Planar maps and continued fractions

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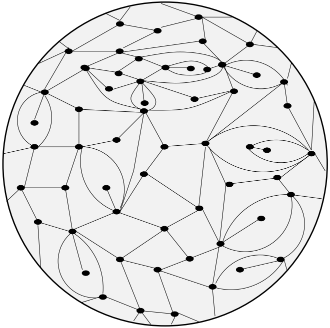
Outline

Maps: graphs embedded in surfaces (sphere in planar case)
considered up to deformation ($\Rightarrow$ finite number of maps with E edges)
a.k.a. planar diagrams, fatgraphs, dynamical random tessellations...



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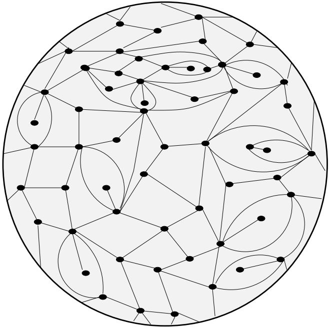

Motivations

- combinatorics [Tutte 1963]
- large N expansion of matrix integrals [Brézin-Itzykson-Parisi-Zuber 1979]
- 2D quantum gravity
- critical phenomena on dynamical (annealed) random surfaces
- probability theory: "Brownian map", connection with conformally-invariant processes

More on motivations

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Continuous results are obtained by taking suitable limits ($n \to \infty$, $g \to g_c$, critical points for matter...).

General model considered here:

Each face of valency k comes with fugacity g_k:

$$Z := \sum_{\text{maps}} \prod_{k \geq 1} g_k^{\#\{k\text{−valent faces}\}}$$

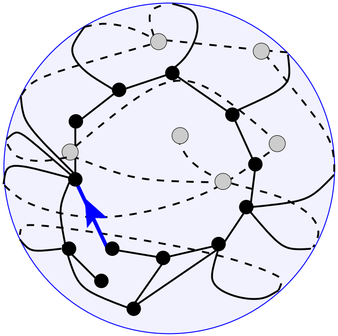
(A priori no matter)

Simple case: triangulations (resp. quadrangulations)

$$g_k = \begin{cases} g \text{ for } k = 3 \quad (\text{resp. } k = 4) \\ 0 \text{ otherwise} \end{cases} \qquad Z = \sum_{\substack{\text{(tri|quadr)-}\\\text{angulations}}} g^{\text{“area”}}$$

Outline

Computing the partition function Z is an enumeration problem. It is simpler to count rooted maps



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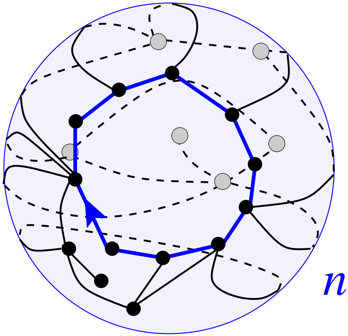

$n = 10$

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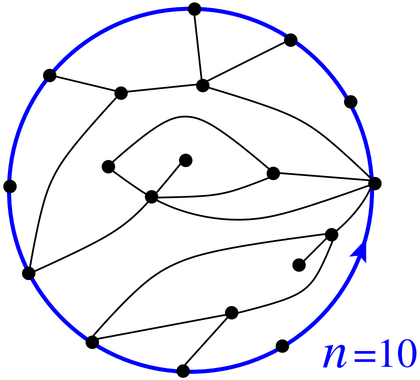

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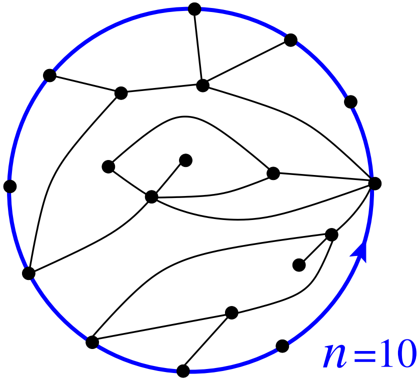

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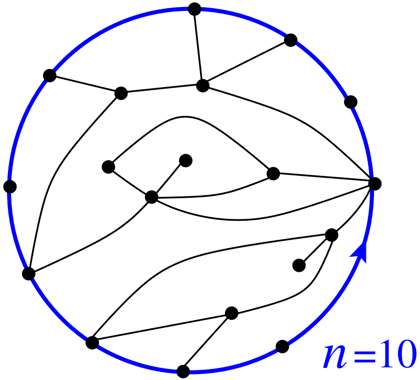
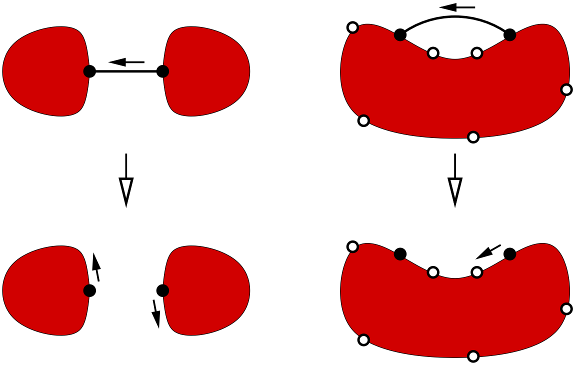
$F(z) := 1 + \sum_{n=1}^{\infty} F_n z^n$ is the disk amplitude.



Tutte's equation (1968) a.k.a. loop equation

The F_n are fully determined by the quadratic equation

$$F_n = \sum_{i=0}^{n-2} F_i F_{n-2-i} + \sum_{k \geq 1} g_k F_{n+k-2} \qquad (n \geq 1, F_0 = 1)$$



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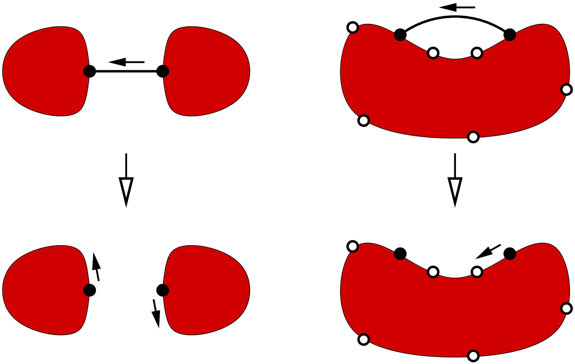
$$F(z) = 1 + z^2 F(z)^2 + \sum_{k\geq 1} g_k z^{2-k} \left(F(z) - \sum_{j=0}^{k-2} z^j F_j \right)$$



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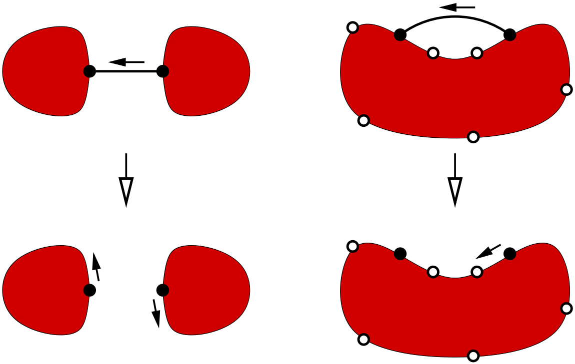

Review of the solution of Tutte's equation

By the previous equation

$$F(z) = \frac{1}{2z^2}\left(1 - \sum_{k\geq 1} g_k z^{2-k} \pm \sqrt{\Delta(z)}\right)$$

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$$F(z) = \frac{1}{2z^2}\left(1 - \sum_{k\geq 1} g_k z^{2-k} - \Gamma(z^{-1})\sqrt{1+\kappa_1 z+\kappa_2 z^2}\right)$$

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But $F(z)$ contains only nonnegative powers of z! This constraint allows to deduce explicit expressions for $\Gamma(z^{-1})$, κ_1, κ_2.

Example: quadrangulations

For $g_k = \begin{cases} g \text{ for } k = 4 \\ 0 \text{ otherwise} \end{cases}$ this method leads to

$$F_{2n} = \sum_{a=0}^{\infty} \frac{(2n)!}{n!(n-1)!} \frac{(2a+n-1)!}{a!(a+n+1)!} (3g)^a \qquad F_{2n+1} = 0$$

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To study large quadrangulations, one must consider the singular expansion around $g_c = 1/12$.

General combinatorial structure of the solution

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$$\kappa(z) := 1 + \kappa_1 z + \kappa_2 z^2 = (1 - Sz)^2 - 4Rz^2$$

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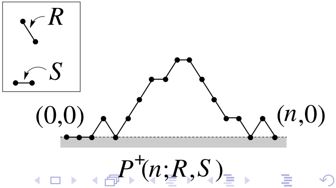

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$$F_n = R \sum_{q \geq 0} \gamma_q P^+(n+q; R, S) \tag{2}$$

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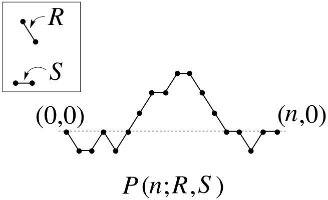
Remark: these may also be given a combinatorial interpretation via

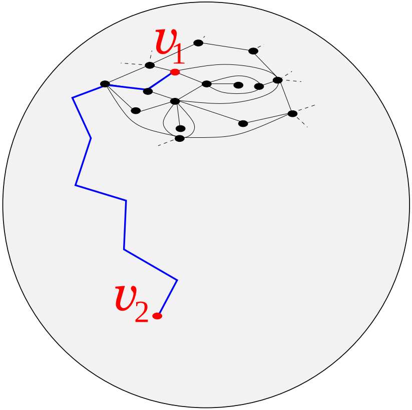
$$1/\sqrt{\kappa(z)} = \sum_{n=0}^{\infty} P(n; R, S) z^n$$



Summary/conclusion on the first problem

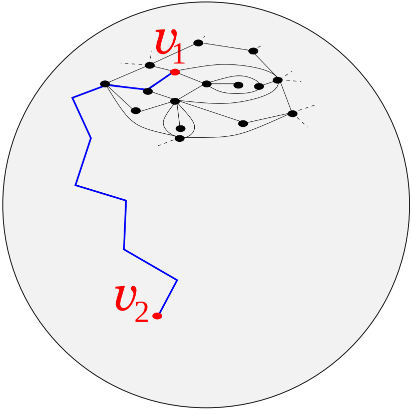
- Maps with a boundary can be enumerated effectively via Tutte's equation.
- A remarkable combinatorial/algebraic structure related to the physical one-cut hypothesis.
- $F(z)$ is a master function in terms of which generating functions for maps with several boundaries and of higher genus ("global observables") can be expressed.
- Generalizations to models with matter are known.

Outline



Geodesic (or graph) distance: minimal number of edges connecting two given vertices

(i.e each edge has length 1)



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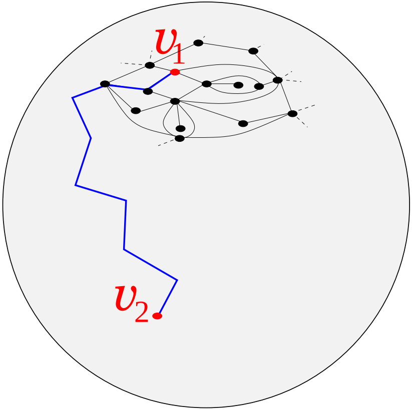
A map may then be viewed as a discrete metric space. What are the metric properties of random planar maps?



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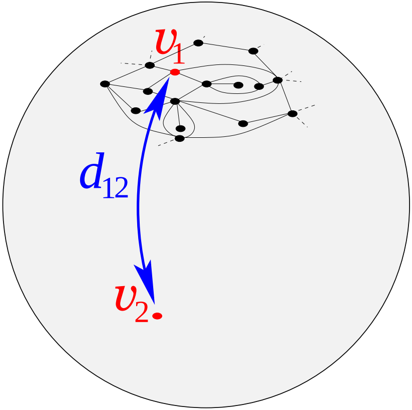
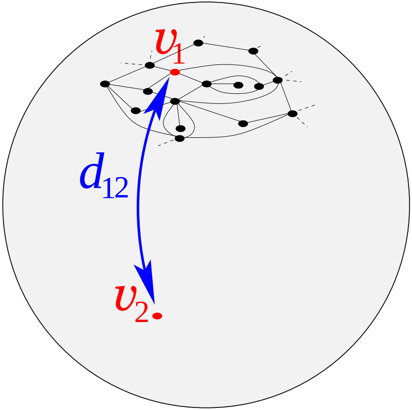
A map may then be viewed as a discrete metric space. What are the metric properties of random planar maps? What can we calculate?

Calculations of interest:

- finite size, exact results
- large size, local limit
- large size, scaling limit



Simple observable: the distance-dependent two-point function [Ambjørn-Watabiki 1996] is the generating function for maps with two marked points at given distance. Computing it is again an enumeration problem!



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Probabilistic interpretation: it encodes the distribution of distances between two uniformly chosen random points.

By a transfer matrix approach, Ambjørn and Watabiki successfully predicted the universal scaling form of the two-point function for pure gravity (a.k.a. Brownian map, the generic scaling limit here).

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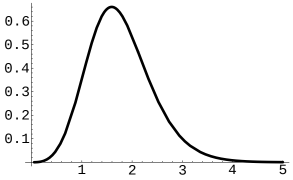
The rescaled distance between two uniform random points admits a limiting distribution as size tends to infinity, with density

$$\rho(d) = \frac{2}{\mathrm{i}\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi\, \xi\, e^{-\xi^2} \mathcal{G}\left(d; \sqrt{\tfrac{-3\mathrm{i}\xi}{2}}\right) \qquad \mathcal{G}(d;\alpha) := 4\alpha^3 \frac{\cosh(\alpha d)}{\sinh^3(\alpha d)}$$



$$\rho(d) \sim d^3 \text{ for } d \to 0$$

$$\rho(d) \sim e^{-Cd^{4/3}} \text{ for } d \to \infty$$

An exact discrete expression whose scaling form agrees with the Ambjørn-Watabiki prediction was found for quadrangulations and, more generally, maps with even face valencies. [B., Di Francesco, Guitter 2003]

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Ingredients:

- coding of maps by trees (Schaeffer's bijection and generalizations)
- identification of the two-point function with tree g.f.
- equation following from recursive decomposition of such trees
- guess of the solution!

Example: quadrangulations

The discrete two-point function is the solution of the equation

$$R_n = 1 + gR_n(R_{n-1} + R_n + R_{n+1}) \qquad (n \geq 1, R_0 = 0)$$

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Explicit solution

$$R_n = R\frac{u_n u_{n+3}}{u_{n+1} u_{n+2}} \tag{3}$$

$$R = 1 + 3gR^2 \qquad u_n = 1 - x^n \qquad x + \frac{1}{x} + 1 = \frac{1}{gR^2}$$

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There are also equations with explicit solutions in more general cases! The form (3) still holds (but u_n gets more complicated). Our explanation for this miracle was discrete integrability of the equations. But is there a more direct, combinatorial, explanation?

New approach [B., Guitter 2010]

The two-point function is encoded in the continued fraction expansion of the disk amplitude $F(z)$!

- Maps with even face valencies: Stieljes fraction

$$F(z) := \sum_{n=0}^{\infty} F_{2n} z^{2n} = \cfrac{1}{1 - \cfrac{R_1 z^2}{1 - \cfrac{R_2 z^2}{1 - \cdots}}}$$

- Maps with arbitrary face valencies: Jacobi fraction

$$F(z) := \sum_{n=0}^{\infty} F_n z^n = \cfrac{1}{1 - S_0 z - \cfrac{R_1 z^2}{1 - S_1 z - \cfrac{R_2 z^2}{1 - \cdots}}} \tag{4}$$

Elements of the proof:

- the combinatorial theory of continued fractions [Flajolet 1980]

Combinatorial interpretation of the Jacobi fraction expansion (4)

F_n is equal to the generating function for Motzkin paths of length n, with weight R_m (resp. S_m) per down-step (resp. level-step) starting at height m.

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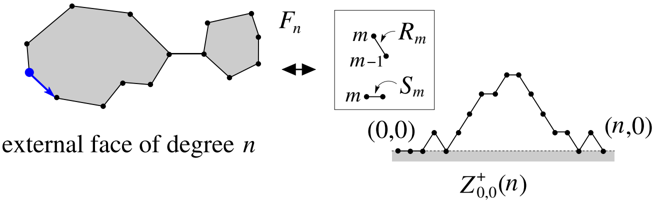
F_n is equal to the generating function for Motzkin paths of length n, with weight R_m (resp. S_m) per down-step (resp. level-step) starting at height m.

- a suitable decomposition of maps with a boundary (via trees or "slices"): Motzkin paths code the distances from the origin to the vertices incident to the root face.



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Even face valencies: because $F_{2n+1} = 0$, we have $S_n = \tilde{H}_n = 0$ and H_n has a natural factorization $H_n = u_{n+2} u_{n+3}$, which yields with the form (3) seen before.

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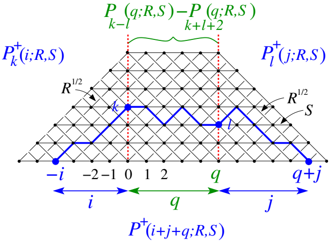
These relations hold in the general theory of continued fractions. In our map model, the specific form of F_n leads to specific Hankel determinants, which are *symplectic Schur functions* $\mathrm{sp}_{2p}(\lambda, \mathbf{x})$.

The general formula for F_n is

$$F_n = \sum_{q=0}^{p} A_q P^+(n+q)$$



Substituting into the Hankel determinant

$$\begin{aligned} H_n &= \det_{0\le i,j\le n} \left(\sum_{q=0}^{p} A_q P^+(i+j+q) \right) \\ &\propto \det_{0\le k,\ell\le n} \left(\sum_{q=0}^{p} A_q (P_{k-\ell}(q) - P_{k+\ell+2}(q)) \right) \\ &\propto \mathrm{sp}_{2p}(\lambda_{p,n+1}, \mathbf{x}) \\ &\propto \det_{1\le i,j\le p} (x_i^{n+j} - x_i^{-n-j}) \end{aligned}$$

The x's are roots of

$$\sum_{r=-p}^{p} \sum_{q=0}^{p} A_q P_r(q) x^r = 0$$

$\lambda_{p,n+1}$ is the "rectangular" partition $\underbrace{(n+1) + \cdots + (n+1)}_{p}$

Remark

We make use of *two* different formulas for F_n involving Motzkin paths:

- as a sum (2) over Motzkin paths of variable length $n, \dots, n+p$ and height-independant weights R, S per step
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Caveat

The expression involving Schur functions assumes that face valencies are bounded: $g_k = 0$ for $k > p+2$. H_n may then be rewritten as a $p \times p$ determinant (rather than $(n+1) \times (n+1)$), easier to study in the limit of large distance n.

Example & combinatorial interpretation: triangulations

Suppose that $g_k = 0$ for $k \neq 3$ (faces are triangles), i.e $p = 1$:

$$F_n = A_0\, P^+(n; R, S) + A_1\, P^+(n + 1; R, S)$$

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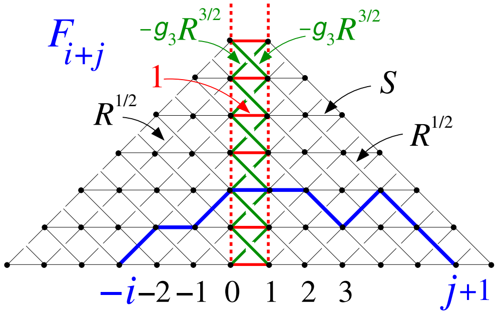

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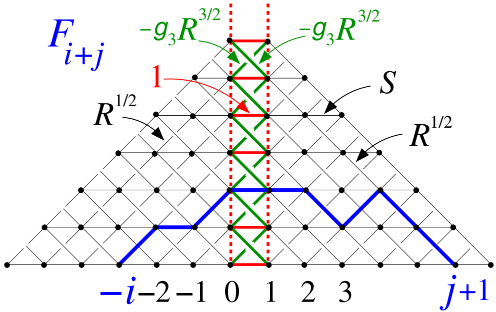
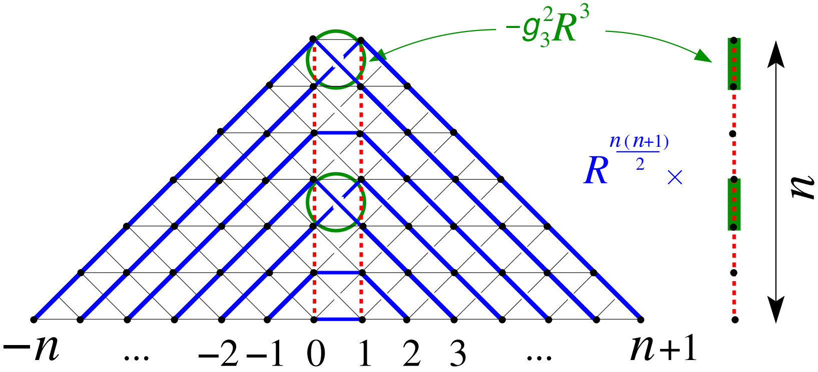


F_{i+j} can be interpreted as paths on a weighted graph. By the Lindström-Gessel-Viennot lemma, the determinant H_n counts configurations of non-intersecting lattice paths on this graph.

Such configurations of non-intersecting lattice paths are highly constrained and, actually, in bijection with configurations of 1D dimers.



Counting 1D dimer configurations is easy, we obtain

$$H_n \propto \frac{1}{(1+y)^{n+1}} \frac{1-y^{n+2}}{1-y}$$

with y related to the dimer weight $-g_3^2 R^3$ by

$$y + \frac{1}{y} + 2 = \frac{1}{g_3^2 R^3}.$$

It yields the simple formula

$$R_n = R \frac{(1-y^n)(1-y^{n+2})}{(1-y^{n+1})^2}$$

and similarly

$$S_n = S - g_3 R^2 y^n \frac{(1-y)(1-y^2)}{(1-y^{n+1})(1-y^{n+2})}$$

Conclusion and outlook

- We have shown that the disk amplitude and the two-point function are encoded in the same function $F(z)$.
- Our results are purely discrete. One may now turn to asymptotic analysis. The generic behaviour is pure gravity ("Brownian map").
- Possible directions:
  - Connections with orthogonal polynomials and matrix models
  - Other distance-related observables (not so many known! radius, three-point function, length of loops, numbers of geodesics...)
  - Generalizations to models with matter
  - Maps with large faces?

References:

- J. Bouttier, P. Di Francesco and E. Guitter, Nucl.Phys. B663 (2003) 535-567, arXiv:cond-mat/0303272,
- J. Bouttier and E. Guitter, arXiv:1007.0419.

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