \quad \begin{tikzcd} \{X\} \arrow[r, "\{f\}"] \arrow[d, "\mathsf{pr}_X"'] & \{Y\} \arrow[d, "\mathsf{pr}_Y"] \\ pX \arrow[r, "p(f)"'] & pY \end{tikzcd} \tag{5.160}$$

Now that pr is defined on both the object and the morphism in a coherent way, its functoriality is obvious from the functoriality of p and $\{-\}$.

Proposition 5.4.2. *The functor* $\mathsf{pr} : \mathcal{E} \longrightarrow \mathrm{Fun}(\mathbf{2}, \mathcal{B})$ *defines a morphism from* p *to* $\mathrm{cod} : \mathrm{Fun}(\mathbf{2}, \mathcal{B}) \longrightarrow \mathcal{B}$ *that preserves cartesian morphisms.*

Proof. It is clear that $\mathrm{cod} \circ \mathsf{pr} = p$, so we focus on the preservation of cartesian morphisms. Suppose that $f : X \longrightarrow Y$ is a cartesian morphism in $\mathcal{E}$, the goal is to prove that the diagram on the right of (5.160) is cartesian for cod, meaning that it is a pullback square. Given $u : C \longrightarrow pX$ and $v : C \longrightarrow \{Y\}$ maps of $\mathcal{B}$ such that $\mathsf{pr}_Y v = p(f)u$, we can construct $v^\natural : \star C \longrightarrow Y$ the adjoint map of v through $\star \dashv \{-\}$. Then

$$p(v^\natural) = p(\varepsilon_Y \circ \star(v)) = \mathsf{pr}_Y v \tag{5.161}$$

and because f is cartesian, there must exist a unique g above u such that $fg = v^\natural$. In turn such a g corresponds bijectively to an adjoint map $g^\flat : C \longrightarrow \{X\}$ such that $\mathsf{pr}_X g^\flat = u$ and $\{f\} g^\flat = v$.

$$\begin{tikzcd} \star C \arrow[drr, bend left, "v^\natural"] \arrow[dr, dashed, "g"'] & & \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \quad \overset{p}{\longmapsto} \quad \begin{tikzcd} C \arrow[drr, bend left, "v"] \arrow[dr, dashed, "g^\flat"] \arrow[ddr, bend right, "u"'] & & \\ & \{X\} \arrow[r, "\{f\}"] \arrow[d, "\mathsf{pr}_X"'] & \{Y\} \arrow[d, "\mathsf{pr}_Y"] \\ & pX \arrow[r, "p(f)"'] & pY \end{tikzcd} \tag{5.162}$$

□

In other words, any comprehension system in the sense of definition 5.4.1 induces a comprehension category as defined by Bart Jacobs in [Jac93].

The pair $(p, \star)$ of a comprehension system is supposed to define an algebraic MLTT $\mathbb{T}_p$ as follows. For each object A of $\mathcal{B}$, there is a type judgment

$$\vdash A \ \mathsf{type} \tag{5.163}$$

and for each object X of $\mathcal{E}$ above A, there is a type judgment

$$x : A \vdash X(x) \ \mathsf{type} \tag{5.164}$$

For each morphism $t : \star A \longrightarrow X$ in the fiber at A, there is a term judgment in the theory:

$$x : A \vdash t(x) : X(x) \tag{5.165}$$

This type theory is very weak: indeed, there is no way to significantly substitute a type along some term. That is because terms we have defined in $\mathbb{T}_p$ are landing in types that are not to appear in any context unless some weakening is applied. So everything is as if they were two level of types, defined either in an empty context (level 0) or in a context reduced to a unique type (level 1), with some fixed terms that cannot interplay with one another. Only working with context of size 0 and 1 is not a restriction if we allow Σ-types though. That is the role devoted to $\{-\}$. More precisely, we add to the theory $\mathbb{T}_p$ the following type equality judgments: for each X in $\mathcal{E}$ above A in $\mathcal{B}$,

$$\vdash \{X\} \equiv \Sigma_{x:A} X(x) \ \mathsf{type} \tag{5.166}$$

The substitution is then obtained through the structure of Grothendieck fibration that p comes with. To a fiber map $t : \star A \longrightarrow X$ in $\mathcal{E}_A$ is associated the adjoint map $t^\flat : A \longrightarrow \{X\}$, and given any object Y above $\{X\}$, we add the type equality of (5.167) to $\mathbb{T}_p$.

$$x : A \vdash ((t^\flat)^* Y)(x) \equiv Y(x, t(x)) \ \mathsf{type} \tag{5.167}$$

Weakening is also obtained by using the structure of Grothendieck fibration of p and pulling over the map of the form pr_X in $\mathcal{B}$: given a cartesian morphism $(\mathsf{pr}_{X'})^* X \longrightarrow X$ above the map $\mathsf{pr}_{X'} : \{X'\} \longrightarrow A$ for some X' in the fiber at A, the theory $\mathbb{T}_p$ gets a new type equality judgment

$$x : \{X'\} \vdash (\mathsf{pr}_{X'})^* X(x) \equiv X(\pi_1(x)) \ \mathsf{type} \tag{5.168}$$

In the same fashion, any morphism $t : \star A \longrightarrow X$ in $\mathcal{E}_A$ give rises to a morphism $t' : \star\{X'\} \longrightarrow (\mathsf{pr}_{X'})^* X$ which interprets the weakening of t, meaning that we add the term equality judgment of (5.169) in $\mathbb{T}_p$.

$$x : \{X'\} \vdash t'(x) \equiv t(\pi_1(x)) : X(\pi_1(x)) \tag{5.169}$$

Remark that (5.168) allows to weaken only the types that are given in a context $x : A$. We, for now, lack the possibility to interpret how to obtain a weakening of a bare type $\vdash B$ type into the type $x : A \vdash B$ type. This issue goes in pair with the question of interpreting Σ-types in context: indeed, (5.166) gives an answer only for the types depending on a bare types $\vdash A$ type, but says nothing about the dependent pair types of the form $\Sigma_{y:X(x)} Y(x, y)$ for a type $X(x)$ defined in context $x : A$. All these reasons compel us to add more structures to our (maybe too simplistic) comprehension systems.

To make the following more readable, let us call *fibrations* those maps of $\mathcal{B}$ which are isomorphic in $\mathrm{Fun}(\mathbf{2}, \mathcal{B})$ to one of the form pr_X for some $X \in \mathcal{E}$. In other words, fibrations are the objects of the essential image of pr.

Definition 5.4.3. A *relative clan* is a comprehension system p with section $\star$ and comprehension functor $\{-\}$ such that

(i) $\mathcal{B}$ have a terminal object 1, and $\star A$ is terminal in the fiber $\mathcal{E}_A$ for every object A of $\mathcal{B}$,

(ii) for every object A in $\mathcal{B}$, the unique map $\tau_A : A \longrightarrow 1$ is a fibration,

(iii) for each fibration $f : A \longrightarrow B$ of $\mathcal{B}$, the substitution functor

$$f^* : \mathcal{E}_B \longrightarrow \mathcal{E}_A \tag{5.170}$$

has a left adjoint denoted Σ_f such that $\{-\} \circ \Sigma_f \cong \{-\}$, in such a way that the Beck-Chevalley condition is satisfied for each (pullback) square of the form (5.171) when $g : X \longrightarrow Y$ is cartesian in $\mathcal{E}$.

$$\begin{tikzcd} \{X\} \arrow[r, "\{g\}"] \arrow[d, "\mathsf{pr}_X"'] & \{Y\} \arrow[d, "\mathsf{pr}_Y"] \\ pX \arrow[r, "p(g)"'] & pY \end{tikzcd} \tag{5.171}$$

Remark 5.4.4. Condition (ii) assures that the requirements of condition (iii) also apply to the arrow $A \longrightarrow 1$ into the terminal object.

By making $\star$ a fiberwise terminal functor, we recover the notion of a D-category introduced by Thomas Ehrhard in [Ehr88], or the equivalent notion of a comprehension category with unit of Bart Jacobs. By doing so, the functor $\star$ becomes fully faithful, making the unit $\mathrm{id}_{\mathcal{B}} \longrightarrow \{\star\}$ of the adjunction $\star \dashv \{-\}$ an isomorphism. However, we would like to advocate that the terminal property of $\star A$ in the fiber $\mathcal{E}_A$ does not seem to be the key ingredient of the following for the interpretation of MLTT's with Σ-types. Rather it seems that the vital property is the following consequence: for any object X and Y in $\mathcal{E}$ such that $pY = \{X\}$, there exists a morphism $k : Y \longrightarrow X$ above pr_X and such that $\{k\} = \mathsf{pr}_Y$. When $\star\{X\}$ is terminal in $\mathcal{E}_{\{X\}}$, one can simply consider the morphism k obtained as the composition

$$k : Y \xrightarrow{!} \star\{X\} \xrightarrow{\varepsilon_X} X \tag{5.172}$$

This is clearly over $\mathsf{pr}_X : \{X\} \longrightarrow pX$ and it respects that $\varepsilon_X \star (\mathsf{pr}_Y) = k\varepsilon_Y$. Playing a little with the triangle identities of $\star \dashv \{-\}$, we can conclude that $\{k\} = \mathsf{pr}_Y$.

Now the weakening rule makes sense for two bare types. Given objects A and B of $\mathcal{B}$, one might add to $\mathbb{T}_p$ the type equality of (5.173) which expresses that the type A viewed in the context $x : B$ should be equal to the type $\Sigma_{\tau_A}(\star A)$ once pulled along $\tau_B : B \longrightarrow 1$.

$$x : B \vdash {\tau_B}^*\Sigma_{\tau_A}(\star A)(x) \equiv A \text{ type} \tag{5.173}$$

Because $\left\{\Sigma_{\tau_A}(\star A)\right\} \cong \{\star A\} \cong A$, any map $u : B \longrightarrow A$ is now a map $u : B \longrightarrow \{A_1\}$, which is equivalent, through the adjunction $\star \dashv \{-\}$, to a map $u^\natural : \star B \longrightarrow \Sigma_{\tau_A}(\star A)$, which in turn is equivalently given by a map $(u^\natural)^\triangleleft : \star B \longrightarrow {\tau_B}^*\Sigma_{\tau_A}(\star A)$. It is then harmless to make the abuse to write $x : B \vdash u(x) : A$ instead of the more rigorous $x : B \vdash (u^\natural)^\triangleleft(x) : A$. Hence any map of the base category $\mathcal{B}$ is identified with a term. Substitution along those should coincide with the cartesian structure of p, meaning that for any $u : B \longrightarrow A$ and any X above A, and for any map $t : \star A \longrightarrow X$ in $\mathcal{E}_A$, the theory $\mathbb{T}_p$ contains the type and term equalities of (5.174)

$$\begin{aligned} x : B &\vdash u^*X(x) \equiv X(u(x)) \text{ type} \\ x : B &\vdash t^*(x) \equiv t(u(x)) : X(u(x)) \end{aligned} \tag{5.174}$$

In order to reconcile these new axioms with (5.168), the theory $\mathbb{T}_p$ needs to derive that the morphisms of the form $\mathsf{pr}_X : \{X\} \longrightarrow A$ are just the projection on the first variable when seen as terms from the type $\{X\}$ to the type A. So we add:

$$x : \{X\} \vdash \mathsf{pr}_X(x) \equiv \pi_1(x) : A \tag{5.175}$$

It remains to show how Σ-types are handled when the sum does not range over a bare type. Given an object X above A and Y above $\{X\}$, there is an object $\Sigma_{\mathsf{pr}_X} Y$ above A, and we add to $\mathbb{T}_p$ the judgment (5.176).

$$x : A \vdash \Sigma_{\mathsf{pr}_X} Y \equiv \Sigma_{y:X(x)} Y(x,y) \text{ type} \tag{5.176}$$

Given a morphism $t : \star A \longrightarrow X$ in the fiber $\mathcal{E}_A$, it induces a morphism $t^\flat : A \longrightarrow \{X\}$, which is furthermore a section of pr_X. Then a $u : \star A \longrightarrow (t^\flat)^* Y$ in $\mathcal{E}_A$ is equivalent to a map $\tilde{u} : \star A \longrightarrow Y$ above $t^\flat$. Denote $\lambda : Y \longrightarrow \Sigma_{\mathsf{pr}_X} Y$ for the cocartesian map above pr_X, and note that $p(\lambda\tilde{u}) = \mathsf{pr}_X \circ t^\flat = \mathrm{id}_A$ so that $\lambda\tilde{u}$ is a map in the fiber $\mathcal{E}_A$. This morphism model the dependent pair (t,u), so that we add the following axiom to $\mathbb{T}_p$:

$$x : A \vdash (\lambda\tilde{u})(x) \equiv (t(x), u(x)) : \Sigma_{y:X} Y(x,y) \tag{5.177}$$

Conversely consider the morphism $f : Y \longrightarrow \star\{X\} \longrightarrow X$ above pr_X, so that we can use the cocartesian property of $\lambda : Y \longrightarrow \Sigma_{\mathsf{pr}_X} Y$ above pr_X to find a fiber morphism $\pi : \Sigma_{\mathsf{pr}_X} Y \longrightarrow X$ in $\mathcal{E}_A$ such that $\pi\lambda = f$. This is the projection on the first component, in the sense that for all $t : \star A \longrightarrow \Sigma_{\mathsf{pr}_X} Y$ we add the following to $\mathbb{T}_p$:

$$x : A \vdash (\pi t)(x) \equiv \pi_1(t(x)) : X(x) \tag{5.178}$$

Any such morphism $t : \star A \longrightarrow \Sigma_{\mathsf{pr}_X} Y$ gives an adjoint map $t^\flat : A \longrightarrow \left\{\Sigma_{\mathsf{pr}_X} Y\right\}$. As $\{Y\} \cong \left\{\Sigma_{\mathsf{pr}_X} Y\right\}$, it is as well a map $A \longrightarrow \{Y\}$ which induces a term $u : \star A \longrightarrow Y$ above $(\pi t)^\flat : A \longrightarrow \{X\}$. Hence we add the axiom (5.179) to the theory $\mathbb{T}_p$.

$$x : A \vdash ((\pi t)^\flat)^* u(x) \equiv \pi_2(t(x)) : Y(\pi_1(t(x))) \tag{5.179}$$

Remark 5.4.5. Keep in mind that remark 5.2.2 applies also to the theory $\mathbb{T}_p$ in the context of relative clans. Again, we can use $\mathbb{T}_p$ and the judgments it derives to foresee properties and constructions in relative clans, but we shall always prove the statements in a purely categorical way, only relying our intuition on the theory $\mathbb{T}_p$.

Example(s) 5.4.6. Given a clan $\mathcal{E}$ with fibration $\mathfrak{F}_{\mathcal{E}}$, the Grothendieck fibration $\mathfrak{p}_{\mathcal{E}} : \mathfrak{F}_{\mathcal{E}} \longrightarrow \mathcal{E}$ is a relative clan. Indeed, the section $\star$ is defined by $A \mapsto \mathrm{id}_A$ and the comprehension functor $\{-\}$ by $p \mapsto \mathrm{dom}(p)$. The base category $\mathcal{E}$ have a terminal object already and the functor $\star$ selects indeed the terminal element id_A in $\mathcal{E}(A)$. The range of pr is precisely $\mathfrak{F}_{\mathcal{E}}$ so that it contains in particular those $A \longrightarrow 1$. Condition (iii) is given by the stability of fibrations under composition. Indeed $f \circ - \dashv f^*$ for each fibration f in the clan $\mathcal{E}$. Moreover $\mathrm{dom}(f \circ -) = \mathrm{dom}$. Finally, the Beck-Chevalley condition for pullback square of the form (5.171) is an immediate consequence of the pasting lemma for pullbacks. Then the MLTT $\mathbb{T}_{\mathfrak{p}_{\mathcal{E}}}$ defined above is equivalent to the MLTT $\mathbb{T}_{\mathcal{E}}$ defined in section 5.2.

5.4.2 A Lawverian approach to tribes

In this section, we propose a structure, drawn on the comprehension systems of the previous section, capable to interpret the identity types of a MLTT. It is build on two primary intuitions:

(i) The prominent example of such a structure should be the functor $\mathfrak{p}_{\mathcal{E}} : \mathfrak{F}_{\mathcal{E}} \to \mathcal{E}$ associated to a (Π-)tribe $\mathcal{E}$ with fibrations $\mathfrak{F}_{\mathcal{E}}$.

(ii) The interpretation of the type $\mathrm{Id}_A(x, y)$ should be obtained by *pushing* in a suitable weakened sense the type $\star A$ above the diagonal $A \longrightarrow A \times A$ in the base.

This fits with the usual way that a Lawvere hyperdoctrines $P : \mathcal{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ deal with extensional equalities, where the predicate $x =_A y$ is defined to be $(\Delta_A)_!(1_A)$, where $(\Delta_A)_!$ is the left adjoint to $P(\Delta_A)$ for $\Delta_A : A \longrightarrow A \times A$ the diagonal of A in the base category $\mathcal{B}$, and where 1_A is the total predicate over A (i.e. the terminal object of $P(A)$). We have conveniently developed a framework in section 5.3 where both strict pushes, like the functor $(\Delta_A)_!$ of an hyperdoctrine, and relative versions of anodyne maps, like the total anodyne maps of a tribe, appear as two avatars of a same notion. We should lean on this framework to define the structure that we are after.

If a relative clan as defined in definition 5.4.3 is to be understood as the relative version of a clan, then we should continue our prospection with a relative version of Π-clans.

Definition 5.4.7. A *relative* Π*-clan* is a relative clan p such that

(i) for each fibration $f : A \longrightarrow B$ of $\mathcal{B}$, the substitution functor

$$\mathsf{pr}_X{}^* : \mathcal{E}_A \longrightarrow \mathcal{E}_{\{X\}} \tag{5.180}$$

has a right adjoint denoted Π_f,

(ii) the Beck-Chevalley condition is satisfied for each (pullback) square of the form (5.181) when $g : X \longrightarrow Y$ is cartesian in $\mathcal{E}$.

$$\begin{tikzcd} \{X\} \arrow[r, "\{g\}"] \arrow[d, "\mathsf{pr}_X"'] & \{Y\} \arrow[d, "\mathsf{pr}_Y"] \\ pX \arrow[r, "p(g)"'] & pY \end{tikzcd} \tag{5.181}$$

Relative Π-types model Π-types in the following sense. For each fibration $\mathsf{pr}_X : \{X\} \longrightarrow A$ and each object Y above $\{X\}$, and for every fiber morphism $t : \star\{X\} \longrightarrow Y$ the theory $\mathbb{T}_p$ now contains the axioms of (5.182).

$$\begin{aligned} x : A &\vdash (\Pi_{\mathsf{pr}_X} Y)(x) \equiv \Pi_{y : X(x)} Y(x, y) \ \mathsf{type} \\ x : A &\vdash (\Pi_{\mathsf{pr}_X} t)(x) \equiv \lambda x.t(x) : \Pi_{y : X(x)} Y(x, y) \end{aligned} \tag{5.182}$$

The second equality judgment takes advantage of Π_{pr_X} being a right adjoint hence preserving terminal object: the morphism $\Pi_{\mathsf{pr}_X} t$ has then domain $\star A$, and as such induces a term in the type theory $\mathbb{T}_p$. Conversely given a term $t : \star A \longrightarrow \Pi_{\mathsf{pr}_X} Y$, we deduce a morphism $(\mathsf{pr}_X)^* t : \star\{X\} \longrightarrow (\mathsf{pr}_X)^* \Pi_{\mathsf{pr}_X} Y$ once we recall that $\star(\mathsf{pr}_X) : \star\{X\} \longrightarrow \star A$ is cartesian above pr_X. Now compose $(\mathsf{pr}_X)^* t$ with the counit of the adjunction $(\mathsf{pr}_X)^* \dashv \Pi_{\mathsf{pr}_X}$ to obtain a morphism $\tilde{t} : \star\{X\} \longrightarrow Y$ in the fiber $\mathcal{E}_{\{X\}}$, and add to $\mathbb{T}_p$ the following:

$$x : A, y : \{X\} \vdash \tilde{t}(x, y) \equiv t(x)(y) : Y(x, y) \tag{5.183}$$

Example(s) 5.4.8. The construction of example 5.4.6 naturally extends to Π-clans. Given such a Π-clan $\mathcal{E}$ with fibrations $\mathfrak{F}_{\mathcal{E}}$, the relative clan $\mathfrak{p}_{\mathcal{E}}$ is readily a relative Π-clans through proposition 5.2.5.

Given a relative clan $p : \mathcal{E} \longrightarrow \mathcal{B}$, recall that we call *fibrations* the maps of the form $\mathsf{pr}_X : \{X\} \longrightarrow pX$ for some X in $\mathcal{E}$. For any object A of $\mathcal{B}$, the maps f of $\mathcal{E}_A$ such that $\{f\}$ is a fibration are called *local fibrations at* A. Finally, call *total fibrations* these maps $f : X \longrightarrow Y$ in $\mathcal{E}$ such that $p(f)$ is a fibration and $f^{\triangleleft}$ is a local fibration at pX. Instantiated for the relative clan $\mathfrak{p}_{\mathcal{E}}$ associated to a clan E, one recovers the vocabulary of section 5.3.3.

Definition 5.4.9. A *relative tribe* is a relative Π-clan p such that there is a right weak factorization system $(\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}, \mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}})$ relative to p where $\mathfrak{L}_{\mathcal{B}}$ contains all morphisms of $\mathcal{B}$ and $\mathfrak{R}_{\mathcal{E}}$ contains exactly the total fibrations.

Remark 5.4.10. Because the fibrations $\{\star A\} \longrightarrow A$ are isomorphisms (with inverse the unit $A \longrightarrow \{\star A\}$), all isomorphisms are fibrations in $\mathcal{B}$. So local fibrations in $\mathcal{E}$ are total fibrations, and so are the cartesian morphisms above fibrations. Moreover composition of total fibrations are total fibrations.

Remark also that for any object X above A, the unique fiber map $X \longrightarrow \star A$ is a local fibration. Hence the unique map $X \longrightarrow \star 1$ in $\mathcal{E}$ is the composition of two total fibrations, namely $X \longrightarrow \star A$ and the cartesian map $\star A \longrightarrow \star 1$. So every object of $\mathcal{E}$ is totally fibrant.

It also shows that in a relative tribe, the class $\mathfrak{R}_{\mathcal{B}}$ has to contain every morphisms.

Terminology 5.4.11. We shall call *totally anodyne* the maps of the class $\mathfrak{L}_{\mathcal{E}}$ in a relative tribe.

A relative tribe p gives us the means to model Id-types. To that effect, recall that for an object A in $\mathcal{B}$, we denote $\tau_A : A \longrightarrow 1$ for the unique map from A to the terminal object, and this is a fibration. Then there is a judgment in $\mathbb{T}_p$ (see (5.173) of the following form:

$$x : A \vdash {\tau_A}^* \Sigma_{\tau_A}(\star A) \equiv A \ \mathsf{type} \tag{5.184}$$

So that if we want to define a Id-type for A, we need it to be above $\{{\tau_A}^*\Sigma_A(\star A)\}$. By proposition 5.4.2, the square (5.185) is a pullback square where $\rho : {\tau_A}^*\Sigma_A(\star A) \longrightarrow \Sigma_A(\star A)$ is the chosen cartesian morphism above τ_A. As also $\{\Sigma_A(\star A)\} \cong A$, on gets that $\{{\tau_A}^*\Sigma_A(\star A)\} \simeq A \times A$.

$$\begin{tikzcd} \{{\tau_A}^*\Sigma_A(\star A)\} \arrow[r, "\{\rho\}"] \arrow[d, "\mathsf{pr}_{{\tau_A}^*\Sigma_A(\star A)}"'] & \{\Sigma_A(\star A)\} \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd} \tag{5.185}$$

Consider the unique map $\star A \longrightarrow \star 1$ in $\mathcal{E}$. Its image by p is $\tau_A : A \longrightarrow 1$ which factors through the diagonal $\Delta_A : A \longrightarrow A \times A$. So by hypothesis, one can factor $\star A \longrightarrow \star 1$ into a totally anodyne map $\iota_A : \star A \longrightarrow P_A$ followed by a total fibration $P_A \longrightarrow \star 1$ such that $p(\iota_A) = \Delta_A$. By remark 5.4.10, the second map offers no information. The first map however is exactly the *weak push* we were looking for. Hence we shall add axioms (5.186) to the theory $\mathbb{T}_p$.

$$\begin{aligned} x : A, y : A &\vdash P_A(x, y) \equiv \mathrm{Id}_A(x, y) \ \mathsf{type} \\ x : A &\vdash \iota_A(x) \equiv \mathsf{refl}_x : \mathrm{Id}_A(x, x) \end{aligned} \tag{5.186}$$

Given an object C of $\mathcal{E}$ above $\{P_A\}$ and a morphism $c : \star A \longrightarrow C$ over $u^\flat : A \longrightarrow \{P_A\}$ (which is equivalently a term $\star A \longrightarrow (u^\flat)^* C$), we shall construct a term $j : \star\{P_A\} \longrightarrow C$ such that $j \circ \star(u^\flat) = c$. Because $p(c) = u^\flat$, and $c = \varepsilon_C \star c^\flat$, one gets that $\mathsf{pr}_C c^\flat = u^\flat$.

Remark that the triangular identities of $\star \dashv \{-\}$ give $\mathsf{pr}_{\star\{P_A\}}\eta_{\{P_A\}} = \mathrm{id}_{\{P_A\}}$; however, $\eta_{\{P_A\}}$ being an isomorphism, it means that $\mathsf{pr}_{\star\{P_A\}}$ also and is its inverse. So if we denote $! : C \longrightarrow \star\{P_A\}$ for the unique fiber map to the terminal of $\mathcal{E}_{\{P_A\}}$, then $\{!\}$ is the composite:

$$\{C\} \xrightarrow{\mathsf{pr}_C} \{P_A\} \xrightarrow[\cong]{\eta_{\{P_A\}}} \{\star\{P_A\}\} \tag{5.187}$$

So that in the end the diagram of (5.188) commutes. Remark that the bottom row is actually just $\star(u^\flat)$ because of the triangular equalities.

$$\begin{tikzcd}
 & & \star\{C\} \arrow[r, "\varepsilon_C"] \arrow[dl, "\star(\mathsf{pr}_C)"'] \arrow[d, "\star\{!\}"] & C \arrow[d, "!"] \\
\star A \arrow[r, "\star(u^\flat)"'] \arrow[urr, bend left, "\star(C^\flat)"] & \star\{P_A\} \arrow[r, "\star\eta_{\{P_A\}}"'] & \star\{\star\{P_A\}\} \arrow[r, "\varepsilon_{\{P_A\}}"'] & \star\{P_A\}
\end{tikzcd} \tag{5.188}$$

Hence the outer diagram states that $! \circ c = \star(u^\flat)$, so that if we call $k = \varepsilon_{P_A} \circ ! : C \longrightarrow P_A$ then we have $kc = u$. Push C and P_A to the fiber 1 to get $\tilde{k} : \Sigma_{\tau_{\{P_A\}}} C \longrightarrow \Sigma_{\tau_{A\times A}} P_A$ such that commutes the diagram:

$$\begin{tikzcd}
\star A \arrow[r, "c"] \arrow[d, "u"'] & C \arrow[r, "\lambda_C"] \arrow[dl, "k"] & \Sigma_{\tau_{\{P_A\}}} C \arrow[d, "\tilde{k}"] \\
P_A \arrow[rr, "\lambda_{P_A}"] & & \Sigma_{\tau_{A\times A}} P_A
\end{tikzcd} \tag{5.189}$$

Moreover, $\{\tilde{k}\} \cong \{k\} \cong \{!\}$ is a fibration, so that $\tilde{k}$ is a local fibration at 1 hence a total fibration. Finally the image of (5.189) through p trivially has a lift $A \times A \longrightarrow 1$, so the totally anodyne character of u ensures that there is a map $\tilde{j} : P_A \longrightarrow \Sigma_{\tau_{\{P_A\}}} C$ such that $\tilde{j}u = \lambda_C c$. It induces $\{\tilde{j}\} : \{P_A\} \longrightarrow \left\{\Sigma_{\tau_{\{P_A\}}} C\right\} \cong \{C\}$ such that $\{\tilde{j}\}\, u^\flat = c^\flat$, which in turn is equivalent to $j = \{\tilde{j}\}^\natural : \star\{P_A\} \longrightarrow C$ such that $j \star (u^\flat) = c$. This the term j we were looking for, hence we add to $\mathbb{T}_p$ the following axiom:

$$x : A, y : A, \alpha : \mathrm{Id}_A(x, y) \vdash j \equiv \mathsf{j}(c(x), x, y, \alpha) : C(x, y, \alpha) \tag{5.190}$$

Example(s) 5.4.12. Corollary 5.3.11 states that a Π-clan $\mathcal{E}$ is a tribe if and only if its associated relative Π-clan $\mathfrak{p}_{\mathcal{E}}$ is a relative tribe.

5.4.3 Toward a simplicial semantics of dependent types

In the new framework of relative tribes we only discussed the interpretation of Id-types for *bare types*, that is those A such that $\vdash A$ type is derivable. Although every dependent type $x : A \vdash X(x)$ type admits a bare counterpart, namely $\Sigma_{x:A}X(x)$, the identity types of this dependent pair type is not the intended Id-type of the dependent type $X(x)$. More precisely, the type $\mathrm{Id}_{X(x)}(y, y')$ is not definitionally equal[5] to $\mathrm{Id}_{\Sigma_{z:A}X(z)}((x, y), (x, y'))$ in the context $x : A, y, y' : X(x)$ as already discussed in section 5.2.3. In a tribe the matter is resolved by interpreting Id-types in contexts in the local tribe over $\Sigma_{x:A}X(x) \times X(x)$. Unfortunately, in the process of generalizing from

[5]Neither are they propositionally equal, but this is not the matter discussed here as we lack universes in the theory $\mathbb{T}_p$ so far anyway.

tribes to relatives tribes, the ability to consider local version of the global structure in the fibers was lost: the fiber $\mathcal{E}_A$ of a relative tribe $p : \mathcal{E} \longrightarrow \mathcal{B}$ at an object $A \in \mathcal{B}$ is not necessarily a relative tribe. Or more precisely, the functor $p_A : \mathrm{P}_2\mathcal{E}(A) \longrightarrow \mathcal{E}_A$ that one would expect to be a relative tribe is not even a relative clan: this functor is obtained as the pullback of p along the restricted comprehension $\{-\} : \mathcal{E}_A \longrightarrow \mathcal{B}$, and it comes with a natural section $X \longmapsto (\star\{X\}, X)$ for which a right adjoint is not guaranteed to exists. The reasons behind such an unfortunate turn of events can be explained through lemma 5.4.13.

Lemma 5.4.13. *Given a pullback square in* Cat *as in (5.191) with the functor p being a comprehension system with section $\star$ and comprehension $\{-\}$, the functor p' is a comprehension system with section $\star' = (\star F, \mathrm{id}_{\mathcal{B}'})$ if and only if F is a Grothendieck fibration restricted to the fibrations of p.*

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d, "p'"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "F"'] & \mathcal{B} \end{tikzcd} \tag{5.191}$$

Proof. We shall use the description of the pullback $\mathcal{E}'$ as the category whose objects are (X, A') with $X \in \mathcal{E}$ and $A' \in \mathcal{B}'$ such that $pX = FA'$ and whose morphisms $(X, A') \longrightarrow (Y, B')$ are the pairs (f, u) with $f : X \longrightarrow Y$ in $\mathcal{E}$ and $u : A' \longrightarrow B'$ in $\mathcal{B}'$ such that $p(f) = F(u)$. Under this description, the functor p' identifies with $(X, A') \longmapsto A'$ and its section $\star'$ with $A' \longmapsto (\star FA', A')$. Hence a right adjoint for $\star'$ is a functor $(X, B') \longmapsto \{(X, B')\}'$ together with a natural transformation $\varepsilon' : \star'\{-\}' \longrightarrow \mathrm{id}_{\mathcal{E}'}$ such that the following is a natural isomorphism:

$$\begin{aligned} \mathcal{B}'\left(A', \{(Y, B')\}'\right) &\cong \mathcal{E}'(\star' A', (Y, B')) \\ &\cong \{f : \star FA' \longrightarrow Y, u : A' \longrightarrow B' \mid p(f) = F(u)\} \\ &\cong \{f^\flat : FA' \longrightarrow \{Y\}, u : A' \longrightarrow B' \mid \mathsf{pr}_Y \circ f^\flat = F(u)\} \end{aligned} \tag{5.192}$$

where the first isomorphism is $\varphi \longmapsto \varepsilon'_{(Y,B')} \circ \star'(\varphi)$. In other words, this is equivalent to $p'(\varepsilon'_{(Y,B')}) : \{(Y, B')\}' \longrightarrow B'$ being cartesian relatively to F over pr_Y for every (Y, B'). □

Consider the pullback square of (5.193) for an object A of $\mathcal{B}$ in a relative tribe p with section $\star$ and comprehension $\{-\}$. The notation for the total category $\mathrm{P}_2\mathcal{E}(A)$ stands for "paths of length 2" in the relative tribe, as it turns out to be exactly that in the case of the relative tribe coming from a Joyal tribe.

$$\begin{tikzcd} \mathrm{P}_2\mathcal{E}(A) \arrow[r] \arrow[d, "p_A"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "\{-\}"'] & \mathcal{B} \end{tikzcd} \tag{5.193}$$

Through lemma 5.4.13, the functor p_A together with the induced section $X \longmapsto (\star\{X\}, X)$ is then a comprehension system if and only if $\{-\} : \mathcal{E}_A \longrightarrow \mathcal{B}$ is a Grothendieck fibration restricted to the fibrations of p. Otherwise put, given X above A and Y above $\{X\}$, we are looking for an object Z above A together with a map $\rho : Z \longrightarrow X$ in the fiber $\mathcal{E}_A$ such that ρ is cartesian relatively to $\{-\}$. Moreover, if we carry away this construction in the case of a tribe $\mathcal{C}$ where X is just a fibration q and Y is a fibration r whose codomain is the domain of p, then the intended Z is given by the composition qr. Hence we don't have much choice: Z should be the object $\Sigma_{\mathsf{pr}_X} Y$ and ρ the second projection $\Sigma_{\mathsf{pr}_X} Y \longrightarrow X$ in the fiber $\mathcal{E}_A$. However there is no reason whatsoever for this map to be cartesian relatively to $\{-\}$. Indeed, suppose given morphisms $f : W \longrightarrow X$ and $u : \{W\} \longrightarrow \{Y\}$ such that commutes the diagram (5.88) in $\mathcal{B}$.

$$\begin{tikzcd} \{W\} \arrow[d, "u"'] \arrow[dr, bend left, "\{f\}"] & \\ \{Y\} \arrow[r, "\mathsf{pr}_Y"'] & \{X\} \end{tikzcd} \tag{5.194}$$

The goal would be to construct a unique morphism $g : W \longrightarrow \Sigma_{\mathsf{pr}_X} Y$ such that $f = g\rho$ (and so $p(f) = p(g)$) and $\{g\} = \{\lambda\}\, u$ (where λ is the canonical map $Y \longrightarrow \Sigma_{\mathsf{pr}_X} Y$, whose image through $\{-\}$ is invertible). In particular the data gives the image through pr of a potential g, as illustrated in diagram (5.195)

$$\begin{tikzcd} \{W\} \arrow[r, "u"] \arrow[d, "\mathsf{pr}_W"'] & \{Y\} \arrow[r, "\{\lambda\}", "\cong"'] \arrow[d, "\mathsf{pr}_Y"] & \left\{\Sigma_{\mathsf{pr}_X} Y\right\} \arrow[dl, "\{\rho\}"] \\ pW \arrow[r, "p(f)"'] & \{X\} & \end{tikzcd} \tag{5.195}$$

Nothing in the structure of a tribe compels g to exists, yet alone to be unique. However through the previous remark such a unique g will exist when $\mathsf{pr} : \mathcal{E} \longrightarrow \mathrm{Fun}\,(\mathbf{2}, \mathcal{B})$ is fully faithful, in which case we are left back with an ordinary tribe seen as a relative one.

We should take this issue as an incentive to extrapolate again on the structure that a model of a MLTT need. Lemma 5.4.13 and the subsequent discussion show that the category of types dependent on a given type does not inherit the structure needed to model MLTT's from the relative tribe. A way to fix this defect is to incorporate directly the data needed to do so into a new type of structure. The main idea is to draw on a *tower* structure as in (5.196) instead of a unique functor as in relative tribes.

$$\ldots \xrightarrow{p_n} \mathcal{C}_n \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_1} \mathcal{C}_1 \xrightarrow{p_0} \mathcal{C}_0 \tag{5.196}$$

The object of the category $\mathcal{C}_n$ is supposed to interpret the types of *height* n, meaning these judgments

$$x_1 : A_1, x_2 : A_2(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash A_{n+1}(x_1, \ldots, x_n) \text{ type} \tag{5.197}$$

and for each $0 \leq k \leq n$, the image of the interpretation of (5.197) through $p_k \circ p_{k+1} \circ \cdots \circ p_{n-1}$ is supposed to interpret the k-truncated context:

$$x_1 : A_1, \ldots, x_k : A_k(x_1, \ldots, x_{k+1}) \vdash A_{k+1}(x_1, \ldots, x_k) \text{ type} \tag{5.198}$$

Each term as in (5.199) then should be interpreted as a fiber morphism $\star_n(A_n) \longrightarrow A_{n+1}$ where $\star_n$ is a section of p_n.

$$x_1 : A_1, x_2 : A_2(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash t(x_1, \ldots, x_n) : A_{n+1}(x_1, \ldots, x_n) \tag{5.199}$$

Following the usual intuition of interpreting substitution as pulling over terms, we should require in addition that p_n is a Grothendieck fibration for each n. Hence now, given B above A_{n+1} and a term $t : \star_n(A_n) \longrightarrow A_{n+1}$, the object t^*B is supposed to interpret the type judgment of (5.200).

$$x_1 : A_1, x_2 : A_2(x_1), \dots, x_n : A_n(x_1, \dots, x_{n-1}) \vdash B(x_1, \dots, x_n, t(x_1, \dots, x_n)) \text{ type} \tag{5.200}$$

However t^*B is an object of $\mathcal{C}_{n+1}$ so it can not *a priori* interpret a type of height $n+1$ as in (5.200). Hence we need a way to get something back in $\mathcal{C}_n$ above A_n from something in $\mathcal{C}_{n+1}$ above $\star_n(A_n)$. This calls for more structure that we shall present more rigorously now.

Recall the description of the simplex category Δ as the category whose objects are the finite linear posets $\mathbf{n} = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$, and whose morphisms are the non decreasing maps between such. It is fairly known that the morphisms of the category Δ are generated by the so-called *faces* and *degeneracies* morphisms given on the left of (5.201) under the relations given on the right.

$$\begin{aligned}
\partial_n^i &: \mathbf{n-1} \longrightarrow \mathbf{n} \\
& j \longmapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{otherwise} \end{cases} \\
\sigma_n^i &: \mathbf{n+1} \longrightarrow \mathbf{n} \\
& j \longmapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{otherwise} \end{cases}
\end{aligned}
\qquad
\begin{aligned}
\partial_{n+1}^j \partial_n^i &= \partial_{n+1}^i \partial_n^{j-1} & i < j \\
\sigma_{n-1}^j \sigma_n^i &= \sigma_{n-1}^i \sigma_n^{j+1} & i \leq j \\
\sigma_n^j \partial_{n+1}^i &= \partial_n^i \sigma_{n-1}^{j-1} & i < j \\
\sigma_n^i \partial_{n+1}^i &= \mathrm{id}_{\mathbf{n}} = \sigma_n^{i-1} \partial_{n+1}^i & \\
\sigma_n^j \partial_{n+1}^i &= \partial_n^{i-1} \sigma_{n-1}^j & i > j+1
\end{aligned}
\tag{5.201}$$

Definition 5.4.14. A *simplicial category* is a functor $\mathcal{C} : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$. The image of $\mathbf{n}$ is a category denoted $\mathcal{C}_n$. The images of the morphisms are determined by the images of ∂_n^i, denoted ${}^{\mathcal{C}}\mathrm{d}_n^i$ or simply d_n^i, and the images of σ_n^i, denoted ${}^{\mathcal{C}}\mathrm{s}_n^i$ or simply s_n^i.

A simplicial category is not to be confused with a (small) category enriched in $\mathcal{S}$. Such an enriched category indeed defines a simplicial category $\Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ such that the postcomposition with the object functor $\mathsf{Cat} \longrightarrow \mathsf{Set}$ yields a constant functor. However not every simplicial category arises in this way obviously. A good mental picture of a simplicial category is a sequence of categories with an increasing number of functors between each term of the sequence as depicted in (5.202) below.

$$\cdots \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \vdots \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{C}_{n+1} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \vdots \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{C}_n \;\substack{\longrightarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow \\ \longrightarrow}\; \cdots \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{C}_1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{C}_0 \tag{5.202}$$

Where from top to bottom between $\mathcal{C}_{k+1}$ and $\mathcal{C}_k$ are appearing the following functors

$$\mathrm{d}_{k+1}^{k+1}, \mathrm{s}_k^k, \mathrm{d}_{k+1}^k, \mathrm{s}_k^{k-1}, \dots, \mathrm{s}_k^0, \mathrm{d}_{k+1}^0 \tag{5.203}$$

Definition 5.4.15. A *simplicial relative clan* is a simplicial category $\mathcal{C}$ such that

(i) Each $\mathcal{C}_n$ has a terminal object 1_n,

(ii) d^0_{n+1} is a Grothendieck fibration preserving the terminal object for all $n \geq 0$,

(iii) $\mathrm{d}^1_{n+2}(f)$ is cartesian for d^0_{n+1} whenever f is cartesian for d^0_{n+2} and $\mathrm{d}^0_{n+2}(f)$ is cartesian for d^0_{n+1}, for all $n \geq 0$,

(iv) for all $n \geq 0$, there are adjunctions

$$\mathrm{d}^i_{n+1} \dashv \mathrm{s}^i_n \qquad \mathrm{s}^i_n \dashv \mathrm{d}^{i+1}_{n+1} \qquad \forall 0 \leq i \leq n \tag{5.204}$$

Axiom (iii) is a version of the Beck-Chevalley condition for this structure. Axiom (iv) is saying that the enumeration of (5.203) is actually a chain of adjunctions for each $k \geq 0$:

$$\mathrm{d}^{k+1}_{k+1} \vdash \mathrm{s}^k_k \vdash \mathrm{d}^k_{k+1} \vdash \mathrm{s}^{k-1}_k \vdash \ldots \vdash \mathrm{s}^0_k \vdash \mathrm{d}^0_{k+1} \tag{5.205}$$

Example(s) 5.4.16. The leading example of such a structure is the 2-categorical nerve of a clan $\mathfrak{C}$ with fibrations $\mathfrak{F}$, by which we mean the simplicial category $\mathrm{N}\,\mathfrak{C} : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ that maps n to the category $\mathrm{Fun}\,(\mathbf{2}, \mathfrak{F})$ of paths of n fibrations.

Each $\mathrm{N}\,\mathfrak{C}_n$ has a terminal object given by the path composed of n times the fibration $\mathrm{id}_1 : 1 \longrightarrow 1$, where 1 is the terminal object of the clan $\mathfrak{C}$. The functor d^0_{n+1} maps a path

$$A_{n+1} \xrightarrow{q_n} A_n \xrightarrow{q_{n-1}} \ldots \xrightarrow{q_0} : A_0 \tag{5.206}$$

to the path $(q_{n-1}, \ldots, q_0)$ where the first morphism is discarded. It clearly preserves the terminal object. This a Grothendieck fibration because fibrations of $\mathfrak{C}$ are stable under pullbacks. And axiom (iii) is exactly the pasting lemma for pullbacks. Now given $0 \leq i \leq n$, the functor s^i_n is the functor that insert an identity at the ith position:

$$(q_{n-1}, \ldots, q_0) \mapsto (q_{n-1}, \ldots, \mathrm{id}_{A_{n-i}}, q_{n-1-i}, \ldots, q_0) \tag{5.207}$$

A map in $\mathrm{N}\,\mathfrak{C}_{n+1}$ from $\vec{q}'$ to $\mathrm{s}^i_n(\vec{q})$ is given by maps f_i in $\mathfrak{C}$ such that commutes the diagram on the left of (5.208).

$$\begin{tikzcd}
{} \arrow[r, "f_{n+1}"] \arrow[d, "q'_n"'] & {} \arrow[d, "q_{n-1}"] & & {} \arrow[r, "f_{n+1}"] \arrow[d, "q_{n-1}"'] & {} \arrow[d, "q'_n"] \\
{} \arrow[r, "f_n"] \arrow[d, "q'_{n-1}"'] & {} \arrow[d, "q_{n-2}"] & & {} \arrow[r, "f_n"] \arrow[d, "q_{n-2}"'] & {} \arrow[d, "q'_{n-1}"] \\
\vdots \arrow[d, "q'_{n-i+1}"'] & \vdots \arrow[d, "q_{n-i}"] & & \vdots \arrow[d, "q_{n-i}"'] & \vdots \arrow[d, "q'_{n-i+1}"] \\
{} \arrow[r, "f_{n-i+1}"] \arrow[d, "q'_{n-i}"'] & {} \arrow[d, equal] & & {} \arrow[r, "f_{n-i+1}"] \arrow[d, equal] & {} \arrow[d, "q'_{n-i}"] \\
{} \arrow[r, "f_{n-i}"] \arrow[d, "q'_{n-i-1}"'] & {} \arrow[d, "q_{n-i-1}"] & & {} \arrow[r, "f_{n-i}"] \arrow[d, "q_{n-i-1}"'] & {} \arrow[d, "q'_{n-i-1}"] \\
\vdots \arrow[d, "q'_0"'] & \vdots \arrow[d, "q_0"] & & \vdots \arrow[d, "q_0"'] & \vdots \arrow[d, "q'_0"] \\
{} \arrow[r, "f_0"] & {} & & {} \arrow[r, "f_0"] & {}
\end{tikzcd} \tag{5.208}$$

This is equivalent to a map

$$(q'_n, q'_{n-1}, \ldots, q'_{n-i} \circ q'_{n-i+1}, q'_{n-i-1}, \ldots, q'_0) \longrightarrow \vec{q} \tag{5.209}$$

in $\mathrm{N}\,\mathcal{C}_n$ because f_{n-i+1} is completely determined by f_{n-i}. But the domain is nothing less than $\mathrm{d}^i_{n+1}(\vec{q}')$. This prove $\mathrm{d}^i_{n+1} \dashv \mathrm{s}^i_n$. Similarly, a map from $\mathrm{s}^i_n(\vec{q})$ to $\vec{q}'$ as in the right of (5.208) is the same as a map in $\mathrm{N}\,\mathcal{C}_n$ from $\vec{q}$ to $\mathrm{d}^{i+1}_{n+1}(\vec{q}')$ (that composes q_{n-i-1} with q_{n-i}) because f_{n-i} is completely determined by f_{n-i+1}.

In particular, the 1-truncation of a simplicial relative clan forms a comprehension system. So it makes sense to talk about the fibration of $\mathcal{C}_0$ in the sense of section 5.4.1.

Definition 5.4.17. Let $\mathcal{C}$ be a simplicial relative clan. A *fibration of level* 0 is a fibration of $\mathcal{C}_0$. A *fibration of level* $n+1$ is defined inductively as a map f of $\mathcal{C}_{n+1}$ such that both d^0_{n+1} and $\mathrm{d}^1_{n+1}(f^{\triangleleft})$ are fibrations of level n.

Define the *anodyne maps of level n* as the morphisms in $\mathcal{C}_n$ that have the left lifting property against all the fibrations of level n relatively to p_{n-1}. (By convention p_{-1} is the functor $\mathcal{C}_0 \longrightarrow 1$.)

Definition 5.4.18. A *simplicial relative tribe* is a simplicial relative clan $\mathcal{C}$ such that

- for each $n \geq 0$ there is a right WFS $(\mathfrak{L}_{n+1}, \mathfrak{R}_{n+1}, \mathfrak{L}_n, \mathfrak{R}_n)$ relative to p_n such that $\mathfrak{L}_n$ contains every morphism of $\mathcal{C}_n$ and $\mathfrak{R}_{n+1}$ contains exactly the fibrations of level n,
- the cartesian map in $\mathcal{C}_{n+1}$ above an anodyne of level n are anodyne of level $n+1$.

EXAMPLE(S) 5.4.19. Drawing on the example $\mathrm{N}\,\mathcal{C}$ of a simplicial relative clan associated with a clan $\mathcal{C}$, we can show that $\mathrm{N}\,\mathcal{C}$ is a simplicial relative tribe when $\mathcal{C}$ is a tribe through the repeated use of proposition 5.3.9.

Figure 5.1: Example of derivation tree for the theory of categories

$$
\dfrac{
  \dfrac{
    \dfrac{\vdash \mathsf{Ob\ type} \quad \vdash \mathsf{Ob\ type}}{x : \mathrm{Ob} \vdash \mathsf{Ob\ type}}
    \quad
    x : \mathrm{Ob} \vdash \mathrm{id}_x : \mathrm{Hom}(x, x)
  }{x, y : \mathrm{Ob} \vdash \mathrm{id}_x : \mathrm{Hom}(x, x)}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{\vdash \mathsf{Ob\ type} \quad \vdash \mathsf{Ob\ type}}{x : \mathrm{Ob} \vdash \mathsf{Ob\ type}}
      \quad
      \dfrac{\vdash \mathsf{Ob\ type}}{x : \mathrm{Ob} \vdash x : \mathrm{Ob}}
    }{x, y : \mathrm{Ob} \vdash x : \mathrm{Ob}}
    \quad
    \dfrac{
      \dfrac{x, y, z : \mathrm{Ob}, f : \mathrm{Hom}(x, y), g : \mathrm{Hom}(y, z) \vdash g \circ f : \mathrm{Hom}(x, z)}{z, x, y : \mathrm{Ob}, g : \mathrm{Hom}(z, x), f : \mathrm{Hom}(x, y) \vdash f \circ g : \mathrm{Hom}(z, y)}
    }{x, y, z : \mathrm{Ob}, g : \mathrm{Hom}(z, x), f : \mathrm{Hom}(x, y) \vdash f \circ g : \mathrm{Hom}(z, y)}
  }{x, y : \mathrm{Ob}, g : \mathrm{Hom}(x, x), f : \mathrm{Hom}(x, y) \vdash f \circ g : \mathrm{Hom}(x, y)}
}{x, y : \mathrm{Ob}, f : \mathrm{Hom}(x, y) \vdash f \circ \mathrm{id}_x : \mathrm{Hom}(x, y)}
$$

Figure 5.2: Diagrammatic proof of proposition 5.2.14

$$
\begin{tikzcd}
 & & (u')^* \mathrm{id}_Y \arrow[d, "\cong"] \arrow[dddddddd, bend left=60, "\varphi(t)"] \arrow[ddddll, equal, bend right] \\
 & & \mathrm{id}_X \arrow[d, "\cong"] \\
 & & (p')^* \mathrm{id}_A \arrow[d, "(p')^*(\cong)"] \\
 & (u')^* p^* \mathrm{id}_B \arrow[r, "\cong"] \arrow[d, "(u')^* p^* \eta_{\mathrm{id}_B}"] \arrow[dl, equal, bend right] & (p')^* u^* \mathrm{id}_B \arrow[d, "(p')^* u^* \eta_{\mathrm{id}_B}"] \\
(u')^* p^* \mathrm{id}_B \arrow[d, "(u')^*(\cong)"'] & (u')^* p^* \Pi_p p^* \mathrm{id}_B \arrow[l, "(u')^*(\varepsilon_{p^* \mathrm{id}_B})"'] \arrow[r, "\cong"] \arrow[d, "(u')^* p^* \Pi_p(\cong)"] & (p')^* u^* \Pi_p p^* \mathrm{id}_B \arrow[d, "(p')^* u^* \Pi_p(\cong)"] \\
(u')^* \mathrm{id}_Y \arrow[rrddd, bend right, "(u')^*(t)"'] & (u')^* p^* \Pi_p \mathrm{id}_Y \arrow[l, "(u')^*(\varepsilon_{\mathrm{id}_Y})"] \arrow[r, "\cong"] \arrow[d, "(u')^* p^* \Pi_p(t)"] & (p')^* u^* \Pi_p \mathrm{id}_Y \arrow[d, "(p')^* u^* \Pi_p(t)"] \\
 & (u')^* p^* \Pi_p q \arrow[r, "\cong"] \arrow[rdd, bend right, "(u')^*(\varepsilon_q)"] & (p')^* u^* \Pi_p q \arrow[d, "(p')^*(\mu_q)"] \\
 & & (p')^* \Pi_{p'} (u')^* q \arrow[d, "\varepsilon_{(u')^* q}"] \\
 & & (u')^* q
\end{tikzcd}
$$

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