Integrable probabilities: around the Longest Increasing Subsequence problem

Lecture notes – cours M2 parcours probabilités

Jérémie Bouttier

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This course is mostly based on the first chapters of the book by Dan Romik [1]. The purpose of these notes is to gather the material which was presented during the lectures and which is not already in Romik's book. **Warning: these notes are incomplete and may contain errors and imprecisions. Please report me if you find any!**

Notations. Here are some notations used throughout the course.

- S_n: the symmetric group of order n, i.e. the set of permutations of $\{1, \dots, n\}$,
- σ_n: a uniform random element of S_n,
- $L(\sigma)$: maximal length of an increasing subsequence of the permutation σ,
- $D(\sigma)$: maximal length of a decreasing subsequence of the permutation σ,
- $\mathcal{P}$: set of all (integer) partitions, i.e. of nonincreasing sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers with finite support,
- $|\lambda|$: size of the partition λ, i.e. the sum of the λ_i,
- $\lambda \vdash n$ is a shorthand notation to say that λ is a partition of size n,
- $\ell(\lambda)$: length of the partition λ, i.e. the smallest i such that $\lambda_{i+1} = 0$,
- λ': the conjugate of the partition λ, i.e. the partition such that $\lambda'_i = \#\{j : \lambda_j \geq i\}$,
- d_λ: the number of standard Young tableaux of shape $\lambda \in \mathcal{P}$,
- $\lambda^{(n)}$: a Plancherel random partition of size n, i.e. a random partition such that

$$\mathbb{P}(\lambda^{(n)} = \lambda) = \begin{cases} \frac{d_\lambda^2}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

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1 Introduction

1.1 Starting point of these notes

In the first part of the course, we established that

$$\frac{L(\sigma_n)}{\sqrt{n}} \to 2, \qquad n \to \infty, \tag{1.1}$$

both in probability and expectation, by following essentially [1, Chapter 1]. Probably the only digression was Viennot's geometric construction of the Robinson-Schensted correspondence (to be written).

These notes aim at covering the second part of the course, in which we study the fluctuations of $L(\sigma_n)$, with the goal of establishing the Baik-Deift-Johansson theorem:

Theorem 1.1. *We have*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L(\sigma_n) - 2\sqrt{n}}{n^{1/6}} \le x\right) = F_2(x) \tag{1.2}$$

where F_2 is the Tracy-Widom distribution of index 2.

Our approach is parallel to that of [1, Chapter 2] with some differences, and we try to be as self-contained as possible.

1.2 An expression for the Tracy-Widom distribution F_2

The Tracy-Widom distribution F_2 may be defined in terms of the Airy function as follows. First we recall that the classical *Airy function* may be defined through the integral representations

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt = \frac{1}{2i\pi}\int_{c+i\mathbb{R}} e^{\frac{z^3}{3} - xz} dz. \tag{1.3}$$

Here the first representation is an improper integral which makes sense for $x \in \mathbb{R}$, while the second representation as an absolutely convergent complex integral makes sense for $x \in \mathbb{C}$ and any $c > 0$, and shows that $\mathrm{Ai}(x)$ is an entire function of x.

Using integration by parts it is easy to check that

$$\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x). \tag{1.4}$$

Also, we note that Ai decays exponentially for $x \to +\infty$ as $\mathrm{Ai}(x) \sim \frac{x^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}$. There is a much slower decay, with oscillations, for $x \to -\infty$. We now define the *Airy kernel* as

$$\mathbf{A}(x,y) := \begin{cases} \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y} & \text{for } x \neq y, \\ \mathrm{Ai}'(x)^2 - x\,\mathrm{Ai}(x)^2 & \text{for } x = y. \end{cases} \tag{1.5}$$

It is a smooth function of x and y by L'Hôpital's rule. We also note the alternate expression

$$\mathbf{A}(x,y) := \int_0^\infty \operatorname{Ai}(x+t)\operatorname{Ai}(y+t)dt \tag{1.6}$$

which is obtained by integrating the relation $\frac{\partial}{\partial t}\mathbf{A}(x+t,y+t) = -\operatorname{Ai}(x+t)\operatorname{Ai}(y+t)$, which itself follows easily from (1.4) and (1.5).

We are now ready to define

$$F_2(t) := 1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!}\int_t^\infty \dots \int_t^\infty \det_{1\le i,j\le n} \mathbf{A}(x_i,x_j)dx_1\cdots dx_n. \tag{1.7}$$

As we shall see, the integral and sum are absolutely convergent for any $t \in \mathbb{R}$ so the definition makes sense. The definition looks rather complicated but it is in fact an instance of *Fredholm determinant*, which is intimately related to the concept of determinantal point process as will be discussed in this section. This leads to the more compact notation

$$F_2(t) = \det(\mathbf{I}-\mathbf{A})_{L^2(t,\infty)}. \tag{1.8}$$

It may be seen that F_2 is indeed a distribution function, i.e. it is an increasing function of t tending to 0 (resp. 1) for $t \to -\infty$ (resp. $t \to \infty$). In fact, the most difficult part is to study the limit at $-\infty$, at $+\infty$ it is a relatively easy consequence of the exponential decay of the Airy function.

2 Determinantal point processes

In this section we give an elementary introduction to determinantal point processes, mostly in the discrete context. For a more general setting, we refer to [2] and references therein.

2.1 Simple point processes

Intuitively speaking, a *simple point process*[1] is a random locally finite subset X of an "ambiant" space Λ. Here, Λ is generally assumed to be a complete separable metric space, and by locally finite we mean that, for any bounded Borel set $B \subset \Lambda$, the number of points

$$\xi(B) := \#\{X \cap B\} \tag{2.1}$$

should be almost surely finite. (The underlying probability space is defined in such a way that $\xi(B)$ is indeed a random variable for any such B.) For $\phi : \Lambda \to \mathbb{C}$ measurable with bounded support, the product $\prod_{x\in X}(1+\phi(x))$ makes sense and is a random variable.

As we do not want to enter into too many measure-theoretical subtleties, we will stick to the case where Λ is a (possibly finite) subset of $\mathbb{Z}$ (*discrete setting*) or an interval of $\mathbb{R}$ (*continuum setting*).

[1]More generally, a point process is a random locally finite *multiset* X, that is an element $x \in \Lambda$ may appear multiple times in X. In this context, one views X as a locally finite counting measure (i.e. a measure taking values in $\mathbb{N}$).

Correlation functions. Let $U = \{u_1, \ldots, u_n\}$ denote a finite subset of Λ. We define the *n-point correlation function* $\rho_n(u_1, \ldots, u_n)$ as

- in the discrete setting:

$$\rho_n(u_1, \ldots, u_n) := \mathbb{P}(U \subset X). \tag{2.2}$$

- in the continuum setting, informally[2]:

$$\rho_n(u_1, \ldots, u_n) du_1 \cdots du_n := \mathbb{P}\left(\xi([u_i, u_i + du_i]) = 1,\ i = 1, \ldots, n\right). \tag{2.3}$$

In this definition it is assumed that the u_i's are distinct, if not $\rho_n(u_1, \ldots, u_n)$ vanishes by convention (which we may intuitively motivate by the fact that the point process is assumed simple). It is a symmetric function of the u_i's since we assume no particular order on them.

For $n = 1$, $\rho_1(u)$ is sometimes called the intensity measure, or density, of the point process. It gives the probability that u is a point of X in the discrete setting, or the infinitesimal probability that X contains a point close to u in the continuum setting.

Example 2.1. The point process is "uncorrelated" if the n-point correlation function factorizes as

$$\rho_n(u_1, \ldots, u_n) = \rho_1(u_1) \cdots \rho_1(u_n) \tag{2.4}$$

when the u_i's are distinct. In the discrete setting, this means that the random variable $\xi(\{u\}) = \mathbb{1}_{u \in X}$ is a Bernoulli random variable of mean $\rho_1(u)$, and is independent of all the $\xi(\{u'\})$ for $u \neq u'$. In the continuum setting, this means that X is a Poisson point process with intensity ρ_1.

Gap probabilities and related functionals. Knowing the correlation functions ρ_n for all n is essentially equivalent to knowing the law of X. To make this clear, let us consider the expectation $E_\phi := \mathbb{E}\left(\prod_{x \in X}(1 + \phi(x))\right)$ with $\phi : \Lambda \to \mathbb{C}$ measurable with bounded support (for $\phi = e^{-\psi} - 1$ with ψ nonnegative, E_ϕ is the so-called Laplace functional of X). Then we claim that:

- knowing E_ϕ for all ϕ amounts to knowing the law of X, in the sense of knowing the joint law of $\xi(B_1), \ldots, \xi(B_n)$ for any collection of disjoint bounded Borel sets $B_1, \ldots, B_n$,
- and E_ϕ may be expressed in terms of the correlation functions.

To justify the first claim, it suffices to take $\phi = \sum (z_i - 1)\mathbb{1}_{B_i}$, so that E_ϕ is equal to the generating function $\mathbb{E}\left(z_1^{\xi(B_1)} \cdots z_n^{\xi(B_n)}\right)$ which, as we vary $z_1, \ldots, z_n$, characterizes the joint law of $\xi(B_1), \ldots, \xi(B_n)$.

[2]The proper notion of correlation function requires to fix a *reference measure* on Λ, see [2, Definition 2.1]. Here the reference measure is the Lebesgue measure on $\mathbb{R}$. In all generality, correlation functions may not exist, or may exist but not characterize the law of X. We will not encounter these subtleties here.

Of particular importance is the case $\phi = -\mathbb{1}_B$, for which $\mathbb{E}_\phi$ is the so-called *gap probability* $\mathbb{P}(X \cap B = \emptyset)$.

We now justify the second claim. Let us first consider the discrete setting, and denote by S the support of ϕ. By assumption S is finite and we may write

$$\prod_{x\in X}(1+\phi(x)) = \prod_{x\in S}(1+\phi(x)\mathbb{1}_{x\in X}) = \sum_{U\subset S}\mathbb{1}_{U\subset X}\prod_{x\in U}\phi(x). \tag{2.5}$$

Upon taking the expectation we obtain

$$\begin{aligned}\mathbb{E}\left(\prod_{x\in X}(1+\phi(x))\right) &= \sum_{U\subset S}\mathbb{P}(U\subset X)\prod_{x\in U}\phi(x)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(u_1,\ldots,u_n)\in\Lambda^n}\rho_n(u_1,\ldots,u_n)\phi(u_1)\cdots\phi(u_n)\end{aligned} \tag{2.6}$$

where, to pass to the second line, we replace the sum over subsets by a sum over ordered tuples of elements of Λ (observe that there are only finitely many nonzero terms, and only tuples of distinct elements contribute).

For the continuum setting, we remain informal and simply observe that the sum over tuples should be replaced by an integral:

$$\mathbb{E}\left(\prod_{x\in X}(1+\phi(x))\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}du_1\cdots du_n\rho_n(u_1,\ldots,u_n)\phi(u_1)\cdots\phi(u_n). \tag{2.7}$$

Here assumptions on the growth of the ρ_n's are necessary to justify that the sum converges, see [2, Propositions 2.2 and 2.3].

2.2 Determinantal point processes: definition

Definition 2.2 (Determinantal point process/DPP)**.** A simple point process is said *determinantal* if there exists a function $K : \Lambda \times \Lambda \to \mathbb{C}$, called *correlation kernel*, such that

$$\rho_n(u_1,\ldots,u_n) = \det_{1\le i,j\le n} K(u_i,u_j). \tag{2.8}$$

Note that the correlation kernel is not unique: for any nonvanishing function $f : \Lambda \to \mathbb{C}$, the kernel $\tilde{K}(x,y) := K(x,y)\frac{f(x)}{f(y)}$ defines the same correlation functions as K.

Example 2.3. Uncorrelated point processes, as defined in Example (2.1), are instances of DPPs as seen by taking $K(x,y) = \rho_1(x)$ for $x = y$, and $= 0$ otherwise.

Example 2.4. Consider the *Hermite polynomials* $(H_n(x))_{n\ge 0}$, defined by applying the Gram-Schmidt orthogonalization procedure to the basis of monomials $(x^n)_{n\ge 0}$ with respect to the scalar product on $\mathbb{C}[x]$ defined by

$$(p(x),q(x)) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{p}(x)q(x)e^{-x^2/2}dx. \tag{2.9}$$

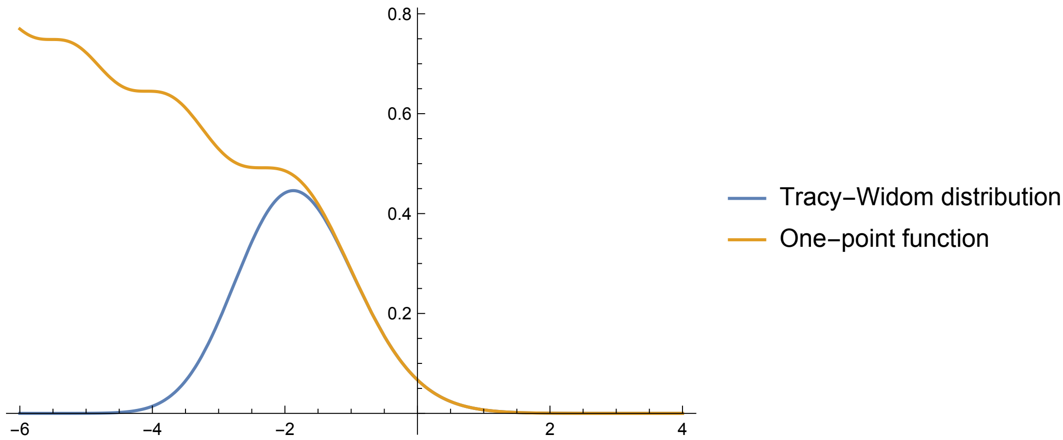

Figure 1: Airy ensemble: the Tracy-Widom distribution (here displayed via its density wrt the Lebesgue measure) vs the one-point function $\mathbf{A}(x, x)$.

In more explicit terms, the first Hermite polynomials read

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x \tag{2.10}$$

and the further ones are given by the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \qquad n \geq 1. \tag{2.11}$$

Then,

$$K_N(x, y) := \sum_{n=0}^{N-1} \frac{H_n(x)H_n(y)}{n!} e^{-(x^2+y^2)/2} \tag{2.12}$$

is the kernel of a DPP which may be shown to coincide with the eigenvalue measure of the GUE of size N.

Determinantal point processes were introduced by Odile Macchi [3] to model the spatial distribution of fermions in a beam. Correlation functions were then called "coincidence probabilities". There is a good reason why we only consider the probability to *contain* a certain subset: observing the full set X is supposed to be experimentally impossible (as this would require doing simultaneous measurements everywhere in space), while for measuring $\rho_n(u_1, \dots, u_n)$ we simply have to put n detectors at $u_1, \dots, u_n$.

It is a nontrivial question whether, for a given kernel $K : \Lambda^2 \to \mathbb{C}$, there exists a determinantal point process for which it is the correlation kernel. The so-called *Macchi-Soshnikov theorem* [3, 4] states that, if $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ is Hermitian and locally trace-class, then it is the correlation kernel of a (unique) DPP if and only if $0 \leq K \leq 1$. We shall not enter into details here, but just note that the Airy kernel $\mathbf{A}$ satisfies the hypotheses of the Macchi-Soshnikov theorem[3], hence it determines a continuous DPP on $\mathbb{R}$ called

[3] $\mathbf{A}$ is manifestly Hermitian from its definition, it is trace-class on $L^2(t, \infty)$ for any $t \in \mathbb{R}$ by the exponential decay at $+\infty$, and it is in fact a projection operator so its eigenvalues are 0 and 1.

the *Airy ensemble* and denoted X_{Airy}. The Airy ensemble admits a maximal element almost surely, and we may therefore write

$$X_{\text{Airy}} = \{\zeta_1 > \zeta_2 > \cdots\}. \tag{2.13}$$

The one-point function of the Airy ensemble is $\mathbf{A}(x,x)$ and, as we shall explain, the distribution of its largest element ζ_1 is the Tracy-Widom distribution F_2. See Figure 1.

In this section we shall actually prove the following extension of Theorem 1.1:

Theorem 2.5 (Edge statistics for Plancherel measure). *For $i \in \mathbb{N}^*$, let $\bar{\lambda}_i^{(n)} := n^{-1/6}(\lambda_i^{(n)} - 2\sqrt{n})$ denote the rescaled i-th part of the Plancherel random partition $\lambda^{(n)}$. Then, for any $k \geq 1$, we have*

$$(\bar{\lambda}_1^{(n)}, \ldots, \bar{\lambda}_k^{(n)}) \xrightarrow[n\to\infty]{(d)} (\zeta_1, \ldots, \zeta_k). \tag{2.14}$$

In other words, the largest parts of a Plancherel random partition converge after rescaling to the Airy ensemble.

This extension was first proved by Okounkov [5] using a beautiful combinatorial argument (mentioned in his Field medal citation), then reproved using DPPs by Borodin, Okounkov and Olshanski [6], and independently Johansson [7]. We follow this latter approach.

2.3 Theory of DPPs on finite spaces

Here we assume Λ to be finite. Therefore, $K : \Lambda \times \Lambda \to \mathbb{C}$ may be viewed as a matrix $\mathbf{K}$ whose rows and columns are indexed by Λ. We start with a "trivial" lemma on finite determinants:

Lemma 2.6. *Let $\mathbf{M}$ be a $n \times n$ matrix, then we have*

$$\det(\mathbf{I} + x\mathbf{M}) = \sum_{U \subset \{1,\ldots,n\}} x^{\#U} \det \mathbf{M}_U \tag{2.15}$$

where $\mathbf{I}$ denotes the identity matrix and $\mathbf{M}_U$ denotes the submatrix obtained by keeping only the rows and columns of $\mathbf{M}$ that are in the set U.

Proof. Follows from expanding the determinant as a sum over permutations. □

We will now use this lemma to show that, in a DPP, the gap probabilities and more generally the expectations E_ϕ discussed in Section 2.1 are given by determinants.

Proposition 2.7. *Let X be a DPP with correlation kernel K. Then, for any $B \subset \Lambda$, we have*

$$\mathbb{P}(X \cap B = \emptyset) = \det(\mathbf{I} - \mathbf{K}_B) \tag{2.16}$$

and

$$\mathbb{E}(z^{\xi(B)}) = \det(\mathbf{I} + (z-1)\mathbf{K}_B). \tag{2.17}$$

Even more generally, for $\phi : \Lambda \to \mathbb{C}$, let $\mathbf{D}$ be the diagonal matrix whose entry (x, x) is $\phi(x)$. Then, we have

$$\mathbb{E}\left(\prod_{x\in X}(1+\phi(x))\right) = \det(\mathbf{I}+\mathbf{DK}). \tag{2.18}$$

Remark 2.8. We commit a slight abuse of notation by viewing $\mathbf{K}_B$ not only as a matrix of size $|B| \times |B|$, but also a matrix of size $n \times n$ whose entries are zero if the row and column indices are not in B. This distinction is immaterial in (2.16) and (2.17): according to the point of view, the determinant in the right-hand side has respectively size $|B|$ or n, but it evaluates to the same quantity since, in the second point of view, $\mathbf{I} - \mathbf{K}_B$ coincides with the identity matrix for rows and columns outside B.

Proof of Proposition 2.7. All the equalities follow from (2.6), the definition of a DPP and Lemma 2.6.

In detail, a one-line derivation of the gap probability is obtained by

$$\mathbb{P}(X\cap B=\emptyset) = \sum_{U\subset B}(-1)^{\#U}\,\mathbb{P}(U\subset X) = \sum_{U\subset B}(-1)^{\#U}\det\mathbf{K}_U = \det(\mathbf{I}-\mathbf{K})_B \tag{2.19}$$

where the first equality comes from the inclusion-exclusion principle (i.e. (2.6) for $\phi = -\mathbb{1}_B$), the second from the definition of a DPP and the third from Lemma 2.6.

For the general expectation E_ϕ, we use the first line of (2.6) and observe that, by the definition of a DPP, we have $\mathbb{P}(U \subset X)\prod_{x\in U}\phi(x) = \det(\mathbf{DK})_U$. This yields (2.18) by Lemma 2.6. Finally we take $\phi = (z-1)\mathbb{1}_B$ to obtain (2.17). □

We now state another proposition which describes a general family of DPPs, that are sometimes called *L-ensembles*:

Proposition 2.9. *Assume that there exists a matrix $\mathbf{L}$ (called* configuration kernel*) such that, for any $A \subset \Lambda$, we have*

$$\mathbb{P}(X=A) = \frac{1}{Z}\det\mathbf{L}_A \tag{2.20}$$

where Z is a normalization factor such that the probabilities sum up to one. Then, X is a determinantal point process with correlation kernel

$$\mathbf{K} = \mathbf{L}(\mathbf{I}+\mathbf{L})^{-1}. \tag{2.21}$$

Proof. From Lemma 2.6 we see that $\sum_{A\subset\Lambda}\mathbb{P}(X = A) = \det(\mathbf{I}+\mathbf{L})/Z$ hence the matrix $\mathbf{I}+\mathbf{L}$ *has* to be invertible for the probabilities to sum up to one, and $Z = \det(\mathbf{I}+\mathbf{L})$. Furthermore, for $\phi : \Lambda \to \mathbb{C}$, the same lemma implies that

$$\mathbb{E}\left(\prod_{x\in X}(1+\phi(x))\right) = \frac{\det(\mathbf{I}+(\mathbf{I}+\mathbf{D})\mathbf{L})}{\det(\mathbf{I}+\mathbf{L})} \tag{2.22}$$

where $\mathbf{D}$ is the diagonal matrix whose entry (x,x) is $\phi(x)$. By (2.21), we see that $(\mathbf{I}+\mathbf{DK})(\mathbf{I}+\mathbf{L}) = \mathbf{I}+\mathbf{L}+\mathbf{DL}$ so that

$$\mathbb{E}\left(\prod_{x\in X}(1+\phi(x))\right) = \det(\mathbf{I}+\mathbf{DK}). \tag{2.23}$$

In view of Proposition 2.7, this says that the functional E_ϕ for X is exactly the same as for the DPP with kernel $\mathbf{K}$, and since this is true for all ϕ the statement follows. □

Remark 2.10. The relation (2.21) may be inverted into

$$\mathbf{L} = \mathbf{K}(\mathbf{I}-\mathbf{K})^{-1}. \tag{2.24}$$

Hence a configuration kernel $\mathbf{L}$ may exist only if $\mathbf{I}-\mathbf{K}$ is invertible. This is not necessarily the case, for instance the trivial DPP which is equal to Λ almost surely has correlation kernel $\mathbf{I}$.

Exercise 2.11. Show that, if X is a DPP whose correlation kernel $\mathbf{K}$ is such that $\mathbf{I}-\mathbf{K}$ is invertible, then X admits a configuration kernel $\mathbf{L}$ in the sense of the Proposition (2.9).

Exercise 2.12. Show that, if X is a DPP, then its complement $\Lambda\setminus X$ is also a DPP. More generally show that, for any $D\subset\Lambda$, the symmetric difference $X\Delta D = (D\setminus X)\cup(X\setminus D)$ is a DPP. (And exhibit a correlation kernel in all cases.)

2.4 DPPs on countable spaces and Fredholm determinants

We now want to extend the theory to the case where Λ is countably infinite (and the bounded sets are the finite sets).

At a superficial level it seems that there are no issues at all: Proposition 2.7 remains valid if we are careful that B should be a *finite* subset of Λ, and ϕ has *finite* support, so that all the determinants involved may be thought as finite determinants. Indeed, it is clear from Definition 2.2 that, if X is a DPP on Λ with correlation kernel K, then its restriction $X\cap B$ to any finite set B is a DPP on B with correlation kernel K_B. (In the previous section we were careful to denote matrices with bold letters but we will be much looser from now on.)

However, the superficial level is not very satisfactory because we would also like to lift the finiteness assumption on B and the support of ϕ. After all, $\xi(B)$ remains a random variable (taking possibly the value ∞) even if B is infinite. The fundamental idea is that Proposition 2.7 remains valid provided that the determinants in the right-hand sides are thought as *Fredholm determinants*.

Definition 2.13. For $M:\Lambda^2\to\mathbb{C}$, we define the *Fredholm determinant* $\det(I+xM)_\Lambda$ by

$$\det(I+xM)_\Lambda := \sum_{\substack{U\subset\Lambda\\ U\text{finite}}} x^{\#U}\det M_U \tag{2.25}$$

provided that the sum in the right-hand side is absolutely convergent.

By comparing with Lemma 2.6, we see that this definition coincides with the usual determinant when Λ is finite. For Λ infinite, it is natural to ask whether there are natural assumptions on M under which $\det(I + xM)_\Lambda$ is well-defined. For this we need bounds on $\det M_U$, which are provided by the following lemma.

Lemma 2.14 (Hadamard's bound)**.** *Let $\mathbf{M} = (m_{i,j})_{i,j=1}^n$ be a $n \times n$ complex matrix:*

1. *if $\mathbf{M}$ is hermitian positive-semidefinite then we have*

$$\det \mathbf{M} \leq \prod_{i=1}^n m_{i,i}, \tag{2.26}$$

2. *if $\mathbf{M}$ is general, we denote by $v_1, \ldots, v_n$ its columns, and we have*

$$|\det \mathbf{M}| \leq \prod_{j=1}^n \|v_j\|_2 \leq n^{n/2} \prod_{j=1}^n \|v_j\|_\infty. \tag{2.27}$$

Proof. 1. Note that $m_{i,i} = e_i^\intercal \mathbf{M} e_i$ where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$. We thus have $m_{i,i} \geq 0$ from the hypothesis. If $\det \mathbf{M} = 0$ there is nothing to prove, otherwise we have $m_{i,i} > 0$ and we consider the matrix $\mathbf{M}'$ defined by

$$m'_{i,j} = \frac{m_{i,j}}{\sqrt{m_{i,i} m_{j,j}}}. \tag{2.28}$$

It is a hermitian positive-definite matrix whose diagonal entries are all 1. If $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues, then

$$\det \mathbf{M}' = \prod \lambda_i \leq \left(\frac{1}{n} \sum \lambda_i \right)^n = \left(\frac{\operatorname{Tr} \mathbf{M}'}{n} \right)^n = 1. \tag{2.29}$$

But $\det \mathbf{M}' = (\det \mathbf{M}) / \prod m_{i,i}$ which implies the wanted result.

2. We set $\mathbf{P} = \mathbf{M}^* \mathbf{M}$ which is hermitian positive-semidefinite, and whose diagonal entries are $p_{j,j} = \|v_j\|_2^2$. Applying the previous inequality to $\det \mathbf{P} = (\det \mathbf{M})^2$ we get the wanted result.

□

Let us now consider a general kernel $M : \Lambda^2 \to \mathbb{C}$. Then, we will use Hadamard's bound to show that $\det(1 + xM)_\Lambda$ is well-defined in the following two situations:

- M is Hermitian positive-semidefinite and trace-class,
- or M has exponential decay.

Definition 2.15. We say that M is *Hermitian* if $M(x, y) = \overline{M(y, x)}$ for all $x, y \in \Lambda$. If so, it is said *positive-semidefinite* if

$$\sum_{(x,y) \in \Lambda^2} \overline{\phi(x)} M(x, y) \phi(y) \geq 0 \tag{2.30}$$

for any $\phi : \Lambda \to \mathbb{C}$ with finite support. If so, it is said *trace-class* if

$$\operatorname{Tr} M := \sum_{x\in\Lambda} M(x,x) \tag{2.31}$$

is finite. (Note that $M(x,x) \geq 0$ since M is assumed positive-semidefinite.)

Remark 2.16. The definition of trace-class may be extended to non Hermitian operators but it is more subtle and we will not use it here.

Definition 2.17. Assume $\Lambda \subset \mathbb{Z}$. We say that M has *exponential decay* if there exist $C, c > 0$ such that

$$M(i,j) \leq Ce^{-c(|i|+|j|)}. \tag{2.32}$$

Proposition 2.18. *If M is Hermitian positive-semidefinite trace-class, or if M has exponential decay, then the Fredholm determinant* $\det(1 + xM)_\Lambda$ *is well-defined for any x, hence is an entire function of x.*

Proof. We start with the Hermitian case. Then, the submatrix M_U is Hermitian positive-semidefinite for all finite U, so we may use (2.26) to bound the modulus of the sum in the right-hand side of (2.25) by

$$\begin{aligned}\sum_{\substack{U\subset\Lambda\\ U\text{finite}}} |x|^{\#U} \prod_{u\in U} M(u,u) &= \sum_{n\geq 0} \frac{|x|^n}{n!} \sum_{\substack{(u_1,\ldots,u_n)\in\Lambda^n\\ \text{all distinct}}} M(u_1,u_1)\cdots M(u_n,u_n) \\ &\leq \sum_{n\geq 0} \frac{|x|^n}{n!} \left(\sum_{u\in\Lambda} M(u,u)\right)^n = e^{|x|\operatorname{Tr}(M)} < \infty.\end{aligned} \tag{2.33}$$

Note that there is an inequality sign in the second line since expanding $\left(\sum M(u,u)\right)^n$ yields a sum over n-tuples $(u_1,\ldots,u_n)$ of nonnecessarily distinct elements of Λ.

For the exponential decay, the proof is similar but we use now the general bound (2.27). See [1, p. 99] for details. $\square$

Remark 2.19. An immediate adaptation of the proof shows that $\det(I + M\phi)_\Lambda$ is well-defined for any bounded function $\phi : \Lambda \to \mathbb{C}$, which we may regard as a diagonal matrix so that $(M\phi)(x,y) := M(x,y)\phi(y)$.

In the following we will only consider the case of kernels with exponential decay: if M and N are two such kernels, so do $M + N$ and MN and xM for any complex x, and it may be seen that

$$\det(I + M)_\Lambda \det(I + N)_\Lambda = \det(I + M + N + MN)_\Lambda \tag{2.34}$$

as we would expect for ordinary determinants. There is however an interest in considering Hermitian kernels (possibly with exponential decay), for the following reason.

Proposition 2.20. *Let $(M^{(k)})_{k\geq 0}$ be a sequence of Hermitian positive-semidefinite trace-class kernels converging as $k \to \infty$ to a trace-class kernel M in the sense that we have both:*

- pointwise convergence*: for any x, y, $M^{(k)}(x, y) \to M(x, y)$,*
- trace convergence*:* $\operatorname{Tr} M^{(k)} \to \operatorname{Tr} M$*.*

Then, for any $x \in \mathbb{C}$, we have $\det(1 + xM^{(k)})_\Lambda \to \det(1 + xM)_\Lambda$ *and the convergence is uniform over bounded sets.*

There is an analogous statement in the case of exponential decay, with trace convergence replaced by some uniform control on the decay of the $M^{(k)}$'s. Arguably, the trace convergence is simpler to state, which is why we prefer it.

Proof of Proposition 2.20. For $\delta > 0$, let us consider a finite subset F whose complement F^c is such that

$$\operatorname{Tr} M_{F^c} < \delta. \tag{2.35}$$

By the pointwise convergence, for $k \to \infty$ we have the convergence of finite determinants

$$\det(I + xM^{(k)})_F \to \det(I + xM)_F. \tag{2.36}$$

But we claim that both sides are respectively close to the Fredholm determinants $\det(I + xM^{(k)})_\Lambda$ and $\det(I + xM)_\Lambda$. Indeed, by the trace convergence, there exists a k_0 such that $\operatorname{Tr} M^{(k)}_{F^c} < \delta$ for all $k \geq k_0$. Proceeding as in (2.33), we have for $k \geq k_0$

$$\begin{aligned}\left|\det(I + xM^{(k)})_\Lambda - \det(I + xM^{(k)})_F\right| &\leq \sum_{\substack{U \subset \Lambda \\ 1 \leq \#(U \cap F^c) < \infty}} |x|^{\#U} \prod_{u \in U} M^{(k)}(u, u) \\ &\leq \sum_{n \geq 0} \sum_{m \geq 1} \frac{|x|^{n+m}}{n!m!} \left(\operatorname{Tr} M^{(k)}_F\right)^n \left(\operatorname{Tr} M^{(k)}_{F^c}\right)^m \leq e^{|x|C}\left(e^{|x|\delta} - 1\right)\end{aligned} \tag{2.37}$$

where $C := \sup_{k \geq 0} \operatorname{Tr} M^{(k)}$, and clearly the same bound holds with $M^{(k)}$ replaced by M. By taking δ small enough we can make the error as small as we want. □

Remark 2.21. An immediate extension of the proof shows that $\det(I + M^{(k)}\phi^{(k)})_\Lambda \to \det(I + M\phi)_\Lambda$ for any uniformly bounded sequence of functions $\phi^{(k)} : \Lambda \to \mathbb{C}$ converging pointwise to ϕ.

Remark 2.22. If we replace the assumption that $\operatorname{Tr} M^{(k)} \to \operatorname{Tr} M$ by the weaker assumption that $\operatorname{Tr} M^{(k)}$ remains bounded, then the proposition becomes false. As a counterexample, take $\Lambda = \mathbb{N}$ and consider the diagonal matrix $M^{(k)}$ whose entry (k, k) is one, and all other entries are zeros.

We now give the analogues of the previous results on DPPs on finite spaces.

Proposition 2.23. *Let X be a DPP with correlation kernel K with exponential decay. Then, for any $B \subset \Lambda$, we have*

$$\mathbb{P}(X \cap B = \emptyset) = \det(I - K)_B \tag{2.38}$$

and

$$\mathbb{E}(z^{\xi(B)}) = \det(I + (z-1)K)_B. \tag{2.39}$$

More generally, if $\phi : \Lambda \to \mathbb{C}$ is bounded then

$$\mathbb{E}\left(\prod_{x \in X}(1 + \phi(x))\right) = \det(I + K\phi)_\Lambda \tag{2.40}$$

where $(K\phi)(x, y) := K(x, y)\phi(y)$.

Remark 2.24. It K has exponential decay then, from (2.39) for $B = \Lambda$, we see that $|X| = \xi(\Lambda)$ is almost surely finite. To obtain a DPP with infinitely many points we should consider kernels which do *not* have exponential decay (nor be trace-class). In the context of the Plancherel measure we will encounter a kernel that has exponential decay at $+\infty$ but not $-\infty$ (i.e. in Definition 2.17 we must remove the absolute values on i and j). The corresponding DPP is bounded above (i.e. $\xi(B) < \infty$ for any B bounded below) but has infinite many points nevertheless.

Proposition 2.25. *Assume that X has a configuration kernel L with exponential decay, i.e. for any $A \subset \Lambda$ finite, we have*

$$\mathbb{P}(X = A) = \frac{1}{Z}\det L_A \tag{2.41}$$

where $Z = \det(I + L)_\Lambda$ (in particular, X is finite almost surely). Then, X is a determinantal point process with correlation kernel

$$K = L(I + L)^{-1}. \tag{2.42}$$

2.5 DPPs on $\mathbb{R}$

We finally discuss briefly, and slightly informally, the case where Λ is an interval of $\mathbb{R}$. Now the correlation kernel should be viewed as an integral operator on $L^2(\Lambda)$:

$$Kf(x) = \int_\Lambda K(x, y)f(y)dy. \tag{2.43}$$

The Fredholm determinant of $I + xK$ is defined as

$$\det(I + xK)_{L^2(\Lambda)} := \sum_{n \geq 0} \frac{x^n}{n!} \int_{\Lambda^n} \det_{1 \leq i,j \leq n} K(x_i, x_j)dx_1 \cdots dx_n \tag{2.44}$$

provided that the rhs is absolutely convergent. It is easy to adapt the previous arguments that involve Hadamard's bound to see that this is the case when:

- K is hermitian positive-semidefinite and trace-class (i.e. $\operatorname{Tr} K := \int_\Lambda K(x, x)dx$ is finite),
- or K has exponential decay, with an immediate adaptation of Definition 2.17.

In this setting, Proposition 2.23 remains valid. Note that the Airy kernel $\mathbf{A}(x,y)$ is Hermitian positive-semidefinite and has exponential decay at $+\infty$, so that we may deduce that

$$F_2(t) := \det(I - \mathbf{A})_{L^2(t,\infty)} \tag{2.45}$$

is indeed equal to $\mathbb{P}(X_{\mathrm{Airy}} \cap (t,\infty) = \emptyset)$.

For later use we also state a proposition about the convergence of Fredholm determinants:

Proposition 2.26. *Let $(K_n)_{n\geq 0}$ and K be hermitian positive-semidefinite trace-class kernels on Λ, and assume that, as $n \to \infty$,*

- *$K_n(x,y)$ converges pointwise to $K(x,y)$,*
- *$\operatorname{Tr} K_n := \int_\Lambda K_n(x,x)dx$ converges to $\operatorname{Tr} K$.*

Then, for any $x \in \mathbb{C}$, we have $\det(I + xK_n)_\Lambda \to \det(I + xK)_\Lambda$ and the convergence is uniform over bounded sets.

Proof. If $x \mapsto \sup_n K_n(x,x)$ is an integrable function on Λ, then we may simply use the definition (2.44) for the Fredholm determinant, and use Hadamard's bound to invoke dominated convergence.

In the general case, the argument is similar to that in the proof of Proposition 2.20. Let us consider the subsets $F_m := \{x \in \Lambda, \sup_n K_n(x,x) \leq me^{-|x|}\}$ for m a nonnegative integer (here $e^{-|x|}$ can be replaced by any positive integrable function on Λ). By the previous argument, we have $\lim_{n\to\infty} \det(I + xK_n)_{F_m} = \det(I + xK)_{F_m}$. Furthermore, the assumptions imply that $\Lambda \setminus \bigcup_{m=1}^{\infty} F_m$ has measure zero. Fix $\delta > 0$ and take m such that $\int_{\Lambda\setminus F_m} K(x,x)dx < \delta$. By the trace convergence it follows that $\int_{\Lambda\setminus F_m} K_n(x,x)dx < \delta$ for n large enough. Then, proceeding as in (2.37), we obtain the same bound

$$|\det(I + xK_n)_\Lambda - \det(I + xK_n)_{F_m}| \leq e^{|x|C}\left(e^{|x|\delta} - 1\right) \tag{2.46}$$

which remains valid with K_n replaced by K, and the same conclusion follows. $\square$

Corollary 2.27 (Discrete to continuum convergence)**.** *Let $(K_n)_{n\geq 0}$ be a sequence of hermitian positive-semidefinite trace class kernels on $\mathbb{N}$, and assume that there exists a positive-semidefinite trace class kernel K on $\mathbb{R}_+$ and some $\alpha > 0$ such that, as $n \to \infty$,*

- *$n^\alpha K_n(\lfloor xn^\alpha \rfloor, \lfloor yn^\alpha \rfloor)$ converges pointwise to $K(x,y)$,*
- *$\operatorname{Tr} K_n := \sum_{k=0}^{\infty} K_n(k,k)$ converges to $\operatorname{Tr} K = \int_0^\infty K(x,x)dx$.*

Then, we have $\det(I + xK_n)_{\mathbb{N}} \to \det(I + xK)_{\mathbb{R}_+}$.

Proof. Set $\tilde{K}_n(x,y) := n^\alpha K_n(\lfloor xn^\alpha \rfloor, \lfloor yn^\alpha \rfloor)$, observe that that $\det(I + xK_n)_{\mathbb{N}} = \det(I + x\tilde{K}_n)_{\mathbb{R}_+}$, and apply the previous proposition. $\square$

Exercise 2.28. Adapt the statements for the convergence of Fredholm determinants of the form $\det(I + K_n\phi_n)$, with suitable hypotheses on ϕ_n.

3 Poissonized Plancherel measure and fermions

We now want to use the DPP technology to establish Theorems 1.1 and 2.5. The global strategy is the following:

1. show that there is a certain discrete DPP associated with the Plancherel measure (PM), in fact we will have to consider the *poissonized* PM,
2. prove that, in the relevant scaling limit, the correlation kernel of that DPP converges to the Airy kernel, and that the hypotheses of Corollary 2.27 are satisfied,
3. deduce the edge statistics for the PPM, and apply a “depoissonization” argument to return to the unpoissonized PM.

In this section we perform the first step. Our approach is different from that followed in Romik's book. His approach is based on the hook-length formula: by some manipulations —see [1, Section 2.3]— he rewrites the probability distribution in a form amenable to Proposition 2.25. Then he computes the associated correlation kernel —see [1, Section 2.5]— before doing asymptotics. Our approach, based on *fermions*[4], is simpler in our opinion, and does not require the knowledge of the hook-length formula (in fact, we obtain a proof of this formula as a byproduct).

3.1 Main result of this section

For $\theta \geq 0$ we define the *poissonized Plancherel measure* of parameter θ as the probability measure on the set $\mathcal{P}$ of partitions given by

$$P_\theta(\lambda) = \frac{\theta^{|\lambda|} d_\lambda^2}{(|\lambda|!)^2} e^{-\theta}. \tag{3.1}$$

It corresponds to a mixture of the Plancherel measures of fixed size. We denote by $\lambda^{\langle\theta\rangle}$ a random partition distributed according to this measure. To $\lambda^{\langle\theta\rangle}$ we will associate a point process as follows.

Let $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ denote the set of *half-integers*[5]. For λ a partition we set

$$S(\lambda) := \left\{ \lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \dots \right\} \subset \mathbb{Z}'. \tag{3.2}$$

We shall see that $S(\lambda^{\langle\theta\rangle})$ is a DPP, whose kernel is defined in terms of the Bessel functions.

For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, the *Bessel function* of order n is given by

$$J_n(z) := \sum_{m=\max(-n,0)}^{\infty} \frac{(-1)^m (z/2)^{2m+n}}{m!(m+n)!} \tag{3.3}$$

[4]To our knowledge, this approach was first developed for the Plancherel measure and generalizations in [8] and [9], but it is a homecoming of sorts since, as mentionned previously, DPP were precisely introduced in connection with fermions [3].

[5]Working with half-integers rather than integers is merely a convention, which makes some formulas more symmetric.

(it is also possible to define the function for noninteger order but we will not need it here). We have the following properties

$$J_{-n}(z) = (-1)^n J_n(z), \tag{3.4}$$

$$J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z} J_n(z). \tag{3.5}$$

The *discrete Bessel kernel* J_θ of parameter θ is defined in terms of the Bessel functions at $z = 2\sqrt{\theta}$. To lighten notation we will set $J_n := J_n(2\sqrt{\theta})$ and set

$$\mathbf{J}_\theta(s,t) := \sqrt{\theta}\frac{J_{s-1/2}J_{t+1/2} - J_{s+1/2}J_{t-1/2}}{s-t} \tag{3.6}$$

$$= \sum_{\ell \in \mathbb{Z}'_{>0}} J_{s+\ell} J_{t+\ell} \tag{3.7}$$

where $s, t \in \mathbb{Z}'$. The first form is a priori defined only for $s \neq t$ but can be extended to $s = t$ by L'Hôpital's rule[6]. The equivalence with the second form follows from (3.5), as we may check that $\mathbf{J}_\theta(s,t) - \mathbf{J}_\theta(s+1,t+1) = J_{s+1/2}J_{t+1/2}$ – note the similarity with the two representations of the Airy kernel.

We may then state the main theorem of this section, which is due to Borodin, Okounkov and Olshanski [6], and independently Johansson [7].

Theorem 3.1. *The point process $S(\lambda^{\langle\theta\rangle})$ is determinantal and has correlation kernel $\mathbf{J}_\theta$.*

We now prove this theorem using "fermions".

3.2 Fermions: basic definitions and properties

Fermionic configurations. We say that that $S \subset \mathbb{Z}'$ is a *fermionic configuration* (or *configuration* for short) if S contains finitely many positive elements, and its complement contains finitely many negative elements. For instance, $S(\lambda)$ as defined above is a fermionic configuration. We call $S(\emptyset) = \mathbb{Z}'_{<0}$ the *vacuum configuration* (where $\emptyset$ denotes the empty partition). Here $\mathbb{Z}'_{<k}$ (resp. $\mathbb{Z}'_{>k}$) denotes of course the set of all half-integers strictly smaller (resp. larger) than k. We denote by $\mathcal{S}$ the set of all fermionic configurations.

For S a configuration, we define its *charge* $c(S) \in \mathbb{Z}$ by

$$c(S) := \#(S \cap \mathbb{Z}'_{>0}) - \#(\mathbb{Z}'_{<0} \setminus S). \tag{3.8}$$

The configurations $S(\lambda)$ associated with partitions all have charge 0. In fact, we may obtain all of them in this way. More generally, the mapping $(\lambda, c) \mapsto S(\lambda) + c$ is a bijection between $\mathcal{P} \times \mathbb{Z}$ and $\mathcal{S}$. This correspondence is illustrated on Figure 2.

If S is a configuration then $S' := \mathbb{Z}' \setminus (-S)$ is also a configuration, with opposite charge[7]. For $S = S(\lambda)$, we have $S' = S(\lambda')$, where λ' is the conjugate partition of λ.

Intuitively, we think of S as a configuration of particles living on $\mathbb{Z}'$: if $s \in S$ then there is a particle at position s.

[6]Upon extending the definition of J_n to noninteger n, and considering its derivative with respect to n.
[7]The convention of working with half-integers is useful here.

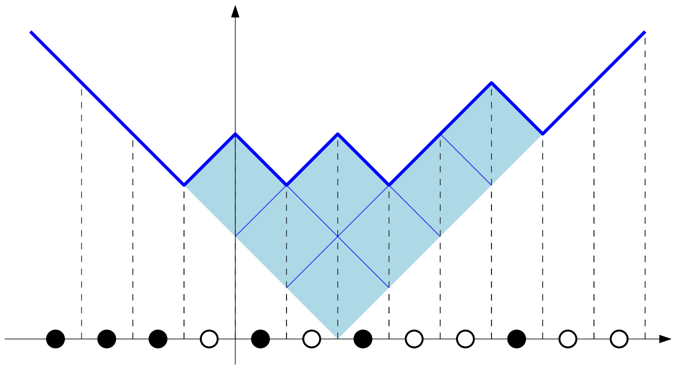

Figure 2: Correspondence between partitions and fermionic configurations: to the partition $\lambda = (4,2,1)$ and the charge $c = 2$ we associate the configuration $S = \{11/2, 5/2, 1/2, -3/2, -5/2, -7/2, \ldots\}$.

Remark 3.2. In physics, s should not be thought of as a spatial position but as an "energy level". The fact that all sites are occupied for $s \ll 0$ corresponds to the concept of "Fermi sea": the vacuum is the configuration with lowest energy, which is obtained by filling all negative energy levels and leaving empty all positive energy levels; a general configuration corresponds to a finite-energy excitation of the vacuum. These concepts play an important role in high-energy and condensed matter theory.

Fermionic Fock space. We then define the *fermionic Fock space* $\mathcal{F}$ as the (infinite-dimensional) vector space whose basis is indexed by fermionic configurations. We denote this basis as $(v_S)_{S\in\mathcal{S}}$ and think of v_S as the column vector indexed by $\mathcal{S}$, with a 1 at position S and 0's elsewhere. This "naive" point of view is sufficient for our discussion. We denote by $v_S^* \in \mathcal{F}^*$ the dual "row vector", so that

$$v_S^* v_{S'} = \delta_{S,S'}. \tag{3.9}$$

Operators on $\mathcal{F}$ will be thought of as infinite matrices with row and columns indexed by $\mathcal{S}$, and product of operators are defined by expanding the matrix products as we would do for finite-dimensional matrices.

Fermionic operators. For $k \in Z'$ we define the *fermion creation operator* ψ_k as the operator acting on $\mathcal{F}$ via

$$\psi_k v_S = \begin{cases} 0 & \text{if } k \in S, \\ (-1)^{\#(S\cap\mathbb{Z}'_{>k})} v_{S\cup\{k\}} & \text{if } k \notin S, \end{cases} \tag{3.10}$$

and the *fermion annihilation operator* ψ_k^* by

$$\psi_k^* v_S = \begin{cases} (-1)^{\#(S\cap\mathbb{Z}'_{>k})} v_{S\setminus\{k\}} & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases} \tag{3.11}$$

Clearly ψ_k^* is the adjoint (transpose) of ψ_k. The role of the sign is to have the following.

Proposition 3.3 (Canonical anticommutation relations)**.** *For $k, \ell \in \mathbb{Z}'$ we have*

$$\psi_k \psi_\ell + \psi_\ell \psi_k = 0, \qquad \psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}, \qquad \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0. \tag{3.12}$$

Proof. We check these identities in the basis of $\mathcal{F}$. For instance we find that $\psi_k \psi_\ell v_S$ is zero unless $k, \ell \notin S$ and $k \neq \ell$, in which case it is equal to a sign times $v_{S \cup \{k,\ell\}}$. But the same is true for $\psi_\ell \psi_k v_S$ except that we get the opposite sign (as seen easily), which establishes the first identity. The others are left as exercises. □

We call

$$N_k := \psi_k \psi_k^* \tag{3.13}$$

a *fermion number operator.* It is diagonal in the basis (v_S), namely $N_k v_S = v_S$ if $k \in S$ and $= 0$ otherwise.

Box operators. We define the *box creation operator* α by

$$\alpha := \sum_{k \in \mathbb{Z}'} \psi_{k+1} \psi_k^*. \tag{3.14}$$

This operator makes sense because, if we act on v_S, only finitely many terms contribute (namely all the k's for which $k \in S$ but $k+1 \notin S$, which are in finite number).

In the partition picture[8], α attempts to "add a box" in all possible ways:

$$\alpha v_\lambda = \sum_{\mu : \lambda \nearrow \mu} v_\mu \tag{3.15}$$

where we recall that the notation $\lambda \nearrow \mu$ means that the Young diagram of μ is obtain by adding one box to that of λ, and where we use the shorthand notation

$$v_\lambda := v_{S(\lambda)}. \tag{3.16}$$

Note that there are no signs involved in (3.15). Dually we define the *box annihilation operator* α^* by

$$\alpha^* := \sum_{k \in \mathbb{Z}'} \psi_{k-1} \psi_k^* \tag{3.17}$$

which is the adjoint (transpose) of α.

Proposition 3.4. *We have the commutation relations*

$$[\alpha, \psi_\ell] = \psi_{\ell+1}, \quad [\alpha^*, \psi_\ell] = \psi_{\ell-1}, \quad [\alpha, \psi_\ell^*] = -\psi_{\ell-1}^*, \quad [\alpha^*, \psi_\ell^*] = -\psi_{\ell+1}^* \tag{3.18}$$

and

$$[\alpha^*, \alpha] = 1. \tag{3.19}$$

[8]Note that α is a charge-preserving operator.

Proof. The first identities are easily obtained as consequences of the canonical anticommutation relations: we have

$$\psi_{k\pm1}\psi_k^*\psi_\ell = \psi_{k\pm1}\left(\delta_{k,\ell} - \psi_\ell\psi_k^*\right) = \delta_{k,\ell}\psi_{k\pm1} + \psi_\ell\psi_{k\pm1}\psi_k^* \tag{3.20}$$

which implies $[\alpha, \psi_\ell] = \psi_{\ell+1}$ and $[\alpha^*, \psi_\ell] = \psi_{\ell-1}$ by summing over k. The other relations are obtained by taking the transpose.

The relation $[\alpha^*, \alpha] = 1$ can be derived in a same way, but the computation is more subtle (exercise). It is much easier to work in the partition picture: we have

$$v_\lambda^*\alpha^*\alpha v_\nu = \begin{cases} \#(\text{outer corners of } \lambda) & \text{if } \lambda = \nu, \\ 1 & \text{if } \lambda \neq \nu \text{ and } \lambda \nearrow \mu,\ \nu \nearrow \mu \text{ for some } \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{3.21}$$

while

$$v_\lambda^*\alpha\alpha^* v_\nu = \begin{cases} \#(\text{inner corners of } \lambda) & \text{if } \lambda = \nu, \\ 1 & \text{if } \lambda \neq \nu \text{ and } \kappa \nearrow \lambda,\ \kappa \nearrow \nu \text{ for some } \kappa. \\ 0 & \text{otherwise,} \end{cases} \tag{3.22}$$

Observing that

- any partition has one more outer corner than inner corner,
- the conditions for the second cases of (3.21) and (3.22) are equivalent (μ being the "union" of λ and ν and κ their "intersection"),

we deduce that $v_\lambda^*[\alpha^*, \alpha]v_\nu = \delta_{\lambda,\mu}$ as wanted. □

3.3 Fermionic expression for the correlation functions of $S(\lambda^{\langle\theta\rangle})$

We may now make the connection with the Plancherel measure:

Proposition 3.5. *We have*

$$\alpha^n v_\emptyset = \sum_{\lambda\vdash n} d_\lambda v_\lambda. \tag{3.23}$$

Proof. By iterating (3.15) we get that

$$\alpha^n v_\emptyset = \sum_{\emptyset\nearrow\lambda^{(1)}\nearrow\cdots\nearrow\lambda^{(n)}} v_{\lambda^{(n)}} \tag{3.24}$$

and we observe that the number of sequences $\emptyset \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)}$ ending with fixed $\lambda^{(n)} = \lambda$ is precisely the number d_λ of standard Young tableaux of shape λ. □

As a consequence, we may write

$$e^{x\alpha}v_\emptyset = \sum_{\lambda\in\mathcal{P}} \frac{x^{|\lambda|}d_\lambda}{|\lambda|!}v_\lambda \tag{3.25}$$

(where we think of x as a formal variable for now) and dually

$$v_\emptyset^* e^{x\alpha^*} = \sum_{\lambda\in\mathcal{P}} \frac{x^{|\lambda|}d_\lambda}{|\lambda|!}v_\lambda^*. \tag{3.26}$$

Taking the product of (3.26) and (3.25) and recalling that $v_\lambda^* v_\mu = \delta_{\lambda,\mu}$, we find that

$$v_\emptyset^* e^{x\alpha^*} e^{x\alpha} v_\emptyset = \sum_{\lambda\in\mathcal{P}} \frac{x^{2|\lambda|}d_\lambda^2}{(|\lambda|!)^2} = e^{x^2} \tag{3.27}$$

which corresponds to the normalization of the poissonized Plancherel measure of parameter

$$\theta = x^2. \tag{3.28}$$

More interestingly, we may insert some fermion number operators $N_{s_1}\cdots N_{s_n}$, as defined in (3.13), between (3.26) and (3.25). Since N_k is diagonal in the fermionic basis, with $N_k v_S = \mathbb{1}_{k\in S} v_S$, we deduce that

$$\frac{v_\emptyset^* e^{x\alpha^*} N_{s_1}\cdots N_{s_n} e^{x\alpha} v_\emptyset}{e^{x^2}} = \mathbb{P}\left(\{s_1,\ldots,s_n\}\subset S(\lambda^{\langle\theta\rangle})\right). \tag{3.29}$$

In other words, we have found a fermionic expression for the correlation functions of $S(\lambda^{\langle\theta\rangle})$.

It remains to prove that the fermionic expression evaluates to a determinant of the discrete Bessel kernel. The first step consists in exploiting the commutation relations of Proposition 3.4. For this we need to pass to their "exponentiated form":

Proposition 3.6. *We have*

$$e^{x\alpha^*}\psi_\ell = \sum_{m\geq 0}\frac{x^m}{m!}\psi_{\ell-m}e^{x\alpha^*}, \qquad \psi_\ell e^{x\alpha} = \sum_{m\geq 0}\frac{(-x)^m}{m!}e^{x\alpha}\psi_{\ell+m} \tag{3.30}$$

and dually

$$\psi_\ell^* e^{x\alpha} = \sum_{m\geq 0}\frac{x^m}{m!}e^{x\alpha}\psi_{\ell-m}^*, \qquad e^{x\alpha^*}\psi_\ell^* = \sum_{m\geq 0}\frac{(-x)^m}{m!}\psi_{\ell+m}^* e^{x\alpha^*}, \tag{3.31}$$

while

$$e^{x\alpha^*}e^{x\alpha} = e^{x^2}e^{x\alpha}e^{x\alpha^*}. \tag{3.32}$$

Proof. Such calculations are routine in the theory of Lie algebras. A self-contained proof of the first identity in (3.30) is as follows: we want to prove that

$$\psi_\ell = \sum_{m\geq 0} \frac{x^m}{m!} e^{-x\alpha^*} \psi_{\ell-m} e^{x\alpha^*}. \tag{3.33}$$

If we differentiate the right-hand side with respect to x, we get

$$\sum_{m\geq 0} \frac{x^{m-1}}{(m-1)!} e^{-x\alpha^*} \psi_{\ell-m} e^{x\alpha^*} + \sum_{m\geq 0} \frac{x^m}{m!} e^{-x\alpha^*} [\psi_{\ell-m}, \alpha^*] e^{x\alpha^*} = 0 \tag{3.34}$$

since $[\psi_{\ell-m}, \alpha^*] = -\psi_{\ell-m-1}$ and we may reindex the first sum. Therefore the right-hand side of (3.33) does not depend on x, and hence equal to ψ_ℓ as seen by taking $x = 0$. (Note that these manipulations make sense with x a formal variable.)

The identity (3.32) may be proved similarly, but we may also recognize it as an instance of the Baker-Campbell-Hausdorff formula, which states that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \tag{3.35}$$

whenever X and Y commute with $[X, Y]$. □

We may now apply these commutations formulas to the fermionic expression (3.29): we move the operator $e^{x\alpha^*}$ to the right, by inserting factors $e^{-x\alpha^*} e^{x\alpha^*}$ between any two factors in the product of operators, which yields

$$\frac{v_\emptyset^* e^{x\alpha^*} N_{s_1} \cdots N_{s_n} e^{x\alpha} v_\emptyset}{e^{x^2}} = \frac{v_\emptyset^* e^{x\alpha^*} N_{s_1} e^{-x\alpha^*} e^{x\alpha^*} \cdots N_{s_n} e^{-x\alpha^*} e^{x\alpha^*} e^{x\alpha} e^{-x\alpha^*} e^{x\alpha^*} v_\emptyset}{e^{x^2}} \tag{3.36}$$

and, using the commutation relations and the relation $e^{x\alpha^*} v_\emptyset = v_\emptyset$ (itself a consequence of $\alpha^* v_\emptyset = 0$), we find

$$\frac{v_\emptyset^* e^{x\alpha^*} N_{s_1} \cdots N_{s_n} e^{x\alpha} v_\emptyset}{e^{x^2}} = v_\emptyset^* \tilde{N}_{s_1} \cdots \tilde{N}_{s_n} e^{x\alpha} v_\emptyset \tag{3.37}$$

where $\tilde{N}_k := e^{x\alpha^*} N_k e^{-x\alpha^*}$. Next we move the operator $e^{x\alpha}$ to the left to obtain

$$v_\emptyset^* \tilde{N}_{s_1} \cdots \tilde{N}_{s_n} e^{x\alpha} v_\emptyset = v_\emptyset^* \hat{N}_{s_1} \cdots \hat{N}_{s_n} v_\emptyset \tag{3.38}$$

where $\hat{N}_k := e^{-x\alpha} \tilde{N}_k e^{x\alpha}$. From Proposition 3.6 we get

$$\hat{N}_k = \hat{\psi}_k \hat{\psi}_k^* \tag{3.39}$$

where

$$\begin{aligned} \hat{\psi}_k &= e^{-x\alpha} e^{x\alpha^*} \psi_k e^{-x\alpha^*} e^{x\alpha} \\ &= \sum_{m,m'\geq 0} \frac{x^m (-x)^{m'}}{m! m'!} \psi_{k-m+m'} \\ &= \sum_{\ell\in\mathbb{Z}} J_\ell(2x) \psi_{k-\ell}. \end{aligned} \tag{3.40}$$

We end up with the expression

$$\mathbb{P}\left(\{s_1,\dots,s_n\}\subset S(\lambda^{\langle\theta\rangle})\right)=\langle\hat\psi_{s_1}\hat\psi^*_{s_1}\cdots\hat\psi_{s_n}\hat\psi^*_{s_n}\rangle \tag{3.41}$$

for the n-point correlation function of $S(\lambda^{\langle\theta\rangle})$, where

$$\langle\mathcal{O}\rangle := v^*_\emptyset \mathcal{O} v_\emptyset \tag{3.42}$$

is the *vacuum expectation value* of the operator $\mathcal{O}$.

3.4 Wick's lemma

The last ingredient we need is Wick's lemma which we may state in the following form:

Lemma 3.7. *Let $\varphi_1,\varphi_3,\dots,\varphi_{2n-1}$ be arbitrary linear combinations of the ψ_k's, and $\varphi^*_2,\varphi^*_4,\dots,\varphi^*_{2n}$ arbitrary linear combinations of the ψ^*_k's. Then, we have*

$$\langle\varphi_1\varphi^*_2\varphi_3\varphi^*_4\cdots\varphi_{2n-1}\varphi^*_{2n}\rangle=\det M \tag{3.43}$$

where M is the $n\times n$ matrix whose entries read

$$M_{i,j}=\begin{cases}\langle\varphi_{2i-1}\varphi^*_{2j}\rangle & \text{if } i\le j,\\ -\langle\varphi^*_{2j}\varphi_{2i-1}\rangle & \text{if } i>j.\end{cases} \tag{3.44}$$

A mnemotechnic rule is that φ_{2i-1} and φ^*_{2j} should appear in $M_{i,j}$ in the same order as in the full product — this is the so-called time-ordered product.

Example 3.8. For $n=2$, we find

$$\langle\varphi_1\varphi^*_2\varphi_3\varphi^*_4\rangle=\langle\varphi_1\varphi^*_2\rangle\langle\varphi_3\varphi^*_4\rangle+\langle\varphi_1\varphi^*_4\rangle\langle\varphi^*_2\varphi_3\rangle. \tag{3.45}$$

Proof. Wick's lemma is a consequence of the canonical anticommutation relations (Proposition 3.3) and may be checked by induction. For $n=1$ there is nothing to prove.

Let us treat the case $n=2$ and leave the case of general n as an exercise[9]. By multilinearity, it suffices to consider the case where $\varphi_i=\psi_{k_i}$ for some k_i. If $k_1>0$, the relation (3.45) is trivially true since all terms are zero, since $v^*_\emptyset\psi_{k_1}=0$. If $k_1<0$ then we move ψ_{k_1} to the right using the canonical anticommutation relations: writing

$$\psi_{k_1}\psi^*_{k_2}\psi_{k_3}\psi^*_{k_4}=\delta_{k_1,k_2}\psi_{k_3}\psi^*_{k_4}+\psi^*_{k_2}\psi_{k_3}\delta_{k_1,k_4}+\psi^*_{k_2}\psi_{k_3}\psi^*_{k_4}\psi_{k_1} \tag{3.46}$$

which itself is a particular case of the "telescopic" identity

$$\begin{aligned}A_1A_2\cdots A_n=&\sum_{i=2}^n(-1)^iA_2\cdots A_{i-1}(A_1A_i+A_iA_1)A_{i+1}\cdots A_n\\&+(-1)^{n+1}A_2\cdots A_nA_1,\end{aligned} \tag{3.47}$$

we obtain by taking the vacuum expectation value the wanted relation (3.45) (noting that $\langle\psi_{k_1}\psi^*_\ell\rangle=\delta_{k_1,\ell}\mathbb{1}_{k_1<0}$). □

[9] See [10, Appendix B] for the solution.

Remark 3.9. Wick's lemma remains true if we replace the vacuum expectation value (3.42) by $v_S^* \mathcal{O} v_S$ for any fermionic configuration S. This is clear from the above "proof": replace "If $k_1 > 0$" (resp. "If $k_1 < 0$") by "If $k_1 \notin S$" (resp. "If $k_1 \in S$").

Exercise 3.10. Prove the hook-length formula using Wick's lemma. The starting point is the relation (3.26), which implies that

$$\frac{d_\lambda}{|\lambda|!} = v_\emptyset^* e^{\alpha^*} v_\lambda. \tag{3.48}$$

Write then that

$$v_\lambda = \psi_{\lambda_1 - 1/2} \psi_{\lambda_2 - 3/2} \cdots \psi_{\lambda_\ell - \ell + 1/2} v_{\mathbb{Z}'_{<-\ell}} \tag{3.49}$$

for ℓ at least the length of λ, and take the dual of this relation with $\lambda = \emptyset$. Use Proposition 3.6 to eliminate e^{α^*} from (3.48), then apply Wick's lemma (using Remark 3.9). We obtain the hook-length formula in the form given in [1, Lemma 2.4].

3.5 Conclusion of the proof of Theorem 3.1

Applying Wick's lemma to the expression (3.41) for the correlation function, we find that

$$\mathbb{P}\left(\{s_1, \ldots, s_n\} \subset S(\lambda^{\langle\theta\rangle})\right) = \det_{1 \le i,j \le n} K(s_i, s_j) \tag{3.50}$$

where

$$K(s,t) := \langle \hat{\psi}_s \hat{\psi}_t^* \rangle. \tag{3.51}$$

(Here we assume the s_i's to be distinct, and the canonical anticommutation relations imply that $\hat{\psi}_s \hat{\psi}_t^* = -\hat{\psi}_t^* \hat{\psi}_s$ for $s \neq t$.)

This shows that $S(\lambda^{\langle\theta\rangle})$ is determinantal, and we conclude the proof by noting that, from (3.40), we have

$$\begin{aligned} K(s,t) &= \sum_{\ell \in \mathbb{Z}} \sum_{\ell' \in \mathbb{Z}} J_\ell(2x) J_{\ell'}(2x) \langle \psi_{s-\ell} \psi_{s-\ell'} \rangle \\ &= \sum_{\ell \in \mathbb{Z}} \sum_{\ell' \in \mathbb{Z}} J_\ell(2x) J_{\ell'}(2x) \sum_{n \in \mathbb{Z}'_{>0}} \delta_{s-\ell,n} \delta_{t-\ell',n} \\ &= \mathbf{J}_\theta(s,t). \end{aligned} \tag{3.52}$$

(In all rigor we have established (3.50) and (3.52) as identities between power series in x. But we may now take x to be any nonnegative real value, since both sides of (3.50) are entire functions of x.) ☐

4 Asymptotics

We now turn to the study of the asymptotics of the discrete Bessel kernel $\mathbf{J}_\theta$, which in turn imply those of the correlation functions.

4.1 Informal discussion of the relevant regimes

Let us first motivate which regimes we are interested in. The size $|\lambda^{\langle\theta\rangle}|$ is a Poisson random variable of parameter θ, therefore $|\lambda^{\langle\theta\rangle}|/\theta \to 1$ as $\theta \to \infty$ in probability and expectation. Therefore, the natural "length scale" for the Young diagram of $\lambda^{\langle\theta\rangle}$ is $\sqrt{\theta} = x$. Since s and t correspond to "lengths" (they correspond to particle positions, measured on the horizontal axis in Figure 2), it is thus natural to consider the limit where

$$\theta = x^2 \to \infty, \qquad \frac{s}{x}, \frac{t}{x} \to A \text{ fixed.} \tag{4.1}$$

We could also the consider the case where s/x and t/x tend to different limits, but it may be seen (adapting the forthcoming computations) that $\mathbf{J}_\theta(s,t)$ tends to 0 in such regime, in other words the correlation length is $o(x)$.

In fact, analyzing the one-point function

$$\rho_1(s) = \mathbb{P}(s \in S(\lambda^{\langle\theta\rangle})) = \mathbf{J}_\theta(s,s) \tag{4.2}$$

is already instructive. In view of the limit shape result established in the first part of the course, we expect to have in the regime (4.1)

$$\rho_1(s) \to \rho(A) = \begin{cases} 0 & \text{if } A \geq 2, \\ 1 & \text{if } A \leq -2, \\ \frac{1}{\pi} \arccos \frac{A}{2} & \text{if } -2 < A < 2. \end{cases} \tag{4.3}$$

Indeed, $\rho_1(s)$ is the probability to find a particle at position s, and should converge to a density $\rho(A)$ depending continuously on A, that is related to the derivate of the limit shape Ω which we computed in the first part of the course by

$$\frac{d}{dA}\Omega(A\sqrt{2}) = 1 - 2\rho(A). \tag{4.4}$$

Here the factor $\sqrt{2}$ arises just because of the slightly different choice of coordinates made in [1, Chapter 1]. We will confirm the limit (4.3) by a rigorous computation.

Beyond the one-point function for which $s = t$, we are also interested in limits for the higher-order correlations, which requires to understand the general relevant scaling for the difference $s-t$, corresponding to the *correlation length.* Here there are two interesting situations:

- *bulk statistics*: $-2 < A < 2$,
- *edge statistics*: $A = \pm 2$.

(In the "frozen" regions $|A| > 2$, nothing much of interest happens.)

In the bulk, the density $\rho(A)$ is nonzero therefore the typical distance between two particles is finite. Therefore we should take $s-t$ to be finite (fixed) to obtain a nontrivial limit for the correlation kernel.

Close to the edge, it is possible to identify the order of magnitude of the correlation length via the following heuristic argument. The expected number of particles between position Ax and ∞ is

$$\sum_{s>Ax} \rho_1(s) \sim x \int_A^\infty \rho(A) dA. \tag{4.5}$$

For $A \to 2^-$, $\rho(A)$ vanishes as $\sqrt{2-A}$ therefore its integral behaves as $(2-A)^{3/2}$. The rightmost particle is expected to be found close to a A such that (4.5) is of order 1, i.e. for $A = 2 - \Theta(x^{-2/3})$, i.e. at a position $2x - \Theta(x^{1/3})$. Note that this scaling is consistent with Theorem 1.1, upon identifying $x = \sqrt{\theta} \sim \sqrt{n}$. Similarly the second, third, and so on, rightmost particles should be at distance $\Theta(x^{1/3})$ of $2x$. Therefore, for the edge statistics we should scale

$$s = \lfloor 2x + \sigma x^{1/3} \rfloor + \frac{1}{2}, \qquad t = \lfloor 2x + \tau x^{1/3} \rfloor + \frac{1}{2} \tag{4.6}$$

where σ, τ are two fixed real numbers, and rescale the density/correlation kernel by $x^{1/3}$ to obtain a nontrivial limit.

4.2 Contour integral representation of the discrete Bessel kernel

The (edge) analysis done in Romik's book is based on the sum representation (3.7) for $\mathbf{J}_\theta$, and the so-called Nicholson's approximation for the Bessel function, see [1, Section 2.7]. Here we present another approach based on contour integral representations, that also applies to the study of bulk statistics (and in fact, allows to *derive* Nicholson's approximation as well).

Our starting point is the expansion

$$e^{x(z-z^{-1})} = \sum_{m,m' \geq 0} \frac{x^m (-x)^{m'}}{m! m'!} z^{m-m'} = \sum_{\ell \in \mathbb{Z}} J_\ell(2x) z^\ell \tag{4.7}$$

which should be understood as the *Laurent series expansion* of the function $z \mapsto e^{x(z-z^{-1})}$, that is analytic in $\mathbb{C} \setminus \{0\}$ so the sum is convergent for any nonzero z. Notice the similarity with (3.40). As a consequence, the Bessel function $J_\ell(2x)$ is given by the contour integral

$$J_\ell(2x) = \frac{1}{2i\pi} \oint \frac{e^{x(z-z^{-1})}}{z^{\ell+1}} dz \tag{4.8}$$

where the integration contour must enclose 0 counterclockwise (for instance, the unit circle does the job). We note that $J_\ell(2x)$ decays superexponentially for $\ell \to \pm\infty$, i.e. it is $O(r^{|\ell|})$ for any $r > 0$ (take the contour as the circle $|z| = r^{\mp 1}$).

We look for a similar representation for the Bessel kernel $\mathbf{J}_\theta(s,t)$. For this we write

$$\begin{aligned}\sum_{s,t\in\mathbb{Z}'} \mathbf{J}_\theta(s,t) z^{s-1/2} w^{-t-1/2} &= \sum_{s,t\in\mathbb{Z}'} \sum_{\ell\in\mathbb{Z}'_{>0}} J_{s+\ell}(2x) J_{t+\ell}(2x) z^{s-1/2} w^{-t-1/2} \\ &= e^{x(z-z^{-1})} e^{x(w^{-1}-w)} \sum_{\ell\in\mathbb{Z}'_{>0}} \frac{w^{\ell-1/2}}{z^{\ell+1/2}} \\ &= \frac{e^{x(z-z^{-1})}}{e^{x(w-w^{-1})}} \cdot \frac{1}{z-w} =: \mathcal{J}_\theta(z,w)\end{aligned} \tag{4.9}$$

where we note that the geometric sum on the second line converges for $|w| < |z|$. The choice for the exponents of z and w is somewhat arbitrary, but convenient for the geometric sum to be "nice".

The above equality makes sense as a *bivariate Laurent series expansion* for the function $\mathcal{J}_\theta(z,w)$ in the annulus $0 < |w| < |z|$. Note that the function is manifestly analytic in the whole domain $\{(z,w) \in (\mathbb{C}\setminus\{0\})^2, z \neq w\}$, and therefore admits another expansion in the annulus $0 < |z| < |w|$, which would be different. The analogue of (4.8) is

$$\mathbf{J}_\theta(s,t) = \frac{1}{(2i\pi)^2} \oint\!\!\oint \frac{e^{x(z-z^{-1})}}{e^{x(w-w^{-1})}} \cdot \frac{1}{z-w} \cdot \frac{dz\,dw}{z^{s+1/2} w^{-t+1/2}} \tag{4.10}$$

where the integration contour for z should encircle that for w, which itself should encircle 0.

We will analyze (4.10) using the *saddle-point method*. Our treatment was basically developed by Okounkov and his collaborators, see the lecture notes [11] and references therein.

4.3 Warmup: the Laplace method

Before discussing the saddle-point method, let us first review its "ancestor" which is the *Laplace method*. The purpose is to estimate the behaviour as $x \to \infty$ of an integral of the form

$$I(x) = \int_a^b e^{xf(t)} dt \tag{4.11}$$

where $f : [a,b] \to \mathbb{R}$ is a real function of a real variable.

Intuitively, we expect the dominant contribution to arise from the vicinity of the point(s) where f attains its maximum, so that $\ln I(x) \sim x \max f$. But, to obtain a full asymptotic equivalent of I, we need to be more precise and consider the behaviour of f around its maximum. Usually one assumes that f attains its maximum at some unique point $t_0 \in (a,b)$, and admits a Taylor expansion

$$f(t) = f(t_0) + \frac{f''(t_0)}{2}(t-t_0)^2 + O\left((t-t_0)^3\right) \tag{4.12}$$

around it, with $f''(t_0) < 0$.

The basic idea is the following: we perform the change of variable

$$t = t_0 + \frac{u}{\sqrt{x}} \tag{4.13}$$

in the integral $I(x)$, where u is to be integrated from $-\infty$ to ∞ in the limit $x \to \infty$, and using the above Taylor expansion we get

$$I(x) \sim e^{xf(t_0)} \int_{-\infty}^{\infty} e^{\frac{f''(t_0)}{2}u^2+o(1)} \frac{du}{\sqrt{x}} \sim \sqrt{\frac{2\pi}{-xf''(t_0)}} e^{xf(t_0)}. \tag{4.14}$$

The above argument is informal but can be made fully rigorous as follows: we split the integral over $[a, b]$ in two parts:

- a *central region* which can be taken as the interval $[t_0 - x^{-1/2+\epsilon}, t_0 + x^{-1/2+\epsilon}]$ where ϵ is some small positive exponent,
- the *tails* corresponding to the rest.

For the integral over the central region, we perform the change of variable (4.13), with u to be integrated over $[-x^\epsilon, x^\epsilon]$. The error term in (4.12) is uniformly $O(x^{-3/2+3\epsilon})$, hence for $\epsilon < 1/6$ we indeed get a uniform approximation of $xf(t)$ over the central region with error $o(1)$. We may then conclude using dominated convergence that the integral over the central region indeed tends to the result of (4.14).

It remains to check that the integral over the tails is negligible, for this it suffices to note that the integrand may be bounded as

$$e^{xf(t)} \leq e^{xf(t_0)-Cx^{2\epsilon}}, \qquad |t - t_0| > x^{-1/2+\epsilon} \tag{4.15}$$

for some constant $C > 0$, as we assume that t_0 is the unique maximum of f.

It is not difficult to adapt the proof to other cases[10] where the prefactor of $e^{xf(t_0)}$ in (4.14) will be modified. We refer to [12, Chapter IV] for a detailed discussion of this topic.

4.4 Saddle-point analysis of the Bessel contour integral

We now turn to the saddle-point method, which is an adaptation of the Laplace method to the complex analytic setting. A general reference for this topic is [13, Chapter VIII]. We consider an integral of the form

$$I(x) = \int_\gamma e^{xf(z)} dz \tag{4.16}$$

where $f : D \to \mathbb{C}$ is now analytic in some domain $D \subset \mathbb{C}$, and γ is some smooth contour in D.

[10]E.g. f attains its maximum at one of the endpoints a or b, f has several maxima, the first nonzero correction in the Taylor expansion involves another exponent than 2, the integrand is multiplied by another function $g(u)$ that may have a zero or singularity at t_0, etc.

The first naive idea consists in looking at the maximum of the modulus of the integrand along γ. Note that

$$|e^{xf(z)}| = e^{x\Re(f(z))} \tag{4.17}$$

therefore we are looking for the maximum of the real part $\Re(f)$ along γ. This maximum is attained at some point z_0 where the gradient of $\Re(f)$ is normal to γ (unless it is attained at the endpoints of γ, but we will disregard this case). However, generically this gradient will be nonzero, and by the Cauchy-Riemann equations, the gradient of the imaginary part $\Im(f)$ is also nonzero and now *tangent* to γ. Therefore, in the Taylor expansion of f around z_0, the first correction is linear and purely imaginary along γ. The effect is that $e^{xf(z)}$ as a rapidly oscillating phase around z_0 along γ, which "kills" the dominant contribution we would expect. Therefore, the naive adaptation of the Laplace method is incorrect.

The solution consists in exploiting analyticity to *deform* the integration contour γ, and have it pass through a point z_* where f' vanishes. Such point is a *saddle point*[11] of $\Re(f)$ and $\Im(f)$, and we have the Taylor expansion

$$f(z) = f(z_*) + \frac{f''(z_*)}{2}(z - z_*)^2 + O\left((z - z_*)^3\right). \tag{4.18}$$

We shall deform the integration contour in such a way that $\Re(f)$ is indeed maximal at z_*, which in particular requires that $\Re(f''(z_*)e^{2i\phi}) \leq 0$, where ϕ is the angle between the real axis and the tangent to γ at z_*. Finding such deformation might be nontrivial, as f' may have several zeros and the proper deformation will depend on the initial position of γ. Still, once the proper deformation is found, we may simply adapt the arguments of the Laplace method and find that the asymptotic equivalent (4.14) remains valid (with t_0 replaced by z_*).

We will now discuss the deformation of the contours for the integral representation (4.8) of the Bessel function J_ℓ, as this will be useful for the analysis of $\mathbf{J}_\theta$. We first rewrite the contour integral as

$$J_\ell(2x) = \frac{1}{2i\pi}\oint e^{xS(z)}dz \tag{4.19}$$

where

$$S(z) = z - z^{-1} - A\ln z \tag{4.20}$$

and $A = \frac{\ell+1}{x}$. (In this discussion we will assume that $\frac{\ell+1}{x}$ is fixed, instead of the more general assumption that $\ell/x \to A$.)

The equation $S'(z) = 0$ has two solutions $z_\pm$ whose position in the complex plane depend on A.

The case $A > 2$. Here $z_\pm$ are two positive real numbers given by

$$z_\pm = e^{\pm u}, \qquad u := \operatorname{arcosh}\frac{A}{2} \qquad (A > 2). \tag{4.21}$$

[11] Since $\Re(f)$ and $\Im(f)$ are both harmonic, they cannot have local maxima nor minima, but only saddle-points.

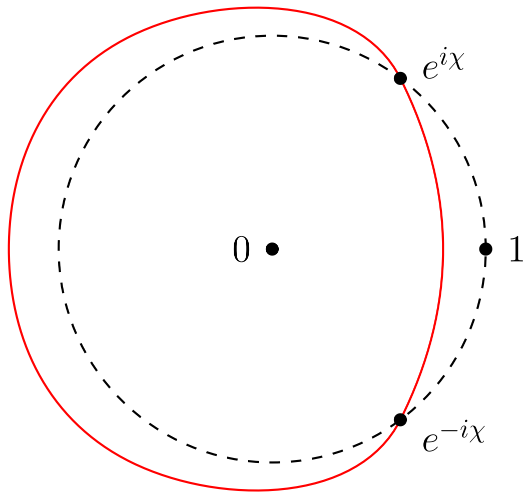

Figure 3: A proper saddle point contour for $S(z) = z - z^{-1} - A \ln z$ when $A \in (-2, 2)$, i.e. $A = 2\cos\chi$ for some $\chi \in (0, \pi)$.

A proper saddle point contour is the circle $|z| = z_+$, indeed it is not difficult to check that, for $z = z_+ e^{i\phi}$, we have

$$\Re(S(z)) = 2\sinh u \cos\phi - 2u \cosh u \tag{4.22}$$

which is indeed maximal at $\phi = 0$. We find that J_ℓ decays exponentially fast as $e^{xS(z_+)}/\sqrt{x}$, where $S(z_+) = 2\sinh u - 2u\cosh u < 0$.

The case $A < -2$. Now $z_\pm$ are two negative real numbers given by

$$z_\pm = -e^{\pm u}, \qquad u := \operatorname{arcosh}\frac{-A}{2} \qquad (A < -2). \tag{4.23}$$

A proper saddle point contour is the circle $|z| = z_-$. In fact, this case is symmetric to the case $A > 2$, by the relation $J_{-\ell} = (-1)^\ell J_\ell$ that follows from the change of variable $z \to -z^{-1}$ in the contour integral representation.

The case $|A| < 2$. Now $z_\pm$ are two conjugate conjugate numbers on the unit circle, given by

$$z_\pm = -e^{\pm i\chi}, \qquad \chi := \arccos\frac{A}{2} \qquad (|A| < 2). \tag{4.24}$$

A proper saddle point contour now passes through *both* saddle points and is displayed on Figure 3. Intuitively, it is obtained by deforming the unit circle in the direction opposite to the gradient of $\Re(S)$. In quantitative terms, for r close to 1 we have

$$\Re\left(S\left(re^{i\phi}\right)\right) = 2\left(\cos\phi - \cos\chi\right)(r-1) + O((r-1)^2) \tag{4.25}$$

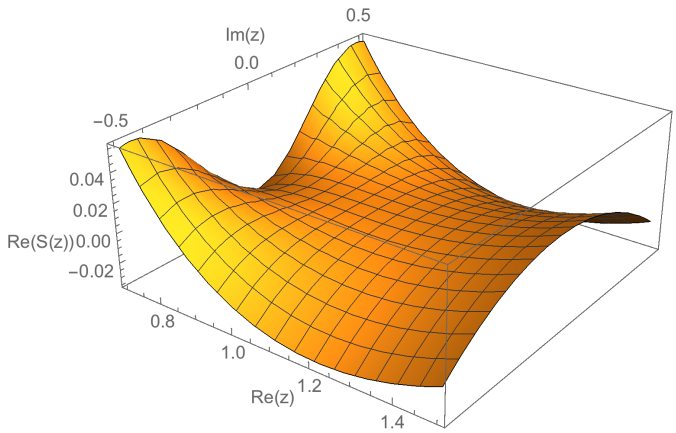

Figure 4: Graph of $\Re(S(z))$ for $A = 2$ around the monkey saddle at $z = 1$. The name derives from the observation that a saddle for a monkey would require three depressions: two for the legs and one for the tail.

and, if we consider a polar parametrization $r(\phi)$ of the contour of Figure 3, then we see that $r(\phi) - 1$ has the sign opposite to $\cos\phi - \cos\chi$ so $\Re(S) \leq 0$ along the contour. For the asymptotics of J_ℓ, we should pay attention to the fact that $z_\pm$ will have contribution of the same order of magnitude (which are, in fact, complex conjugates). Hence $J_\ell(2x)$ behaves as $\Re(e^{xS(z_+)+i\varphi})/\sqrt{x}$, where φ is some phase and $S(z_+) = 2i(\sin\chi - \chi\cos\chi)$ is purely imaginary. So there is a much slower decay than in the case $|A| > 2$, and oscillations.

The case $A = \pm 2$. In this critical situation, S' admits a double zero at 1 (if $A = 2$) or -1 (if $A = -2$). Such saddle point is sometimes called a *monkey saddle*, see Figure 4. Let us discuss the case $A = 2$, the other case $A = -2$ being easily deduced by symmetry. The main change is that we now have the Taylor expansion

$$S(z) = \frac{(z-1)^3}{3} + O((z-1)^4) \tag{4.26}$$

which is different from the generic situation discussed in Section 4.3: the change of variable (4.13) should be replaced by $z = 1 + \frac{y}{x^{1/3}}$ in order for $e^{xS(z)}$ to have a nontrivial limit in the central region. Therefore, the subleading correction $1/\sqrt{x}$ appearing in (4.14), which comes from the change of variable, will be replaced by a $1/x^{1/3}$.

Let us now discuss the deformation of the integration contour: since it has to encircle 0, a proper saddle point contour must follow the two symmetric depressions on Figure 4 and avoid the third depression which is in the direction of 0. In fact, it is also possible to work simply with the circle $|z| = e^{x^{-1/3}}$, which does not quite pass through the saddle point, but on which we have

$$\Re(S(z)) = 2\,(\cos\phi - 1)\,x^{-1/3} + O(x^{-2/3}), \qquad \phi = \arg(z) \tag{4.27}$$

which is maximal for $\phi = 0$.

We now work out the precise asymptotics of J_ℓ in this regime. In fact, it costs little to consider the more general situation where

$$\ell = 2x + sx^{1/3} + o(x^{1/3}) \tag{4.28}$$

with s a fixed real number, as we then have the contour integral representation

$$J_\ell(2x) = \frac{1}{2i\pi} \oint \frac{e^{xS(z)}}{z^{\ell-2x+1}} dz \tag{4.29}$$

with S keeping the same expression as before with $A = 2$. From (4.27) we see that the integral is exponentially dominated by the vicinity of $z = 1$, and it is convenient to perform the change of variable

$$z = e^{\zeta x^{-1/3}} \tag{4.30}$$

where ζ is ultimately to be integrated over $1 + i\mathbb{R}$. From (4.26) we find

$$J_\ell(2x) \sim \frac{1}{2i\pi x^{1/3}} \int_{1+i\mathbb{R}} e^{\frac{\zeta^3}{3} - s\zeta} d\zeta. \tag{4.31}$$

Recognizing the complex integral representation (1.3) for the Airy function, we conclude that

$$\lim_{\substack{x\to\infty \\ \ell-2x\sim sx^{1/3}}} x^{1/3} J_\ell(2x) = \mathrm{Ai}(s) \tag{4.32}$$

which is *Nicholson's approximation*, see [1, Theorem 2.27] (whose full rederivation is left as an exercise).

4.5 Analysis of $\mathbf{J}_\theta$: bulk statistics

We now turn to the discrete Bessel kernel $\mathbf{J}_\theta(s,t)$, given by the double contour integral (4.10). We first consider the limit (4.1) with general A, and rewrite the integral as

$$\mathbf{J}_\theta(s,t) = \frac{1}{(2i\pi)^2} \oiint \frac{e^{x(S(z)-S(w))}}{z^{s'+1/2} w^{-t'+1/2}} \cdot \frac{dz\, dw}{z-w} \tag{4.33}$$

where S is as in (4.20), and $s' := s - Ax$, $t' := t - Ax$. Here the dominant factor as $x \to \infty$ is the exponential $e^{x(S(z)-S(w))}$ so, inspired by our previous discussion, we know we should move the integration contours to the saddle point contours for S. For z we should take precisely the same contours, while for w, using the symmetry relation $S(w) = -S(w^{-1})$, we should take their images through the inversion $w \to w^{-1}$.

The case $A > 2$. Here the saddle point contours for z and w are the respective circles $|z| = z_+$ and $|w| = z_-$. There is no issue in deforming the contours form their initial position, and in view of the discussion for J_ℓ we conclude that $\mathbf{J}_\theta(s,t)$ decays exponentially to 0, consistently with the idea that we are in a "frozen" region completely void of particles.

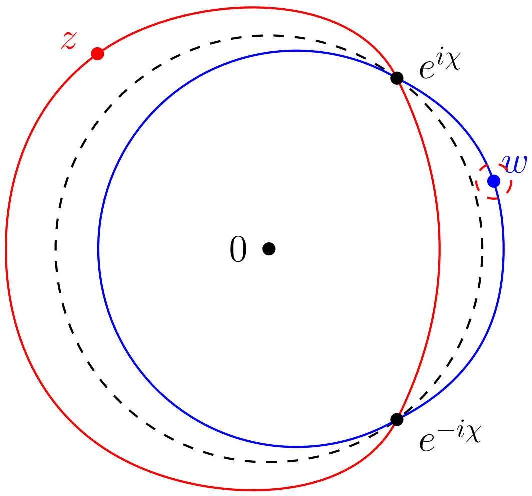

Figure 5: Saddle point contours for z and w in the analysis of $\mathbf{J}_\theta(s,t)$, in the case $|A|<2$. When deforming the contours from their initial position where the z-contour encircles the w-contour, we should be careful that we pick a contribution from the residue at $z=w$ when $|\arg(w)|<\chi$.

The case $A<-2$. Now the saddle point contours for z and w are the respective circles $|z|=z_-$ and $|w|=z_+$, i.e. they are *switched* with respect to the previous case. This raises the question of how we may deform the contours from their initial position, where the z-contour must encircle the w-contour. We start by deforming the w-contour into the circle $|w|=z_+$, keeping $|z|$ strictly larger than z_+, say on the circle $|z|=2z_+$. Then, we observe that the integrand $\Phi(z,w)$ of (4.33) has a simple pole at $z=w$ with residue $1/w^{s-t+1}$, so that we may write

$$\oint_{|z|=2z_+}\Phi(z,w)dz=\oint_{|z|=z_-}\Phi(z,w)dz+\frac{2i\pi}{w^{s-t+1}} \tag{4.34}$$

and therefore

$$\mathbf{J}_\theta(s,t)=\frac{1}{(2i\pi)^2}\oint_{|w|=z_+}\oint_{|z|=z_-}\Phi(z,w)dzdw+\underbrace{\frac{1}{2i\pi}\oint_{|w|=z_+}\frac{dw}{w^{s-t+1}}}_{\delta_{s,t}}. \tag{4.35}$$

In view of the discussion for J_ℓ we again see that the first term decays exponentially and we conclude that $\mathbf{J}_\theta(s,t)\to\delta_{s,t}$, consistently with the idea that we are in a "frozen" region completely filled with particles.

The case $|A|<2$. Now the saddle point contour for z is the same as that displayed on Figure 3, and that w is its image via an inversion. Note that the integrand as a pole at $z=w$ but this leads to an integrable singularity when we integrate over both z and w.

As in the case $A<-2$, there is the issue of the deformation of the contours from their initial position, which we treat in the same way by picking the residue at $z=w$. The difference is that we now only pick the residue for the w's that are outside the unit circle, see Figure 5. Therefore, we find

$$\mathbf{J}_\theta(s,t) = \frac{1}{(2i\pi)^2} \oiint_{\text{saddle point contours}} \Phi(z,w)dzdw + \frac{1}{2i\pi}\int_{e^{-i\chi}}^{e^{i\chi}} \frac{dw}{w^{s-t+1}}. \tag{4.36}$$

We need not evaluate precisely the first integral, we only need to get convinced that it tends to zero[12]: clearly the contribution of z's and w's away from the saddle points are exponentially small, and for the contribution coming from the vicinities of the saddle points, changes of variables of the form (4.13) show that the integral is $O(1/\sqrt{x})$ (which arises from the factor $\frac{dzdw}{z-w}$). The second integral is easily seen to be equal to

$$\mathbf{K}_{\sin}(s-t;\chi) := \begin{cases} \frac{\chi}{\pi} & \text{if } s=t, \\ \frac{\sin(\chi(s-t))}{\pi(s-t)} & \text{if } s \neq t. \end{cases} \tag{4.37}$$

so we conclude that, if $s-t$ is kept fixed as $x \to \infty$, $\mathbf{J}_\theta(s,t)$ tends to $\mathbf{K}_{\sin}(s-t;\chi)$. In fact the conclusion also holds if $s-t \to \pm\infty$, where it is understood that $\mathbf{K}_{\sin}(\pm\infty;\chi) = 0$. We may also incorporate the cases $A \geq 2$ and $A \leq -2$, by setting $\chi = 0$ if $A \geq 2$ (we have $\mathbf{K}_{\sin}(s-t;0) = 0$) and $\chi = \pi$ if $A \leq -2$ (we have $\mathbf{K}_{\sin}(s-t;\pi) = \delta_{s,t}$). In all rigor we have not treated the cases $A = \pm 2$ but it is also quite easy to get convinced that the corresponding integral over saddle point contours tends to 0, showing that we get the same "trivial" limits as for $|A| > 2$.

The kernel $\mathbf{K}_{\sin}(\cdot - \cdot;\chi)$ is known as the *discrete sine kernel* of parameter χ. It defines a translation-invariant determinantal point process on $\mathbb{Z}$ (or $\mathbb{Z}'$) whose density is χ/π, which is called the *discrete sine process.* In fact, there also exists a continuous DPP on $\mathbb{R}$ whose kernel is also $\mathbf{K}_{\sin}(\cdot - \cdot;\chi)$ (which is now viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}$). It is known as the *sine process* and arises in the bulk statistics of random matrices.

We end up with the following theorem which is due to Borodin, Okounkov and Olshanski [6], and which confirms the informal discussion of Section 4.1, and in particular the formula (4.3) for the limiting one-point function.

Theorem 4.1 (Bulk statistics of the Plancherel measure)**.** *The point process $S(\lambda^{\langle\theta\rangle})$ converges locally to the discrete sine process, in the sense of the convergence of the correlation kernel and therefore of the correlation functions.*

More precisely, in the limit where $\theta = x^2 \to \infty$, where s/x and t/x tend to $A \in \mathbb{R}$ and where $s-t$ tends to a limit in $\mathbb{Z} \cup \{\pm\infty\}$, we have

$$\mathbf{J}_\theta(s,t) \to \mathbf{K}_{\sin}(s-t;\chi), \qquad \chi = \begin{cases} \arccos(A/2) & \text{if } |A| < 2, \\ 0 & \text{if } A \geq 2, \\ \pi & \text{if } A \leq -2. \end{cases} \tag{4.38}$$

[12] Okounkov [11] as the appealing simple argument that integrals of the form $\int_\gamma e^{xf(z)}dz$ never have finite limit as $x \to \infty$. But this argument does not apply stricto sensu to the situation where the integrand contains an extra factor with a singularity at the saddle point, such as in the present case, or more simply in the case of $\int_{\mathbb{R}-i} e^{-xz^2}\frac{dz}{z}$ which is constant equal to $i\pi$ so, a fortiori, has a finite limit as $x \to \infty$.

Therefore, if $s_1, \ldots, s_n$ are such that $s_i/x \to A$ and $s_i - s_j$ has a limit (finite or not) as $x \to \infty$, then

$$\mathbb{P}\left(\{s_1, \ldots, s_n\} \subset S(\lambda^{\langle\theta\rangle})\right) \to \det_{1\le i,j\le n} \mathbf{K}_{\sin}(s_i - s_j; \chi) \tag{4.39}$$

which is the n-point correlation function of the discrete sine process.

Remark 4.2. We may partition the set $\{s_1, \ldots, s_n\}$ of the theorem into blocks such that $s_i - s_j$ has a finite limit if and only s_i and s_j belong to the same block. The theorem says that there is an independent copy of the discrete sine process for each such block. We may even write a generalized statement in the case where the limit A_i of s_i/x depends on the block (A_i giving the parameter of the corresponding discrete sine process).

In a more physical language, all this says is that the correlation function is finite in the bulk.

4.6 Analysis of $\mathbf{J}_\theta$: edge statistics

According to the previous discussion, we know that, for s/x and t/x tending to 2, the discrete Bessel kernel $\mathbf{J}_\theta(s,t)$ tends to 0. But we want now a precise asymptotic estimate. From the informal discussion of Section 4.1, we expect $\mathbf{J}_\theta(s,t)$ to decay as $x^{-1/3}$ when s, t are scaled as in (4.6). The precise result is the following:

Proposition 4.3 (Convergence to the Airy kernel at the edge)**.** *In the limit where $\theta = x^2 \to \infty$ and s, t are given by* (4.6)*, we have*

$$x^{1/3}\,\mathbf{J}_\theta(s,t) \to \mathbf{A}(\sigma, \tau). \tag{4.40}$$

We also have the trace convergence

$$\sum_{u=s}^{\infty} \mathbf{J}_\theta(u,u) \to \int_\sigma^\infty \mathbf{A}(v,v)dv. \tag{4.41}$$

Remark 4.4. The sum $\sum_{u=s}^{\infty} \mathbf{J}_\theta(u,u)$ is finite since $\mathbf{J}_\theta$ has superexponential decay for $s \to +\infty$, as a consequence of the decay of J_ℓ. By (4.10) we have the integral representation

$$\sum_{u=s}^{\infty} \mathbf{J}_\theta(s,s) = \frac{1}{(2i\pi)^2} \oint\!\!\oint \frac{e^{x(z-z^{-1})}}{e^{x(w-w^{-1})}} \cdot \frac{(w/z)^{u-1/2}}{(z-w)^2} dzdw. \tag{4.42}$$

The reader might notice that we are setting the stage for an application of Corollary 2.27, for which the last assumption to check is that $\mathbf{J}_\theta$ is hermitian positive-semidefinite, which is immediate from the sum representation (3.7).

Proof of Proposition 4.3. We of course perform another saddle point analysis, and the calculation is in fact a simple variant of that done for the Bessel function J_ℓ in Section 4.4 in the case $A = 2$, which led to Nicholson's approximation (4.32). We provide a bit more detail as we rely on the result to establish the main theorem of the course.

We start from the integral representation (4.33) where we take the integration contours to be the circles $|z| = e^{x^{1/3}}$ and $|w| = e^{-x^{1/3}}$. Fix an exponent $\epsilon \in (0, \frac{1}{12})$ (we could take an explicit value but keeping it arbitrary will show its role more clearly), then we split the double integral in two parts:

- the *central region* where both $|\arg(z)|$ and $|\arg(w)|$ are smaller than $x^{-1/3+\epsilon}$,
- and the *tails* corresponding to the rest.

The reader will notice that we follow the rigorous justification of the Laplace method given in Section 4.3 (adapted to the case of a monkey saddle).

We first consider the tails: from the estimate (4.27), the symmetry $S(w) = -S(w^{-1})$ and the inequality $\cos\phi \leq 1 - \frac{\phi^2}{8}$ valid for $\phi \in [-\pi, \pi]$, we find that the exponential factor $e^{x(S(z)-S(w))}$ is uniformly $O(e^{x^{2\epsilon}/4})$, while all other factors in (4.3) are bounded (note that $\frac{1}{|z-w|} \leq \frac{x^{1/3}}{2}$ on the chosen integration circles). Therefore, the integral over the tails decays exponentially.

We now consider the central region, where we perform the change of variables

$$z = e^{\zeta x^{-1/3}}, \qquad w = e^{\omega x^{-1/3}} \tag{4.43}$$

with ζ and ω running over respectively $1 + i[-\pi x^\epsilon, \pi x^\epsilon]$ and $-1 + i[-\pi x^\epsilon, \pi x^\epsilon]$. From the Taylor expansion (4.26) we see that the exponential factor may be uniformly approximated as

$$e^{x(S(z)-S(w))} = e^{\frac{\zeta^3}{3} - \frac{\omega^3}{3} + O(x^{4\epsilon - 1/3})} \tag{4.44}$$

and, noting that $s'x^{-1/3} \to \sigma$ and $t'x^{-1/3} \to \tau$, the other factors appearing in (4.33) may be uniformly approximated as

$$\frac{1}{z^{s'+1/2}w^{-t'+1/2}} \frac{dz\,dw}{z-w} \sim e^{-\sigma\zeta + \tau\omega} \frac{d\zeta d\omega}{\zeta - \omega} x^{1/3}. \tag{4.45}$$

We may then deduce by dominated convergence that

$$x^{1/3}\, \mathbf{J}_\theta(s,t) \to \frac{1}{(2i\pi)^2} \int_{1+i\mathbb{R}} d\zeta \int_{-1+i\mathbb{R}} d\omega \frac{e^{\frac{\zeta^3}{3} - \sigma\zeta - \frac{\omega^3}{3} + \tau\omega}}{\zeta - \omega}. \tag{4.46}$$

The last step is to use the integral representation $\frac{1}{\zeta-\omega} = \int_0^\infty e^{-y(\zeta-\omega)} dy$ (which is valid since $\Re(\zeta - \omega) = 2 > 0$), interchange the order of integrals and recognize the complex integral representation (1.3) of the Airy function. We then precisely obtain the integral representation of the Airy kernel (1.6) (with t replaced by y). This establishes the first statement of the proposition.

For the second statement, we start from the integral representation (4.42), and the arguments are completely similar. □

Remark 4.5. Our choice of integration contours (circles of radii $e^{\pm x^{1/3}}$) is different from the "standard" saddle point contours used for instance in [11]. Our motivation is that, not only are circles easier to describe, but also they allow to treat the "finite temperature" variant of the discrete Bessel kernel [14].

4.7 Conclusion of the proof of Theorems 1.1 and 2.5

For $x, \tau \in \mathbb{R}$, set

$$t := \lfloor 2x + \tau x^{1/3} \rfloor \tag{4.47}$$

define the kernel $K_{x,t}$ by

$$K_{x,t}(i,j) = \mathbf{J}_{x^2}(t+i+1/2, t+j+1/2), \qquad i,j \in \mathbb{N}. \tag{4.48}$$

Then, by Corollary 2.27, we see that

$$\det(I - K_{x,t})_{\mathbb{N}} \to \det(I - \mathbf{A}(\tau + \cdot, \tau + \cdot))_{L^2(\mathbb{R}_+)} \tag{4.49}$$

which we may rewrite as

$$\det(I - \mathbf{J}_\theta)_{\mathbb{Z}'_{>t}} \to \det(I - \mathbf{A})_{L^2(\tau,\infty)} = F_2(\tau). \tag{4.50}$$

But the Fredholm determinant $\det(I - \mathbf{J}_\theta)_{\mathbb{Z}'_{>t}}$ is nothing but the probability that $S(\lambda^{\langle\theta\rangle})$ does not intersect $\mathbb{Z}'_{>t}$, i.e. the probability that $\lambda_1^{\langle\theta\rangle} \leq t$. Therefore we obtain the following:

Theorem 4.6. *Let $\theta = x^2$. Then, the first part of the poissonized Plancherel random partition $\lambda^{\langle\theta\rangle}$ converges after rescaling to the maximum of the Airy ensemble, namely*

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{\lambda_1^{\langle\theta\rangle} - 2x}{x^{1/3}} \leq \tau\right) = F_2(\tau). \tag{4.51}$$

In comparison with Theorem 1.1 (where we recall that $L(\sigma_n) = \lambda_1^{(n)}$), we see that the only difference is that we work with the poissonized Plancherel measure instead of the ordinary (fixed size) Plancherel measure. Since, furthermore, $|\lambda^{\langle\theta\rangle}|$ being a Poisson random variable of parameter θ has expectation θ and variance θ, it is natural to expect that we may perform "depoissonization" and deduce Theorem 1.1 from Theorem 4.6 (with the correspondence $\theta \to n$, $x \to n^{1/2}$ and $\lambda^{\langle\theta\rangle} \to \lambda^{(n)}$). In practice this requires some "regularity assumptions". Here we may use monotonicity, as in the following lemma (which is just [1, Lemma 2.31] rewritten differently).

Lemma 4.7. *For $\theta \leq 0$, denote by N_θ a Poisson random variable of parameter θ. Let $P(\cdot) : \mathbb{N} \to [0,1]$ be nonincreasing and, for $n \in \mathbb{N}$, let $\theta_n^\pm := n \pm 4\sqrt{n \ln n}$. Then we have*

$$\mathbb{E}\left(P(N_{\theta_n^+})\right) - \frac{1}{9 \ln n} \leq P(n) \leq \mathbb{E}\left(P(N_{\theta_n^-})\right) + \frac{1}{9 \ln n}. \tag{4.52}$$

Proof. It is easy to check that $n < \theta_n^+ - 3\sqrt{\theta_n^+ \ln \theta_n^+}$ so, by the assumptions on P,

$$\mathbb{E}\left(P(N_{\theta_n^+})\right) \leq P(n) + \mathbb{P}\left(N_{\theta_n^+} < \theta_n^+ - 3\sqrt{\theta_n^+ \ln \theta_n^+}\right) \tag{4.53}$$

and, by Chebyshev's inequality, the second term is bounded by $\frac{1}{9 \ln n}$ as wanted. Similarly we have $n > \theta_n^- + 3\sqrt{\theta_n^- \ln \theta_n^-}$ and

$$\mathbb{E}\left(P(N_{\theta_n^-})\right) \geq P(n)\left(1 - \mathbb{P}\left(N_{\theta_n^-} > \theta_n^- + 3\sqrt{\theta_n^- \ln \theta_n^-}\right)\right). \tag{4.54}$$

$\square$

We now apply the lemma with

$$P_t(n) = \mathbb{P}(\lambda_1^{(n)} \le t) \tag{4.55}$$

for a fixed $t \in \mathbb{R}$. Indeed, it follows from the existence of the Plancherel growth process (an increasing coupling of the $\lambda^{(n)}$'s for all n), discussed in the first part of the course, that $n \mapsto P_t(n)$ is nonincreasing. Therefore we have

$$Q_t(\theta_n^+) - \frac{1}{9\ln n} \le P_t(n) \le Q_t(\theta_n^-) + \frac{1}{9\ln n} \tag{4.56}$$

where

$$Q_t(\theta) := \mathbb{P}(\lambda_1^{\langle\theta\rangle} \le t) = \mathbb{E}\left(P_t(N_\theta)\right). \tag{4.57}$$

We now take $t = 2\sqrt{n} + \tau n^{1/6}$. It is not difficult to check that

$$\frac{t - 2\sqrt{\theta_n^\pm}}{(\theta_n^\pm)^{1/6}} \to \tau \tag{4.58}$$

(as $\sqrt{\theta_n^\pm} = \sqrt{n} \cdot \sqrt{1 \pm \frac{4\ln n}{n}} = \sqrt{n} \pm 4\ln n + o(1)$ and $(\theta_n^\pm)^{1/6} = n^{1/6} + o(1)$). Therefore, by Theorem 4.6, we conclude that $Q_t(\theta_n^\pm) \to F_2(\tau)$ and Theorem 1.1 follows from (4.56).

To obtain Theorem 2.5, we proceed in the same way starting from its poissonized version, which we have not established yet.

Theorem 4.8. *For $i \in \mathbb{N}^*$, let $\bar\lambda_i^{\langle\theta\rangle} := x^{-1/3}(\lambda^{\langle\theta\rangle} - 2x)$ denote the rescaled i-th part of the poissonized Plancherel random partition $\lambda^{\langle\theta\rangle}$ with $\theta = x^2$. Then, for any $k \ge 1$, we have*

$$(\bar\lambda_1^{\langle\theta\rangle}, \ldots, \bar\lambda_k^{\langle\theta\rangle}) \xrightarrow[x\to\infty]{(d)} (\zeta_1, \ldots, \zeta_k). \tag{4.59}$$

In other words, the largest parts of a poissonized Plancherel random partition converge after rescaling to the Airy ensemble.

Proof. We have to use the generalized version of Corollary 2.27 suggested in Exercise 2.28. Let $\tau_1 > \ldots > \tau_k$ be real numbers, $z_1, \ldots, z_k$ complex variables, and set $t_i = \lfloor 2x + \tau_i x^{1/3} \rfloor$. Let us consider the functionals $\phi : \mathbb{Z}' \to \mathbb{C}$ and $\tilde\phi : \mathbb{R} \to \mathbb{C}$ defined by respectively

$$\phi(s) = \sum_{i=1}^{k} (z_i - 1)\mathbb{1}_{t_i < s < t_{i-1}}, \qquad \tilde\phi(\sigma) = \sum_{i=1}^{k} (z_i - 1)\mathbb{1}_{\tau_i < s < \tau_{i-1}} \tag{4.60}$$

where it is understood that $t_0 = \tau_0 = +\infty$. Then it is clear that

$$\phi(\lfloor 2x + \sigma x^{1/3} \rfloor + 1/2) \to \tilde\phi(\sigma) \tag{4.61}$$

and by the generalized version of Corollary 2.27 we have

$$\det(I + \mathbf{J}_\theta\, \phi)_{\mathbb{Z}'_{>t_k}} \to \det(I + \mathbf{A}\, \tilde\phi)_{L^2(t_k,\infty)}. \tag{4.62}$$

But the left-hand size is nothing but the expectation $\mathbb{E}(z_1^{n_1}\cdots z_k^{n_k})$ where $n_i := S(\lambda^{\langle\theta\rangle})\cap[t_i,t_{i-1}]$. And the right-hand side is the same with $\tilde{n}_i := X_{\text{Airy}}\cap(\tau_i,\tau_{i-1})$. Since the convergence holds for any $z_1,\ldots,z_k$ (uniformly in bounded sets), we deduce that

$$\mathbb{P}(n_1=\cdots=n_k=1)\to\mathbb{P}(\tilde{n}_1=\cdots=\tilde{n}_k=1) \tag{4.63}$$

which amounts to the convergence in distribution we are looking for (since the τ_i's are arbitrary). □

We now deduce Theorem 2.5 by taking

$$P_{t_1,\ldots,t_n}(n)=\mathbb{P}\left(\lambda_1^{(n)}\le t_1,\ldots,\lambda_k^{(n)}\le t_k\right) \tag{4.64}$$

which is nonincreasing in n, and applying Lemma 4.7 as before. Details are left to the reader.

References

[1] Dan Romik. *The surprising mathematics of longest increasing subsequences.* Institute of Mathematical Statistics Textbooks. Cambridge University Press, New York, 2015.

[2] Kurt Johansson. Random matrices and determinantal processes. In *Mathematical statistical physics*, pages 1–55. Elsevier B. V., Amsterdam, 2006. arXiv:math-ph/0510038.

[3] Odile Macchi. The coincidence approach to stochastic point processes. *Advances in Appl. Probability*, 7:83–122, 1975.

[4] A. Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55(5(335)):107–160, 2000.

[5] Andrei Okounkov. Random matrices and random permutations. *Internat. Math. Res. Notices*, (20):1043–1095, 2000.

[6] Alexei Borodin, Andrei Okounkov, and Grigori Olshanski. Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13(3):481–515, 2000.

[7] Kurt Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math. (2)*, 153(1):259–296, 2001.

[8] Andrei Okounkov. Infinite wedge and random partitions. *Selecta Math. (N.S.)*, 7(1):57–81, 2001.

[9] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108(5-6):1071–1106, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[10] Cédric Boutillier, Jérémie Bouttier, Guillaume Chapuy, Sylvie Corteel, and Sanjay Ramassamy. Dimers on rail yard graphs. *Ann. Inst. Henri Poincaré D*, 4(4):479–539, 2017. arXiv:1504.05176 [math-ph].

[11] Andrei Okounkov. Symmetric functions and random partitions. In *Symmetric functions 2001: surveys of developments and perspectives*, volume 74 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 223–252. Kluwer Acad. Publ., Dordrecht, 2002. arXiv:math/0309074 [math.CO].

[12] Jean Dieudonné. *Calcul infinitésimal.* Hermann, Paris, 1968.

[13] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* Cambridge University Press, Cambridge, 2009. Available online at `http://algo.inria.fr/flajolet/Publications/books.html`.

[14] Dan Betea and Jérémie Bouttier. The Periodic Schur Process and Free Fermions at Finite Temperature. *Math. Phys. Anal. Geom.*, 22(1):22:3, 2019.