Toward an asymptotic behaviour of the ABC dynamo

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Abstract – The ABC flow was originally introduced by Arnol'd to investigate Lagrangian chaos. It soon became the prototype example to illustrate magnetic field amplification via fast dynamo action, i.e. dynamo action exhibiting magnetic field amplification on a typical timescale independent of the electrical resistivity of the medium. Even though this flow is the most classical example for this important class of dynamos (with application to large scale astrophysical objects), it was recently pointed out [BD13] that the fast dynamo nature of this flow was unclear, as the growth rate still depended on the magnetic Reynold number at the largest values available so far (Rm = 25000). Using state of the art high performance computing, we present high resolution simulations (up to 4096^3) and extend the value of Rm up to $5 \cdot 10^5$. Interestingly, even at these huge values, the growth rate of the leading eigenmode still depends on the controlling parameter and an asymptotic regime is not reached yet. We show that the maximum growth rate is a decreasing function of Rm for the largest values of Rm we could achieve (as anticipated in the above-mentioned paper). Slowly damped oscillations might indicate either a new mode crossing or that the system is approaching the limit of an essential spectrum.

Introduction. – Fifty years after the ABC flow has been introduced in the seminal work of Arnol'd [Arn65], as a prototype for Lagrangian chaos, its properties as a fast dynamo are still unclear. In a recent study [BD13], we stressed that contrary to earlier expectations, this flow still does not act as a fast dynamo for $\mathrm{Rm} \simeq 25000$. The same year [JG14], introduced a detailed study of the symmetries of the various dynamo branches up to $\mathrm{Rm} = 10^4$. Here, we investigate the kinematic dynamo action associated with the ABC-flow up to $\mathrm{Rm} = 5 \cdot 10^5$. Such extreme values require very high spectral resolutions (up to 4096^3 modes) and state of the art parallel computing.

Governing equations. – The time evolution of the magnetic field in a conducting medium (such as an ionized astrophysical plasma) is governed by the magnetohydrodynamics equations. If one assumes that the magnetic field is weak enough not to influence the fluid flow, a single equation, known as the induction equation, governs the time evolution of the solenoidal magnetic field under a prescribed flow

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\mathbf{u} \times \mathbf{B} - \mathrm{Rm}^{-1}\, \boldsymbol{\nabla} \times \mathbf{B} \right) . \qquad (1)$$

Finding exponentially growing solutions to this equation is known as the kinematic dynamo problem. We consider here the ABC-flow ([Arn65,Hén66]), which takes the form

$$\begin{aligned} \mathbf{u} = & (A \sin z + C \cos y)\, \mathbf{e}_x \\ & + (B \sin x + A \cos z)\, \mathbf{e}_y \\ & + (C \sin y + B \cos x)\, \mathbf{e}_z \,. \end{aligned} \qquad (2)$$

We want to assess its fast dynamo property, i.e. the independence of the growthrate on Rm in the limit $\mathrm{Rm} \to \infty$. We restrict our attention to configurations in which the magnetic field has the same periodicity as the flow (i.e. 2π-periodic in all directions of space, see [ADN03] for ex-

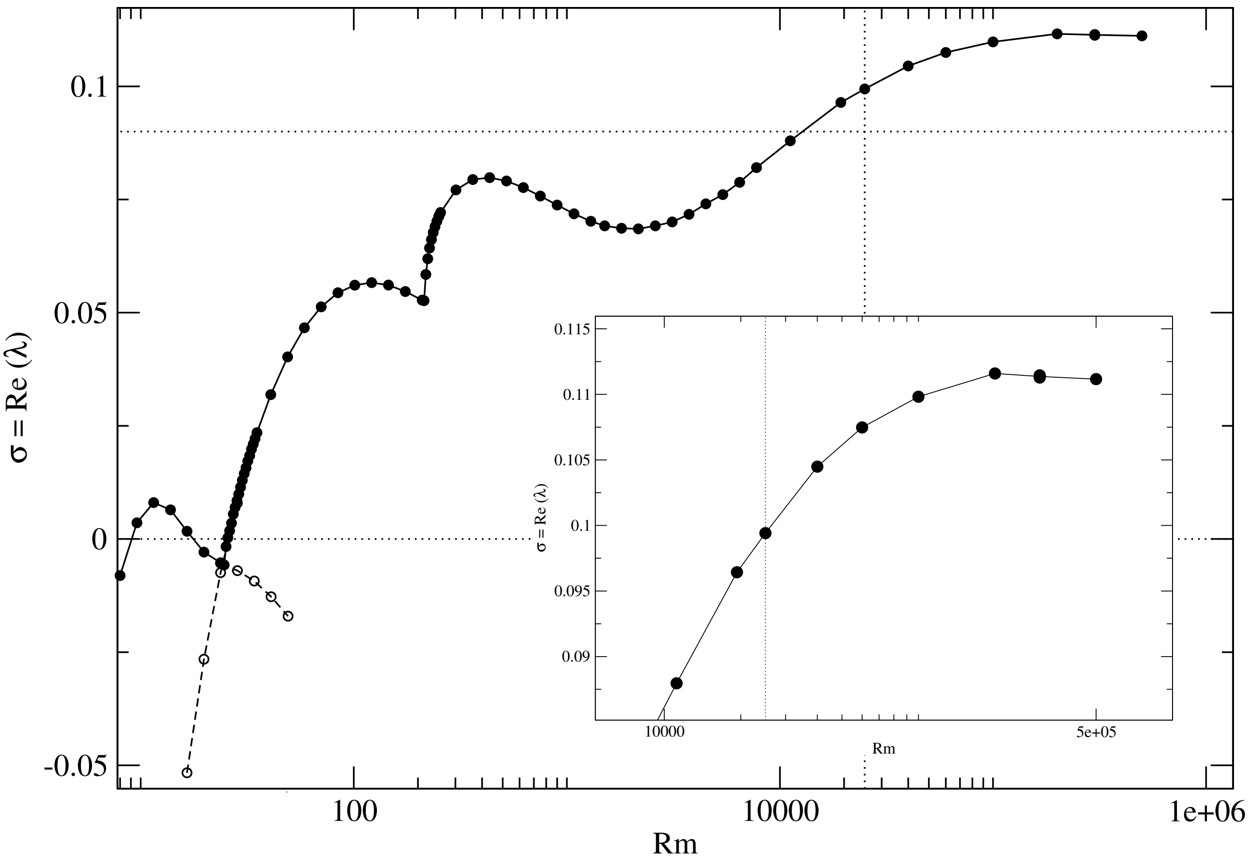

Fig. 1: Plot of the magnetic field growth rate as a function of Rm, up to $\mathrm{Rm} = 5 \cdot 10^5$. The inset presents a closer view on the range $2 \cdot 10^5 - 2 \cdot 10^6$, which stresses the decrease of the growth rate at our larger values of Rm. The horizontal dotted line at $\sigma \simeq 0.09$ corresponds to the theoretical upper bound provided by the topological entropy (h_{line}).

tensions) and the weights of the three symmetric Beltrami components are of equal strength ($A = B = C \equiv 1$).

The simulations presented in this paper were performed using a modified version of a code originally developed by [GF84]. It uses a fully spectral method with explicit mode coupling, which we parallelized using domain decomposition in the spectral space (see [BD13]).

Numerical simulations up to $\mathrm{Rm} = 5 \cdot 10^5$**.** – In order to try to approach an asymptotic behaviour, we extend our previous study of the variation of the fastest growth rate as a function of the magnetic Reynolds number [BD13] up to $\mathrm{Rm} = 5 \cdot 10^5$. Each simulation involves N^3 Fourier modes. Simulations up to $\mathrm{Rm} = 6 \cdot 10^4$ were performed with resolutions $N = 512$ and $N = 1024$ in order to check convergence. Simulations up to $\mathrm{Rm} = 3 \cdot 10^5$ were performed with resolutions $N = 1024$ and $N = 2048$ and the highest Rm we were able to perform, $\mathrm{Rm} = 5 \cdot 10^5$ was validated using $N = 2048$ and $N = 4096$. All simulations were initialized with a random divergence free initial seed field, with the exception our our largest and most expensive simulation, $\mathrm{Rm} = 5 \cdot 10^5$, which was started using the final stage of $\mathrm{Rm} = 3 \cdot 10^5$. The resulting plot of the fastest growth rate is displayed in Figure 1.

Our former study, up to $\mathrm{Rm} = 25000$ revealed a growth rate $\sigma \simeq 0.1$ for the magnetic energy. This value is in excess of the theoretical upper bound provided by the so-called topological entropy ($h_{\text{line}} \simeq 0.09$) [KY95, CG95]. We therefore anticipated the necessity of a decrease of the growth rate at larger (not yet available) values of Rm.

Enlarging the graph of the evolution of the growth rate for large Rm (see the inset in Figure 1) clearly highlights that the maximum growth rate, indeed reaches a local maximum around $\mathrm{Rm} \simeq 2 \cdot 10^5$, and then decreases with Rm above this value. The imprecision on the growth rate is associated with slowly damped oscillations, which are present at large magnetic Reynold number (see below).

It is striking to note that even for $\mathrm{Rm} = 5 \cdot 10^5$, the growth rate has not settled to an asymptotic value. Not only does it still vary with the controlling parameter Rm, but it is also still significantly larger ($\sigma \simeq 0.11$) than the theoretical upper-bound ($h_{\text{line}} \simeq 0.09$).

Cross sections of the $(x - y)$ **plane with rescaled coordinates.** – In the asymptotic limit of large Rm, it is expected that the magnetic structures will scale as

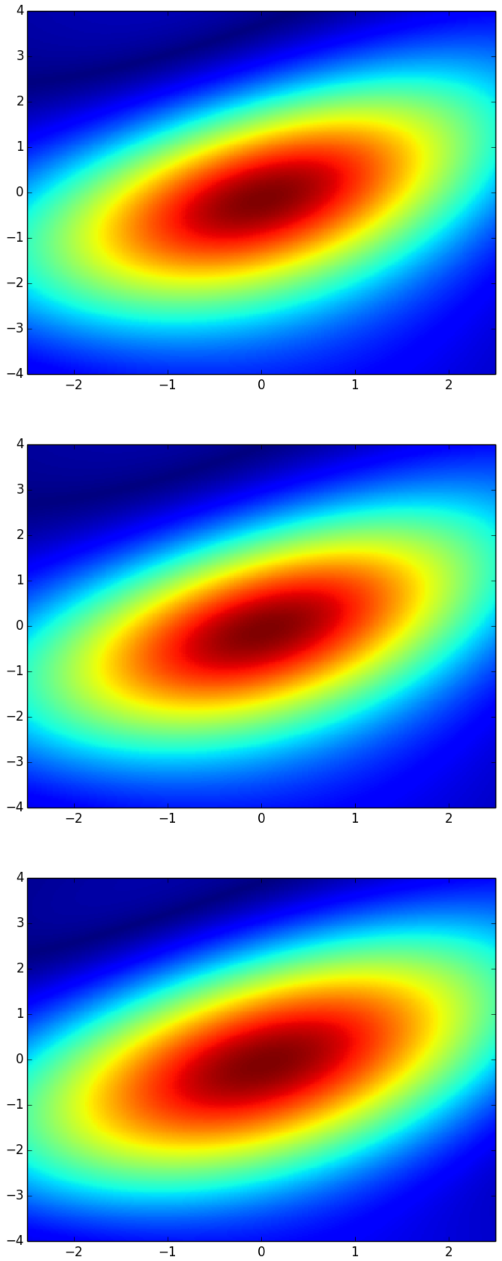

Fig. 2: Cross sections of the magnetic field amplitude at $z = 0$ in the rescaled boundary layer coordinates ζ_x, ζ_y for $\mathrm{Rm} = 4 \cdot 10^4, 2 \cdot 10^5, 5 \cdot 10^5$.

$\mathrm{Rm}^{-1/2}$ [MP85]. In order to validate this dependency in our direct numerical simulations, but also to test any additional variation of the leading eigenmode with Rm, we produce cross sections through the solution at $z = 0$ for varying values of Rm. The *loci* of large magnetic field, corresponding to the traces of the "cigare" shaped structures on this plane, are then peaks of magnetic energy. One of these is centred on $(x = 0,\, y = 0)$, the section of this structure in the plane is expected to have a characteristic length-scale which behaves as $\mathrm{Rm}^{-1/2}$. The magnetic

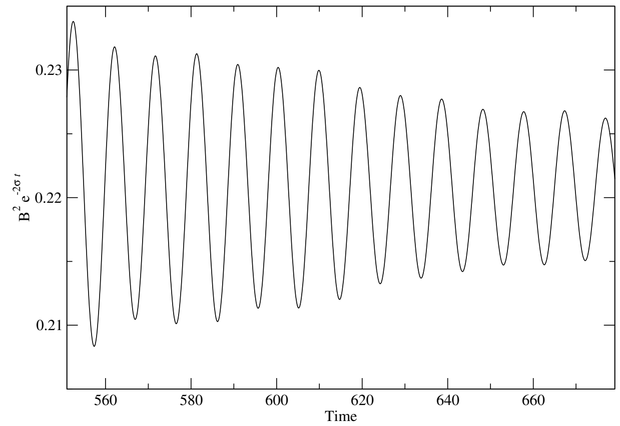

Fig. 3: Damped oscillations in the time evolution of the magnetic energy corrected for the averaged exponential growth rate at $\mathrm{Rm} = 2 \cdot 10^5$.

Reynolds numbers considered here are extremely large, and the structure is thus sharply localised. In order to compare the structures obtained at various values of Rm, we therefore introduce rescaled coordinates relevant to the asymptotic limit of large Rm

$$\zeta_x = x\,\mathrm{Rm}^{1/2}\,, \qquad \zeta_y = y\,\mathrm{Rm}^{1/2}\,. \tag{3}$$

The magnetic field amplitude is represented versus $(\zeta_x,\, \zeta_y)$ for increasing values of the magnetic Reynolds number in figure 2. This figure provides instantaneous cross-sections through the cigares, using rescaled coordinates. The leading eigenmode represented in these rescaled coordinates does not exhibit any significant variation when Rm is varied from $4 \cdot 10^4$ to $5 \cdot 10^5$. This suggests that the system might be approaching an asymptotic behaviour. We can however not rule out, on the basis of these numerical simulations, a remaining slow dependency of the leading eigenmode structure on Rm (other than the length-scale shortening accounted for via the rescaled coordinates).

Damped oscillations. – A noticeable new feature emerges from the large Rm simulations. Whereas the leading eigenvalue was reported to be purely real for $\mathrm{Rm} > 215$ (a critical value denoted Rm_2 in [BD13]), damped oscillations appear for $\mathrm{Rm} > 10^5$ (see figure 3). The presence of oscillations suggests the existence of a complex eigenvalue. Yet the fact that these oscillations are damped indicates that the leading eigenvalue is still real.

The decomposition in symmetry classes introduced by [JG14] highlighted the families corresponding to the first and the second dynamo window of the ABC flow, respectively denoted II and V. Figure 4 suggests that the submode corresponding to this complex, non-dominant, eigenvalue, could belong to the symmetry class II.

This may be an indication of a new eigenvalue crossing, which could occur at larger Rm and which would result in the reappearance of time oscillations. Indeed, some models of dynamo action in steady flow suggest the possibility

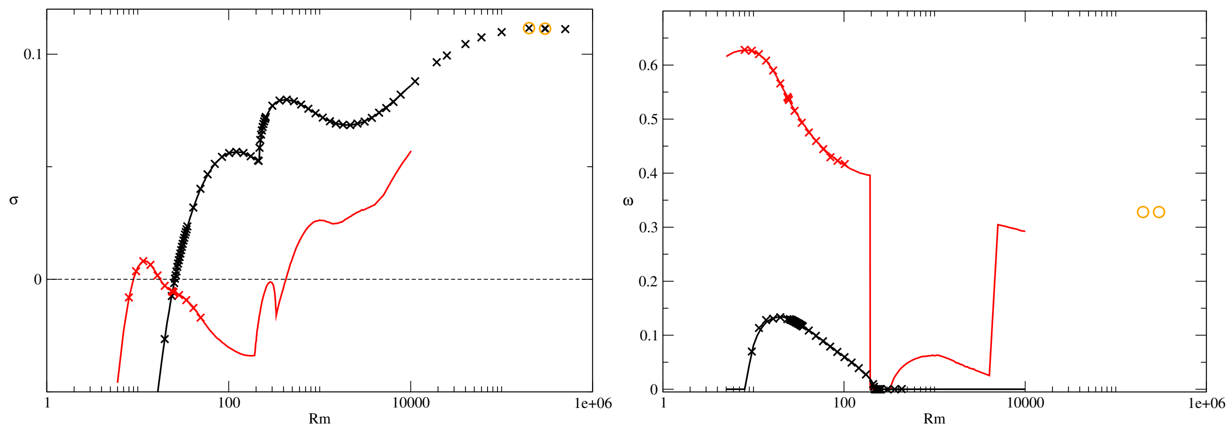

Fig. 4: Real (left) and imaginary (right) part of the leading eigenvalue associated to the V symmetry family (black) and the II symmetry family (red), in the classification introduced by [JG14]. Continuous lines correspond to the results published in [JG14]. Crosses present our numerical results in [BD13] and in the present study. Circles denote the transient behaviour.

of repeated mode crossings as Rm $\to \infty$, while the actual growth rate itself saturates [FO88]. An alternative scenario, could be that as one is approaching an essential spectrum in the limit Rm $\to \infty$, i.e. the complementary to the discrete spectrum (isolated eigenvalues with finite multiplicity) see [EE86,Kat95]. The growth rate (real part of the eigenvalue) of all the eigenmodes then tends to the same value. This somewhat more optimistic interpretation might suggest that the asymptotic behaviour of the $1:1:1$ ABC dynamo, while not yet obtained numerically could be tackled in a near future.

Discussion. – The values of Rm achieved in this study (up to $5 \cdot 10^5$) are the largest numerically investigated so far. They require a very significant numerical resolution (up to 4096^3 Fourier modes) and were performed using state of the art computational resources.

We have first shown that the branch identified in the so called "second window" of the ABC-dynamo (see [GF84]) remains the leading eigenmode up to Rm $= 5 \cdot 10^5$. This branch corresponds to the V symmetry class introduced by [JG14], and is associated to a purely real leading eigenvalue in this parameter range (i.e. Rm $\in [215, 5 \cdot 10^5]$). We show that the growth rate is a decreasing function of Rm for the largest values we could tackle. Furthermore, we demonstrate that the leading eigenmode follows the anticipated spatial scaling as Rm$^{-1/2}$. Finally, we indentify slowly damped oscillations occuring at large values of Rm.

Several aspects of our simulations indicate that the ABC-dynamo is approaching an asymptotic behaviour for Rm $\simeq 10^5$. Namely, the fact that the cross section through the eigenmode does not reveal any change in its structure in the rescaled coordinates. This is also supported by the fact that the growthrate is only slightly above the theoretical upper bound and is now decreasing with Rm. The occurance of damped oscillations points to an approaching eigenvalue. This could be relevant to the asymptotic behaviour, for which an essential spectrum is expected.

However the asymptotic behaviour is not yet established and several issues indicate that one must be cautious in interpreting the numerical results. The occurence of slowly damped oscillations, could also be interpreted as a possible hint for an approaching eigenvalue crossing (as the one observed near Rm $\simeq 24$). This would result in a change of leading eigenmode. Besides, such high values may still be considered small in some asymptotic problems. Such is the case, for example in [Sow87], which reveals a decrease of the growth rate as $\log(\log(\mathrm{Rm}))/\log(\mathrm{Rm})$. If this was the case for the $1:1:1$ ABC dynamo, its asymptotic behaviour could remain out of reach of direct numerical simulations for still a long period of time.

Despite its simple analytical form, the ABC-flow dynamo remains remarkably challenging from a computational point of view. The simulations presented here are extreme in terms of parameter value (fast dynamo limit), in terms of numerical resolutions (up to 4096^3) and in terms of computational resources (more than 10^6 CPU hours for the five new values of Rm calculated in this study).

The two main scenario that emerge from our study, are either the possibility of an approaching new mode crossing which implies that the asymptotic behaviour is not yet met, or that the real parts of all eigenvalues are approaching the same limit, which would instead indicate an essential spectrum. Simulations at larger values of Rm may shed some light on those issues.

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