Generating Series of the Trace Group

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Abstract. We prove an analog for trace groups of the Möbius inversion formula for trace monoids (Cartier-Foata, 1969). A by-product is to obtain an explicit and combinatorial formula for the growth series of a trace group. This is used to study the average height of traces.

1 Introduction

A trace group (monoid) is the quotient of a free group (monoid) by relations of commutation between some pairs of generators. Trace monoids are often used in computer science to model the occurrence of events in concurrent systems, see [7] and the references therein. Trace groups have been studied from several viewpoints (and under various names), see for instance [8, 10, 17]. An important motivation is that trace goups can 'approximate' braid groups [17].

A *decomposition* of an element m of a monoid M is a n-uple $(m_1, \ldots, m_n)$ satisfying $m_1 \cdots m_n = m$ with $m_i \in M \backslash \{1\}$ (where 1 is the unit of the monoid). The decomposition is *even* (*odd*) if n is even (odd). The monoid M has the *finite decomposition property* if any element has a finite number of decompositions. Assume M has the property and set $(\mu_M|m)$ to be the number of even decompositions of m minus the number of odd decompositions. View μ_M as a formal series of $\mathbb{Z}\langle\langle M\rangle\rangle$ and call it the *Möbius series of* M. A one line computation shows that μ_M is the formal inverse of the characteristic series:

$$\text{in } \mathbb{Z}\langle\langle M\rangle\rangle, \quad (\sum_{m\in M} m) \cdot \mu_M = \mu_M \cdot (\sum_{m\in M} m) = 1 . \qquad (1)$$

This identity is called a *Möbius inversion formula*, see Cartier and Foata [3], Lallement [15], or Rota [16] in a different setting. The classical Möbius inversion principle in number theory (see [13], Chapter XVI) is a special instance of the identity. However, it is generally difficult, given a monoid with the finite decomposition property, to effectively compute the Möbius series.

It is easily checked that a trace monoid has the finite decomposition property. An element m of the trace monoid is a *clique* if any letter appears at most once in m and if all the letters of m are commuting. Let $\mathfrak{C}$ denote the set of cliques. Let $|m|$ be the length of m, see (3). In [3, Theorem 2.4], it is proved that the Möbius series of a trace monoid M is the polynomial:

$$\mu_M = \sum_{c\in\mathfrak{C}} (-1)^{|c|} c .$$

This is the starting point of the combinatorial study of the trace monoid [5, 7].

Like any non-trivial group, a trace group does *not* have the finite decomposition property. Hence, one cannot follow the above path to get the formal inverse of the characteristic series. Nevertheless, we obtain in this paper an identity for the trace group which has the flavor of a Möbius inversion formula. Let F be a trace group over the alphabet $\Sigma \cup \overline{\Sigma}$ ($\overline{a} \in \overline{\Sigma}$ is the inverse of $a \in \Sigma$). Let M be the trace monoid over $\Sigma \cup \overline{\Sigma}$ whose monoid presentation is obtained from the one of F by removing the relations $a\overline{a} = \overline{a}a = 1, a \in \Sigma$. Let ϕ be the canonical injection from F to M. (More precisely, $\phi(t), t \in F$, is the projection on M of any minimal length representative of t over $(\Sigma \cup \overline{\Sigma})^*$.) We have (Theorem 4.1):

$$\text{in } \mathbb{Z}\langle\langle M \rangle\rangle, \quad (\sum_{t \in \phi(F)} t) \cdot (\sum_{d \in \mathfrak{D}} (-1)^{|d|} d) = (\sum_{d \in \mathfrak{D}} (-1)^{|d|} d) \cdot (\sum_{t \in \phi(F)} t) = 1 \,, \tag{2}$$

where $\mathfrak{D} = T(\mathfrak{C})$ and $T : M \to \mathcal{P}(M)$ is the map defined by $T(a) = a(\bar{a}a)^*(1 + \bar{a}), a \in \Sigma \cup \bar{\Sigma}$, and extended by morphism. As opposed to $\mathfrak{C}$, the set $\mathfrak{D}$ is infinite. However, $\mathfrak{D}$ is a rational subset of M. The identity (2) can be lifted to the free monoid under some conditions, extending a result of Diekert [5].

A nice consequence of the identity (2) is to get precise information about the growth of the trace group. The growth is probably the most popular way of quantifying infinite monoids or groups. Consider a monoid M with a fixed and finite set of generators S (when M is a group, we assume that the generating set contains the inverse of each of its elements). The *length (with respect to S)* of an element m in M is by definition

$$|m| = \min\{k \mid m = m_1 \cdots m_k, m_i \in S\} \,. \tag{3}$$

The *growth series* of M is the series $\mathrm{Gr}M \in \mathbb{Z}[\![x]\!]$ defined by

$$\mathrm{Gr}M = \sum_{m \in M} x^{|m|} = \sum_{n \in \mathbb{N}} \#\{m \in M \mid |m| = n\}\, x^n \,. \tag{4}$$

Set $a_n = \#\{m \in M \mid |m| = n\}$. The *growth rate* of M is $\rho_M = \lim_n a_n^{1/n} \geqslant 1$ (the limit exists by sub-additivity of the sequence $(\log a_n)_n$). For M and F defined as in the paragraph preceding (2), we obtain as a by-product of (2) that

$$\mathrm{Gr}F(x) = \mathrm{Gr}M(x(1+x)^{-1}) \,. \tag{5}$$

This allows to transfer to the trace group several results initially proved for the trace monoid. In particular, $\mathrm{Gr}F$ is rational with a computable growth rate and we have $a_n \sim cn^k \rho_F^n$, where c is a constant and where k depends on the connected subgraphs of the dependence graph of F (Proposition 4.1).

Finally, we use the identity (2) to study the height series in §5.

Let us comment on the article [17]. For proving only the rationality of the growth series, an easier way consists of considering lexicographic normal forms, see §3. In [17], Vershik and al. study particular trace groups: the generators $a_1, \ldots, a_k$ satisfy $a_i a_j = a_j a_i$ if $|i - j| > 1$. The authors compute Gr using lexicographic normal forms, and they derive a precise asymptotic analysis of the series. In particular, they prove that the limit value of the growth rate is 7 (resp. 4) when the number k of generators of the trace group (resp. trace monoid) tends to infinity.

2 Preliminaries

Let Σ be a non-empty finite set and denote by Σ^* the free monoid over Σ. The empty word is denoted by 1. Consider a relation $R \subset \Sigma^* \times \Sigma^*$, and let $\sim_R$ be the least congruence on Σ^* such that $u \sim_R v$ if $(u, v) \in R$. The quotient monoid $(\Sigma^*/\sim_R)$ is called the *monoid presented by Σ and R* and is denoted $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$.

Let $\overline{\Sigma}$ be a copy of Σ, that is a set which is in bijection with Σ and disjoint from Σ; to each letter $a \in \Sigma$ corresponds the letter $\overline{a}$ in $\overline{\Sigma}$. Set $\widetilde{\Sigma} = \Sigma \cup \overline{\Sigma}$. We extend the bar notation to $\widetilde{\Sigma}$, by setting $\overline{\overline{a}} = a$. Denote by $\mathbb{F}(\Sigma)$ the free group over Σ, with $\overline{\Sigma}$ as the set of inverses of the generators. A monoid presentation of the free group is

$$\mathbb{F}(\Sigma) = \langle\, \widetilde{\Sigma} \mid a\overline{a} = 1, \overline{a}a = 1,\ \forall a \in \Sigma \,\rangle\,. \tag{6}$$

Let $I \subset \Sigma \times \Sigma$ be an anti-reflexive and symmetric relation, called an *independence* (or *commutation*) relation. The *trace monoid* (or *free partially commutative monoid*) $\mathbb{M}(\Sigma, I)$ is defined by the monoid presentation

$$\mathbb{M}(\Sigma, I) = \langle\, \Sigma \mid ab = ba,\ \forall (a, b) \in I \,\rangle\,. \tag{7}$$

The elements of $\mathbb{M}(\Sigma, I)$ are called *traces*. Two words are representatives of the same trace if they can be obtained one from the other by repeatedly commuting independent adjacent letters. The graph (Σ, I) is the *independence graph* of $\mathbb{M}(\Sigma, I)$. The *dependence graph* of $\mathbb{M}(\Sigma, I)$ is (Σ, D) where $D = (\Sigma \times \Sigma)\backslash I$.

Let $\mathbb{F}(\Sigma, I)$ be defined by the monoid presentation

$$\mathbb{F}(\Sigma, I) = \langle\, \widetilde{\Sigma} \mid a\overline{a} = 1, \overline{a}a = 1,\ \forall a \in \Sigma;\quad ab = ba,\ \forall (a, b) \in I \,\rangle\,. \tag{8}$$

It is easily checked that $\mathbb{F}(\Sigma, I)$ is a group, called the *trace group* (or *free partially commutative group*). Notice that a *group presentation* of the trace group is $\mathbb{F}(\Sigma, I) = \langle\, \Sigma \mid ab = ba,\ \forall (a, b) \in I \,\rangle$. The elements of $\mathbb{F}(\Sigma, I)$ are also called traces.

Let $\iota : \widetilde{\Sigma} \longrightarrow \Sigma$ be defined by: $\forall a \in \Sigma, \iota(a) = \iota(\overline{a}) = a$. We associate with the graph (Σ, I), the graph $(\widetilde{\Sigma}, \widetilde{I})$ where the relation $\widetilde{I} \subset \widetilde{\Sigma} \times \widetilde{\Sigma}$ is defined by $(u, v) \in \widetilde{I} \iff (\iota(u), \iota(v)) \in I$. The graph $(\widetilde{\Sigma}, \widetilde{I})$ is the *independence graph* of $\mathbb{F}(\Sigma, I)$. The *dependence graph* of $\mathbb{F}(\Sigma, I)$ is $(\widetilde{\Sigma}, \widetilde{D})$ where $\widetilde{D} = (\widetilde{\Sigma} \times \widetilde{\Sigma})\backslash \widetilde{I}$.

The diagram below, where the applications are the canonical surjective morphisms, is commutative.

$$\begin{tikzcd}
\widetilde{\Sigma}^* \arrow[r, "\pi_{\mathbb{M}}"] \arrow[d] \arrow[dr, "\pi_{\mathbb{F}}"] & \mathbb{M}(\widetilde{\Sigma}, \widetilde{I}) \arrow[d, "\varphi"] \\
\mathbb{F}(\Sigma) \arrow[r] & \mathbb{F}(\Sigma, I)
\end{tikzcd}$$

The notations $\pi_{\mathbb{F}}$ and $\pi_{\mathbb{M}}$ are shortened to π when there is no possible confusion. In the sequel, we often simplify the notations by denoting a trace by any of its representatives, that is by identifying w and $\pi(w)$. For instance, 1 denotes the empty trace.

When $I = \varnothing$, the corresponding trace monoid is the free monoid Σ^* and the corresponding trace group is the free group $\mathbb{F}(\Sigma)$. When $I = \Sigma \times \Sigma \backslash \{(a,a), a \in \Sigma\}$, the corresponding trace monoid is the free commutative monoid $\mathbb{N}^{\Sigma}$ and the corresponding trace group is the free commutative group $\mathbb{Z}^{\Sigma}$.

Given a word $w \in \Sigma^*$ and a letter $a \in \Sigma$, denote by $|w|_a \in \mathbb{N}$ the number of occurrences of the letter in the word.

We now give several definitions which are stated in order to hold both in a trace monoid and in a trace group. For the trace monoid only, the same definitions take a simpler form, as the reader can convince himself.

The *length* of a trace t, denoted $|t|$, is the minimum of the lengths of its representatives: $|t| = \min\{|x|, x \in \pi^{-1}(t)\}$. This definition of the length is consistent with the one given in (3). Given a trace t and a letter a, define $|t|_a = \min_{w \in \pi^{-1}(t)} |w|_a$, and define the *alphabet* of t as $\mathrm{alph}(t) = \{a \mid |t|_a > 0\}$.

A trace t is a *clique* if

$$\forall a \in \mathrm{alph}(t), |t|_a = 1 \quad \text{and} \quad \forall a, b \in \mathrm{alph}(t), ab = ba\,.$$

We denote the set of cliques by $\mathfrak{C}$. Cliques are in one-to-one correspondence with the complete subgraphs (also called cliques in graph theory) of the independence graph.

An ordered pair $(u,v) \in \mathfrak{C}\backslash\{1\} \times \mathfrak{C}\backslash\{1\}$ is *Cartier-Foata (CF-) admissible* if

$$\forall b \in \mathrm{alph}(v), \exists a \in \mathrm{alph}(u),\ a = b \text{ or } ab \neq ba\,.$$

Observe that the above condition implies that $|uv| = |u| + |v|$.

Given a trace t, there is a uniquely defined sequence of non-empty cliques $(c_1, c_2, \ldots, c_m)$ such that $t = c_1 c_2 \cdots c_m$, and the ordered pair (c_j, c_{j+1}) is CF-admissible for all j in $\{1, \ldots, m-1\}$. This sequence is called the *Cartier-Foata (CF) decomposition* of the trace. For a proof in the trace monoid, see [3, Chap. I], and for an illustration, see Example 2.1.

We visualize traces using Viennot's representation as heaps of pieces [18, 17]. In the trace monoid, a trace corresponds to a *heap*. In the trace group, a trace corresponds to a *colored heap* (letters of Σ are associated with light gray pieces, letters of $\overline{\Sigma}$ with dark gray pieces, and, in the colored heap, consecutive pieces of type a and $\overline{a}$ cancel each other). Observe that the existence of the heap representation implies that the word problem is solvable in polynomial time in a trace group or monoid.

Example 2.1. The *Basic Example* to be used as illustration throughout the paper consists of $\Sigma = \{a, b, c\}$ and $I = \{(a,b), (b,a)\}$. We have represented in Figure 1, the traces in $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ and $\mathbb{F}(\Sigma, I)$ corresponding to the word $u = aa\bar{b}\bar{c}c\bar{a}\bar{b}$. We have $|\pi_{\mathbb{M}}(u)| = 7$ and $|\pi_{\mathbb{F}}(u)| = 3$. The CF decomposition of $\pi_{\mathbb{M}}(u)$ is $(a\bar{b}, a, \bar{c}, c, \bar{a}\bar{b})$ and the one of $\pi_{\mathbb{F}}(u)$ is $(a\bar{b}, \bar{b})$.

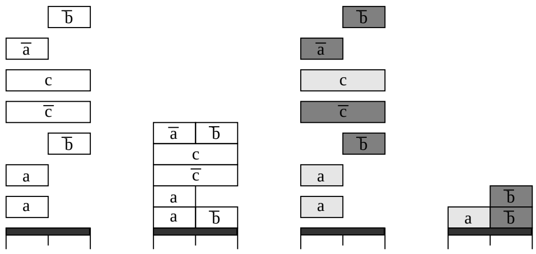


Fig. 1. A heap in $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ (left) and a colored heap in $\mathbb{F}(\Sigma, I)$ (right).

Let M be either a trace monoid or a trace group. A function T from M to $\mathbb{Z}$ is called a *(formal power) series (over M)*. It is convenient to use the notation $T = \sum_{m \in M} (T|m)m$. The set of series is denoted by $\mathbb{Z}\langle\langle M \rangle\rangle$. When $M = \mathbb{N}^{\Sigma}$, the set of series is denoted by $\mathbb{Z}[\![\Sigma]\!]$. As recalled in §1, the trace monoid $\mathbb{M}(\Sigma, I)$ has the finite decomposition property. It enables us to define the *Cauchy product* TU of two series T and U of $\mathbb{Z}\langle\langle \mathbb{M}(\Sigma, I) \rangle\rangle$ by $(TU|m) = \sum_{tu=m} (T|t)(U|u)$. The set $\mathbb{Z}\langle\langle \mathbb{M}(\Sigma, I) \rangle\rangle$ equipped with the usual sum and the Cauchy product forms a ring. We denote the unit of this ring by 1. This ring is cancellative, in particular a right or left inverse is an inverse. A series $T \in \mathbb{Z}\langle\langle \mathbb{M}(\Sigma, I) \rangle\rangle$ is inversible if and only if $(T|1) = 1$ or -1. If $(T|1) = 1$, the inverse of T is $(1-T)^* = \sum_{n \in \mathbb{N}} (1-T)^n$.

3 Rationality of the Growth Series

The results in this section are classical, at least for the trace monoid.

Let M be a monoid with a finite set of generators S and let π be the canonical surjection from S^* onto M. A set of *normal forms* of M in S^* is a subset L of S^* such that for each element m of M, we have $\#(\pi^{-1}(m) \cap L) = 1$. Assume that the set of normal forms L is a regular language of S^* and consists of minimal length representatives, i.e. $\forall w \in L, |w| = |\pi(w)|$. Then the growth series of M is rational. Indeed, project an unambiguous regular expression of L into $\mathbb{Z}[\![x]\!]$ by sending all the letters to x. The resulting rational series is $\mathrm{Gr}M$.

We now prove the rationality of the growth series for both the trace monoid and the trace group. It is convenient to denote by S the set of generators, that is $S = \Sigma$ for the trace monoid and $S = \widetilde{\Sigma}$ for the trace group.

Consider a total order $\leqslant$ on Σ, and the induced total order $\leqslant$ on $\widetilde{\Sigma}$ given by $a \leqslant \overline{a} \leqslant b \leqslant \overline{b}$ if $a \leqslant b, a \neq b$. Denote by $\leqslant_{\mathrm{lex}}$ the corresponding lexicographic order of S^*. For a trace t, define $\mathrm{Lex}(t) \in S^*$ as follows

$$\mathrm{Lex}(t) \in \pi^{-1}(t),\ |\mathrm{Lex}(t)| = |t|,\ \forall u \in \pi^{-1}(t), [|u| = |t|] \implies [\mathrm{Lex}(t) \leqslant_{\mathrm{lex}} u]\,.$$

In words, $\mathrm{Lex}(t)$ is the representative of t which is minimal for $\leqslant_{\mathrm{lex}}$ among all the representatives of minimal length. The set $\mathrm{Lex}(\mathbb{M}(\Sigma, I))$, resp. $\mathrm{Lex}(\mathbb{F}(\Sigma, I))$, is a set of normal forms of $\mathbb{M}(\Sigma, I)$, resp $\mathbb{F}(\Sigma, I)$, called the set of *lexicographic* *normal forms.* This set is a regular language. Define $I_< = \{(a,b) \mid (a,b) \in I, a < b\}$. For $a \in \Sigma$, set $I(a) = \{c \mid (a,c) \in I\}$. Then we have

$$\begin{aligned} \mathrm{Lex}(\mathbb{M}(\Sigma,I)) &= \Sigma^* - \bigcup_{(a,b)\in I_<} \Sigma^* b I(a)^* a \Sigma^* \\ \mathrm{Lex}(\mathbb{F}(\Sigma,I)) &= \widetilde{\Sigma}^* - [\bigcup_{(a,b)\in \widetilde{I}_<} \widetilde{\Sigma}^* b \widetilde{I}(a)^* a \widetilde{\Sigma}^* \cup \bigcup_{a\in\widetilde{\Sigma}} \widetilde{\Sigma}^* a\bar{a} \widetilde{\Sigma}^*]\,. \end{aligned} \tag{9}$$

The justification is simple; for the trace monoid it can be found in [5, Chapter 1.2]. We conclude that $\mathrm{Gr}\mathbb{M}(\Sigma,I)$ and $\mathrm{Gr}\mathbb{F}(\Sigma,I)$ are rational.

Given a trace t with CF decomposition $(c_1, c_2, \ldots, c_k)$, define $\mathrm{CF}(t) \in S^*$ by $\mathrm{CF}(t) = \mathrm{Lex}(c_1)\mathrm{Lex}(c_2)\cdots\mathrm{Lex}(c_k)$. The set $\mathrm{CF}(\mathbb{M}(\Sigma,I))$, resp. $\mathrm{CF}(\mathbb{F}(\Sigma,I))$, is a set of normal forms called the set of *Cartier-Foata normal forms.* This set is a regular language of S^*. Indeed, consider the deterministic automaton $\mathcal{CF}$ with states $\mathfrak{C}$, initial state 1, final states $\mathfrak{C}$, and with transitions

$$1 \xrightarrow{u} u \text{ if } u \in \mathfrak{C}\backslash\{1\}, \quad \text{and} \quad u \xrightarrow{v} v \text{ if } (u,v) \text{ is CF-admissible}\,. \tag{10}$$

The automaton $\mathcal{CF}$ recognizes a language L over $\mathfrak{C}^*$. Let μ be the morphism from $\mathfrak{C}^*$ to S^* defined by $\mu(c) = \mathrm{Lex}(c)$ for $c \in \mathfrak{C}$. The set $\mu(L)$ is clearly the set of Cartier-Foata normal forms. This provides an alternative proof of the rationality of the growth series.

Remark. Using either the Cartier-Foata or the lexicographic normal forms, it is not difficult to prove that trace groups (monoids) are automatic groups (monoids) in the sense of [11, 2].

Now let us restate some results from [3] already given in §1. Let $\mathbb{M}(\Sigma,I)$ be a trace monoid. In $\mathbb{Z}\langle\langle\mathbb{M}(\Sigma,I)\rangle\rangle$, the following identity holds

$$(\sum_{t\in\mathbb{M}(\Sigma,I)} t)\cdot(\sum_{c\in\mathfrak{C}}(-1)^{|c|}c) = 1\,. \tag{11}$$

To be consistent with §1, we call this identity a Möbius inversion formula for the trace monoid. Let $\mu_{\mathbb{M}(\Sigma,I)}(x) \in \mathbb{Z}[\![x]\!]$ be defined by

$$\mu_{\mathbb{M}(\Sigma,I)}(x) = \sum_{c\in\mathfrak{C}}(-1)^{|c|}x^{|c|}\,. \tag{12}$$

An easy consequence of (11) is the following identity in $\mathbb{Z}[\![x]\!]$:

$$\mathrm{Gr}\mathbb{M}(\Sigma,I)\cdot\mu_{\mathbb{M}(\Sigma,I)}(x) = 1\,. \tag{13}$$

In words, $\mu_{\mathbb{M}(\Sigma,I)}(x)$ is the formal inverse of the growth series. Hence, we get a third proof of the rationality of $\mathrm{Gr}\mathbb{M}(\Sigma,I)$, but also a much stronger result due to the explicit and combinatorial expression for $\mu_{\mathbb{M}(\Sigma,I)}(x)$.

The objective is to get analogs of (11) and (13) for trace groups.

4 Möbius Formula for the Trace Group

Given $u, v \in \widetilde{\Sigma}^*$ such that $\pi_{\mathbb{F}}(u) = \pi_{\mathbb{F}}(v)$ and $|u| = |v| = |\pi_{\mathbb{F}}(u)|$, then we have $\pi_{\mathbb{M}}(u) = \pi_{\mathbb{M}}(v)$. In other words, a trace of $\mathbb{F}(\Sigma, I)$ admits a unique representative of minimal length in $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$. We denote by

$$\phi : \mathbb{F}(\Sigma, I) \longrightarrow \mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$$

the corresponding map. Using the language of heaps, if t is a colored heap then $\phi(t)$ is the corresponding (non-colored) heap obtained by forgetting the colors. For a formal proof of this point, see [9, Prop. 2.4.7].

Define the set of *alternate traces* as the subset $\mathfrak{D}$ of $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ given by the regular expression

$$\mathfrak{D} = \sum_{c \in \mathfrak{C}} \Big[\prod_{x \in \mathrm{alph}(c)} x(\bar{x}x)^*(1+\bar{x}) \Big] , \tag{14}$$

where $\mathfrak{C}$ is the set of cliques of $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$. Figure 2 shows some alternate traces for the Basic Example.

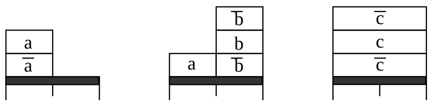


Fig. 2. Examples of elements of $\mathfrak{D}$.

Theorem 4.1. *In $\mathbb{Z}\langle\langle \mathbb{M}(\widetilde{\Sigma}, \widetilde{I}) \rangle\rangle$, we have the following identity:*

$$\Big(\sum_{t \in \phi(\mathbb{F}(\Sigma, I))} t \Big) \cdot \Big(\sum_{d \in \mathfrak{D}} (-1)^{|d|} d \Big) = 1 . \tag{15}$$

Proof. We are going to use the same type of *bijective* argument as in [18, Remark 5.2]. For a trace $t \in \mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$, define

$$\mathrm{Top}(t) = \{ a \in \mathrm{alph}(t) \mid \exists u, t = ua \} . \tag{16}$$

The set $\mathrm{Top}(t)$ contains all the pieces which are fully visible when the heap t is viewed from above. Given $t \in \mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ and $h \in \mathfrak{D}$, define $M(t,h) \subset \Sigma$ by $M(t,h) = \iota(\mathrm{Top}(h) \cup \{a \in \mathrm{Top}(t) \mid \forall b \in \mathrm{Top}(h),\ a\widetilde{I}b\})$. Observe that $M(t,h) = \varnothing$ if and only if $(t,h) = (1,1)$. We now are going to define an application

$$\psi : \phi(\mathbb{F}(\Sigma, I)) \times \mathfrak{D} \longrightarrow \phi(\mathbb{F}(\Sigma, I)) \times \mathfrak{D} .$$

We have $\psi(1,1) = (1,1)$. For $(t,h) \in \phi(\mathbb{F}(\Sigma, I)) \times \mathfrak{D}$ with $(t,h) \neq (1,1)$, we distinguish between four cases. Let a be the minimal letter of $M(t,h)$ (we assume that $\widetilde{\Sigma}$ is equipped with a total order defined as in §3).

1. If we have $t = t_1 a_1$ with $a_1 \in \{a, \overline{a}\}$ and $a, \overline{a} \notin \text{alph}(h)$, then $\psi(t,h) = (t_1, a_1 h)$.
2. If we have $h = a_1 h_1$ with $a_1 \in \{a, \overline{a}\}$ and $a, \overline{a} \notin \text{Top}(t)$, then $\psi(t,h) = (ta_1, h_1)$.
3. If we have $t = t_1 a_1$ with $a_1 \in \{a, \overline{a}\}$ and $h = a_1 h_1$, then $\psi(t,h) = (ta_1, h_1)$.
4. If we have $t = t_1 a_1$ with $a_1 \in \{a, \overline{a}\}$ and $h = \overline{a}_1 h_1$, then $\psi(t,h) = (t_1, a_1 h)$.

The dynamics of ψ for the Basic Example is illustrated in Figure 3. Here the order is $a \leqslant \bar{a} \leqslant b \leqslant \bar{b} \leqslant c \leqslant \bar{c}$.

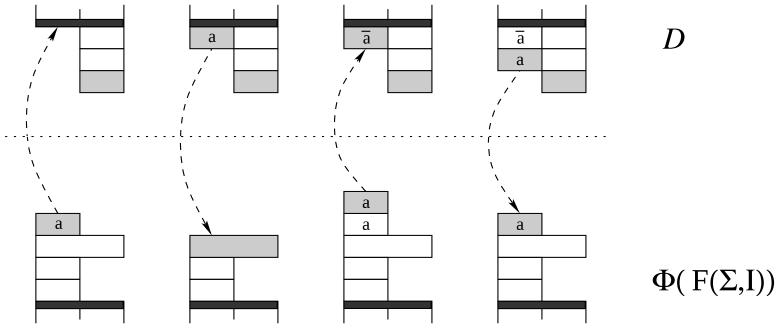


Fig. 3. The involution ψ. From left to right, the cases are 1,2,4, and 3.

Observe that the configurations of type aa are allowed in $\phi(\mathbb{F}(\Sigma, I))$ but not in $\mathfrak{D}$, and the other way round the configurations of type $a\bar{a}$ are allowed in $\mathfrak{D}$ but not in $\phi(\mathbb{F}(\Sigma, I))$. Notice also that the minimal letters of $M(t,h)$ and $M\psi(t,h)$ are the same. It follows easily from these two points that $\psi^2 = \text{Id}$. It is also clear that the only fixed point of ψ is $(1,1)$. Furthermore, if $(t,h) \neq (1,1)$ and $\psi(t,h) = (t',h')$, then we have $th = t'h'$ and $|h'| = |h| \pm 1$. So, we have

$$(-1)^{|h|} th = -(-1)^{|h'|} t'h' \ .$$

Set $S = \phi(\mathbb{F}(\Sigma, I)) \times \mathfrak{D}$. By decomposing the sum over the orbits of ψ, we get

$$\Big(\sum_{t \in \phi(\mathbb{F}(\Sigma,I))} t \Big) \cdot \Big(\sum_{h \in \mathfrak{D}} (-1)^{|h|} h \Big) = 1 + \sum_{(t,h) \in S \setminus \{(1,1)\}} (-1)^{|h|} th \ = \ 1 \, . \qquad \square$$

Corollary 4.1. *In $\mathbb{Z}[\![x]\!]$, the following identity holds:*

$$\text{Gr}\mathbb{F}(\Sigma, I) \cdot \mu_{\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})}(x/(1+x)) = 1 \, . \tag{17}$$

Observe that $\mu_{\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})}(x) = \mu_{\mathbb{M}(\Sigma, I)}(2x)$. Since the growth rate is the inverse of the modulus of a dominant singularity, we obtain immediately the following relations

$$\rho_{\mathbb{F}(\Sigma,I)} = \rho_{\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})} - 1 = 2\rho_{\mathbb{M}(\Sigma,I)} - 1 \, .$$

A consequence is that the results originally proved for $\rho_{\mathbb{M}(\Sigma,I)}$ in [12, 14] can be directly transferred to $\rho_{\mathbb{F}(\Sigma,I)}$. For instance, $\rho_{\mathbb{F}(\Sigma,I)} = 1$ iff $\mathbb{F}(\Sigma,I) = \mathbb{Z}^{\Sigma}$. Furthermore, we get the following proposition.

Proposition 4.1. *Set* $a_n = \#\{t \in \mathbb{F}(\Sigma,I) \mid |t| = n\}$.

1. If the dependence graph of $\mathbb{F}(\Sigma,I)$ *is connected, then* $\mathrm{Gr}\mathbb{F}(\Sigma,I)$ *has a unique dominant singularity which is positive real and of order 1. Consequently* $a_n \sim c\rho^n_{\mathbb{F}(\Sigma,I)}$.

2. If the dependence graph of $\mathbb{F}(\Sigma,I)$ *is non-connected, denote by* $(\mathbb{F}_s)_s$ *the trace groups whose dependence graphs are the maximal connected subgraphs. The series* $\mathrm{Gr}\mathbb{F}(\Sigma,I)$ *has a unique dominant singularity equal to* $\rho_{\mathbb{F}(\Sigma,I)} = \min_s \rho_{\mathbb{F}_s}$, *and whose order is* $k = \#\{s, \rho_{\mathbb{F}_s} = \rho_{\mathbb{F}(\Sigma,I)}\}$. *Consequently* $a_n \sim cn^{k-1}\rho^n_{\mathbb{F}(\Sigma,I)}$.

Example 4.1. For the free group, we have $\mathrm{Gr}\mathbb{F}(\Sigma) = (1+x)/(1-(2\#\Sigma-1)x)$ and the growth rate is $\rho_{\mathbb{F}(\Sigma)} = (2\#\Sigma - 1)$. For the free commutative group, we have $\mathrm{Gr}\mathbb{Z}^{\Sigma} = (1+x)^{\#\Sigma}/(1-x)^{\#\Sigma}$ and $\rho_{\mathbb{Z}^\Sigma} = 1$.

Consider now the Basic Example. Applying Theorem 4.1, the formal inverse of the characteristic series $(\sum_{t\in\phi(\mathbb{F}(\Sigma,I))} t)$ is

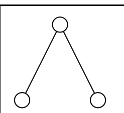
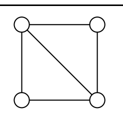
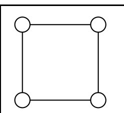
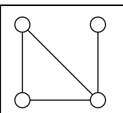
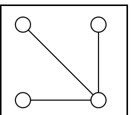
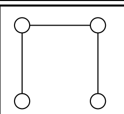
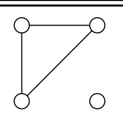
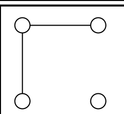
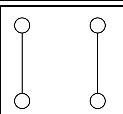
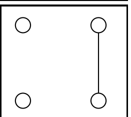
$$1 - \sum_{x\in\widetilde{\Sigma}} x(\overline{x}x)^*(\overline{x}-1) + \sum_{\iota(x)=a,\iota(y)=b} x(\overline{x}x)^*(\overline{x}-1)y(\overline{y}y)^*(\overline{y}-1)\,.$$

Using (13), we have $\mathrm{Gr}\mathbb{M}(\widetilde{\Sigma},\widetilde{I}) = (1-6x+4x^2)^{-1}$. Applying Corollary 4.1, we get the growth series

$$\mathrm{Gr}\mathbb{F}(\Sigma,I) = (1+x)^2/(1-4x-x^2)\,.$$

The Taylor expansion of the series around 0 is $\mathrm{Gr}\mathbb{F}(\Sigma,I) = 1 + 6x + 26x^2 + 110x^3 + \cdots + 635622x^9 + O(x^{10})$. For instance, there are 635622 traces of length 9. The growth rate of the trace group is $(\sqrt{5}+2) = 4.236\ldots$

In the table below, we give the growth rate for the other trace groups such that $\#\Sigma \leqslant 4$. The value α in the table is the inverse of the smallest root of $1 - 5x - x^2 - 3x^3$. Numerically, we have $\alpha = 5.29\ldots$

(Σ, I)					
$\rho_{\mathbb{F}}$	3	3	3	$\sqrt{5}+2$	5
(Σ, I)					
$\rho_{\mathbb{F}}$	5	α	$3+2\sqrt{2}$	$3+2\sqrt{2}$	$2\sqrt{3}+3$

Lifting to the free monoid. Set $M = \mathbb{F}(\Sigma, I)$ or $\mathbb{M}(\Sigma, I)$ and $S = \widetilde{\Sigma}$ or Σ respectively. Recall that Lex was defined in §3 and define the application $\overline{\mathrm{Lex}} : M \to S^*$ as follows:

$$\overline{\mathrm{Lex}}(t) \in \pi^{-1}(t),\ |\overline{\mathrm{Lex}}(t)| = |t|,\ \forall u \in \pi^{-1}(t),\ [|u| = |\overline{\mathrm{Lex}}(t)|] \Rightarrow [u \leqslant_{\mathrm{lex}} \overline{\mathrm{Lex}}(t)].$$

In words, $\overline{\mathrm{Lex}}(t)$ is the maximal representative of minimal length of t. Recall that $\mathrm{Lex}(M)$ is a regular language whose complement in S^* is a two-sided ideal, see (9). We denote by B the basis of this ideal. An *overlapping word* is a word $w \in S^*$ which can be written as $w = w_1 \cdots w_n$ with $w_1 \in S, w_i \in \mathrm{Lex}(M), w_i w_{i+1} \in S^*B \backslash S^*BS^+$. The set of overlapping words is denoted by $\mathcal{O}(M)$. It is proved in [5, Theorem 4.4.2] that, in $\mathbb{Z}\langle\!\langle \Sigma^* \rangle\!\rangle$, we have

$$\Big(\sum_{u \in \mathrm{Lex}(M)} u\Big) \cdot \Big(\sum_{v \in \mathcal{O}(M)} (-1)^{|v|} v\Big) = 1\,. \tag{18}$$

In [5, Chapter 4.3] (see also [4]), more precise results are given for the trace monoid only: $\mathcal{O}(\mathbb{M}(\Sigma, I))$ is finite if and only if the relation $I_<$ is transitive, and in this case, $\mathcal{O}(\mathbb{M}(\Sigma, I)) = \overline{\mathrm{Lex}}(\mathfrak{C})$. Then, the identity (18) can be viewed as a *lifting* to the free monoid of the Möbius inversion formula (11). Here we propose an analog of this last result for trace groups. Recall that the set $\mathfrak{D}$ of alternate traces was defined in (14).

Proposition 4.2. *If the relation $I_<$ is transitive, then $\mathcal{O}(\mathbb{F}(\Sigma, I)) = \overline{\mathrm{Lex}}(\mathfrak{D})$.*

The proof is easy and left as an exercise. In particular, in $\mathbb{Z}\langle\!\langle \widetilde{\Sigma}^* \rangle\!\rangle$, we have

$$\Big(\sum_{u \in \mathrm{Lex}(\mathbb{F}(\Sigma, I))} u\Big) \cdot \Big(\sum_{v \in \overline{\mathrm{Lex}}(\mathfrak{D})} (-1)^{|v|} v\Big) = 1\,. \tag{19}$$

This identity is a lifting to the free monoid of the identity (15). Observe that it is not always possible to find a total order $\leqslant$ such that $I_<$ is transitive.

The monoid $\mathbb{MI}(\widetilde{\Sigma}, \widetilde{I})$. In between $\mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ and $\mathbb{F}(\Sigma, I)$ lies the *involutive trace monoid* $\mathbb{MI}(\widetilde{\Sigma}, \widetilde{I}) = \langle\, \widetilde{\Sigma} \mid \bar{a}a = 1, \forall a \in \Sigma;\ ab = ba, \forall (a, b) \in \widetilde{I}\,\rangle$. Consider the trace monoid $\mathbb{M}(\widetilde{\Sigma}, \overline{I})$ where $\overline{I} = \widetilde{I} \cup \{(a, \bar{a}), a \in \widetilde{\Sigma}\}$. Consider the canonical injection $f : \mathbb{MI}(\widetilde{\Sigma}, \widetilde{I}) \longrightarrow \mathbb{M}(\widetilde{\Sigma}, \widetilde{I})$ and the canonical surjection $g : \mathbb{M}(\widetilde{\Sigma}, \widetilde{I}) \longrightarrow \mathbb{M}(\widetilde{\Sigma}, \overline{I})$. Clearly $g \circ f$ is a bijection from $\mathbb{MI}(\widetilde{\Sigma}, \widetilde{I})$ into $\mathbb{M}(\widetilde{\Sigma}, \overline{I})$. Hence, the generating series of $\mathbb{MI}(\widetilde{\Sigma}, \widetilde{I})$ are obtained directly using $\mathbb{M}(\widetilde{\Sigma}, \overline{I})$.

5 The Height Generating Series

The set $\mathfrak{C}\backslash\{1\}$ is a set of generators for the trace monoid (group) since it contains Σ ($\widetilde{\Sigma}$). To avoid confusions, we denote the length according to $\mathfrak{C}\backslash\{1\}$ (see (3)) by $h(.)$ and we call it the *height*. Consider a trace t with CF decomposition $(c_1, c_2, \ldots, c_m)$, then the height of t is $h(t) = m$. In the visualization of traces using heaps of pieces, the height corresponds precisely to the height of the heap. The motivation for studying the height is explained at the end of the section.

The *height series* is the growth series with respect to $\mathfrak{C}\setminus\{1\}$, i.e. the series of $\mathbb{Z}[\![x]\!]$ defined by $\sum_t x^{h(t)}$. This series is rational since it is recognized by the automaton $\mathcal{CF}$ (see (10)) modified by setting the labels to x. Consider a trace group $\mathbb{F}(\Sigma, I)$. Let $\mathbb{F}_n$, respectively ${}_n\mathbb{F}$, be the set of traces of length, respectively height, n. Assuming existence of the limits, define

$$\lambda_{\mathbb{F}(\Sigma,I)} = \lim_n \frac{\sum_{t\in\mathbb{F}_n} h(t)}{n\#\mathbb{F}_n}, \quad \gamma_{\mathbb{F}(\Sigma,I)} = \lim_n \frac{\sum_{t\in{}_n\mathbb{F}} |t|}{\#_n\mathbb{F}} \tag{20}$$

$$\lambda^*_{\mathbb{F}(\Sigma,I)} = \lim_n \frac{\sum_{w\in\tilde{\Sigma}^n} h\circ\pi(w)}{n\#\tilde{\Sigma}^n}\,. \tag{21}$$

Observe that $\lambda_{\mathbb{F}(\Sigma,I)}$ is the asymptotic average height with respect to the uniform distribution over $\mathbb{F}_n$. An analog interpretation holds for $\gamma_{\mathbb{F}(\Sigma,I)}$ and $\lambda^*_{\mathbb{F}(\Sigma,I)}$ with respect to uniform distributions over ${}_n\mathbb{F}$ and $\tilde{\Sigma}^n$. Analogs of these quantities for the trace monoid have been studied by several authors, see [14] and the references therein.

Proposition 5.1. *The limits in (20) exist and the numbers $\lambda_{\mathbb{F}(\Sigma,I)}$ and $\gamma_{\mathbb{F}(\Sigma,I)}$ are algebraic.*

The proof is omitted here (it is given in [1]) since it is an adaptation of the argument given in [14, Proposition 5.1] for the trace monoid. The two starting ingredients of the proof are the Möbius formula of Corollary 4.1 and the rationality of the series $F = \sum_{t\in\mathbb{F}(\Sigma,I)} x^{|t|}y^{h(t)}$. Observe that F is indeed rational since it is recognized by the automaton $\mathcal{CF}$ (see (10)) modified by setting the label of $u\to v$ to $x^{|v|}y$. The proof is constructive and provides an exact formula for $\lambda_{\mathbb{F}(\Sigma,I)}$ and $\gamma_{\mathbb{F}(\Sigma,I)}$:

$$\lambda_{\mathbb{F}(\Sigma,I)} = \frac{[G(x)(\rho-x)^{k+1}]_{|x=\rho}}{k\rho\cdot[F(x,1)(\rho-x)^k]_{|x=\rho}} \quad \gamma_{\mathbb{F}(\Sigma,I)} = \frac{[\tilde{G}(y)(\tilde{\rho}-y)^{\tilde{k}+1}]_{|y=\tilde{\rho}}}{\tilde{k}\tilde{\rho}\cdot[F(1,y)(\tilde{\rho}-y)^{\tilde{k}}]_{|y=\tilde{\rho}}}, \tag{22}$$

where $G=(\partial F/\partial y)(x,1)$, $\tilde{G}=(\partial F/\partial x)(1,y)$, where $\rho=\rho_{\mathbb{F}(\Sigma,I)}$ (resp. $\tilde{\rho}$) is the unique dominant singularity of $F(x,1)=\mathrm{Gr}\mathbb{F}(\Sigma,I)$ (resp. $F(1,y)$), and where k (resp. $\tilde{k}$) is the order of ρ (resp. $\tilde{\rho}$).

The existence of the limit in (21) follows directly from a subadditive argument. However $\lambda^*_{\mathbb{F}(\Sigma,I)}$ is in general not algebraic and its exact value can not be computed, except for trace groups which are direct products of free groups. In this last case, it is easily proved that $\lambda^*_{\mathbb{F}(\Sigma,I)} = \lambda^*_{\mathbb{M}(\Sigma,I)} - 1/\#\Sigma$ and that $\lambda^*_{\mathbb{M}(\Sigma,I)} = (\max_i \#\Sigma_i)/(\sum_i \#\Sigma_i)$ if $\mathbb{M}(\Sigma,I) = \prod_i \Sigma_i^*$. Otherwise, even for the Basic Example, the exact computation of $\lambda^*_{\mathbb{F}(\Sigma,I)}$ seems out of reach. In trace monoids, the situation is basically the same except that there are more cases for which $\lambda^*_{\mathbb{M}(\Sigma,I)}$ is exactly computable, see [14]. Let us mention as a curiosity that in all cases that we have checked numerically, we have $\lambda^*_{\mathbb{M}(\Sigma,I)} - \lambda^*_{\mathbb{F}(\Sigma,I)} \in [1/\#\Sigma - 2/100, 1/\#\Sigma]$.

A Discrete Event System application. It is natural to model the occurrence of events in a concurrent system using a trace monoid $\mathbb{M}(\Sigma, I)$ with $(a, b) \in I$ if the two events a and b can occur simultaneously (see [7]). Then, the two basic performance measures associated with a trace t are its length $|t|$ (the 'sequential' execution time) and its height $h(t)$ (the 'parallel' execution time). The quantity $\lambda^*_{\mathbb{M}(\Sigma,I)}$ has been singled out by various authors as a good measure of the efficiency of the concurrent system in a random environment. To go further, one may want to quantify the robustness of the system with respect to breakdowns or cancellations. In [1], a model is proposed which leads to this unexpected interpretation: the resistance to cancellations can be parametrized by $\rho_{\mathbb{F}(\Sigma,I)}$.

References

1. A. Bouillard. Rapport de DEA : Le groupe de traces. LIAFA research report 2002-13, Université Paris 7, 2002.
2. C. Campbell, E. Robertson, N. Ruškuc, and R. Thomas. Automatic semigroups. *Theoret. Comput. Sci.*, 250(1-2):365–391, 2001.
3. P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements.* Number 85 in Lecture Notes in Mathematics. Springer, 1969.
4. C. Choffrut and M. Goldwurm. Determinants and Möbius functions in trace monoids. *Discrete Math.*, 194(1-3):239–247, 1999.
5. V. Diekert. *Combinatorics on traces.* Number 454 in LNCS. Springer Verlag, 1990.
6. V. Diekert and A. Muscholl. Solvability of equations in free partially commutative groups is decidable. In *ICALP'01*, n. 2076 in LNCS, pages 543–554. Springer, 2001.
7. V. Diekert and G. Rozenberg, editors. *The Book of Traces.* World Scientific, 1995.
8. C. Droms, B. Servatius, and H. Servatius. Groups assembled from free and direct products. *Discrete Math.*, 109:69–75, 1992.
9. C. Duboc. *Commutations dans les monoïdes libres : un cadre théorique pour l'étude du parallélisme.* PhD thesis, Univ. Rouen, 1986.
10. G. Duchamp and D. Krob. Partially commutative Magnus transformations. *Internat. J. Algebra Comput.*, 3:15–41, 1993.
11. D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston. *Word processing in groups.* Jones and Bartlett, Boston, 1992.
12. M. Goldwurm and M. Santini. Clique polynomials have a unique root of smallest modulus. *Information Processing Letters*, 75(3):127–132, 2000.
13. G. Hardy and E. Wright. *An introduction to the theory of numbers.* Oxford at the Clarendon Press, 1979. 5-th edition.
14. D. Krob, J. Mairesse, and I. Michos. Computing the average parallelism in trace monoids. *Discrete Math.*, 2002. Accepted for publication. An abridged version appears in STACS'02, LNCS 2285, p.477-488, Springer.
15. G. Lallement. *Semigroups and Combinatorial Applications.* Wiley, New-York, 1979.
16. G.-C. Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 2:340–368, 1964.
17. A. Vershik, S. Nechaev, and R. Bikbov. Statistical properties of locally free groups with applications to braid groups and growth of random heaps. *Commun. Math. Phys.*, 212(2):469–501, 2000.
18. G.X. Viennot. Heaps of pieces, I: Basic definitions and combinatorial lemmas. In Labelle and Leroux, editors, *Combinatoire Énumérative*, number 1234 in Lect. Notes in Math., pages 321–350. Springer, 1986.