Equations with one catalytic variable in enumerative combinatorics

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July 2, 2018

These are lecture notes for a minicourse given at the workshop *Random matrices, maps and gauge theories* held at ENS de Lyon on June 25-28, 2018. I try to present the general approach from [1] and illustrate it on some examples (1D lattice walks and planar maps). Due to lack of time and competence I do not discuss equations with several catalytic variables, on which there has been some important progress. These notes contain probably typos, imprecisions or mistakes, please let me know if you find any.

1 Introduction: two illustrative examples

1.1 Simple walks on the half-line

As a very simple introductory example, we consider the classical problem of counting simple walks on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with n steps that start and end at 0. By such a walk we mean a sequence $(x_0, x_1, \ldots, x_n) \in \mathbb{N}^{n+1}$ such that $x_0 = x_n = 0$ and $x_t - x_{t-1} = \pm 1$ for any $t = 1, \ldots, n$.

Suppose that we are very naive and do not know about the reflection method nor any other clever combinatorial tricks. If we attempt to enumerate walks by constructing them step by step, we are naturally led to generalize the counting problem and consider the number $a_{n,k}$ of walks which have n steps, still start at 0 but end at an arbitrary position $k \geq 0$. This number is determined by the recurrence relation

$$a_{n,k} = \begin{cases} a_{n-1,k-1} + a_{n-1,k+1} & \text{for } n \geq 1 \text{ and } k \geq 1, \\ a_{n-1,k+1} & \text{for } n \geq 1 \text{ and } k = 0, \\ 1 & \text{for } n = 0 \text{ and } k = 0, \\ 0 & \text{for } n = 0 \text{ and } k \geq 1. \end{cases} \tag{1}$$

This recurrence relation appeared essentially in Bertrand's 1887 paper on the classical ballot problem, hence the $a_{n,k}$ are sometimes called the ballot numbers. Bertrand solved the recurrence by conjecturing a general formula[1] for $a_{n,k}$ and checking it by induction, but how could we proceed even without a guess? A possible approach is to introduce the univariate series

$$A_k(t) := \sum_{n \geq 0} a_{n,k} t^n \tag{2}$$

which is convergent for $|t| < 1/2$ (as we clearly have $a_{n,k} \leq 2^n$), and satisfies

$$A_k = \begin{cases} t\,(A_{k-1} + A_{k+1}) & k \geq 1, \\ 1 + tA_1 & k = 0. \end{cases} \tag{3}$$

(For brevity we write A_k instead of $A_k(t)$, and we will very often do similar elisions in the following.)

We see that A_k satisfies a homogeneous linear recurrence with constant coefficients, and as such we look for a solution of the form

$$A_k = \alpha X^k + \beta X^{-k} \tag{4}$$

where X and X^{-1} are the two roots of the *characteristic equation*

$$1 = t(X + X^{-1}). \tag{5}$$

We assume X to be the root with smaller modulus, which is actually a series in t:

$$X = \frac{1 - \sqrt{1 - 4t^2}}{2t} = t + t^3 + 2t^5 + 5t^7 + 14t^9 + o(t^9). \tag{6}$$

The constants α and β are fixed by the boundary conditions. It seems at first that we only have one equation $A_0 = 1 + tA_1$ for two unknowns, but in fact we should take $\beta = 0$: colloquially this is because A_k should not "blow up" for $k \to \infty$ (and we have $|X| < 1$ for any $t < |1/2|$), more precisely it is because A_k should be a formal power series in t for all k, but X^{-k} is a Laurent series containing negative powers of t, hence must be ruled out (we will be more precise about such "formalities" later). We then find $\alpha = X/t$ so that $A_k = X^{k+1}/t$. In particular, the Taylor expansion of $A_0 = X/t$ involves the famous *Catalan numbers*

$$a_{2n,0} = \frac{1}{n+1}\binom{2n}{n} \tag{7}$$

which yields the solution to our initial counting problem.

It is instructive to redo the same computation using the *bivariate* generating function

$$A(t,u) := \sum_{n \geq 0} \sum_{k \geq 0} a_{n,k} t^n u^k. \tag{8}$$

Then, the recurrence relation (1) amounts to the functional equation

$$A(t,u) = 1 + tuA(t,u) + t\frac{A(t,u) - A(t,0)}{u}. \tag{9}$$

[1] Which was in fact known to Moivre as early as 1708.

Here, the second term corresponds to the contribution of the $a_{n-1,k+1}$ in the rhs of (1):

$$\sum_{n\geq 1}\sum_{k\geq 0} a_{n-1,k+1} t^n u^k = \frac{t}{u}\sum_{n\geq 0}\sum_{k\geq 1} a_{n,k} t^n u^k = t\frac{A(t,u)-A(t,0)}{u}. \tag{10}$$

Note that we are missing the term $k=0$ when doing the reindexing, and this is crucial in order to get a series without negative powers of u. Such divided differences will be very common in the functional equations that we will encounter, and are what makes them non trivial. In particular, taking $u=0$ in (9) does not seem to bring useful information. Following the terminology of Zeilberger, u is called a *catalytic variable*.

The following method is basically due to Knuth [2, p.537] and is now called the *kernel method*. We rewrite (9) in the form

$$K(t,u)A(t,u) = R(t,u) \tag{11}$$

where

$$K(t,u) := 1 - t(u+u^{-1}), \qquad R(t,u) := 1 - tu^{-1}A_0(t) \tag{12}$$

(K being called the *kernel*).

The key observation is that the kernel vanishes when we take $u=X$, with X the root of the characteristic equation considered above[2]. Therefore, if we substitute $u=X$ in (11), we deduce that $R(t,X)$ vanishes too, which yields a short proof of the formula $A_0(t) = X/t$. An important technical detail is that the substitution $u=X$ in (11) must be well-defined, which is the case because X is a series in t with only positive powers of t so that $A(t,X)$ makes sense. In contrast, substituting $u=X^{-1}$, the other root of the characteristic equation, is not allowed.

Furthermore, in constrast with the previous approach, the kernel method brings directly to an expression for $A(t,0)$ (which solves our initial counting problem). Computing the full bivariate series $A(t,u)$ (i.e. solving the extended counting problem) is not a prerequisite but may be done in a second step: using again (11) and factoring the kernel we get

$$A(t,u) = \frac{R(t,u)}{K(t,u)} = \frac{1-X/u}{\frac{X}{t}(1-Xu)(1-X/u)} = \frac{X}{t}\cdot\frac{1}{1-Xu} = \sum_{k\geq 0}\frac{X^{k+1}}{t}u^k. \tag{13}$$

1.2 Rooted planar maps

A *planar map* is a connected graph embedded in the sphere (without edge crossings), and considered up to continuous deformation. Loops and multiple edges are allowed. A map is *rooted* if one of its edges is marked and oriented. We denote by m_n the number of rooted planar maps with n edges.

It is not clear how to write down a recurrence equation for m_n but, following Tutte, we introduce the number $m_{n,k}$ of rooted planar maps with n edges and outer degree k, where the *outer degree* is defined as the number of edges incident to the face on the right

[2] Of course it is no coincidence that $K(t,X)=0$ is precisely the characteristic equation.

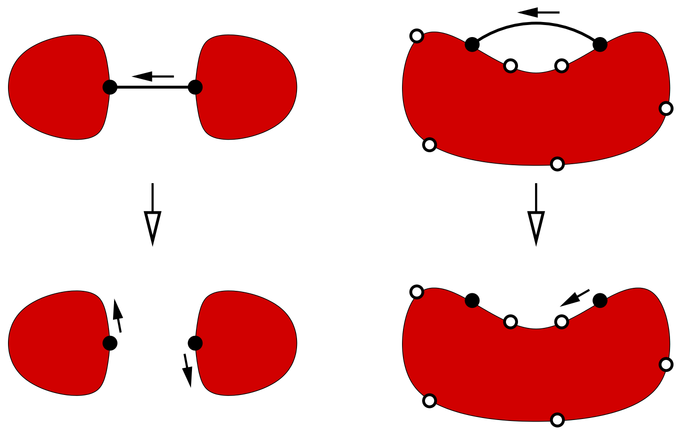

Figure 1: Tutte's recursive decomposition of a rooted planar map.

of the root, counted with multiplicity (a bridge contributes 2 to the outer degree). By convention we set $m_{0,0} = 1$ (correspond to the vertex-map reduced to a single vertex with no edge) and $m_{0,k} = 0$ for $k \neq 0$. Then, for $n \geq 1$, we have the recurrence relation

$$m_{n,k} = \sum_{n'=0}^{n-1} \sum_{k'=0}^{k-2} m_{n',k'} m_{n-n'-1,k-k'-2} + \sum_{k' \geq k-1} m_{n-1,k'}. \tag{14}$$

The interpretation of this relation is displayed on Figure 1: if we remove the root edge in a map counted by $m_{n,k}$, two situations may occur:

- if the root edge is a bridge (which happens only for $k \geq 2$), then we obtain two maps (both canonically rooted) whose numbers of edges add up to $n-1$ and outer degrees add up to $k-2$,
- otherwise, we obtain a single map with $n-1$ edges and outer degree $\geq k-1$ (with equality if the root edge is a loop).

By passing to the generating function

$$M(t,u) := \sum_{n \geq 0} \sum_{k \geq 0} m_{n,k} t^n u^k \tag{15}$$

we obtain the functional equation

$$M(t,u) = 1 + tu^2 M(t,u)^2 + tu \frac{uM(t,u) - M(t,1)}{u-1}. \tag{16}$$

Note that $M(t,1) = \sum_{n \geq 0} m_n t^n$ is the series we want to determine.

In contrast with the functional equation (9) encounted previously, the main difference is that (16) is not linear but quadratic (the fact that it involves a specialization at $u = 1$ rather than $u = 0$ is inessential). We may rewrite it in the form

$$P(M(t,u), M(t,1), t, u) = 0 \tag{17}$$

with P the polynomial

$$P(M, \mu, t, u) = tu^2(1-u)M^2 + (u - 1 - tu^2)M + 1 - u + tu\mu. \tag{18}$$

The trick to solve such an equation is the following. Let us differentiate (17) with respect to the variable u: by the chain rule we get

$$\frac{\partial P}{\partial M}(M(t,u), M(t,1), t, u)\frac{\partial M}{\partial u}(t,u) + \frac{\partial P}{\partial u}(M(t,u), M(t,1), t, u) = 0. \tag{19}$$

The quantity $\frac{\partial P}{\partial M}(M(t,u), M(t,1), t, u)$ is the nonlinear analogue of the kernel (we recover the same notion for P linear in M). Pursuing the analogy, let us assume[3] that we may find a series U in the variable t such that the substitution $M(t,U)$ makes sense, and such that

$$\frac{\partial P}{\partial M}(M(t,U), M(t,1), t, U) = 0. \tag{20}$$

Then it follows from (19) that we have also

$$\frac{\partial P}{\partial u}(M(t,U), M(t,1), t, U) = 0. \tag{21}$$

This relation is analogous to the equation $R(t,X) = 0$ of the previous section. But the new feature is that (20) and (21) still involve the unknown series $M(t,U)$. However, if we add the relation (17) at $u = U$ (which is not a consequence of the two others), then we obtain a system of three polynomial equations for the three "unknowns" $M(t,U)$, $M(t,1)$, U and the variable t. With some chance, it is possible to solve this system and deduce an expression for $M(t,1)$. This turns out to be the case, and we leave the actual computation as an exercise (which, as often here, is more easily done with a computer algebra system).

Exercise 1. Show that $M(t,1)$ is given by

$$M(t,1) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n. \tag{22}$$

(Hint: use the fact that (20) and (21) are respectively linear in $M(t,U)$ and $M(t,1)$ to express them in terms of t and U. Deduce from (17) a polynomial equation satisfied by U. Pick the correct solution from the requirement that $M(t,1)$ is a series in t without negative powers.)

[3] It is actually not difficult to justify this assumption, using the considerations from the next section.

2 Solving polynomial equations with one catalytic variable

2.1 Formalities on formal power series

In this section, R denotes an integral domain[4] and t a formal variable. A *formal power series* in t with coefficients in R is an expression of the form

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 X + a_2 X^2 + \cdots \tag{23}$$

where $(a_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of elements of R. The summation symbol is here formal and it usually makes no sense to substitute t with a nonzero element of R. We denote by $[t^n]A(t) := a_n$ the coefficient of t^n in $A(t)$ and, by analogy with Taylor series, we often write

$$a_0 = A(0), \qquad a_1 = A'(0), \qquad a_2 = \frac{A''(0)}{2}, \qquad \text{etc.} \tag{24}$$

Ring structure Let $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be two formal power series. We define their sum and product by

$$A(t) + B(t) := \sum_{n=0}^{\infty} (a_n + b_n) t^n, \qquad A(t) \times B(t) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) t^n. \tag{25}$$

These operations are the same as addition and multiplication for polynomials in t, but in a formal power series we drop the requirement that only a finite number of coefficients are nonzero.

We denote by $R[[t]]$ the set of formal power series in t with coefficients in R, equipped with the above addition and multiplication. It is a commutative ring (with unit 1), containing the ring $R[t]$ of polynomials in t as a subring.

Example 1. The series $1 + t + t^2 + t^3 + \cdots$ is the multiplicative inverse of $1 - t$. It is an example of a *rational* formal power series, namely a series $A(t)$ such that $q(t)A(t) = p(t)$ for some nonzero polynomials $p(t)$ and $q(t)$.

Example 2. The series $X(t)$ of (6) satisfies $X(t) = t + X(t)^2$. It is an example of an *algebraic* formal power series, namely a series $A(t)$ such that $p_0(t) + p_1(t)A(t) + \cdots + p_r(t)A(t)^r = 0$ for some $r > 0$ and some nonzero polynomials $p_0(t), p_1(t), \ldots, p_r(t)$. Algebraic formal power series enjoy many nice properties and, in particular, studying the asymptotic behaviour of their coefficients (for $R = \mathbb{C}$) can be done in a systematic way. (One may consider the more general classes of D-finite and D-algebraic series, but we will not encounter them here.)

[4] Commutative ring with unit 1 and without nonzero zero divisors.

Distance, topology, composition, Lagrange inversion The *valuation* of a nonzero series $A(t)$ is the smallest n such that $[t^n]A(t) \neq 0$. It is denoted $\nu(A)$ and, by convention, $\nu(0) = +\infty$. We have $\nu(AB) = \nu(A) + \nu(B)$ and $\nu(A + B) \geq \min(\nu(A), \nu(B))$. Given two series A and B, we define their distance as

$$d(A, B) := 2^{-\nu(A-B)} \tag{26}$$

with the convention $2^{-\infty} = 0$. This endows $R[[t]]$ with the structure of a complete ultrametric space. Let us note the following facts:

- a sequence of series $(A_m)_{m\geq 0}$ converges iff, for any n, the sequence $([t^n]A_m)_{m\geq 0}$ is eventually constant,
- in particular A is indeed the limit of its "polynomial approximations" $\sum_{n=0}^{m} a_n t^n$ as $m \to \infty$,
- a mapping $A \mapsto \Phi(A)$ is contracting iff, for any n, the coefficient of t^n in $\Phi(A)$ depends only of those of smaller degree in A,
- an infinite sum of series $\sum_m A_m(t)$ converges iff $A_m(t) \to 0$.

Let $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t)$ be two series. We define their *composition* by the formula

$$A(B(t)) := \sum_{n=0}^{\infty} a_n (B(t))^n \tag{27}$$

provided that the sum is convergent. By the previous convergence criterion, this is the case if $A(t)$ is a polynomial, or if $\nu(B) > 0$.

Let $\Phi(t) = \sum_{n=0}^{\infty} \phi_n t^n$. By the Banach fixed point theorem, there exists a unique series $Z(t)$ with zero constant coefficient such that

$$Z(t) = t\,\Phi(Z(t)). \tag{28}$$

Indeed, the mapping $A(t) \mapsto t\Phi(A(t))$ is contracting with respect to d in the closed subspace of series with zero constant coefficient. The *Lagrange-Bürmann* inversion formula gives an explicit expression for the coefficients of $Z(t)$ and its powers, namely

$$[t^n]Z(t)^k = \frac{k}{n}[w^{n-k}]\Phi(w)^n, \qquad n, k \geq 0. \tag{29}$$

Three different proofs (two of them combinatorial) are given in [3].

Remark 3. If ϕ_0 is a unit of R then $A(t) = t/\Phi(t)$ is a well-defined series, and we have $A(Z(t)) = t$, in other words Z is the *compositional inverse* of A. Conversely any series A with $A(0) = 0$ and $A'(0)$ invertible can be written in the form $A(t) = t/\Phi(t)$, hence the Lagrange-Bürmann formula is a way to perform *series reversion*.

Series in two (or more) variables Let u be another formal variable. A *formal power series* in t and u with coefficients in R is an expression of the form

$$A(t,u) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k} t^n u^k \tag{30}$$

where the $a_{n,k}$ are arbitrary elements of R. Addition and multiplication are easy to define, and we denote by $R[[t,u]]$ the corresponding ring. Note that we may identify

$$R[[t,u]] \simeq R[[t]][[u]] \simeq R[[u]][[t]]. \tag{31}$$

We denote by $[t^n]$, $[u^k]$, $[t^n u^k]$ the partial or complete extraction of coefficients.

We may define a natural topology[5] on $R[[t,u]]$, but the notion of distance (useful for contractivity arguments) is less clear. However, most of the bivariate series that we consider lie actually in the subring $R[u][[t]]$ of series in t whose coefficients are polynomials in u. This is the case if the coefficients in (30) are such that, for fixed n, only a finite number of $a_{n,k}$ are nonzero. This situation naturally occurs in enumeration when considering a "refined" counting problem. We have the proper inclusions

$$R[[t]][u] \subset R[u][[t]] \subset R[[t,u]]. \tag{32}$$

The natural notion of valuation and distances for series in $R[u][[t]]$ is that corresponding to the variable t. For any series $A(t,u) \in R[u][[t]]$, it makes sense to substitute u with any element of R. More generally, for $U(t) \in R[[t]]$, the univariate series

$$A(t,U(t)) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n,k} U(t)^k \right) t^n \tag{33}$$

is well-defined and, furthermore, the mapping $U(t) \mapsto A(t,U(t))$ is 1-Lipschitz.

Exercise 2. Prove that the equation (20) considered for the enumeration of rooted planar maps indeed admits a unique solution $U(t) \in \mathbb{Z}[[t]]$. (Hint: rewrite the equation as a fixed point equation of a manifestly contractive mapping.)

Fractional power series Returning to the case of one variable, we may actually encounter generalized series of the form

$$A(t) = \sum_{n=n_0}^{\infty} a_n t^{n/d} \tag{34}$$

with d a positive integer and n_0 a possibly negative integer. If $a_{n_0} \neq 0$ then n_0/d is the valuation of $A(t)$. These are called *Puiseux series* and form a ring denoted $R^{fr}((t))$. It has two subrings of interest:

[5] A sequence $(A_i(t,u))_{i\geq 0}$ converges iff, for any n,k, the sequence of coefficients $[t^n u^k]A_i(t,u)$ is eventually constant.

- the ring $R((t))$ of *formal Laurent series* (series of the form (34) with $d = 1$),
- the ring $R^{fr}[[t]]$ of *fractional power series* (series of the form (34) with $n_0 \geq 0$).

Let us mention two well-known facts:

- if K is a field, then $K((t))$ is the field of fractions of $K[[t]]$,
- if K is an algebraically closed field of characteristic zero, then $K^{fr}((t))$ is the algebraic closure of $K((t))$ (this is the *Newton-Puiseux theorem*).

We also mention the following useful fact.

Proposition 4. *Let $\Phi(t, u) \in K[u]^{fr}[[t]]$ (i.e. a fractional power series in t with coefficients polynomial in u), with K an algebraically closed field of characteristic 0. If the polynomial $\Phi(0, u)$ is nonzero and has degree k, then there exists k fractional power series $U_1, \ldots, U_k \in K^{fr}[[t]]$ such that*

$$\Phi(t, u) = (u - U_1) \cdots (u - U_k)\Psi(t, u) \tag{35}$$

with $\Psi(0, u)$ a nonzero constant. In particular we have $\Phi(t, U_i) = 0$ for all $i = 1, \ldots, k$.

(This is a slight adaptation of the Newton-Puiseux theorem that only applies to the case where $\Phi(t, u) \in K^{fr}[[t]][u]$.)

2.2 General strategy

We now unify the two examples considered in Section 1 within a general framework. Here, K denotes a field of characteristic 0 (typically $\mathbb{C}$ or $\mathbb{C}(x, y, \ldots)$ if we have extra parameters $x, y, \ldots$), and $\bar{K}$ denotes its algebraic closure.

In the general framework, we consider a $(k+1)$-tuple of series $(F(t, u), F_1(t), \ldots, F_k(t)) \in K[u][[t]] \times K[t]^k$ which is *completely* determined by an equation of the form

$$P(F(u), F_1, \ldots, F_k, t, u) = 0 \tag{36}$$

with $P(x_0, x_1, \ldots, x_k, t, v)$ a polynomial in $k + 3$ variables with coefficients in K (in this section we will often drop the explicit dependence on t for concision). We call an equation of the form (36) a *polynomial equation with one catalytic variable*, and equality should hold in the ring $K[u][[t]]$.

It is often the case that the F_i are specializations of $F(u)$ or its derivatives at $u = 0$ or 1, but we need not assume it a priori. Actually, it is typically a *consequence* of (36), for instance in our second introductory example, at $u = 1$ we have $P(x_0, x_1, t, 1) = t(x_1 - x_0)$ which *implies* that $F_1 = F(1)$.

We redo the same observation as in Section 1.2: if we differentiate (36) with respect to u we get

$$F'(u)\frac{\partial P}{\partial x_0}(F(u), F_1, \ldots, F_k, t, u) + \frac{\partial P}{\partial v}(F(u), F_1, \ldots, F_k, t, u) = 0 \tag{37}$$

and, if we substitute $u = U(t)$ a series such that

$$\frac{\partial P}{\partial x_0}(F(U), F_1, \ldots, F_k, t, U) = 0 \tag{38}$$

then we also have $\frac{\partial P}{\partial v}(F(U), F_1, \ldots, F_k, t, U) = 0$ and $P(F(U), F_1, \ldots, F_k, t, U) = 0$. In general, we should look for such fractional power series in $\bar{K}^{fr}[[t]]$ and there might be *several* distinct solutions $U_1, \ldots, U_\ell$, so that we get a system of 3ℓ polynomial equations

$$\begin{aligned} P(F(U_i), F_1, \ldots, F_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial x_0}(F(U_i), F_1, \ldots, F_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial v}(F(U_i), F_1, \ldots, F_k, t, U_i) &= 0. \end{aligned} \tag{39}$$

The unknowns in this system are $F_1, \ldots, F_k, U_1, \ldots, U_k, F(U_1), \ldots, F(U_k)$. If we are "lucky", we will have $\ell = k$ so that there are as many equations as unknowns, and the system will uniquely determine $F_1, \ldots, F_k$.

More precisely, we are in a *good situation* if the two following conditions are satisfied:

1. (existence) there exists k *distinct* series $U_1, \ldots, U_k \in \bar{K}^{fr}[[t]]$ such that (38) holds,
2. (unicity) the solution in $(\bar{K}^{fr}[[t]])^{3n}$ of the system (39) with distinct U_i's, which exists by the previous condition, is unique up to a permutation of them.

If so, we may conclude that the $F_1, \ldots, F_k$ determined by (36) are necessarily equal to those determined by (39).

In practice, the first existence condition should be proved without knowing the solution of (39). This may be done using contractivity (see Exercise 2) or algebraic (Proposition 4) arguments.

An applicability result We now state a general sufficient condition under which the above scheme works and we are in a good situation. (Of course, we may already apply it to specific examples and verify that it works on a case-by-case basis.)

We denote by Δ the divided difference operator with respect to u:

$$\Delta F(u) := \frac{F(u) - F(0)}{u}. \tag{40}$$

The action of its iterates read

$$\Delta^i F(u) = \frac{F(u) - F(0) - uF'(0) - \cdots - \frac{u^{i-1}}{(i-1)!}F^{(i-1)}(0)}{u^i}. \tag{41}$$

Note that Δ is 1-Lipschitz on $K[u][[t]]$.

We then consider a functional equation of the form

$$F(t, u) = F_0(u) + t\, Q\left(F(u), \Delta F(u), \ldots, \Delta^k F(u), t, u\right) \tag{42}$$

with $F_0(u)$ and $Q(y_0, y_1, \ldots, y_k, t, u)$ given polynomials in respectively 1 and $k+3$ variables. By contractivity this equation determines a unique series $F(t, u)$, and using (41) and multiplying by a large enough power of u the functional equation can be put in the form (36) with $F_i = F^{(i-1)}(0)$. Then, it is shown in [1, Section 4] that we are basically in a good situation, which leads to the following result.

Theorem 5 ([1])**.** *The series $F(t, u)$ is algebraic over $K(t, u)$.*

We do not provide a detailed proof here. Let us just remark that the U_i must be roots of

$$u^k = t \sum_{j=0}^{k} u^{k-j} \frac{\partial Q}{\partial y_j} \Big(F(u), \Delta F(u), \ldots, \Delta^k F(u), t, u \Big) \tag{43}$$

hence, by Proposition 4 (both sides belonging to $K[u][[t]]$), there are indeed k solutions.

Remark 6. If Q is divisible by u then (42) implies that $F(t, 0) = F_0(t)$. Hence we readily "get rid" of one unknown. Correspondingly, the equation (43) admits $U = 0$ as a trivial root. We still have as many nontrivial roots as nontrivial unknowns.

3 Applications

3.1 1D lattice walks

We now return to the case of walks on the half-line $\mathbb{N}$ but we now allow for wider steps. Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}$ containing at least one positive and one negative element. We wish to enumerate sequences $(x_0, \ldots, x_n) \in \mathbb{N}^{n+1}$ such that $x_0 = x_n = 0$ and $x_t - x_{t-1} \in \mathcal{S}$ for any $t = 1, \ldots, n$. Following the terminology of [4] these walks are called *excursions*. As in Section 1.1, we shall consider the more general family of *meanders* where the endpoint x_n is not necessarily at 0: we denote by $a_{n,k}$ the number of meanders with n steps ending at k. (We could of course consider weighted paths but we will not do it here for simplicity.)

It is a simple exercise to check that the bivariate generating function $A(t, u) := \sum a_{n,k} t^n u^k$ is determined by the functional equation

$$A(t, u) = 1 + t \sum_{\substack{i \in S \\ i \geq 0}} u^i A(t, u) + t \sum_{\substack{i \in S \\ i < 0}} \Delta_u^{-i} A(t, u). \tag{44}$$

This equation is of the form (42) so we are in a good situation. The kernel equation reads

$$1 = t \sum_{i \in S} u^i \tag{45}$$

or in polynomial form

$$u^c = t \sum_{i \in S} u^{i+c}, \qquad c := -\min S. \tag{46}$$

By Proposition 4 we will find exactly c roots $U_1, \dots, U_c$ in $\mathbb{C}^{fr}[[t]]$ and actually they are given by $U_i(t) = V(\omega_i t^{1/d})$ with ω_i a c-th root of unity and $V(\tau)$ the unique series in $\mathbb{C}[[\tau]]$ such that

$$V = \tau \left(\sum_{i \in S} V^{i+c} \right)^{1/c} . \tag{47}$$

Because we are in a linear situation we may proceed more directly than solving the general system (39). The functional equation can be rewritten in the form

$$\left(u^c - t \sum_{i \in S} u^{i+c} \right) A(t,u) = R(t,u) \tag{48}$$

where $R(t,u) \in \mathbb{C}[[t]][u]$ is a monic[6] polynomial of degree c in u, which therefore immediately factorizes as

$$R(t,u) = \prod_{i=1}^{c} (u - U_i(t)) . \tag{49}$$

We deduce that

$$A(t,u) = \frac{\prod_{i=1}^{c} (u - U_i(t))}{u^c - t \sum_{i \in S} u^{i+c}} \tag{50}$$

and in particular excursions are counted by

$$A(t,0) = \frac{(-1)^{c-1}}{t} \prod_{i=1} U_i(t). \tag{51}$$

Exercise 3 (The libertine club). Take $S = \{-2, 3\}$. Show that $X = A(t,0)$ satisfies

$$X = 1 + t^5(2X^5 - X^6 + X^7) + t^{10} X^{10}. \tag{52}$$

It is possible to obtain an explicit expression for the general term, see [4, Example 5].

3.2 Rooted planar maps

We now return to the case of rooted planar maps considered in Section 1.2 but we now want to add face weights: let $(z_i)_{0 \geq 1}$ be a set of variables (which may be formal or complex), and consider the generating function

$$F(t,u) = \sum_{\mathcal{M}} t^{e(\mathcal{M})} u^{d(\mathcal{M})} z_0^{v(\mathcal{M})} \prod_{i \geq 1} z_i^{f_i(\mathcal{M})} \tag{53}$$

where the sum is over all rooted planar maps, and $d(\mathcal{M})$, $e(\mathcal{M})$, , $v(\mathcal{M})$, $f_i(\mathcal{M})$ denote respectively the outer degree, the number of edges, the number of vertices, and the number of inner faces of degree i of $\mathcal{M}$.

[6]This comes immediately from the constant term 1 in (44).

By the Tutte recursive decomposition explained in Section 1.2, we find the functional equation

$$F(t,u) = z_0 + tu^2 F(t,u)^2 + tu \sum_{i\geq 1} z_i \Delta^{i-1} F(t,u) \tag{54}$$

where Δ is the divided difference operator (40) acting on the variable u. This is a refinement of (16) which we recover when all z_i's are equal to 1.

We now concentrate on the case of *bounded degrees*, i.e. we fix $m \geq 3$ and assume that $z_i = 0$ for $i > m$. Then we have a functional equation of the good form (42) hence we know that $F(t,u)$ is algebraic. We are furthermore in the situation of Remark 6 so the equation $\frac{\partial P}{\partial x_0}(F(U), F_1, \ldots, F_{m-2}, t, U) = 0$ admits $m-2$ nontrivial roots $U_1, \ldots, U_m$. We note that the functional equation is *quadratic* in $F(t,u)$ hence may be rewritten in the form

$$(2aF + b)^2 = D \tag{55}$$

where $D = b^2 - 4ac$ is the discriminant and

$$a = tu^m, \qquad b = t\sum_{i=1}^{m} z_i u^{m-i} - u^{m-2}, \qquad c = z_0 u^{m-2} - t\sum_{i=2}^{m} z_i u^{m-i}(\cdots) \tag{56}$$

which are polynomials in u of respective degrees $m, m-1, m-2$ (with coefficients in $K[[t]]$) so that D has degree $2m-2$. But note that the roots U_i cancel by definition the factor $2aF + b$ hence are *double* roots of D. We conclude that D may be factored as

$$D(t,u) = C(t)\kappa(t,u)\prod_{i=1}^{m-2}(u - U_i(t))^2 \tag{57}$$

where $\kappa(t,u)$ is a polynomial of degree 2 in u. But, since $D(0,u)$ has degree only $2m-4$, we see that $\kappa(0,u)$ is of degree 0 in u, and may be taken equal to 1 up to a redefinition of $C(t)$, so that we may write

$$D(t,u) = (1 + \kappa_1(t)u + \kappa_2(t)u^2)Q(t,u)^2 \tag{58}$$

with Q of degree $m-2$ in u. By taking the square root of (55) we conclude that $F(t,u)$ takes the *one-cut* form

$$F(t,u) = \frac{1}{2}\left(\frac{1}{tu^2} - \sum_{i=1}^{m}\frac{z_i}{tu^i} + \frac{Q(t,u)}{tu^m}\sqrt{1 + \kappa_1(t)u + \kappa_2(t)u^2}\right). \tag{59}$$

But recall that $F(t,u)$ must have coefficients polynomial in u so all the terms with negative powers of u in (59) must vanish, and in addition $F(t,0) = 1$: this yields a system of $m+1$ equations determining Q and $\kappa_{1,2}$. It is possible to perform this computation in a very explicit manner and arrive at a general formula for $[u^k]F(t,u)$ with a nice combinatorial interpretation, see [5, Section 3.2] for details.

Let us do this computation in the *bipartite* case where all face degrees are even ($z_i = 0$ for i odd and $m \to 2m$). Then, $F(t,u)$ is a series in $v = u^2$ so that the solution may be rewritten in the simpler form

$$F(t,v) = \frac{1}{2}\left(\frac{1}{tv} - \sum_{i=1}^{m} \frac{z_{2i}}{tv^i} - \frac{Q(t,v)}{tv^m}\sqrt{1-4Rv}\right) \tag{60}$$

with Q now of degree $m-1$ in v. Note the series expansions

$$\sqrt{1-4Rv} = 1 - 2\sum_{k\geq 0} Cat_n (Rv)^{k+1}, \qquad \frac{1}{\sqrt{1-4Rv}} = \sum_{k\geq 0}\binom{2k}{k}(Rv)^k. \tag{61}$$

Dividing (60) by $\sqrt{1-4Rv}$ we get

$$\frac{F(t,v^{1/2})}{\sqrt{1-4Rv}} = \frac{1}{2}\left(\frac{1}{tv} - \sum_{i=1}^{m}\frac{z_{2i}}{tv^i}\right)\left(\sum_{k\geq 0}\binom{2k}{k}(Rv)^k\right) - \frac{Q(t,v)}{2tv^m}. \tag{62}$$

But the lhs does not contain negative powers of v and its constant coefficient is z_0: this yields a total of $m+1$ conditions which determine the m coefficients of Q (from the no negative powers condition) and give an equation for R (constant term 1):

$$z_0 = \frac{1}{2}\left(\frac{2R}{t} - \sum_{i=1}^{m}\frac{z_{2i}}{t}\binom{2i}{i}R^i\right) \tag{63}$$

hence

$$R = tz_0 + t\sum_{i=1}^{m} z_{2i}\frac{1}{2}\binom{2i}{i}R^i. \tag{64}$$

By plugging the expression for Q into (60) and expanding, we obtain an explicit expression of $[u^k]F(t,v)$ as a polynomial in R involving binomial coefficients and Catalan numbers, having nice combinatorial interpretation and applications [5]. But it turns out that R has all the information we want, indeed it may be shown that

$$F^\bullet(t,v) = \frac{1}{\sqrt{1-4Rv}} = \sum_{k\geq 0}\binom{2k}{k}R^k v^k \tag{65}$$

is the generating function for *pointed* rooted maps (i.e. an extra vertex is marked). By the Lagrange inversion and the multinomial formula we get that

$$\frac{1}{v}[t^n z_0^v z_2^{f_2}\cdots z_{2m}^{f_{2m}}]\binom{2k}{k}R^k = \frac{k}{nv}\binom{2k}{k}\binom{n-k}{v, f_2, \cdots, f_{2m}}\prod_{i=1}^{m}\binom{2i-1}{i}^{f_{2i}}\delta_{v-n+\sum f_i,1} \tag{66}$$

which is the number of rooted bipartite planar maps with outer degree $2k$, n edges, v vertices and f_i faces of degree $2i$ for each i.

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