Enumeration of planar maps: from the quadratic method to quantum gravity

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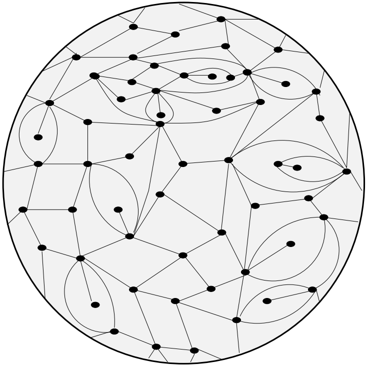
Outline

1. The beginning: Tutte's recursive approach
2. A detour in physics: matrix models and 2D quantum gravity
3. The bijective approach: labeled trees and the Brownian map
4. Beyond the Brownian map

Planar map: definition



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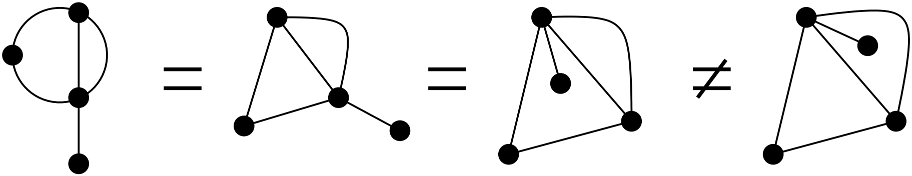

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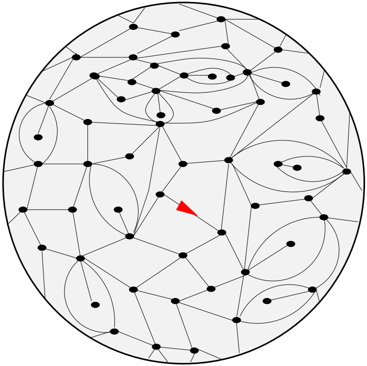

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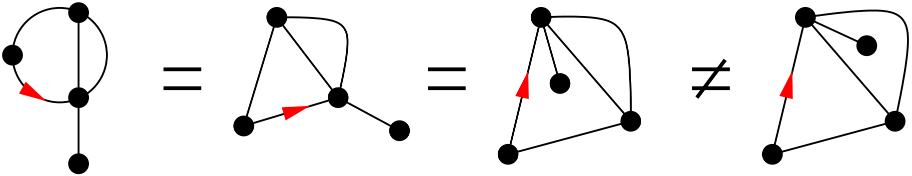
A rooted map has a distinguished oriented edge. A pointed map has a distinguished vertex.



Outline

Tutte's "Census" papers (1962-63)

A CENSUS OF PLANAR TRIANGULATIONS

W. T. TUTTE

1. Triangulations. Let P be a closed region in the plane bounded by a simple closed curve, and let S be a simplicial dissection of P. We may say that S is a dissection of P into a finite number α of triangles so that no vertex of any one triangle is an interior point of an edge of another. The triangles are "topological" triangles and their edges are closed arcs which need not be straight segments. No two distinct edges of the dissection join the same two vertices, and no two triangles have more than two vertices in common.

There are $k \geqslant 3$ vertices of S in the boundary of P, and they subdivide this boundary into k edges of S. We call these edges *external* and the remaining edges of S, if any, *internal*. If r is the number of internal edges we have

(1.1) $$3\alpha = 2r + k,$$

(1.2) $$r \equiv k \pmod 3.$$

Let us call S a *triangulation* of P if it satisfies the following condition: *no internal edge of S has both its ends in the boundary of P*. We note that in the case $k = 3$ every simplicial dissection is a triangulation.

A CENSUS OF SLICINGS

W. T. TUTTE

1. Introduction. A *band* is a closed connected set in the 2-sphere, bounded by one or more disjoint simple closed curves.

Consider a band B with bounding curves $J_1, J_2, \ldots, J_k$. On each curve J_i let there be chosen $m_i \geqslant 0$ points to be called *vertices*, with the restriction that the sum of the k integers m_i is to be even. Write

(1) $$\sum_{i=1}^{k} m_i = 2n.$$

Next consider a set of n disjoint open arcs in the interior of B which join the $2n$ vertices in pairs and partition the remainder of the interior of B into simply connected domains. We call the resulting dissection of B a *slicing* with respect to the given set of vertices. The arcs are the *internal edges* of the slicing and the simply connected domains are its *internal faces*, or *slices*.

A CENSUS OF HAMILTONIAN POLYGONS

W. T. TUTTE

Summary. In this paper we deal with trivalent planar maps in which the boundary of each country (or "face") is a simple closed curve. One vertex is distinguished as the *root* and its three incident edges are distinguished as the first, second, and third *major edges*. We determine the average number of Hamiltonian polygons, passing through the first and second major edges, in such a "rooted map" of $2n$ vertices. Next we consider the corresponding problem for 3-connected rooted maps. In this case we obtain a functional equation from which the average can be computed for small values of n.

1. Rooted maps. For the purposes of this paper a *planar map* M is a representation of the 2-sphere (or closed plane) as a union of a finite number of disjoint point-sets called *cells*. The cells are of three kinds, *vertices*, *edges*, and *faces*, said to have dimension 0, 1, and 2 respectively. Each vertex consists of a single point. Each edge is an open arc whose ends are distinct vertices. Each face is a simply connected domain whose boundary is a simple closed curve made up of edges and vertices. We denote the numbers of cells, vertices, edges, and faces of M by $C(M)$, $V(M)$, $E(M)$, and $F(M)$ respectively.

A CENSUS OF PLANAR MAPS

W. T. TUTTE

1. Introduction. In the series of "Census" papers, of which this is the fourth, we attempt to lay the groundwork of an enumerative theory of planar maps **(12, 13, 14)**. The maps concerned are *rooted* in the sense that some edge is fixed as the *root*, and a positive sense of description and right and left sides are specified for it. This device simplifies the theory by ruling out the possibility of a map being symmetrical.

In this paper formulae are obtained for the number of rooted maps (with n edges), the number of non-separable rooted maps, and the number of 3-connected rooted maps without multiple joins (called c-nets). Some similar enumerations, supplementing the results of earlier papers, are given for triangulations and bicubic maps.

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We will find and solve an equation satisfied by the generating function

$$M(x,y) := \sum_{n,k\geq 0} M_{n,k} x^n y^k$$

though we are ultimately interested only in $M(x,1)$ (y is a "catalytic" variable).

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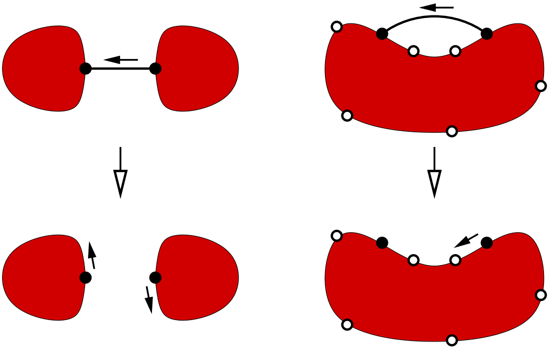

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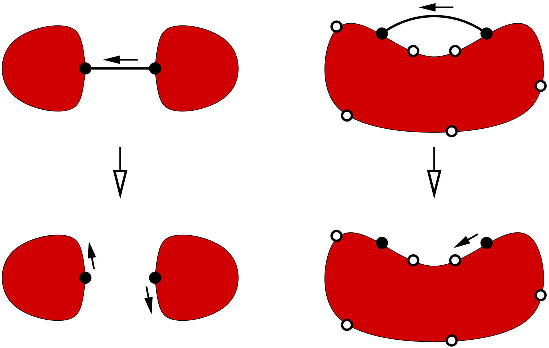

$$M_{n,k} = \sum_{\substack{n_1+n_2=n-1 \\ k_1+k_2=k-2}} M_{n_1,k_1} M_{n_2,k_2} \qquad + \qquad \sum_{k' \geq k'-1} M_{n-1,k'}$$

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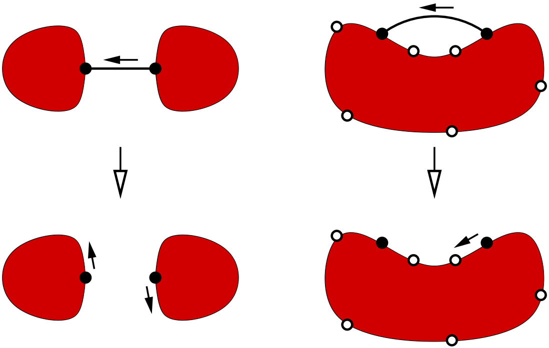

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(holds for $n \geq 1$, with $M_{0,0} = 1$, $M_{0,k} = 0$ for $k \geq 1$)

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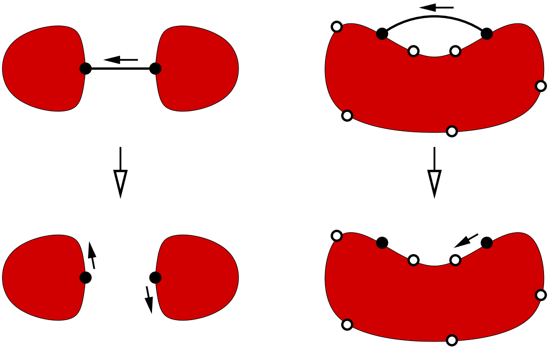

$$M(x,y) = 1 + xy^2 M(x,y)^2 + xy \frac{yM(x,y) - M(x,1)}{y-1}$$

.

Tutte's recursive approach to counting planar maps

$M(x, y)$ satisfies

$$\Phi(M(x, y), M(x, 1), x, y) = 0 \quad (1)$$

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$$\Phi(M, m, x, y) = xy^2(1 - y)M^2 + (y - 1 - xy^2)M + 1 - y + xym.$$

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By differentiating (1) wrt y, we get

$$\underbrace{\frac{\partial \Phi}{\partial M}(M(x,y), M(x,1), x, y)}_{(2)} \frac{\partial M}{\partial y}(x,y) + \underbrace{\frac{\partial \Phi}{\partial y}(M(x,y), M(x,1), x, y)}_{(3)} = 0.$$

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Let $Y(x)$ be a series such that (2) vanishes when we substitute $y = Y(x)$ (it exists and is unique!). Then (3) vanishes too. Combined with (1), we get 3 equations for 4 parameters, hence by elimination we may deduce an algebraic equation relating $M(x, 1)$ and x.

Tutte's recursive approach to counting planar maps

In practice, we even get a rational parametrization

$$x = \frac{(Y-1)(3-2Y)}{Y^2}, \qquad M(x,1) = \frac{Y(4-3Y)}{(3-2Y)^2}$$

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which allows to apply the Lagrange inversion theorem and get:

Theorem (Census of planar maps, 1963)

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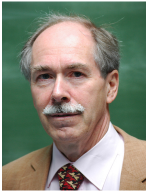
Some remarks:

- for many classes of maps, we get a similar quadratic equation $\Longrightarrow$ "quadratic method" (Brown, 1965)
- nowadays there is a general method to solve polynomial equations with one catalytic variable (Bousquet-Mélou and Jehanne, 2006)
- asymptotically $M_n \sim C 12^n n^{-5/2}$, the $-5/2$ exponent is universal.

Outline

Matrix models



G. 't Hooft

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Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi*, and J. B. Zuber
Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, F-91190 Gif-sur-Yvette, France

Abstract. We investigate the planar approximation to field theory through the limit of a large internal symmetry group. This yields an alternative and powerful method to count planar diagrams. Results are presented for cubic and quartic vertices, some of which appear to be new. Quantum mechanics treated in this approximation is shown to be equivalent to a free Fermi gas system.

In 1978, BIPZ considered the special case of zero spatial dimensions, where the gauge field reduces to a simple $N \times N$ matrix.

Matrix models

The problem is to estimate the large N asymptotics of an integral over $N \times N$ hermitian matrices.

3. Combinatorics of Quartic Vertices

1) Vacuum Diagrams

Setting each diagram equal to unity, apart from the overall weight, is equivalent to treat a field theory in zero dimension, in which space-time is reduced to one or to a finite number of points. It means that

$$\exp - N^2 E^{(0)}(g) = \lim_{N\to\infty} \int d^{N^2}M \exp - \left[\tfrac{1}{2}\operatorname{tr} M^2 + \frac{g}{N}\operatorname{tr} M^4\right]. \tag{3}$$

The integration measure on hermitian matrices is

$$d^{N^2}M \equiv \prod_i dM_{ii} \prod_{i<j} d(\operatorname{Re} M_{ij}) d(\operatorname{Im} M_{ij}) \tag{4}$$

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Such an integral may be viewed as a perturbation of a Gaussian integral (GUE partition function). By a formal expansion of the non Gaussian term, we get a perturbative series which is standardly represented via Feynman diagrams.

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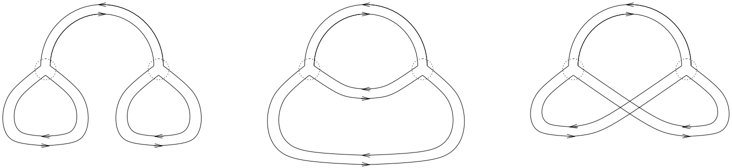
In the context of $N \times N$ matrices, the Feynman diagrams can be represented as "fatgraphs" or "ribbon graphs" (because a matrix carries two indices). For instance, the diagrams contributing to the expansion of $\langle (\mathrm{Tr} M^3)^2 \rangle$ are:



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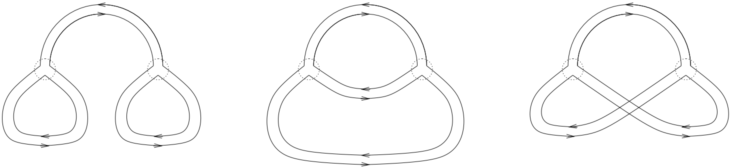

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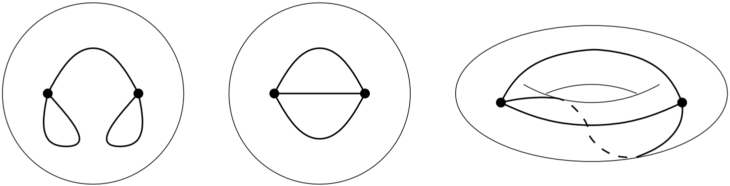

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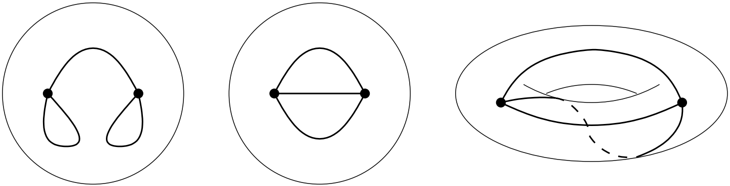

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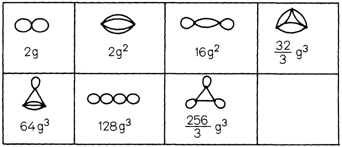
When N gets large, the dominant diagrams correspond to planar maps!

Table 1. Counting rules for the vacuum amplitude $E^{(0)}(g)$ in the planar limit, up to order three



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Yes we can, by exploiting the $U(N)$ invariance of the integrand:

Since the integrand (3) depends only on the eigenvalues λ_i, this allows us to integrate over U and, up of to a g-independent normalizing factor we obtain

$$\exp - N^2 E^{(0)}(g) = \lim_{N\to\infty} \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp - \left[\tfrac{1}{2} \sum \lambda_i^2 + \frac{g}{N} \sum \lambda_i^4 \right]. \quad (6)$$

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The constant term in this series ($g = 0$) is the k-th moment of the semicircle distribution, which vanishes for k odd, while for $k = 2m$ it is:

- the m-th Catalan number $c_m = \frac{(2m)!}{m!(m+1)!}$,
- i.e. the number of rooted plane trees with m edges,
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To count general (non unicellular) maps we need to study the non Gaussian ($g \neq 0$) case. Nowadays, this is routinely done using the so-called loop or Schwinger-Dyson equations for the resolvent

$$F(z) := \int_{\mathrm{supp}\mu} \frac{d\mu(x)}{x - z} = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int x^k d\mu(x)$$

which are basically equivalent to Tutte's recursive equations (z is the catalytic variable).

Random surfaces and 2D quantum gravity

"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces."
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His paper introduces Liouville quantum gravity:

$$Z = \int D\varphi(\xi) \exp\left(-\frac{26-D}{48\pi} \int_{\xi} \left[\tfrac{1}{2}(\partial_\mu \varphi)^2 + \mu^2 e^{\varphi} \right] \right). \quad (25)$$

Here $D\varphi(\xi)$ denotes the (ill-defined!) functional integration over a field $\varphi : \Sigma \to \mathbb{R}$, with Σ a Riemann surface.

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Here $D\varphi(\xi)$ denotes the (ill-defined!) functional integration over a field $\varphi : \Sigma \to \mathbb{R}$, with Σ a Riemann surface. For $\mu = 0$ we can make sense of it as a *Gaussian free field*.

Random surfaces and 2D quantum gravity

"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces."
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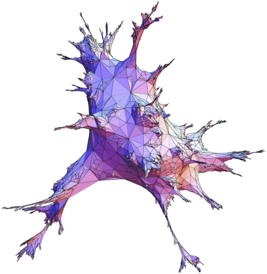
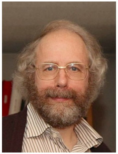
Random surfaces and 2D quantum gravity

In 1984-1985, several papers suggested another approach: discretize quantum gravity using random triangulations.
Liouville quantum gravity should emerge for a very large number of vertices and a very small mesh size.
The computations require being able to enumerate maps with various constraints, and matrix models were used as an efficient tool.



Picture by N. Curien

V. Kazakov



F. David

J. Ambjørn, B. Durhuus, J. Fröhlich

Random surfaces and 2D quantum gravity

A highlight of this approach is the "solution" of the 2D Ising model on random triangulations by Kazakov in 1986, using a two-matrix model.
Combinatorially, it corresponds to enumerating planar triangulations endowed with a (non necessarily proper) 2-coloring, fixing the number of vertices and the number of monochromatic edges. (For proper colorings, we recover Tutte's "chromatic sum" at $\lambda = 2$.)

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Quite surprisingly, the resulting critical exponents were in complete agreement with predictions from Liouville quantum gravity.
The Potts model (q-colorings for general q) was similarly "solved" via matrix models a few years later (Daul, Zinn-Justin, Bonnet, Eynard...), still consistently with LQG.

Outline

The bijective approach

Let us return to pure enumerative combinatorics and ask whether we can find a bijective proof of Tutte's enumeration formula for rooted planar maps with n edges

$$M_n = 2 \cdot 3^n \cdot \frac{(2n)!}{n!(n+2)!}.$$

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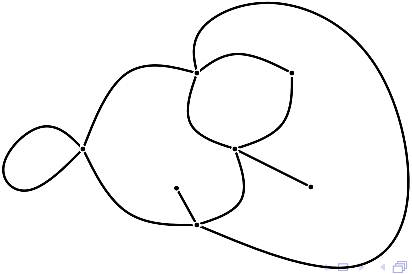

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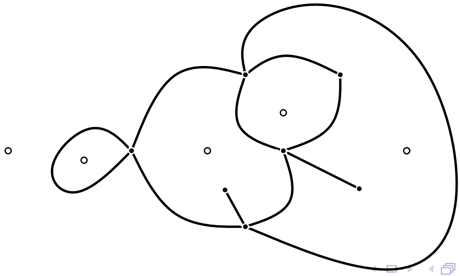

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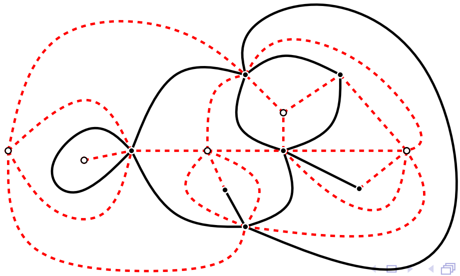

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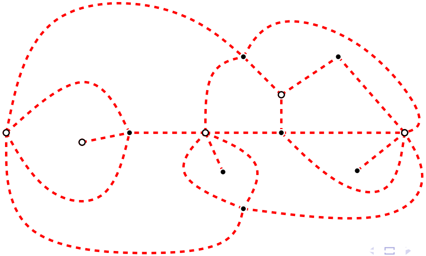

Cori-Vauquelin-Schaeffer bijection

Let us also observe that a planar quadrangulation with n edges has $n+2$ vertices, therefore

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Idea: let us consider a labeling ℓ of the vertices of the tree by integers, such that

- $\ell(\text{root}) = 0$,
- $|\ell(u) - \ell(v)| \neq 1$ if u, v are neighbours.

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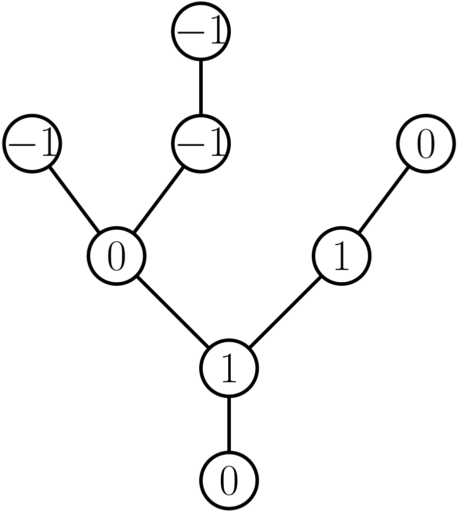

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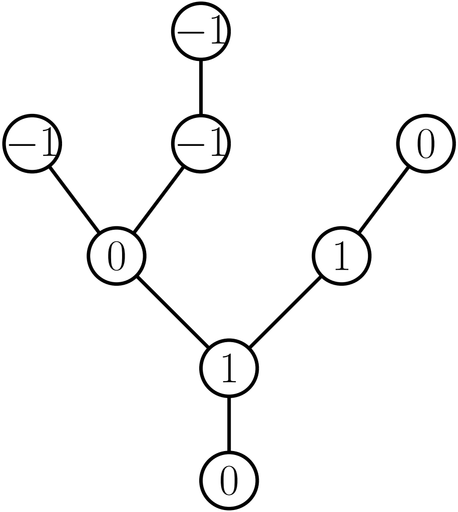

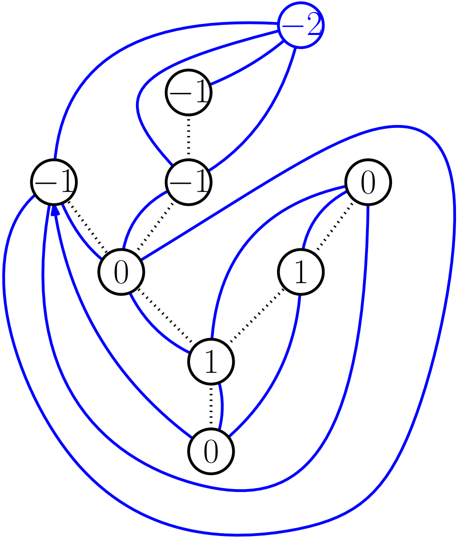

Cori-Vauquelin-Schaeffer bijection

The CVS bijection has the following interesting property: let v^* be the added vertex, then for any vertex v of the tree we have

$$d_Q(v, v^*) = \ell(v) - \min \ell + 1$$

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It is not difficult to see that, for a uniform random quadrangulation with n faces, the distances should be of order $n^{1/4}$ (the branches of the trees have length $\propto n^{1/2}$, on each branch the labels form a lazy random walk on $\mathbb{Z}$).

Consequences of the CVS bijection

This was made more precise in several works. Chassaing and Schaeffer proved in 2002 that

$$\frac{1}{n^{1/4}} \max d_{Q_n}(\cdot, v^*) \xrightarrow[n\to\infty]{(d)} w_{\mathrm{ISE}}$$

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Of course, v^* plays no special role, it is just a uniformly chosen vertex in the quadrangulation.

Consequences of the CVS bijection

With Di Francesco and Guitter, I obtained in 2003 an exact expression for the generating function $R_\ell(t)$ of planar quadrangulations with two marked vertices at distance $\leq \ell$. It reads

$$R_\ell = R\frac{(1-x^\ell)(1-x^{\ell+3})}{(1-x^{\ell+1})(1-x^{\ell+2})}$$

where the series $R = R(t)$ and $x = x(t)$ satisfy

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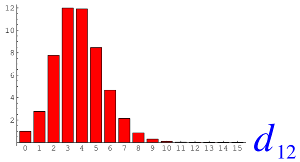
This encodes the exact law for the distance between two uniform points in a random quadrangulation of size n.



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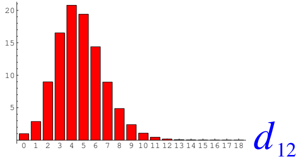

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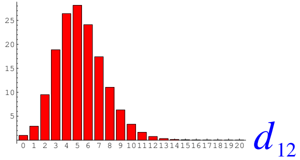

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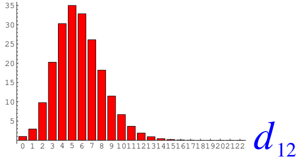

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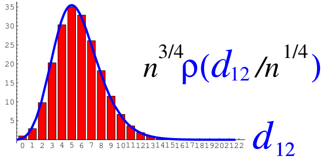
This encodes the exact law for the distance between two uniform points in a random quadrangulation of size n.
The limit $n \to \infty$ can be analyzed by standard analytical methods, and we confirm a prediction of Ambjørn and Watabiki (1996).



$d_{12} \sim n^{1/4}$

Beyond the CVS bijection

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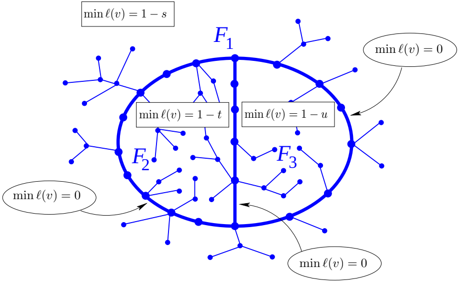

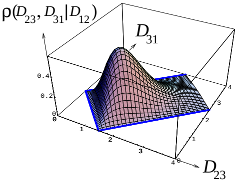

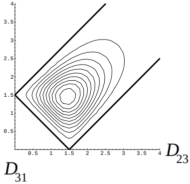

Towards the Brownian map

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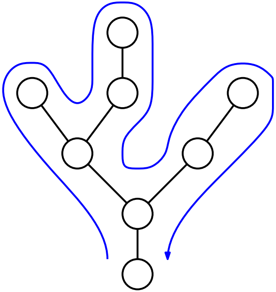
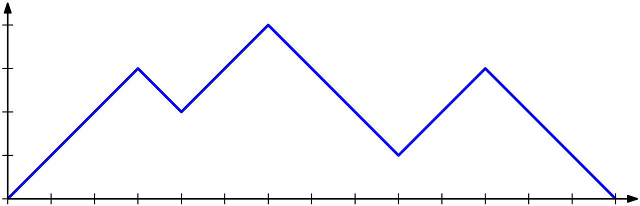
Recall that a rooted plane tree with n edges can be described by its contour process, which is a Dyck path of length $2n$.





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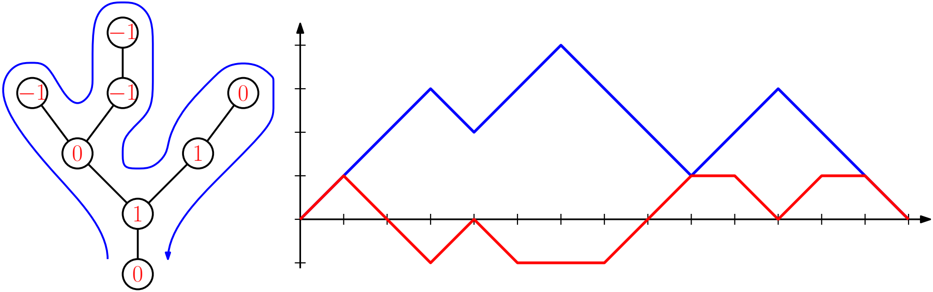

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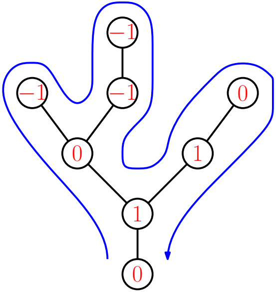

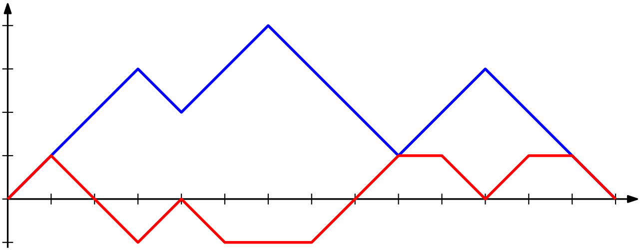

Theorem (Chassaing-Schaeffer, 2002)

By appropriate rescalings, the contour and the label processes of a uniform random labeled tree with n edges converge jointly to a continuous process (e, Z) called the Brownian snake.

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(Several properties proved in the meantime: BM is homeomorphic to the sphere, Hausdorff dimension is 4, geodesics, etc.)

Outline

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The Brownian map is believed to be the "universal" scaling limit for any "reasonable" family of planar maps, i.e. basically any family whose enumeration by size goes as

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Is there nothing else to see? Since Kazakov's solution, we know that for maps endowed with a critical Ising model,

$$Z_n^{\mathrm{Ising}}(\nu_c) \sim C\kappa^n n^{-7/3}.$$

(For the q-state Potts model, the exponent varies with q.)

Slicings

We again find inspiration in the work of Tutte:

Theorem (Census of slicings, 1962)

The number of bipartite rooted planar maps with e edges and f faces, among which f_i have degree $2i$ for all i, is

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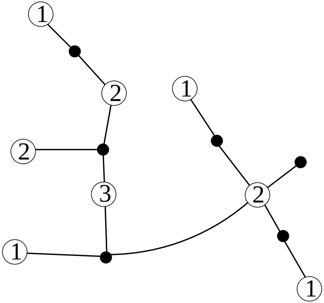
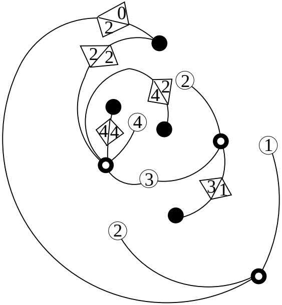
Schaeffer gave a bijective proof of this formula in 1997, using a different sort of trees. In 2004, with Di Francesco and Guitter, we generalized the CVS bijection to this setting (and more). This is now known as the BDG bijection.

Mobiles





Beyond the Brownian map

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where $(q_i)_{i \geq 1}$ is a given sequence of face weights. By the BDG bijection, we obtain a certain distribution over mobiles (labeled trees), where q_i controls the proportion of vertices of degree i. The structure of such a tree is best described by its Lukasiewicz path (instead of contour), as q_i controls the probability to jump up by $i - 1$.

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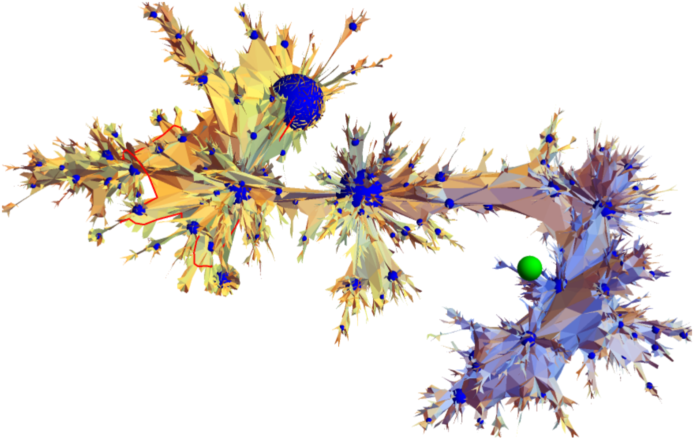
Theorem (Le Gall-Miermont, 2011)

For any $d \in (2, 4)$, there exists nongeneric sequences of face weights $(q_i)_{i \geq 1}$ for which the corresponding Boltzmann random maps of size n have distances of order $n^{1/d}$.
After rescaling, these maps converge (along subsequences) to a random compact metric space of Hausdorff dimension d.

By a continuous analogue of the BDG bijection, Le Gall and Miermont also constructed the conjecturally unique limit, the stable map.

Beyond the Brownian map



Picture by T. Budd of the dual of a stable map

Beyond the Brownian map

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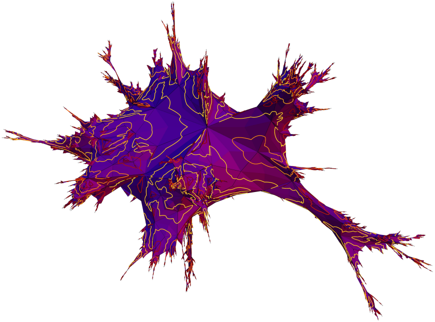
Yes, Le Gall and Miermont conjectured that stable maps are connected with the so-called $O(n)$ loop model on random maps (related to FK percolation, $n = 1$ corresponds to percolation and the Ising model).



Picture by J. Bettinelli

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Latest developments: results about the nestings of loops, that connect with Liouville quantum gravity (Borot-B-Duplantier 2016, Chen-Curien-Maillard 2017).

Conclusion

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This is just my own perspective and I certainly forgot many other deep aspects, e.g.:

- enumeration of maps of higher genus, that led to the so-called "topological recursion" in algebraic geometry,
- much work on the connection between random maps and Liouville quantum gravity – see the recent preprint of Gwynne-Miller-Sheffield on Tutte embedding of the mated-CRT map.

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To conclude let me quote a colleague of Tutte: "*If I have seen further it is by standing on the shoulders of giants.*"