Distances in random maps and discrete integrability

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Based on joint works with Emmanuel Guitter and Philippe Di Francesco

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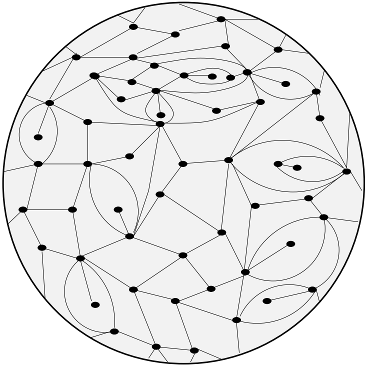
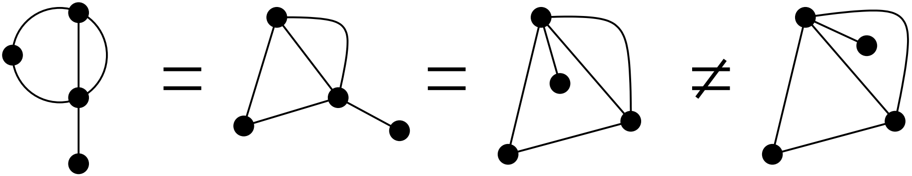
Introduction



A planar map is a connected (multi)graph embedded in the sphere, considered up to continuous deformation. It is made of vertices, edges and faces.
When all faces have degree 4, the map is a quadrangulation. We similarly define triangulations, etc.



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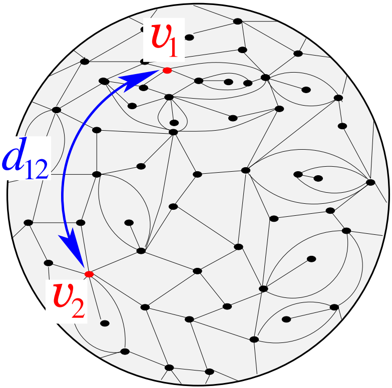
- two-dimensional quantum gravity (*"develop an art of handling sums over random surfaces"*)
- algebraic geometry and representation theory
- random geometry (random metric spaces, measures, conformal properties...)

The two-point function of quadrangulations



Basic question

Consider a uniformly distributed random planar quadrangulation with n faces (and $n+2$ vertices). Pick two uniformly distributed random vertices v_1 and v_2. What is the law of the graph distance d_{12} between them ?

Equivalent counting problem

Count the number of planar quadrangulations with n faces and two marked vertices at a prescribed distance d_{12}.

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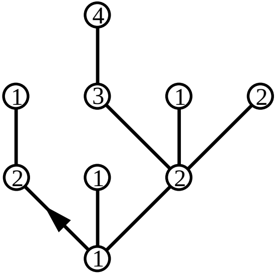

A well-labeled tree is a plane tree with integers labels on vertices, such that labels on adjacent vertices differ by at most 1.

Theorem (Cori-Vauquelin '81, Schaeffer '98, see also Chassaing-Schaeffer '02, loosely stated)

There is a one-to-one correspondence between planar quadrangulations with n faces and well-labeled trees with n edges.

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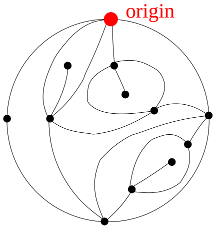

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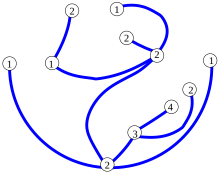

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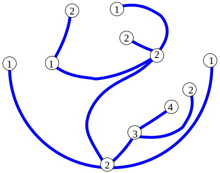

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There is a one-to-one correspondence between planar quadrangulations with n faces and well-labeled trees with n edges.

Schaeffer pointed out that labels encode graph distances to an origin in the quadrangulation. Precisely we have the following bijections:

pointed quad. $\leftrightarrow$ unrooted tree with positive labels and a label 1

rooted quad. $\leftrightarrow$ rooted tree with positive labels and root label 1

pointed rooted quad. $\leftrightarrow$ rooted tree with unconstrained labels considered up to a global shift

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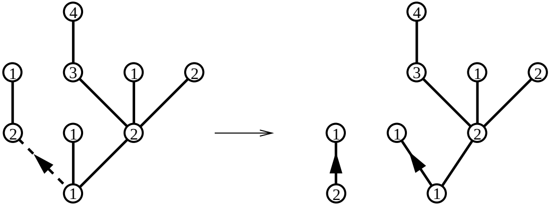
A well-labeled tree with positive labels and root label $\ell \geq 1$ corresponds (essentially) to a quadrangulation with two marked points at distance at most ℓ. It is quite simple to write down an equation for the generating function of such objects:

$$R_\ell := \sum_{n \geq 0} g^n \#\{\text{positive w.-l. trees with } n \text{ edges and root label } \ell\}$$

$$\text{satisfies } R_\ell = \begin{cases} 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1}), & \ell \geq 1 \\ 0 & \ell = 0. \end{cases}$$



(see also B.-Di Francesco-Guitter '03 for an alternate derivation)

The two-point function of quadrangulations

Interestingly, this equation admits the explicit solution

$$R_\ell = R\frac{(1-x^\ell)(1-x^{\ell+3})}{(1-x^{\ell+1})(1-x^{\ell+2})}$$

where the power series R, x are determined via

$$R = 1 + 3gR^2, \qquad x + \frac{1}{x} + 1 = \frac{1}{gR^2}.$$

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The equation is discrete integrable in the sense that it admits a conserved quantity: $\psi(R_n, R_{n+1})$ is independant of n with

$$\psi(x,y) := (1 - gx - gy)(1 + gxy).$$

Here we pick a convergent solution, $\psi(R_n, R_{n+1}) = \psi(R, R)$, $R_0 = 0$.

(see also B.-Di Francesco-Guitter '03 for the general solution)

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- Local limit: estimate $[g^n]R_\ell$ for $n \to \infty$, ℓ fixed:

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By normalizing properly we deduce the expected volume of the ball of radius ℓ centered at the origin in the Uniform Infinite Planar Quadrangulation (Chassaing-Durhuus '03, Krikun '05...)

$$\mathbb{E} V_\ell = \frac{C_\ell + C_{\ell+1}}{C_1} = \frac{3(\ell+2)^2(5\ell^4 + 40\ell^3 + 104\ell^2 + 96\ell + 35)}{140(\ell+1)(\ell+3)} \sim \frac{3\ell^4}{28}$$

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- Scaling limit: estimate $[g^n]R_\ell$ for $n \to \infty$, $L := \ell \cdot n^{-1/4}$ fixed:

$$\frac{\mathbb{E}_n V_\ell}{n+2} \to \Phi(L) := \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, \xi^2 e^{-\xi^2} \left(1 + \frac{3}{\sinh^2(L\sqrt{-3i\xi/2})} \right)$$

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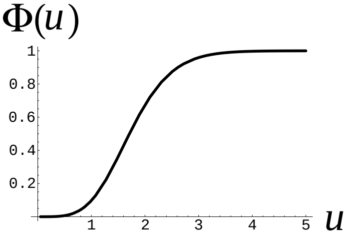

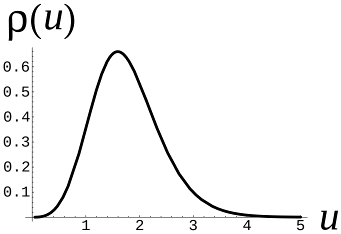

$$\Phi(L) \sim \frac{3D^4}{28}, \; L \to 0 \qquad \log(1-\Phi(L)) \sim \mathrm{cst}.L^{4/3}, L \to \infty.$$

Generalized two-point function

We may consider the same question in more general classes of maps. A favorable setting is given by maps with controlled face degrees

$$\mathbb{P}(\{\mathfrak{m}\}) = \frac{1}{Z} \prod_{k \geq 1} g_k^{\#\{\text{faces of degree } k \text{ in } \mathfrak{m}\}}$$

(we recover quadrangulations, triangulations, etc, by specialization).

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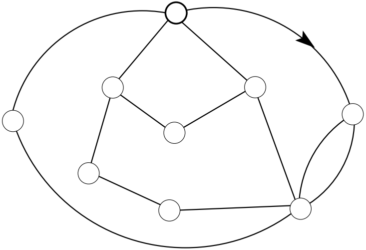

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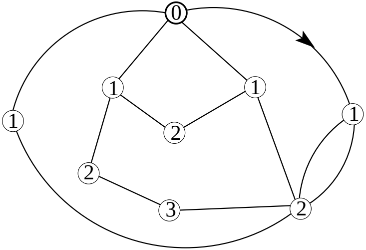

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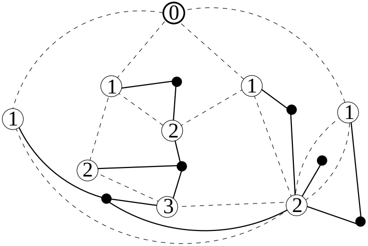

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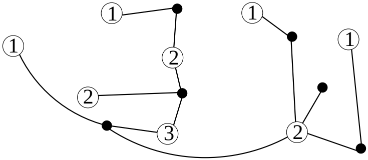

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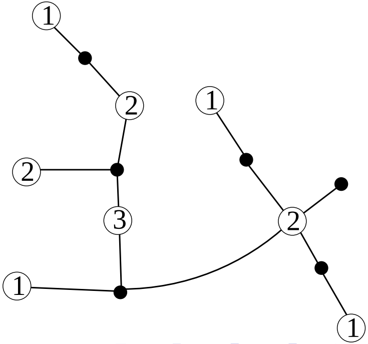

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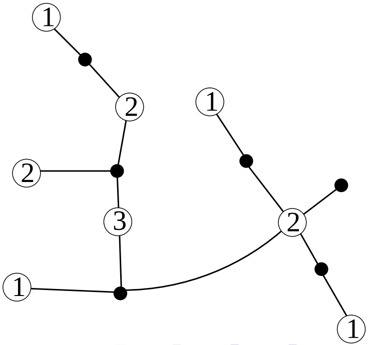

Easier case: bipartite maps ($g_k = 0$ for k odd). Map-tree dictionary:

- vertex at distance $\ell \leftrightarrow$ vertex labeled ℓ
- face of degree $2k \leftrightarrow$ unlabeled vertex of degree k

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A mobile with positive labels and root label $\ell \geq 1$ corresponds (essentially) to a map with two marked points at distance at most ℓ. The generating function R_ℓ of such objects obeys recursive equations.

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Example: squares and hexagons ($g_k = 0$ unless $k = 4$ or 6)

$$\begin{aligned} R_\ell = 1 + g_4 R_\ell (R_{\ell+1} + R_\ell + R_{\ell-1}) + \\ g_6 R_\ell \big(R_{\ell+2} R_{\ell+1} + R_{\ell+1}^2 + 2 R_{\ell+1} R_\ell + R_{\ell+1} R_{\ell-1} + \\ 2 R_\ell R_{\ell-1} + R_{\ell-1}^2 + 2 R_{\ell-1} R_{\ell-2} \big) \end{aligned}$$

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for $\ell \geq 1$, $R_\ell = 0$ otherwise. There is still an explicit solution

$$R_\ell = R \frac{u_\ell u_{\ell+3}}{u_{\ell+1} u_{\ell+2}}, \qquad u_\ell = 1 - \lambda_1 x_1^\ell - \lambda_2 x_2^\ell + c_{12} \lambda_1 \lambda_2 (x_1 x_2)^\ell$$

where $R, x_1, x_2, \ldots$ are determined by some algebraic equations. Also there are now several independent conserved quantities.

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where $R, x_1, x_2, \ldots$ are determined by some algebraic equations. Also there are now several independent conserved quantities. The same phenomenon occurs if we allow for an arbitrary finite number of face degrees.

(B.-Di Francesco-Guitter '03, DG '05, BG '10)

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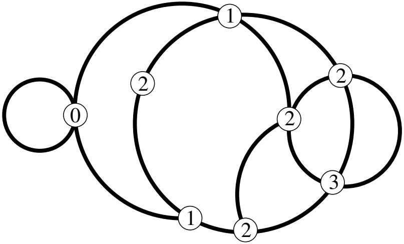
More involved case: arbitrary face degrees.



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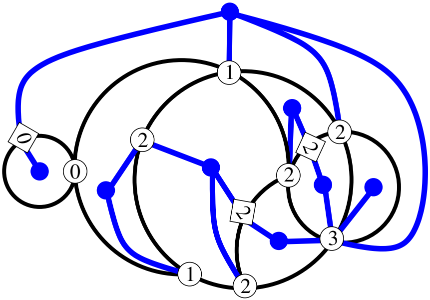

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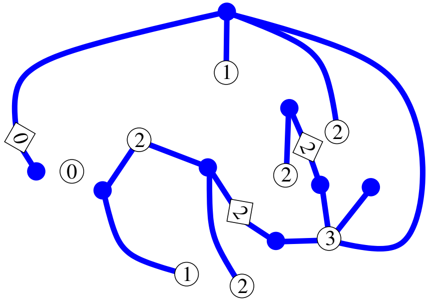

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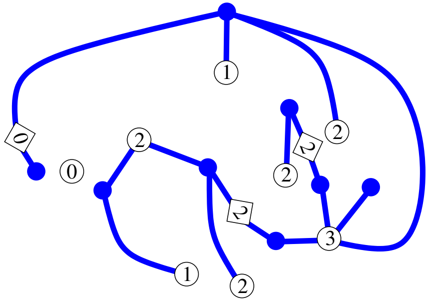
More involved case: arbitrary face degrees. Mobiles now have "flagged" edges too.



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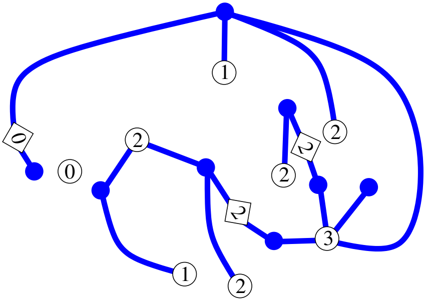
Introduce g.f. R_ℓ and S_ℓ of mobiles rooted respectively on a label $\ell \geq 1$ or on a flag $\ell \geq 0$, get recursive equations, reinterpret in terms of maps.



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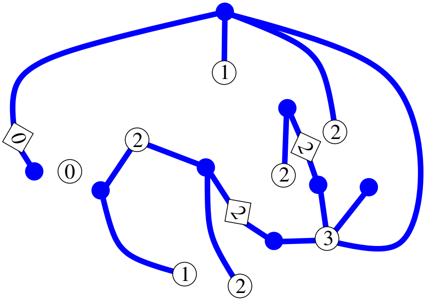

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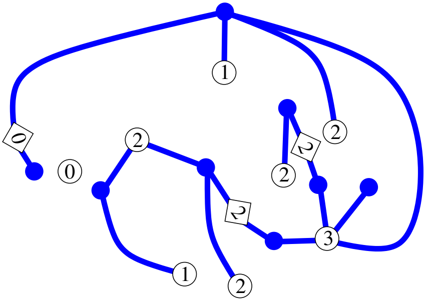

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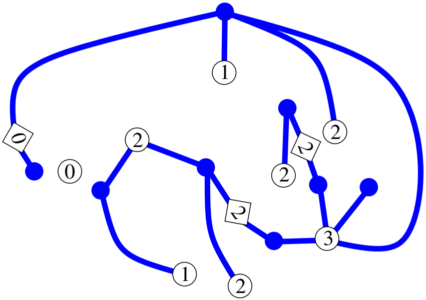
More involved case: arbitrary face degrees. Mobiles now have “flagged” edges too.

Introduce g.f. R_ℓ and S_ℓ of mobiles rooted respectively on a label $\ell \geq 1$ or on a flag $\ell \geq 0$, get recursive equations, reinterpret in terms of maps.



Example: triangulations ($g_k = 0$ unless $k = 3$)

$$R_\ell = \begin{cases} 1 + g_3 R_\ell (S_\ell + S_{\ell-1}), & \ell \geq 1 \\ 0, & \ell = 0 \end{cases} \qquad S_\ell = g_3 (S_\ell^2 + R_\ell + R_{\ell+1}), \quad \ell \geq 0$$

Still an explicit solution, conserved quantities... (here $y + y^{-1} + 2 = 1/(g_3^2 R^3)$)

$$R_\ell = R \frac{(1-y^\ell)(1-y^{\ell+2})}{(1-y^{\ell+1})^2} \qquad S_\ell = S - g_3\, R^2 y^\ell \frac{(1-y)(1-y^2)}{(1-y^{\ell+1})(1-y^{\ell+2})}$$

Generalized two-point function

Applications:

- local limit: computations of expected ball volumes in infinite maps,

Generalized two-point function

Example: the expected volume of the ball of radius ℓ centered at the origin in the Uniform Infinite Planar Triangulation (Angel-Schramm '02) reads

$$\mathbb{E}V_\ell = \frac{2(5\ell^6 + 45\ell^5 + 163\ell^4 + 303\ell^3 + 305\ell^2 + 159\ell + 35)}{35(\ell+1)(\ell+2)} \sim \frac{2}{7}\ell^4$$

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Remark: the radius of quadrangulations (Chassaing-Schaeffer '02) can be studied by analyzing the same equation with different boundary conditions (BDG '03, Drmota '09)

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Bottom line

A combinatorial miracle happens.

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Bottom line

A combinatorial miracle happens. More? Why?

More in our combinatorial toolbox

From now on we restrict to the case of quadrangulations.

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Schaeffer's bijection only encodes distances to one special vertex.
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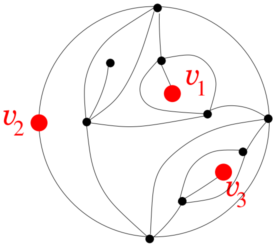

- Input: quadrangulation with p distinct marked vertices $v_1, \ldots, v_p$

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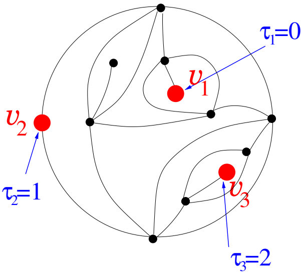

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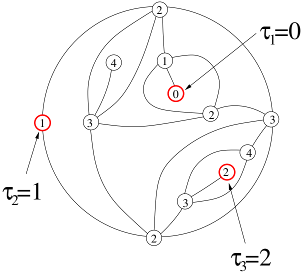

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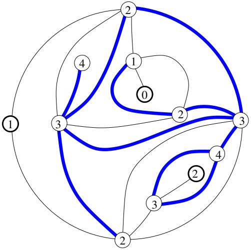

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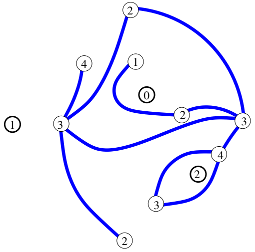

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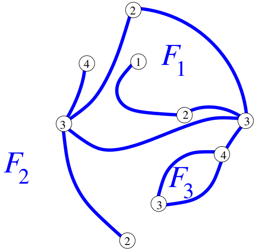

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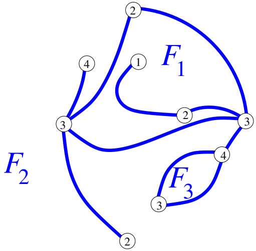

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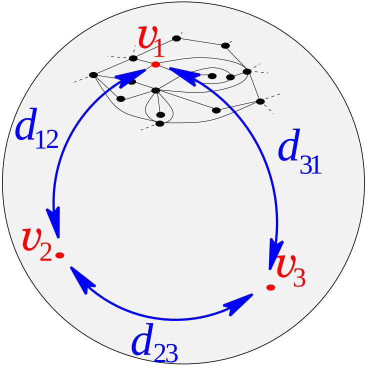
Property: $\ell(v) = d(v, v_i) + \tau_i$ if v is incident to F_i

The three-point function of planar quadrangulations

We may apply this bijection to compute the three-point function of quadrangulations.

(B.-Guitter '08)



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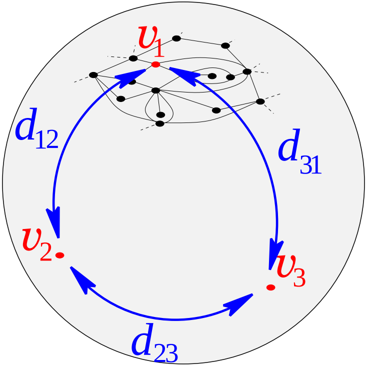

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Trick: apply the Miermont bijection with delays $\tau_1 = -s, \tau_2 = -t, \tau_3 = -u$ where

$$d_{12} = s + t$$
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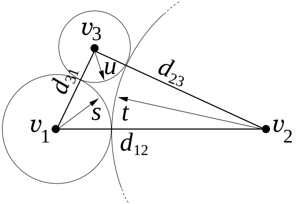

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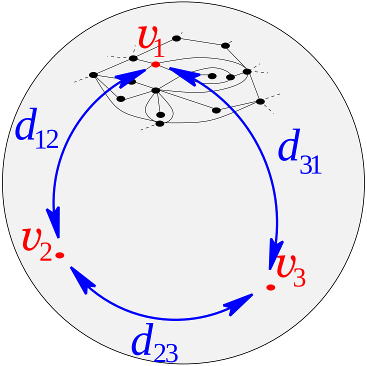

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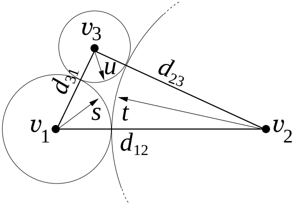
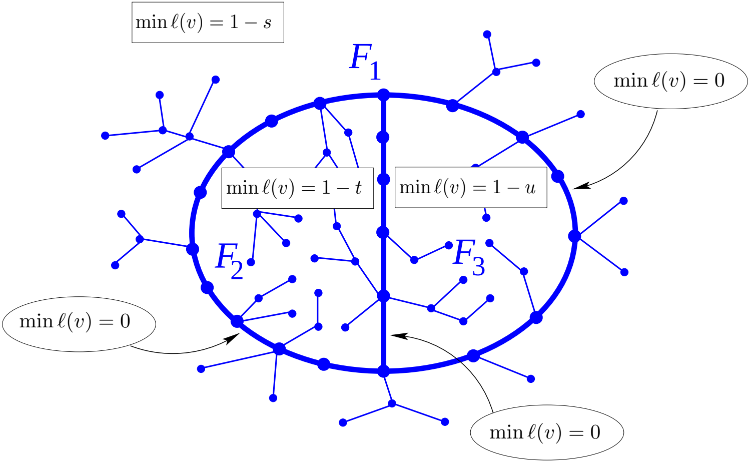


Get a bijection between planar quadrangulations with three marked points at prescribed distances and some well-labeled maps with three faces...

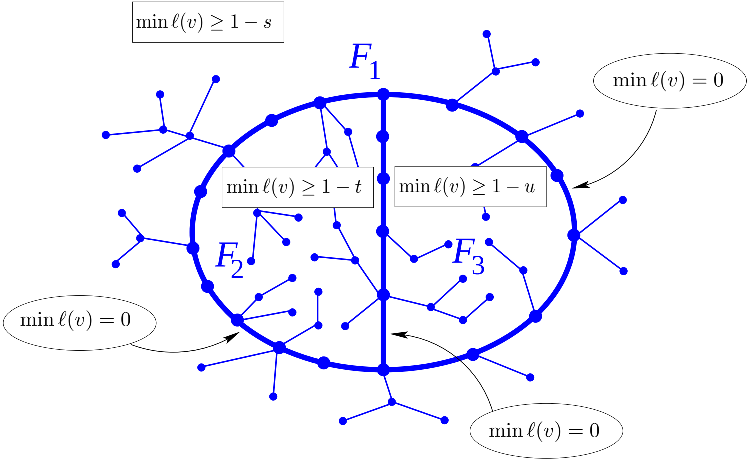
The three-point function of planar quadrangulations



Constraints on the corresponding well-labeled maps.

Generating function: $G_{s,t,u}(g)$ with g weight per edge

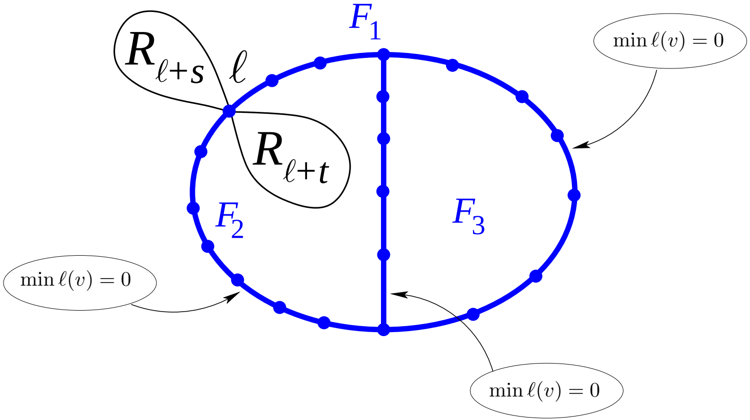
The three-point function of planar quadrangulations



Replace some equality constraints by bounds (easier to count).

Generating function: $F_{s,t,u} = \sum_{s' \leq s} \sum_{t' \leq t} \sum_{u' \leq u} G_{s',t',u'}$

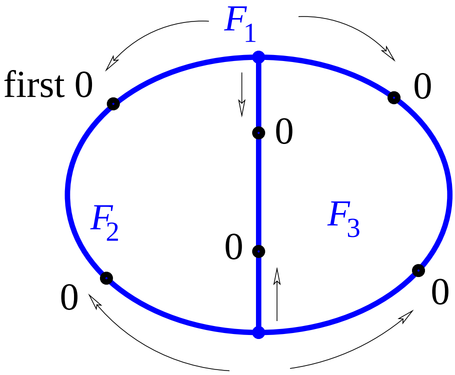
The three-point function of planar quadrangulations



The map is made of well-labeled trees attached to a skeleton.

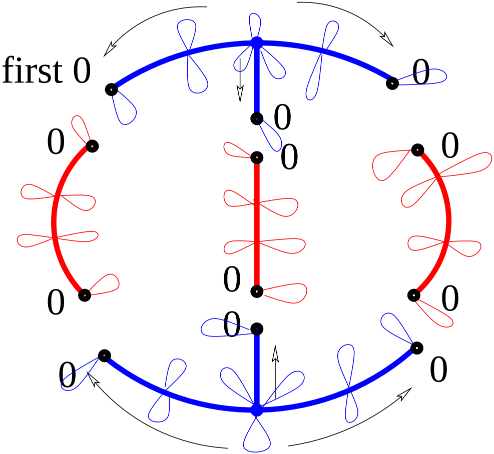
(Recall the previous expression for the well-labeled trees g.f. R_ℓ)

The three-point function of planar quadrangulations



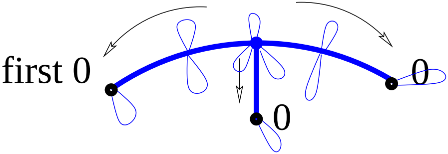
Decompose the skeleton at the first and last label 0 along each branch.

The three-point function of planar quadrangulations

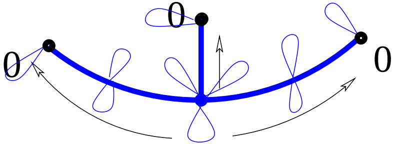


Obtain acyclic components.

The three-point function of planar quadrangulations



$X_{s,t}$ $X_{t,u}$ $X_{u,s}$



"Chains" depends on two indices only.

The three-point function of planar quadrangulations

$$Y_{s,t,u}$$

$$X_{s,t} \qquad X_{t,u} \qquad X_{u,s}$$

$$Y_{s,t,u}$$

"Stars" depend on all three indices.

The three-point function of planar quadrangulations

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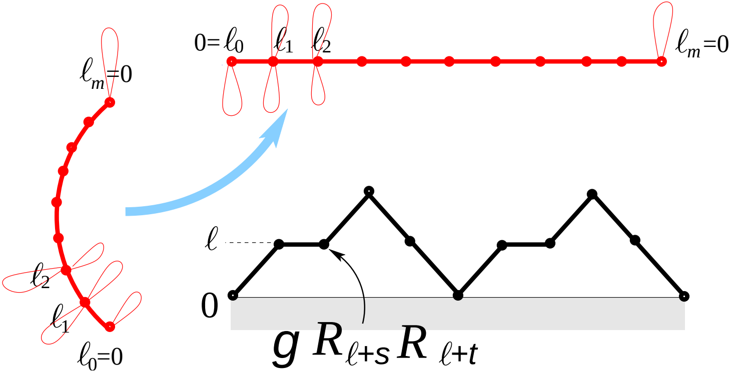
"Stars" depend on all three indices.

$$F_{s,t,u} = X_{s,t}X_{t,u}X_{u,s}(Y_{s,t,u})^2$$

The three-point function of planar quadrangulations

Consider the generating function $X_{s,t}$ for well-labeled chains.



$$X_{s,t} = \sum_{m \geq 0} \sum_{\substack{\text{Motzkin paths of length } m \\ \mathcal{M}=(0=\ell_0,\ell_1,\dots,\ell_m=0)}} \prod_{k=0}^{m-1} g\, R_{\ell_k+s}\, R_{\ell_k+t}$$

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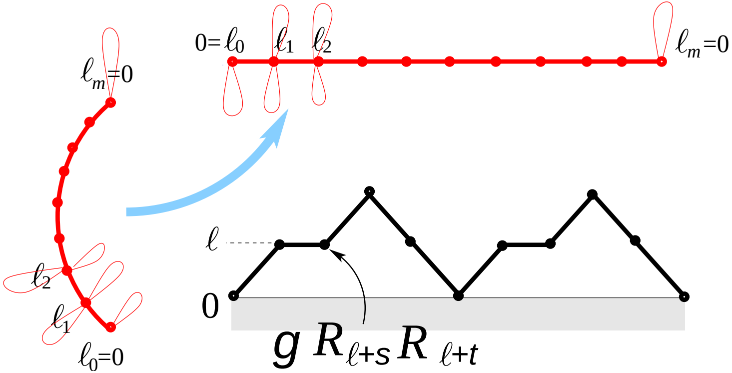

$$X_{s,t} = 1 + gR_sR_tX_{s,t}(1 + R_{s+1}R_{t+1}X_{s+1,t+1})$$

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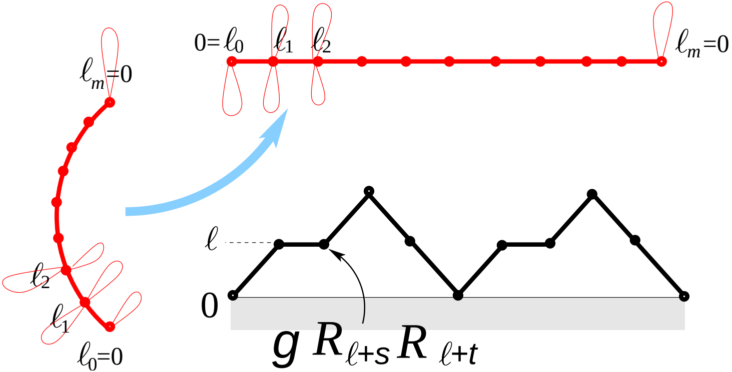

$$X_{s,t} = \frac{(1-x^3)}{(1-x)} \frac{(1-x^{s+1})}{(1-x^{s+3})} \frac{(1-x^{t+1})}{(1-x^{t+3})} \frac{(1-x^{s+t+3})}{(1-x^{s+t+1})}$$

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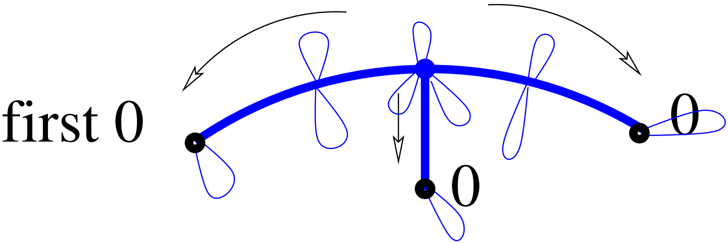

$$Y_{s,t,u} = 1 + g^3 R_s R_t R_u R_{s+1} R_{t+1} R_{u+1} X_{s+1,t+1} X_{t+1,u+1} X_{u+1,s+1} Y_{s+1,t+1,u+1}$$

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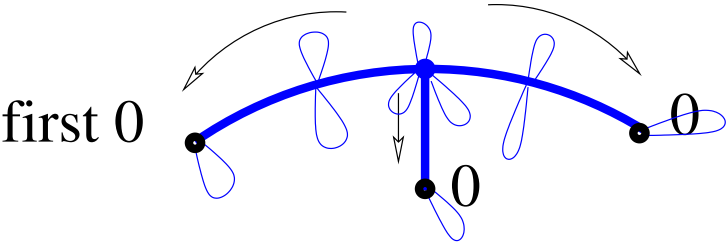

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$$= \frac{(1-x^{s+3})(1-x^{t+3})(1-x^{u+3})(1-x^{s+t+u+3})}{(1-x^3)(1-x^{s+t+3})(1-x^{t+u+3})(1-x^{u+s+3})}$$

The three-point function of planar quadrangulations

Gathering all expressions we get (B.-Guitter '08)

$$F_{s,t,u} = \frac{[3]\left([s+1][t+1][u+1][s+t+u+3]\right)^2}{[1]^3[s+t+1][s+t+3][t+u+1][t+u+3][u+s+1][u+s+3]}$$

where

$$[\ell] := \frac{(1-x^\ell)}{(1-x)}.$$

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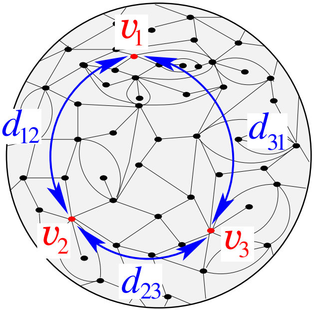

$G_{s,t,u} = \Delta_s \Delta_t \Delta_u F_{s,t,u}$ is the generating function for quadrangulations with three marked vertices at distances $d_{12} = s+t, d_{23} = t+u, d_{31} = u+s$.

It encodes the joint law of the distances $d_{12}^{(n)}, d_{23}^{(n)}, d_{31}^{(n)}$ between three uniform random vertices in a uniform random planar quadrangulation of size n.

The three-point function of planar quadrangulations

Scaling limit: for $n \to \infty$ we have

$$n^{-1/4} \cdot (d_{12}^{(n)}, d_{23}^{(n)}, d_{31}^{(n)}) \xrightarrow{d} (D_{12}, D_{23}, D_{31})$$

with an explicit analytical expression for the density of the limit (three-point function of the Brownian map).

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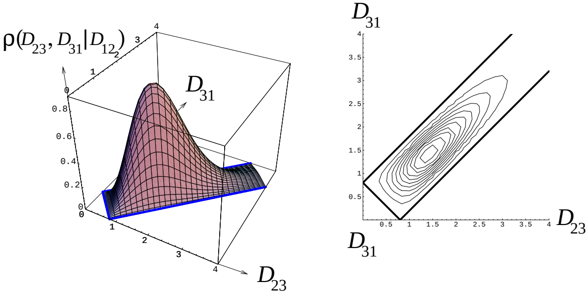

Density of two rescaled distances conditionnally on the third.

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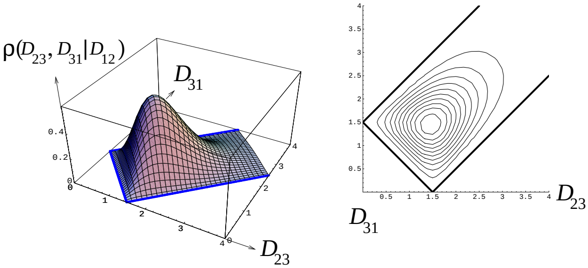
$D_{12} = 1.5$



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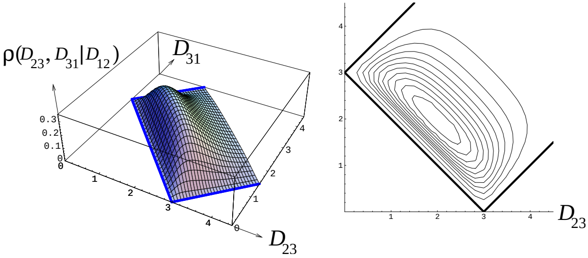
$D_{12} = 3.0$



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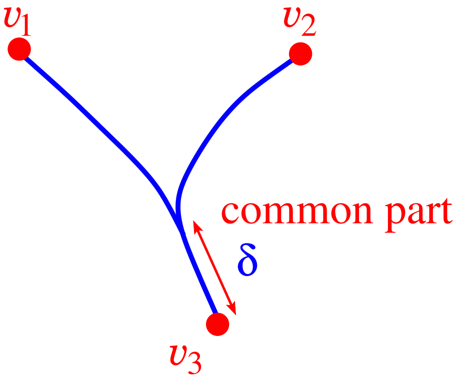
Other related results (B.-Guitter '08)

Le Gall ('08) has shown the phenomenon of confluence of geodesics.



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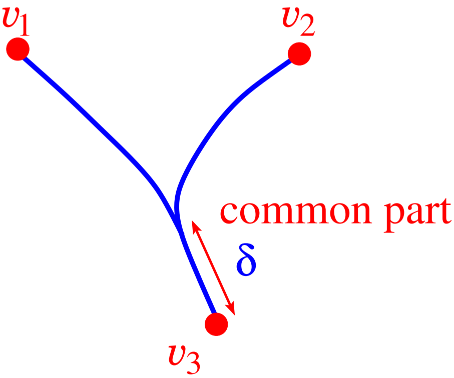

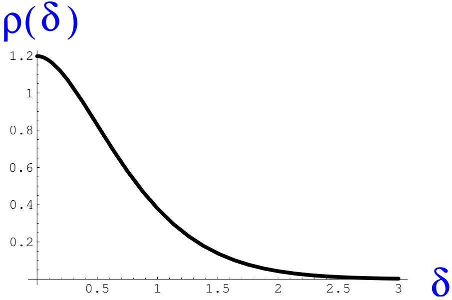

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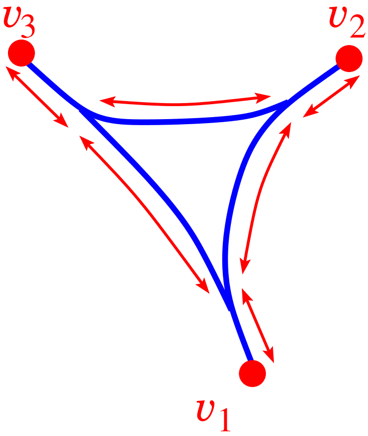

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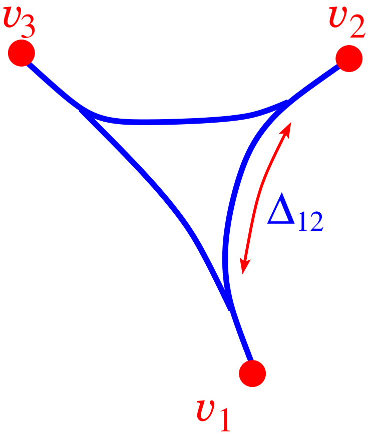

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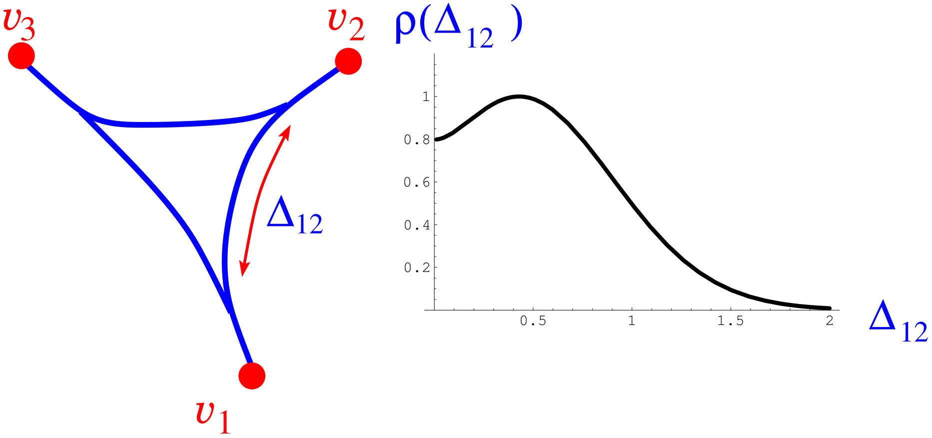

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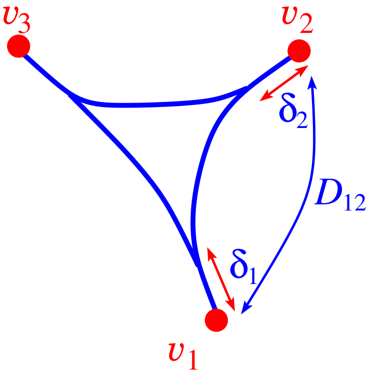

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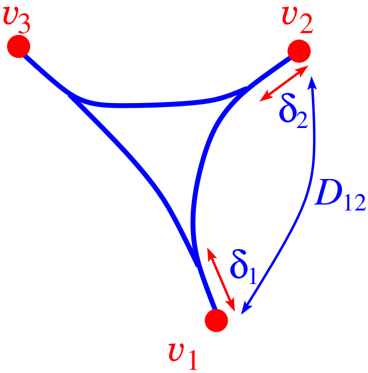

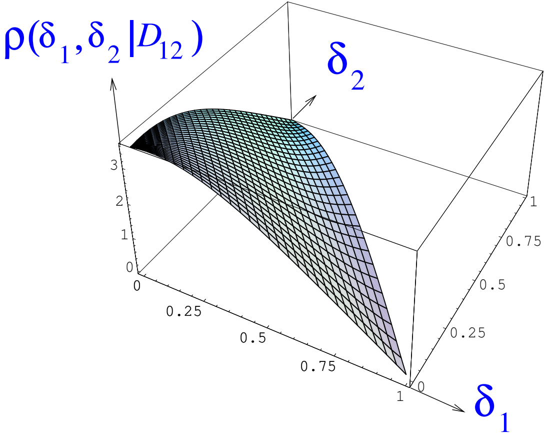

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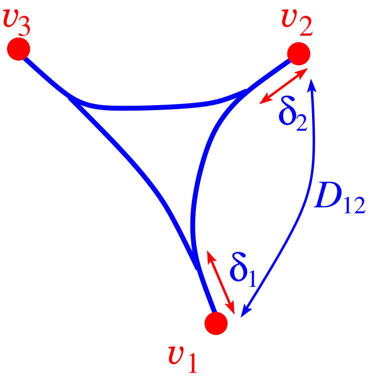

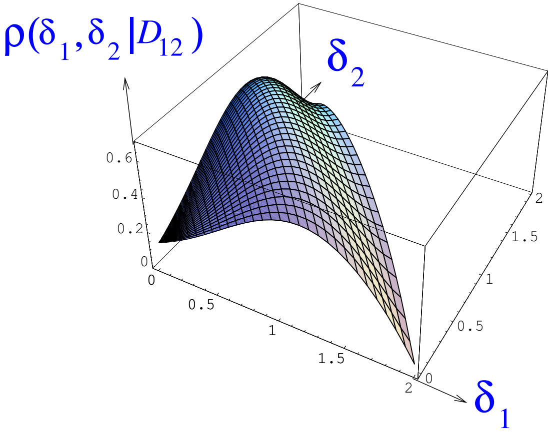

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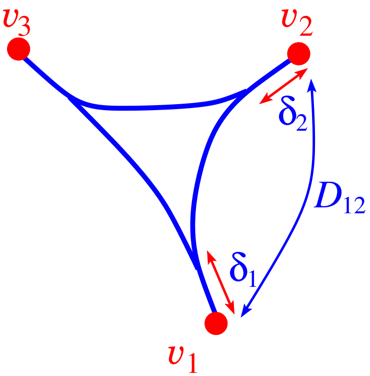

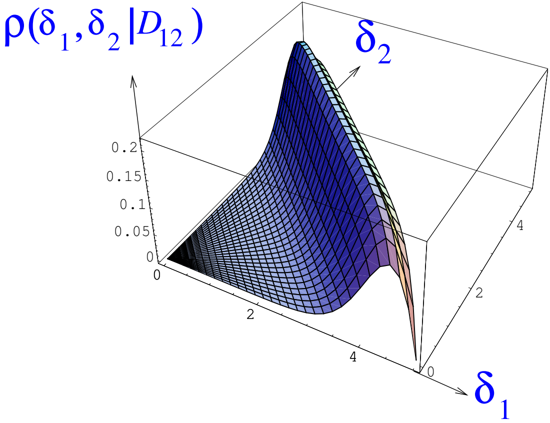

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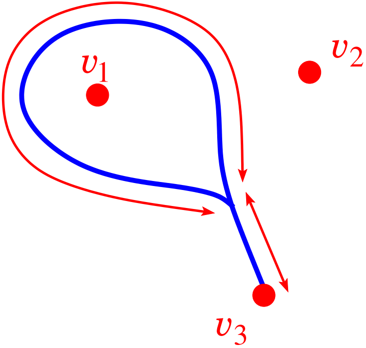
We may also study separating loops.



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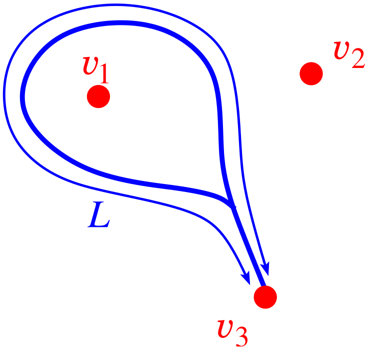

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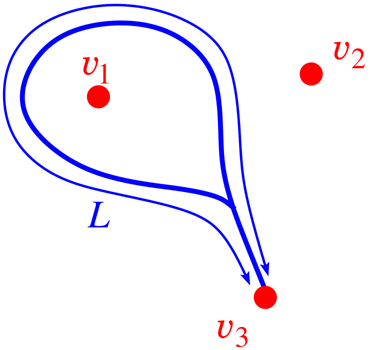

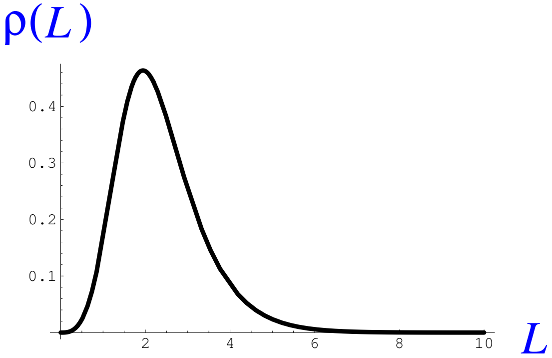

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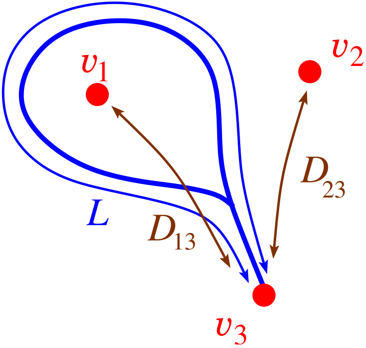

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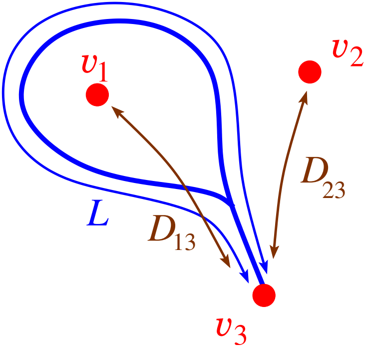

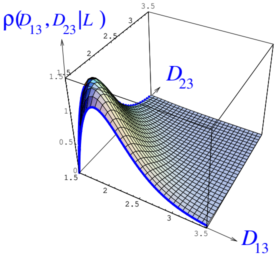

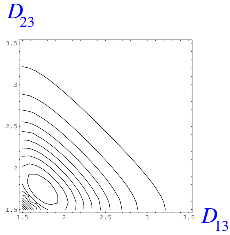

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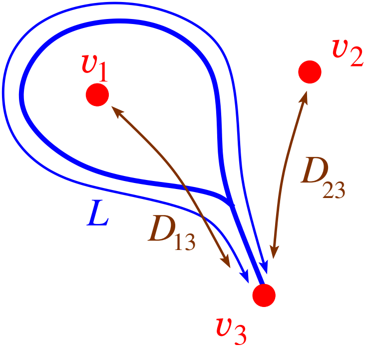

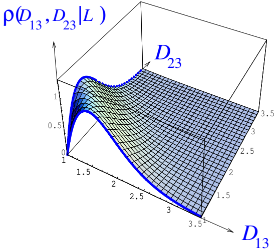

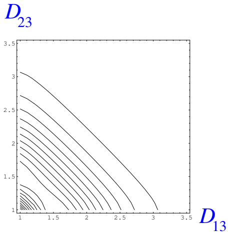

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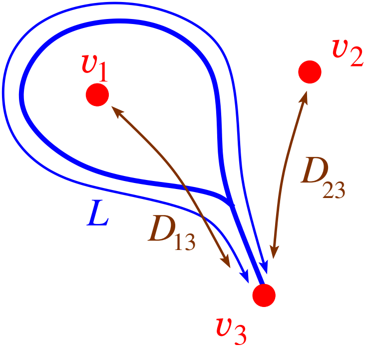

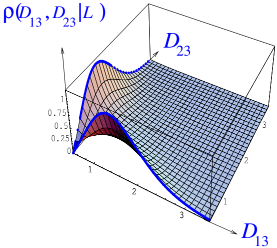

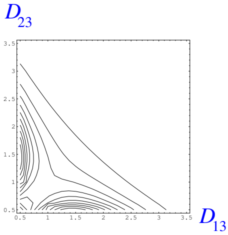

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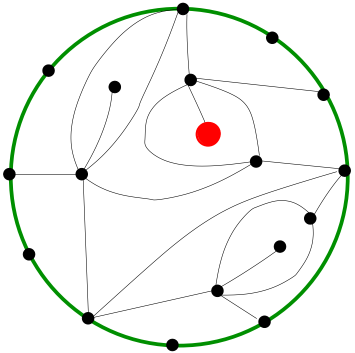
Quadrangulations with a boundary (B.-Guitter '09)

Let us now consider a pointed quadrangulation with a boundary where the origin-boundary distance is at most d.



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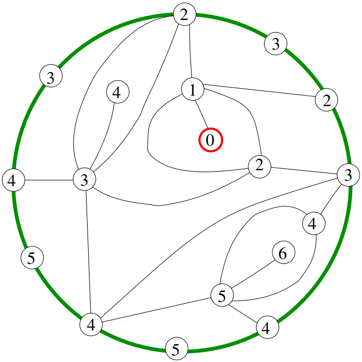

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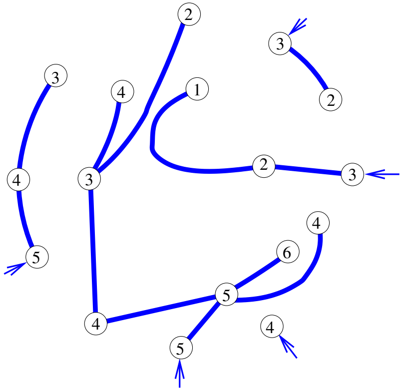

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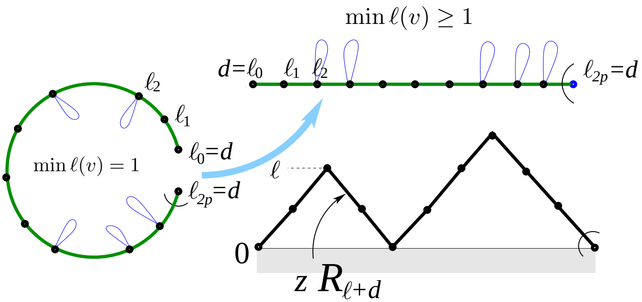

Bivariate generating function of well-labeled forests (z per outer edge):

$$W_d = \sum_{m \geq 0} \sum_{\substack{\text{Dyck paths of length } 2m \\ \mathcal{D}=(0=\ell_0,\ell_1,\ldots,\ell_{2m}=0)}} \prod_{\text{down steps } \ell \to \ell-1} z^2 R_{\ell+d}$$

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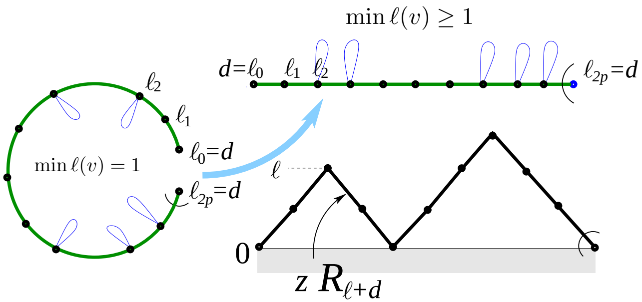

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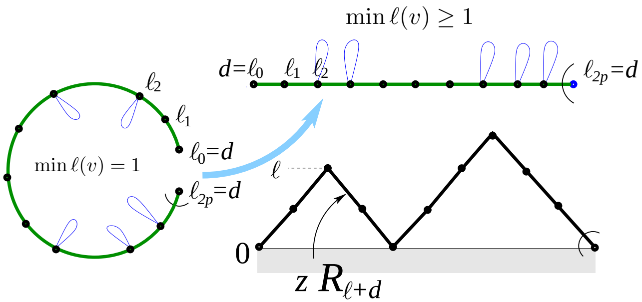

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$$W_d = \cfrac{1}{1 - \cfrac{z^2 R_{d+1}}{1 - \cfrac{z^2 R_{d+2}}{\ddots}}}$$

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but ω is also the generating function of quadrangulations of a polygon, a "well-known" quantity (e.g. resolvent of a one-matrix model):

$$[g^n z^{2p}]\omega = \frac{3^n (2p)!}{p!(p-1)!} \frac{(2n+p-1)!}{n!(n+p+1)!}$$

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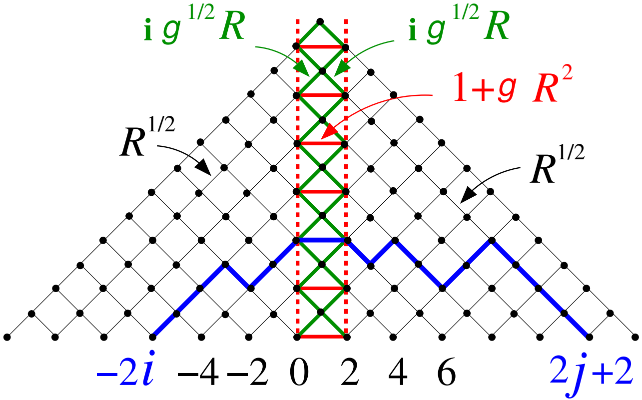
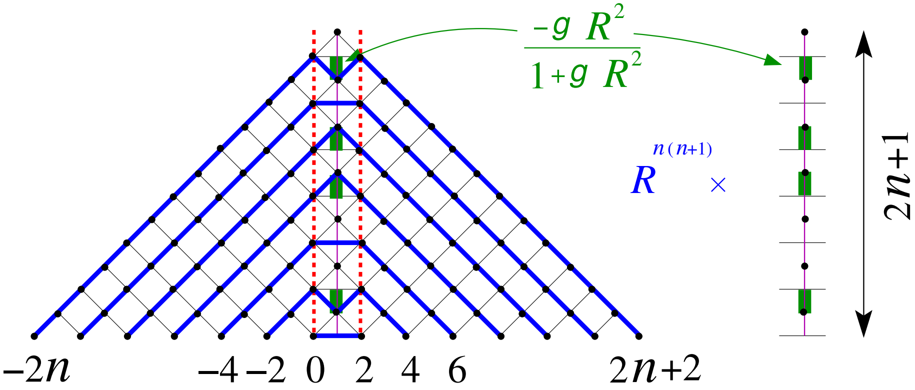
A combinatorial explanation for the form of R_ℓ follows by the Lindström-Gessel-Viennot lemma!

Continued fractions



ω_{i+j} counts "perturbed" Dyck paths.

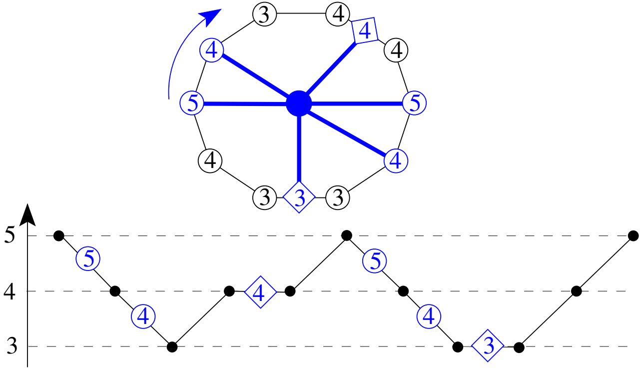
Continued fractions



The Hankel determinant count configurations of non-intersecting paths, in bijection with configurations of 1D dimers. By elementary combinatorics, our explicit expression for R_ℓ follows.

Continued fractions

The same coincidence happens in the setting of maps with controlled face degrees, by the bijection with mobiles.



Continued fractions

- Bipartite maps: Stieljes fraction

$$\omega = \cfrac{1}{1 - \cfrac{R_1 z^2}{1 - \cfrac{R_2 z^2}{1 - \cdots}}}$$

- Arbitrary maps: Jacobi fraction

$$\omega = \cfrac{1}{1 - S_0 z - \cfrac{R_1 z^2}{1 - S_1 z - \cfrac{R_2 z^2}{1 - \cdots}}}$$

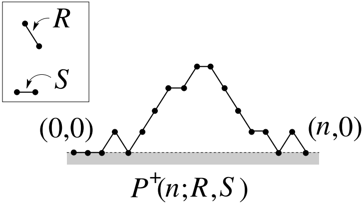
(B.-Guitter '10)

Continued fractions

But, again, ω is the g.f. of rooted maps with a boundary and is well studied. For a fixed boundary length its coefficient takes the general form

$$\omega_p = R \sum_{q \geq 0} \gamma_q P^+(p+q; R, S)$$

where R, S, γ_q are algebraic power series in the face weights $g_1, g_2, \ldots$.



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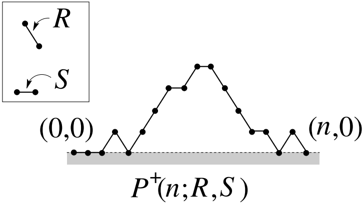

where R, S, γ_q are algebraic power series in the face weights $g_1, g_2, \ldots$.

In turn the coefficients in the continued fraction expansion are expressed via Hankel determinants:

$$R_\ell = \frac{H_\ell H_{\ell-2}}{H_{\ell-1}^2} \qquad H_\ell := \det_{0 \leq i,j \leq \ell} \omega_{i+j}$$

$$S_\ell = \frac{\tilde{H}_\ell}{H_\ell} - \frac{\tilde{H}_{\ell-1}}{H_{\ell-1}} \qquad \tilde{H}_\ell := \det_{0 \leq i,j \leq \ell} \omega_{i+j+\delta_{j,\ell}}.$$

Continued fractions

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$$R_\ell = R \, \frac{\det\limits_{1\le m,n\le M}[\ell+1+n]_m \det\limits_{1\le m,n\le M}[\ell-1+n]_m}{\left(\det\limits_{1\le m,n\le M}[\ell+n]_m\right)^2}$$

$$S_\ell = S - \sqrt{R}\left(\frac{\det\limits_{1\le m,n\le M}[\ell+1+n-\delta_{n,1}]_m}{\det\limits_{1\le m,n\le M}[\ell+1+n]_m} - \frac{\det\limits_{1\le m,n\le M}[\ell+n-\delta_{n,1}]_m}{\det\limits_{1\le m,n\le M}[\ell+n]_m}\right)$$

where the size of the determinants is independent of ℓ. Here $[\ell]_m \equiv \dfrac{y_m^{-\ell} - y_m^{\ell}}{y_m^{-1} - y_m}$ with y_m roots of $\mathcal{P}_p\left(y + \frac{1}{y}\right) = 0$, hence algebraic power series in the face weights $g_1, g_2, \ldots$.

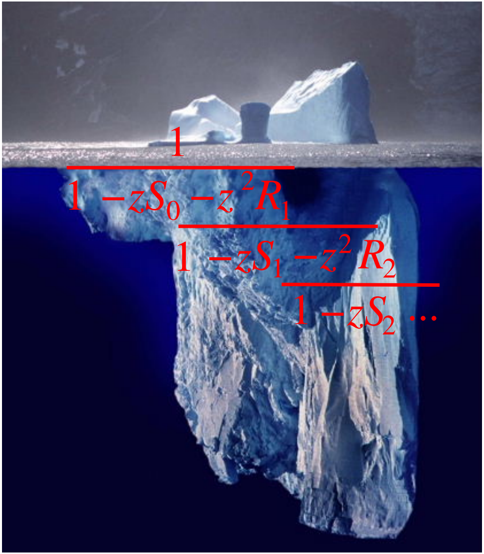
Continued fractions

Some remarks:

- we also have a combinatorial understanding of the conserved quantities (the ω_p themselves),
- bijections with trees may be replaced by a more intuitive "slice" decomposition of maps,
- orthogonal polynomials are lurking behind, but these are different from the usual ones encountered in random matrix theory (potential vs spectral density),
- a still mysterious connection with the KP integrable hierarchy (our symplectic Schur functions are related to N-soliton tau-functions),
- three-point function in the general setting still not understood.

Continued fractions

Continued fractions



General conclusion

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  (first attempts: Le Gall & Miermont '09, Borot-B.-Guitter '11-'12)
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Thanks for your attention!