THÈSE de DOCTORAT DE SORBONNE UNIVERSITÉ
Opérée au sein de :
l'IMJ-PRG

Ecole Doctorale 386
Sciences Mathématiques de Paris Centre

Spécialité de doctorat : Mathématiques
Discipline : Mathématiques

Soutenue publiquement le 03/07/2020, par :
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Computation of Refined Enumerative Invariants
in Real and Tropical Geometry

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Remerciements

Avant toute chose, il convient de remercier les personnes sans qui rien de tout cela n'aurait été possible. Parmi elles, nombreuses sont celles à avoir eu sur moi une influence mathématique, et ce à différentes périodes du parcours qui m'a permis d'en arriver là. Comme il est parfois difficile de séparer les mathématiques du reste, même lorsqu'il n'est pas explicitement question de mathématiques, encore plus nombreuses sont les personnes qui auront la surprise d'apparaître dans ces remerciements grâce l'impact détourné qu'ils auront eu sur le contenu mathématique de ces dernières années, le plus souvent à travers la musique. Mention spéciale aux quelques personnes qui arrivent à se retrouver dans les deux catégories. Je tiens donc à remercier chaleureusement toutes ces personnes.

- Commençons par le début, les membres de ma famille, PB, RB, BB et MB pour l'accompagnement psychologique qu'une thèse exige, et avoir suscité chez moi la curiosité scientifique qui m'a poussé à m'orienter dans les maths.
- Je suis très reconnaissant envers Omid Amini et Hannah Markwig qui ont accepté d'être les rapporteurs, et les remercie de l'intérêt porté à mes travaux. Je remercie également Antoine Ducros, Grigory Mikhalkin, Johannes Nicaise et Alexandru Oancea d'avoir accepté d'être membre de mon jury de thèse et suis très honoré de leur présence.
- J'ai eu la chance en tant que moniteur à l'ENS de pouvoir faire parti de deux laboratoires. Je tiens donc à remercier l'ensemble des membres de l'équipe du DMA où j'ai pu profiter d'un cadre idéal, plus particulièrement les occupants des dessous du toit NB, NT, ST, OB, OdGF, RB, AD, ZC et AM, ainsi que mes cobureau TO-M, VB, BS, EC, NdR. Je remercie aussi LG même si elle bosse au rez-de-chaussée. Je tiens également à remercier l'ensemble des membres de l'équipe Analyse Algébrique de l'IMJ que j'ai pû cotoyer au cours des dernières années, ainsi que les personnels administratifs de chacune de ces équipes, tout particulièrement AT dont la bonne humeur et la réactivité ont illuminé le DMA.
- Je remercie l'ensemble des professeurs de mathématiques que j'ai pu avoir et rencontrer du lycée à la prépa, tout particulièrement VL, BR et FB qui ont su susciter en moi le goût des maths.
- Je remercie l'ensemble des personnes avec qui j'ai pu avoir des discussions mathématiques ces dernières années, et tout particulièrement l'ensemble de la communauté tropicale qui font de ce domaine quelque chose de vivant et un milieu sympathique dans lequel il est extrêmement plaisant et intéressant de travailler. Tout particulièrement II, PG, EB, AR, AJ, AN, BB, GM, ES, MM, KS, ClT, CA.
- Je remercie ST pour m'avoir fait découvrir l'organisation de séminaires avec le Réga, ainsi que les personnes avec qui j'ai eu le plaisir de l'organiser : SG, YC et HI.
- Comme les mathématiques ne se limitent pas à la recherche, je tiens également à remercier les élèves du DMA des trois dernières années pour les fantastiques TDs qu'ils m'ont procuré, et plus particulièrement ceux que j'ai eu la chance d'encadrer en mémoire : L-PC et VC, CB et MR, AP et HM-D, MT et VB.

- Enfin, comme il n'est pas toujours question de mathématiques, je tiens à remercier l'ensemble des personnes ayant accompagné les aventures musicales qui ont pu égayer ces trois dernières années. Cela inclut
  — Tous les membres passés et présents de BrassENS : FFR, A-CC, JF, VD, YM, CS, RP, J-EC, LP, ED, JB, AA, MC, CJ-B, RR, PM, PT, PN, CP, AT, LG, ML, OB.
  — Tous les membres de Brassage, et plus particulièrement LD, JR, CL, OC, MC, DG, SG, OD, J-DL, PL, PL, DG, MD et mes comparses trombonistes DG, PW, JC, CL, ainsi que tous les autres (la liste serait trop longue).
  — Les membres de la Bandarredurgens, plus particulièrement ceux qui n'ont pas déjà eu de remerciement personnalisé SG, AT, LG et MV, dont les capacités pianistiques continuent de m'impressioner.
  — Les trop nombreux membres de l'Ernestophone, et plus particulièrement les membres passés par la Blommobile, notamment AR, LS, AV qui ont réussi à supporter la musique de Brass Band et mes piètres compétences de copilote.
- Un grand merci à AA et PT pour ces trois années de colocation.
- Pour finir,la personne sans doute la plus importante à de nombreux points de vue, celle que je ne saurai jamais assez remercier pour les nombreuses années passées à travailler avec, les innombrables discussions mathématiques toujours éclairantes et les nombreux conseils, sa disponibilité, sa bienveillance et son inépuisable énergie, notamment pour les multiples re(re)lectures de tout ce que j'ai pu lui envoyer, et le choix d'un sujet de thèse riche et passionnant et une grande liberté dans la manière de le traiter : Ilia Itenberg, un directeur de thèse incroyable auprès duquel j'espère encore beaucoup apprendre.

Table des matières

Chapitre 1

Introduction

Sommaire

1.1 Historique

1.1.1 De l'énumération des courbes complexes

L'énumeration de courbes algébriques, bien qu'alors peu connue sous ce nom, remonte à l'antiquité. Elle consiste alors à résoudre des problèmes simples comme *"Combien de droites passent par deux points du plan ?"*, ou bien *"Combien de coniques passent par cinq points du plan ?"*. De telles questions, auxquelles la réponse est ici facile, peuvent aisément être généralisées à des questions auxquelles la réponse l'est beaucoup moins. Cela peut se faire de plusieurs manières : changer la nature des courbes que l'on regarde, changer l'espace dans lequel elles vivent, et changer la nature des contraintes que l'on impose. La première des généralisations d'un tel problème, qualifié d'*énumératif*, se fait à travers la donnée de plusieurs quantités : un degré d et un genre g. On essaie alors de compter combien de courbes de degré d et de genre g passent par un certain nombre de points prédéfinis du plan. Dans le cas du genre 0, le problème prend la forme suivante.

Problem 1.1.1. Combien de courbes rationnelles irréductibles de degré d passent par $3d-1$ points du plan ?

Pour les courbes irréductibles de genre plus élevé, le problème est le suivant.

Problem 1.1.2. Combien de courbes de degré d et genre g passent par $3d-1+g$ points du plan ?

Le nombre $3d-1+g$ de points choisis peut sembler arbitraire. En réalité, il s'agit du nombre de points à choisir pour obtenir une réponse qui soit un nombre fini différent de 0.

A priori, la réponse à cette question pourrait dépendre de la configuration de points choisie. Toutefois, à condition que l'on travaille sur le corps des nombres complexes $\mathbb{C}$, il n'en est rien. On note cette réponse N_d dans le cas du genre 0 et $N_{d,g}$ dans le cas général. L'indépendance vis-à-vis du choix de la configuration de points provient du fait que l'on cherche ici à calculer le degré de certaines variétés de *Séveri*. Cela n'a pas empêché la réponse de demeurer un mystère pendant fort longtemps. Les réponses pour $d=1$ ou 2 étaient connues dans l'antiquité, mais s'écoule presque deux mille ans avant que par exemple la valeur $d=4$ ne soit calculée. Il faut attendre les années 90 pour que M. Kontsevich donne une formule récursive [KM94] qui permet de calculer toutes les valeurs de N_d.

Theorem 1.1.3 (Kontsevich[KM94])
Les N_d satisfont la relation de récurrence

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-1} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}.$$

Les $N_{d,g}$ dans toute leur généralité sont quant à eux calculés par la formule de Caporaso-Harris [CH98], mais la formule est d'une complexité algorithmique qui la rend difficile à utiliser dans la pratique. De plus, contrairement à la formule de Kontsevich, cette formule ne fait pas intervenir que les $N_{d,g}$: ceux-ci font partie d'une famille d'invariants plus grande, calculés également par la formule, mais ce qui en rallonge l'application.

Cette première famille de problèmes énumératifs, traitant des courbes de degré d dans le plan projectif $\mathbb{C}P^2$, peut être généralisée à n'importe quelle surface torique, une famille de surfaces dont le plan projectif est un exemple particulier. Un autre exemple en est $\mathbb{C}P^1 \times \mathbb{C}P^1$.

1.1.2 Les débuts de la géométrie énumérative tropicale

Bien que les valeurs des $N_{d,g}$ fussent déjà connues, ne serait-ce qu'à travers la formule de Kontsevich dans le cas spécifique des N_d, le calcul de leurs valeurs est à nouveau rendu possible en 2002 par le théorème de correspondance de Mikhalkin [Mik05], utilisant cette fois-ci des méthodes de la géométrie tropicale. L'article où ce théorème est démontré marque les débuts de la géométrie énumérative tropicale et des relations que cette dernière peut entretenir avec la géométrie énumérative classique. La géométrie tropicale peut être vue naïvement comme la géométrie obtenue en remplaçant les opérations de base ” $+$ ” et ” $\times$ ” par ” max ” et ” $+$ ”, qui sont les opérations images par la valuation sur un corps valué. Dès lors, la géométrie tropicale peut être vue comme la géométrie résiduelle de la géométrie sur un corps qualifié de *non-archimédien*. La découverte du théorème de correspondance marque le départ d'un essor considérable du domaine tropical. La preuve du théorème de correspondance de Mikhalkin fait appel à une version sophistiquée de la méthode du patchwork de Viro [Vir84], lui aussi pouvant d'ailleurs être interprété de manière tropicale. Le théorème de correspondance fournit une manière de reconstruire les courbes d'un degré donné passant par une collection fixée de points proche de ce que l'on appelle la *limite tropicale*.

Les résultats de Mikhalkin, valables pour les courbes planes d'un genre quelconque, ont depuis fait l'objet de nombreuses généralisations et nouvelles preuves dans des cas plus spécifiques. Ceux-ci ont été étendus en dimension plus grande dans le cas du genre 0 par T. Nishinou et B. Siebert [NS06], et ont également été retrouvés plusieurs fois à travers diverses approches, comme par E. Shustin [Shu06a], ou par I. Tyomkin [Tyo12] ; [Tyo17].

Le théorème de correspondance de Nishinou et Siebert traite en outre une généralisation des problèmes énumératifs considérés précédemment dans des variétés toriques de dimension arbitraire. De ce fait, les contraintes sont également plus générales, ne se restreignant plus à la simple contrainte de passer par un point. Toutefois, si l'on désire conserver l'approche tropicale, nous sommes maintenant contraints de nous limiter à des courbes de genre 0, à cause de l'apparition en genre supérieure de courbes qualifiées de *super-abondantes*, pour lesquelles les méthodes des théorèmes de correspondance usuels ne s'appliquent pas.

1.1.3 L'énumération des courbes réelles

Bien que loin d'avoir épuisé les possibilités de l'énumération de courbes sur le corps des nombres complexes, il est tout à fait naturel de se poser les mêmes problèmes sur le corps $\mathbb{R}$ des réels. Hélas, contrairement au cas complexe, le nombre de courbes algébriques réelles passant par une configuration de points réels dépend de cette dernière. Cela n'est pas surprenant : en effet, déjà le nombre de racines d'un polynôme complexe générique de degré fixé est constant, mais celui d'un polynôme réel varie. Dès lors, il est vain d'espérer obtenir des invariants en comptant naïvement les solutions d'un problème énumératif sur le corps des réels.

Toutefois, J-Y. Welschinger [Wel05b] a montré que dans le cadre des courbes rationnelles dans une surface de del Pezzo, si celles-ci sont comptées convenablement, la réponse est à nouveau un nombre indépendant de la configuration de points réels choisis. De plus, en cas de non nullité, un tel invariant permet de fournir une borne inférieure au nombre de courbes rationnelles réelles passant par une configuration quelconque de points. Ce résultat se précise dans la situation suivante : si les courbes recherchées sont réelles, *i.e.* invariantes par la conjugaison complexe, il convient de choisir la configuration de points également invariante par conjugaison. On parle alors de *configuration réelle de points* et non plus de configuration de points réels. Il existe plusieurs manières de choisir une telle configuration : les points de la configuration peuvent soit être réels, soit venir par paire avec leur conjugué. Dans le cas où les points sont tous réels, on parle donc de *configuration de points réels*. Dans le cas du plan projectif, le résultat de Welschinger affirme que le compte des courbes d'un degré fixé et avec signe ne dépend que du nombre de points réels et paires de points complexes conjugués, pas de leur choix. Le signe des courbes est défini comme suit : une courbe rationnelle générique est nodale, ce qui signifie que ses singularités sont des points doubles non dégénérés. Pour les courbes réels, ces points doubles peuvent être de trois types :

- les paires de points doubles complexes conjugués,
- les points doubles hyperboliques, qui résultent de l'intersection de deux branches réelles, localement $x^2 - y^2 = 0$,
- les points doubles elliptiques, résultant de l'intersection de deux branches complexes conjuguées, localement $x^2 + y^2 = 0$.

Le signe de Welschinger d'une courbe réelle $\mathbb{R}C$ est défini comme étant $(-1)^{m(\mathbb{R}C)}$, où $m(\mathbb{R}C)$ désigne le nombre de points doubles elliptiques. Donnons maintenant la définition des invariants de Welschinger dans le cas particulier du plan projectif. Soient r et s tels que $r + 2s = 3d - 1$. Soit $\mathcal{P}$ une configuration générique de r points réels de $\mathbb{R}P^2$, et s paires de points complexes conjugués dans $\mathbb{C}P^2 \backslash \mathbb{R}P^2$. On pose alors

$$N_d^{\mathbb{R}}(\mathcal{P}) = \sum_{C \supset \mathcal{P}} (-1)^{m(\mathbb{R}C)},$$

où la somme a lieu sur les courbes rationnelles réelles de degré d dans $\mathbb{C}P^2$, qui passent par la configuration de points $\mathcal{P}$. Le théorème de Welschinger [Wel05b] dans le cas du plan projectif s'énonce alors comme suit.

Theorem 1.1.4 (Welschinger [Wel05b])
La valeur de $N_d^{\mathbb{R}}(\mathcal{P})$ ne dépend que de d et s, pas du choix de la configuration de points $\mathcal{P}$ tant que celle-ci est générique.

La valeur de cet invariant, appelé invariant de Welschinger, est dénotée $W_{d,s}$.

Le calcul des invariants de Welschinger est également rendu possible par la géométrie tropicale et le théorème de correspondance de Mikhalkin. Les invariants $W_{d,0}$, correspondant au cas où les points sont tous réels, sont calculés dans [Mik05]. Dans le cas où la configuration comporte aussi des paires de points complexes conjugués, cela est fait par E. Shustin dans [Shu06b].

Les résultats de Welschinger ont fait l'objet de quelques généralisations en dimension supérieure, comme dans [Wel05a]. Sans parler de généralisation, ils ont également fait l'objet de nombreux travaux, ne serait-ce que pour les calculer. Ainsi, dans les découvertes qui ont suivi leur introduction, on peut compter les travaux de I. Itenberg, V. Kharlamov et E. Shustin, qui ont pu démontrer une formule dans le style de la formule de Caporaso-Harris qui permet de calculer les invariants de Welschinger de certaines surfaces de del Pezzo [IKS09], et donner des estimations asymptotiques des invariants [IKS04], prouvant ainsi leur non-nullité. Cependant, comme remarqué dans [IKS03], le compte des courbes réelles avec signe de Welschinger ne semble pas fournir d'invariant au delà du genre 0. Les invariants de Welschinger sont encore un sujet fertile, comme en attestent les travaux d'E. Brugallé [Bru] ; [Bru18], traitant d'une invariance plus générale des invariants de Welschinger, ou ceux de E. Brugallé et N. Puignau [BP15]. De plus, suivant une idée de J. Solomon, X. Chen a pu démontrer des formules récursives [Che18a] ; [Che18b] entre les invariants de Welschinger.

1.1.4 Manipulations tropicales et invariants raffinés

Les méthodes de la géométrie tropicale utilisées dans [Mik05], [Shu06a], [Shu06b] et [NS06] permettent de ramener le calcul de certains invariants algébriques à divers problèmes combinatoires relevant uniquement de la géométrie tropicale. Ces problèmes tropicaux sont souvent plus simples à appréhender, puisque les difficultés techniques se sont cachées dans le théorème de correspondance. Cependant, ils sont parfois combinatorialement difficile à résoudre. Ainsi, on peut se demander s'il est possible de retrouver les preuves de certains résultats à l'aide de méthodes tropicales. C'est ainsi que les formules de Caporaso-Harris [GM07a] et la formule de Kontsevich [GM08], qui toutes deux permettent le calcul des réponses aux problèmes évoqués précédemment, ont pu être redémontrées, modulo les théorèmes de correspondance, de manière purement tropicale. De même, on peut se demander s'il est possible de retrouver l'invariance des comptes de courbes tropicales autrement qu'en important l'existence d'un invariant du côté complexe ou réel. Ainsi, l'invariance des comptes tropicaux aboutissant au calcul de $N_{d,g}$, auparavant déduite de leur invariance complexe, a pu être redémontrée [GM07b] par des méthodes uniquement tropicales. Les preuves tropicales se font parfois au prix d'un jeu combinatoire compliqué. Par exemple, la preuve tropicale de l'invariance des invariants de Welschinger dans le cas où la configuration de points comporte des paires de points complexes conjugués ne s'est faite que de manière détournée à l'aide de l'introduction des courbes dites *broccoli* [GMS13].

Au cours de la recherche de telles preuves d'invariance tropicale, il n'est pas rare de trouver de nouvelles quantités invariantes qui ne proviennent pas de géométrie classique. Par exemple, en restant sur les problèmes évoqués précédemment, la méthode tropicale de calcul des invariants de Welschinger, qui n'existent que pour les courbes rationnelles, s'applique également aux courbes tropicales de genre supérieur et fournit un invariant tropical. Dès lors, la manipulation des objets tropicaux permet d'obtenir des familles d'invariants dont la signification classique n'est pas toujours chose aisée. C'est également le cas des invariants *raffinés* et des multiplicités quantiques.

Pour comprendre l'introduction des invariants dits raffinés, il faut savoir que dans le théorème de correspondance de Mikhalkin, afin de calculer les invariants $N_{d,g}$ et $W_{d,0}$, on compte les courbes tropicales avec deux multiplicités qui sont chacune un produit sur les sommets de la courbe tropicale. Dans le premier cas, on tombe sur l'invariant complexe $N_{d,g}$, dans le second cas, si le genre est nul, on retrouve l'invariant de Welschinger $W_{d,0}$. Dans [BG16], F. Block et L. Göttsche proposent de rassembler ces multiplicités en une seule multiplicité polynomiale, également produit sur les sommets de la courbe. Si cette multiplicité polynomiale est évaluée en ± 1, elle redonne les deux multiplicités précédentes. Cette multiplicité, sortant a priori d'une manipulation combinatoire, semble apparaître à plusieurs endroits. Dans le problème des courbes de degré d et genre g passant par $3d-1+g$ points, Itenberg et Mikhalkin ont montré que le compte des courbes tropicales avec cette multiplicité fournit un invariant [IM13].

Depuis, de nombreux autres problèmes énumératifs tropicaux, jouissant ou non d'un théorème de correspondance les liants à un problème *classique*, se sont montrés être le théâtre d'un raffinement, sans que celui-ci ne possède pourtant d'explication par la géométrie classique. On obtient ainsi dans plusieurs situations un invariant polynomial tropical qui interpole entre deux invariants connus en géométrie classique : un invariant complexe pour sa valeur en 1, et un invariant réel en -1. Par exemple, les courbes broccoli ont elles aussi eu droit à leur raffinement [GS19], obtenant ainsi un invariant polynomial qui interpole entre les invariants de Welschinger $W_{d,s}$ avec paires de points complexes et les invariants dits descendants de genre 0, qui ont également un raffinement [BS19], tout comme les invariants descendants des courbes elliptiques [SS18].

Parmi les tentatives d'interprétation de ces mystérieux invariants qui ont été faites, plusieurs restent encore conjecturales. De plus, le lien entre ces pour l'instant diverses intérprétations reste inconnu. La principale conjecture attendue est celle de Göttsche et Shende [GS14]. Celle-ci stipule que le compte des courbes tropicales avec les multiplicités raffinées permet le calcul de la série génératrice des genre de Hirzebruch de certains schémas de Hilbert relatifs. En d'autre termes, le raffinement de la multiplicité complexe par la multiplicité raffinée du côté tropical doit correspondre au raffinement de la caractéristique d'Euler par le genre de Hirzebruch du côté classique. Certains travaux de J. Nicaise, S. Payne et F. Schroeter [NPS18] semblent aller dans cette direction. Cependant, cette interprétation n'est pas la seule, et les invariants raffinés semblent intervenir à d'autres endroits, comme dans le calcul de certains invariants de Gromov-Witten faisant intervenir des λ-classes [Bou19], ou en présentant certaines ressemblances avec d'autres invariants, comme les invariants de wall-crossing de type Donaldson-Thomas considérés par M. Kontsevich et Y. Soibelman [KS08].

L'un de ces autres endroits majeurs où les invariants raffinés apparaissent est une situation spécifique du premier problème considéré plus haut : combien de courbes rationnelles réelles de degré d passent par $3d-1$ points situés sur les axes de coordonnées ? Bien que la configuration de points puisse être considérée comme étant assez peu générique, le théorème de correspondance de [Mik05] s'applique et il est possible de calculer le nombre de solutions complexes. De plus, il est possible de définir un *indice quantique* [Mik17], et le compte des courbes peut être raffiné en fonction de la valeur de cet indice. En modifiant légèrement le problème énumératif, en comptant également les courbes passant par les configurations de points obtenues en symétrisant les points, le résultat de ce compte raffiné s'avère être indépendant du choix des points sur les axes. Le problème est détaillé dans la section suivante. Enfin, si ceux-ci sont tous réels, il peut être calculé à l'aide du théorème de correspondance [Mik05], ce qui fait alors curieusement apparaître les multiplicités raffinées.

Ces invariants, dont la définition et les résultats s'étendent aisément à n'importe quelle surface torique, sont des versions raffinées des problèmes énumératifs considérés par M. Gross, R. Padharipande et B. Siebert dans [GPS+10]. Dans cet article, on cherche à calculer des commutateurs dans un groupe appelé le *Tropical Vertex Group* afin d'effectuer l'équilibrage de certains objets appelés *Scattering Diagrams*, ou diagrammes de dispersion. Le calcul des commutateurs dans ce groupe se fait à travers le compte de courbes tropicales rationnelles ayant des directions à l'infinies prescrites. Ce problème est l'exacte *tropicalisation* du problème considéré plus tard par Mikhalkin dans [Mik17], ce qui signifie qu'ils sont reliés par un théorème de correspondance. Dans le cas non raffiné, l'existence du théorème de correspondance permettait déjà d'interpréter les invariants tropicaux apparaissant comme certains comptes de courbes algébriques dans les surfaces toriques. Le raffinement obtenu en calculant les mêmes commutateurs dans une version déformée du tropical vertex group considérée par S. Filippini et J. Stoppa [FS15] ainsi que T. Mandel [Man15] peut alors également être interprété du côté algébrique à l'aide d'indices quantiques.

Contrairement à la conjecture de Göttsche et Shende, ces dernières considérations de diagrammes de dispersions se généralisent partiellement aux courbes en dimension plus grande. Cela permet d'obtenir une tentative de raffinement du dénombrement des courbes en dimension plus grande, où la multiplicité des courbes tropicales donnée par les versions en plus grande dimension du théorème de correspondance n'est plus nécessairement un produit sur les sommets de la courbe, ce qui était une condition nécessaire au raffinement des multiplicités dans le cas planaire.

1.2 Cadre et Motivations

On s'intéresse particulièrement aux résultats exposés par Mikhalkin dans [Mik17], puisque les résultats exposés dans cette thèse les généralisent, et permettent le calcul de certains des invariants mentionnés dans ce même article. Afin de garder un cadre familier, on se restreindra dans cette introduction au cas du plan projectif $\mathbb{C}P^2$, mais les résultats sont valables dans n'importe quelle surface torique.

1.2.1 Invariants raffinés dans les surfaces toriques

Commençons par rappeler la définition des invariants introduits par Mikhalkin dans [Mik17]. Leur définition dans le cas général d'une surface torique est rappelée dans le chapitre 4. Voici brièvement comment ils sont définis dans le cas particulier du plan projectif $\mathbb{C}P^2$. On considère les courbes rationnelles de degré d dans $\mathbb{C}P^2$, et on choisit trois décompositions de d comme suit :

$$d = r_1 + 2s_1 = r_2 + 2s_2 = r_3 + 2s_3,$$

avec $r_1 \geqslant 1$. Sur chacun des trois axes de coordonnées, indexés $1, 2, 3$, on choisit respectivement r_i paires de points réels opposés, et s_i paires opposées de paires de points complexes conjugués, excepté sur le premier axe, où on choisit seulement $r_1 - 1$ paires de points réels opposés au lieu de r_1. On note $\mathcal{P}$ cette configuration de points, dite symétrique. Si l'on ne prend qu'un point réel dans chacune des paires de points opposés, et une paire de points complexes conjugués dans chacune des paires opposées de paires, on obtient une configuration de $3d-1$ points. On cherche alors les courbes rationnelles réelles de degré d, c'est à dire possédant une paramétrisation de la forme

$$t \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \longmapsto [X(t) : Y(t) : Z(t)],$$

avec X, Y et Z trois polynômes de degré d, passant par au moins un point de chaque paire opposée. Une telle courbe passe donc par un point de chaque paire réelle, et une des deux paires opposées de points complexes conjugués.

En enlevant les points envoyés sur les axes de coordonnées, on obtient une courbe rationnelle dans le tore complexe $(\mathbb{C}^*)^2$, qui est pourvu d'une application $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, obtenue en prenant le logarithme du module coordonnée par coordonnée. L'image de la courbe par cette application est appelée *amibe* de la courbe. Il est possible de calculer l'aire de la rétrotirette de ω la forme volume de $\mathbb{R}^2$ par l'application logarithmique sur l'une des deux composantes connexes S de $\mathbb{C}P^1 \backslash \mathbb{R}P^1$. Cela correspond à l'aire signée de l'amibe de la courbe. Mikhalkin montre que sous réserve que la courbe ait des intersection réelles ou imaginaires pures avec les axes de coordonnée, cette aire est un multiple demi-entier de π^2.

Theorem 1.2.1 (Mikhalkin[Mik17])
Soit $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^2$ une courbe rationnelle réelle, dont les points d'intersections avec les diviseurs toriques ont des coordonnées réelles ou imaginaires pures. Alors l'aire logarithmique

$$\mathcal{A}_{\mathrm{Log}}(S) = \int_{\varphi(S)} \mathrm{Log}^* \omega,$$

est un multiple demi-entier de π^2.

Le résultat s'étend aussi à certaines courbes qui ne sont pas rationnelles mais conservent des intersections réelles ou imaginaires pures avec les axes de coordonnée. Dans le cas où les points d'intersection complexes avec le bord torique sont imaginaires purs, ce multiple s'appelle l'*indice quantique* de la courbe rationnelle réelle orientée S. Si les intersections ne sont plus imaginaires pures, l'indice quantique est obtenu à partir de l'aire logarithmique grâce à un terme correctif dépendant des arguments des points d'intersection avec le bord torique. Cependant, la définition générale de l'indice quantique, bien que nécessaire pour l'introduction des invariants raffinés, n'est pas utile pour leur compréhension, ni pour le calcul qui est le notre puisque les points d'intersection complexes choisis sont imaginaires purs, et l'indice quantique coïncide donc avec l'aire logarithmique dans ce cas-ci. En comptant les courbes rationnelles réelles passant par $\mathcal{P}$, avec un signe $\sigma(S)$ dont la définition est rappelée dans le chapitre 4, et en fonction de la valeur de cet indice quantique, on obtient un polynôme $R_d(\mathcal{P})$:

$$R_d(\mathcal{P}) = \frac{1}{4}\sum \sigma(S) q^{k(S)} \in \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

Ce polynôme s'avère ne pas dépendre du choix de la configuration symétrique $\mathcal{P}$ mais seulement du nombre de paires de points complexes conjugués sur chacun des axes de coordonnées dans celle-ci.

Theorem 1.2.2 (Mikhalkin[Mik17])
La valeur de $R_d(\mathcal{P})$ ne dépend pas du choix de la configuration $\mathcal{P}$ mais seulement du nombre de paires de points complexes sur chacun des axes de coordonnée.

On note alors cet invariant dit raffiné $R_{d,(s_1,s_2,s_3)}$.

1.2.2 Invariants raffinés tropicaux

Du côté tropical, il est également possible de définir un invariant polynomial $N_d^{\partial,\mathrm{trop}}$. Pour cela, donnons brièvement la définition d'une courbe tropicale, les détails plus précis pouvant être trouvés dans le chapitre 3. Une *courbe tropicale* dans $\mathbb{R}^2$ est un graphe rectiligne dont les arêtes pondérées, éventuellement non bornées, sont à pente dans $\mathbb{Z}^2$, et dont les sommets vérifient la condition d'équilibre : la somme des pentes sortantes en chaque sommet multipliée par le poids de l'arête sortante doit faire 0. On peut également définir une notion de courbe rationnelle pour les courbes tropicales, et une notion de degré, donnée par la famille des pentes des arêtes non bornées. Les arêtes non bornées sont également appelées *arêtes infinies*.

On considère un degré de courbe tropicale, c'est à dire une famille de somme nulle $\Delta \subset N$, par exemple le degré d : $\{(-1,0)^d, (0,-1)^d, (1,1)^d\}$, où l'exposant d signifie que l'ensemble contient d copies de chacun des vecteurs $(0,-1)$, $-1,0)$ et $(1,1)$. Pour chacun des vecteurs n_e dans Δ sauf un, on choisit une droite D_e de pente n_e. On peut chercher les courbes tropicales rationnelles dont chacune des arêtes infinies e de pente n_e est contenue dans la droite D_e. On compte les courbes avec les multiplicités raffinées proposées par Block et Göttsche dans [BG16], rappelée en définition 3.2.14 :

$$N_\Delta^{\partial,\mathrm{trop}}(\mathcal{D}) = \sum m_\Gamma^q,$$

où la somme se fait sur les courbes tropicales rationnelles dont les arêtes infinies sont contenues dans les droites choisies D_e. Le résultat ne dépend pas du choix des droites D_e. Ce résultat, qui est un corollaire du théorème de correspondance, est prouvé de manière purement tropicale dans la sous-section 5.1.2. Cet invariant tropical est dénoté $N_\Delta^{\partial,\mathrm{trop}}$. Dans le cas des courbes de degré d, on le note simplement $N_d^{\partial,\mathrm{trop}}$.

Dans [Mik17], grâce au théorème de correspondance, Mikhalkin relie l'invariant raffiné classique $R_{d,(0,0,0)}$ à l'invariant raffiné tropical $N_d^{\partial,\mathrm{trop}}$.

Theorem 1.2.3 (Mikhalkin[Mik17])
On a la relation

$$R_{d,(0,0,0)} = (q^{1/2} - q^{-1/2})^{3d-2} N_d^{\partial,trop}.$$

Le résultat s'applique en réalité dans n'importe quelle surface torique pourvu que tous les points de la configuration symétrique $\mathcal{P}$ soient choisis réels.

1.3 Résultats

Les résultats démontrés ici généralisent ceux énoncés dans la section précédente, et ce dans plusieurs directions.

1.3.1 Calcul des invariants raffinés dans un cadre plus général

La relation entre invariants raffinés toriques et invariants raffinés tropicaux démontrée par Mikhalkin n'est valable que dans le cas où les points de la configuration symétrique $\mathcal{P}$ sont tous réels. On étend ce résultat au cas de paires de points complexes conjuguées se trouvant sur un même diviseur torique, ce qui dans le cas du plan projectif revient à supposer que le triplet (s_1, s_2, s_3) est de la forme $(s, 0, 0)$. Le résultat s'énonce alors comme suit. On note $N_{d,s}^{\partial,\mathrm{trop}}$ l'invariant raffiné tropical associé au degré $\{(-1,0)^{d-2s}, (-2,0)^s, (0,-1)^d, (1,1)^d\}$.

Theorem
(4.4.1) On a la relation

$$R_{d,(s,0,0)} = 2^s \frac{(q^{1/2} - q^{-1/2})^{3d-2-s}}{(q - q^{-1})^s} N_{d,s}^{\partial,trop}.$$

Pour ce faire, on utilise un théorème de correspondance tropical qui permet de compter effectivement les courbes passant par une configuration de points proche de la limite tropicale, où les points complexes sont choisis imaginaires purs. Dans ce cas particulier, l'indice quantique est égal à l'aire logarithmique. Toutefois, comme une paire de points imaginaires purs conjugués est égale à sa paire opposée, le compte raffiné est multiplié par 2^s pour obtenir l'invariant raffiné défini dans le cas de points complexes génériques.

Dans le cas général, *i.e.* où les paires de points complexes ne sont pas toutes sur le même diviseur, les invariants raffinés de Mikhalkin sont également reliés aux invariants raffinés tropicaux de degré correspondant. En notant

$$\Delta(s) = \{(0,-1)^{r_1}, (0,-2)^{s_1}, (-1,0)^{r_2}, (-2,0)^{s_2}, (1,1)^{r_3}, (2,2)^{s_3}\},$$

on a

$$R_{d,(s_1,s_2,s_3)} = 2^{s_1+s_2+s_3} \frac{(q^{1/2}-q^{-1/2})^{3d-2-s_1-s_2-s_3}}{(q-q^{-1})^s} N_{\Delta(s)}^{\partial,\mathrm{trop}}.$$

Cependant, comme le calcul ne peut a priori pas être mené avec des points complexes imaginaires purs, le calcul s'en trouve plus subtile, bien que fonctionnant avec les mêmes techniques. Pour plus de détails, voir [Blo20].

1.3.2 Formule récursive

Le théorème 4.4.1, tout comme celui de Mikhalkin dans le cas où les points sont tous réels, permet de ramener le calcul des invariants raffinés de [Mik17] à un calcul tropical associé à un problème énumératif tropical qu'il convient encore de résoudre.

Dans le chapitre 5, on démontre une formule récursive dans le style de la formule de Caporaso et Harris. Cette formule permet un calcul explicite des invariants tropicaux $N_\Delta^{\partial,\mathrm{trop}}$, et mène également à des formules récursives entre les invariants algébriques via le théorème de correspondance.

L'idée de la preuve de la formule est dans les grandes lignes la même que celle de la preuve tropicale de la formule de Caporaso et Harris [GM07a] : il s'agit de choisir un ensemble dégénéré de contraintes, de manière à connaître exactement la forme des courbes tropicales solutions du problème énumératif et ainsi le résoudre partiellement. Les courbes tropicales solutions se scindent alors en plusieurs morceaux qui correspondent aux différents termes de la formule. Dans le cas de la formule de Caporaso-Harris, la configuration est dégénérée car un point est beaucoup plus bas que tous les autres. Dans le cas de la formule du théorème 5.1.4, les droites sont choisies avec une droite qui passe loin de tous les points d'intersection entre les autres droites.

Étant donné la définition des invariants raffinés $N_\Delta^{\partial,\mathrm{trop}}$ comme nombre de courbes tropicales rationnelles ayant leurs arêtes infinies contenues dans des droites fixées à l'infini, la dégénération des contraintes, c'est-à-dire ici des droites, est la suivante. On choisit deux des droites, et on les envoie loin à l'infini. De loin, on a alors l'impression que les autres droites choisies sont concourantes puisqu'elles passent toutes par 0. (En réalité, 0 n'est pas sur les droites mais seulement très proche.) Les courbes tropicales solution du problème se scindent alors en plusieurs parties : une corde qui relie les deux arêtes infinies appartenant aux deux droites choisies, et des courbes tropicales d'un degré "plus petit" reliées à la corde par des arêtes radiales, comme on peut le voir sur les figures 5.7, 5.8 et 5.9.

La formule en elle-même pouvant sembler quelque peu rebutante, au moins autant que celle de Caporaso-Harris, l'énoncé exact peut être trouvé dans le théorème 5.1.4.

La formule a également une complexité assez importante, ce qui rend assez fastidieuse l'obtention de résultats grâce à cette dernière. Il est néanmoins possible d'obtenir des valeurs explicites pour les petits degrés, donnée dans le cas du plan projectif dans la section 5.3.

1.3.3 Courbes en dimension plus grande

Dans le dernier chapitre, on s'intéresse aux possibles généralisations des résultats de [Mik17] en dimension plus grande. On considère donc naturellement des courbes rationnelles dans des variétés toriques de dimension arbitraire, donc $\mathbb{C}P^n$ dans cette introduction. Tout d'abord, l'hypothèse que les points d'intersection entre la courbe et les diviseurs toriques aient des coordonnées réelles ou imaginaires pures, nécessaire à la définition de l'indice quantique dans [Mik17], conduit de la même manière à la définition d'une classe d'homologie quantique qui permet le calcul de l'aire signée de l'amibe de la courbe pour n'importe quelle 2-forme $\omega \in \Lambda^2(\mathbb{R}^n)^*$. Tout comme l'indice quantique de Mikhalkin, cette classe quantique peut être calculée dans le cas des courbes rationnelles, et des courbes dites de type torique I.

Aux vues de ses hypothèses d'existence, n'importe quel compte de courbes rationnelles peut a priori être raffiné par la valeur de la classe quantique sous réserve que les points d'intersection des courbes avec les diviseurs toriques soient réels. C'est par exemple le cas des problèmes énumératifs considérés dans le chapitre 6. Pour autant, le résultat ne conduit pas forcément à un invariant. On peut toutefois s'attendre à ce que ce soit le cas si le compte près de la limite tropicale, qui peut-être réalisé à l'aide de théorèmes de correspondance, est lui invariant. C'est effectivement le cas des problèmes énumératifs tropicaux considérés dans le chapitre 6.

On considère les courbes tropicales rationnelles d'un degré fixé $\Delta \subset \mathbb{Z}^n$. On fixe une 2-forme ω générique sur le réseau $\mathbb{Z}^n$, étendue au $\mathbb{R}^n$ dans lequel vivent les courbes tropicales. Pour chaque arête infinie e dirigée par n_e, on fixe un hyperplan H_e affine dirigé par $\omega(n_e, -)$. On cherche alors les courbes tropicales rationnelles de degré d telles que chaque arête infinie e se trouve dans l'hyperplan H_e. Ces conditions ne sont pas suffisantes pour n'avoir qu'un nombre fini de courbes tropicales. Pour cela, on ajoute des contraintes additionnelles. Dans chacun des cas considérés, le compte des courbes avec des multiplicités raffinées adéquates conduit à l'obtention d'un invariant raffiné tropical. Les résultats traitent des cadres suivants.

— Courbes tropicales dont l'une des arêtes infinies se trouve dans un plan fixé à l'avance.
— Courbe tropicale rencontrant une autre courbe tropicale fixée à l'avance.

Dans chacun de ces cas, on donne une multiplicité raffinée qui généralise la multiplicité de Block-Göttsche, et on prouve l'invariance du compte des courbes tropicales avec cette même multiplicité. Enfin, l'application du théorème de correspondance permet d'obtenir l'invariance d'un certain compte signé des courbes réelles proche de la limite tropicale.

1.4 Plan du manuscrit

Le chapitre 2 pose les notations relatives aux surfaces toriques utilisées dans le reste du manuscrit, tandis que le chapitre 3 est dévolu aux rappels et notations propres à la géométrie tropicale. On y définit les notions de courbe tropicale abstraite, paramétrée et réelle. On y définit également l'espace des modules des courbes tropicales rationnelles, qui permet de les manipuler de manière plus confortable et de formuler les problèmes énumératifs auxquels on s'intéresse. De tels problèmes énumératifs permettent d'associer des multiplicités naturelles aux courbes tropicales. On rappelle leur définition et fournit un algorithme permettant leur calcul. Enfin, on rappelle les procédés dits de *tropicalisation*, au coeur des théorèmes de correspondance, qui permettent d'obtenir une courbe tropicale à partir d'une famille de courbes algébriques classiques.

Le chapitre 4 rappelle la définition des indices quantiques et le dénombrement raffiné effectué dans [Mik17]. On donne ensuite une méthode de calcul de l'indice quantique dans le cas des courbes rationnelles, ainsi qu'un théorème de correspondance qui permet le calcul des invariants raffinés de Mikhalkin dans le cas du théorème 4.4.1.

Le chapitre 5 est un chapitre purement tropical qui fournit un algorithme de calcul pour les invariants tropicaux $N_\Delta^{\partial,\mathrm{trop}}$ à travers la formule récursive du théorème 5.1.4. Après avoir donné quelques exemples obtenus par l'application de la formule, on en déduit quelques formules récursives ayant traits aux invariants algébriques.

Enfin, le chapitre 6 traite la généralisation de certains des résultats précédents dans le cadre des variétés toriques de dimension arbitraire. On y aborde plusieurs variantes d'un même problème énumératif tropical, ainsi que son analogue classique, auquel il est relié par un théorème de correspondance. Pour le problème classique, on définit la notion de *classe quantique* pour certaines courbes algébriques, et on montre l'invariance proche de la limite tropicale pour le compte des courbes rationnelles raffiné par la valeur de la classe quantique dans le cas des problèmes énumératifs considérés.

Chapitre 2

Généralités et notations sur les variétés toriques

Dans ce chapitre on rappelle la définition des variétés toriques afin de préciser les notations qui interviennent dans le reste du manuscrit.

Sommaire

2.1 Variétés toriques

Toute l'énumération dont il est question dans cette thèse prend place dans les variétés dites toriques. Pour construire les variétés toriques, on considère un réseau N, appelé réseau des co-caractères, ou réseau des poids, ainsi que son réseau dual $M = \mathrm{Hom}(N, \mathbb{Z})$, appelé réseau des caractères. Le tore complexe associé au réseau N est $N \otimes \mathbb{C}^*$, qui est isomorphe à $(\mathbb{C}^*)^p$, où p est le rang commun de M et N. À cause de la structure multiplicative de $\mathbb{C}^*$, on notera z^n plutôt que $n \otimes z \in N \otimes \mathbb{C}^*$. Dès lors, les éléments de M sont des fonctions multiplicatives sur le tore complexe $N \otimes \mathbb{C}^*$. En effet, si $m \in M$, on notera χ^m la fonction

$$\chi^m(z^n) = z^{\langle m,n \rangle}.$$

La donnée d'un éventail $\Sigma \subset N_\mathbb{R}$, dont les directions sont données par des éléments de N, permet de définir une compactification partielle du tore $N \otimes \mathbb{C}^*$. Cette compactification est une vraie compactification si le support de l'éventail est $N_\mathbb{R}$ tout entier. La construction de la variété torique associée à l'éventail Σ, détaillée dans [Ful93], s'effectue comme suit. Soit $\sigma \subset \Sigma$ un cône de l'éventail. On considère le cône dual $\hat{\sigma} \subset M_\mathbb{R}$:

$$\hat{\sigma} = \{m \in M_\mathbb{R} \ : \ \forall n \in \sigma \ \langle m, n \rangle \geqslant 0\}.$$

On pose alors $U_\sigma = \mathrm{Spec}\ \mathbb{C}[\hat{\sigma} \cap M]$.

Example 2.1.1. - Si le cône σ est engendré par une base de N, son cône dual l'est également, et l'espace U_σ est une copie de l'espace affine $\mathbb{C}^p$.

- Si le cône σ est engendré par une partie $e_1, \ldots, e_k$ d'une base $(e_1, \ldots, e_p)$ de N, alors $\hat{\sigma}$ est le cône engendré par $(e_1^*, \ldots, e_k^*, \pm e_{k+1}^*, \ldots, \pm e_p^*)$. L'ouvert U_σ est alors isomorphe à $\mathbb{C}^k \times (\mathbb{C}^*)^{p-k}$. En particulier, si $\sigma = \{0\}$, U_σ est Spec $\mathbb{C}[M] \simeq N \otimes \mathbb{C}^*$.

$\lozenge$

Si ρ est un rayon de l'éventail Σ, l'ouvert U_ρ correspondant est isomorphe à $\mathbb{C} \times (\mathbb{C}^*)^{p-1}$. Il lui est associé un diviseur torique $\{0\} \times (\mathbb{C}^*)^{p-1}$. Intuitivement, la compactification de $N \otimes \mathbb{C}^*$ peut se voir par adjonction de diviseurs à l'infini.

Les U_σ sont ensuite recollés selon les applications suivantes : pour chaque inclusion de faces $\tau \subset \sigma$, on a une application ouverte $U_\tau \to U_\sigma$. La variété torique associée à l'éventail Σ est le recollement des U_σ le long de ces applications.

Example 2.1.2. - Si Σ est l'éventail engendré par les rayons dirigés par $(1,1)$, $(0,-1)$ et $(-1,0)$, on obtient le plan projectif $\mathbb{C}P^2$. Ses trois diviseurs toriques sont les axes de coordonnées donnés dans les coordonnées usuelles par $\{x=0\}$, $\{y=0\}$ et $\{z=0\}$.

- Si Σ est l'éventail engendré par les rayons dirigés par $(0,1)$, $(1,0)$, $(0,-1)$ et $(-1,0)$, on obtient la quadrique $\mathbb{C}P^1 \times \mathbb{C}P^1$. Celle-ci possède quatre diviseurs toriques, qui sont $\{0\} \times \mathbb{C}P^1$, $\{\infty\} \times \mathbb{C}P^1$, $\mathbb{C}P^1 \times \{0\}$ et $\mathbb{C}P^1 \times \{\infty\}$.
- Soit (e_i) la base canonique de $\mathbb{Z}^n$. Si Σ est l'éventail engendré par les $-e_i$ et $\sum_1^n e_i$, la variété torique associée est l'espace projectif $\mathbb{C}P^n$.

$\lozenge$

2.2 Courbes paramétrées dans les variétés toriques

On appelle courbe paramétrée dans une variété torique une application rationnelle de la forme

$$\varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*,$$

où $\mathbb{C}C$ est une surface de Riemann. Étant donné le choix d'une base de N, une telle application consiste en la donnée de p fonctions méromorphes sur $\mathbb{C}C$, où p est le rang de N. Le genre de la courbe paramétrée est le genre de la surface $\mathbb{C}C$. On dit que la courbe est *rationnelle* si elle est de genre 0, *i.e.* $\mathbb{C}C = \mathbb{C}P^1$.

Soit $x \in \mathbb{C}C$ un point où l'application rationnelle φ n'est pas définie. Le point x peut donc être un zéro ou un pôle de chacun des monômes χ^m. L'application

$$m \in M \longmapsto \mathrm{val}_x \varphi^* \chi^m,$$

qui associe à m l'ordre du zéro ou du pôle correspondant en x est un morphisme. Il correspond donc à un élément de N, appelé vecteur de poids de φ en x. Le vecteur de poids est nul si φ est définie en x. La famille Δ des vecteurs de poids non nul est appelée degré de la courbe paramétrée φ. Le théorème des résidus garantit que Δ est une famille de somme nulle.

Si $N = \mathbb{Z}^p$, on dit qu'une courbe est de degré d si les vecteurs de poids sont dans l'une des directions $-e_i$ ou $\sum_1^p e_i$, et que la somme des vecteurs poids dans chacune de ces directions fait d fois le vecteur en question.

Dans le cas particulier des courbes rationnelles, c'est à dire où la courbe est paramétrée par la droite projective $\mathbb{C}P^1$, la paramétrisation prend la forme suivante :

$$\varphi : t \longmapsto \chi \prod_{i=1}^{m} (t - \alpha_i)^{n_i} \in N \otimes \mathbb{C}^*.$$

Ici, χ désigne un élément de $N \otimes \mathbb{C}^*$, les α_i sont les points de $\mathbb{C}P^1$ où φ n'est pas définie, le vecteur de poids correspondant est n_i. Si l'on choisit une base (e_j) de N, et que l'on note (e_j^*) la base duale de M, l'application φ est simplement donnée en coordonnées par

$$t \longmapsto \left(\chi(e_j^*) \prod_{i=1}^{m} (t - \alpha_i)^{\langle e_j^*, n_i \rangle} \right)_j \in (\mathbb{C}^*)^p.$$

On peut observer que chacune des coordonnées est bien une fraction rationnelle sur $\mathbb{C}P^1$.

Étant donné une courbe de degré Δ, le degré Δ permet de définir un éventail Σ_Δ, qui est l'éventail dont les rayons sont dirigés par les vecteurs de Δ. Cet éventail n'est évidemment pas complet, mais il est toujours possible de compléter ce dernier en rajoutant des cônes de manière à le rendre complet. On obtient alors une variété torique compacte $\mathbb{C}\Delta$, éventuellement singulière.

En chaque point x de non définition de φ, il est possible de la prolonger en acceptant qu'elle soit à valeurs dans $\mathbb{C}\Delta$ plutôt que $N \otimes \mathbb{C}^*$. En effet, le point x est associé à un vecteur de poids n_x de Δ, lui-même associé à un rayon ρ de Σ_Δ, dirigé par n_x. On choisit une base de $\hat{\rho}$ de la forme $(m_1, \pm m_2, \dots, \pm m_p)$, où $(m_2, \dots, m_p)$ est une base de l'orthogonal $\rho^\perp$, et m_1 la complète en une base de M, choisi tel que $\langle m_1, n_x \rangle > 0$. En coordonnées, si t est une coordonnée locale au voisinage de x, φ est donc donnée par

$$t \longmapsto (\chi^{m_1}(\varphi(t)), \chi^{m_2}(\varphi(t)), \dots, \chi^{m_p}(\varphi(t))).$$

Cette application se prolonge en x par $(0, \chi^{m_2}(\varphi(x)), \dots, \chi^{m_p}(\varphi(x)))$, qui est bien définie car si $i \geqslant 2$, $\langle m_i, n_x \rangle = 0$, donc les limites $\chi^{m_i}(\varphi(x))$ sont bien définies et non nulles, et l'ordre de $\varphi^* \chi^{m_1}$ en x est $\langle m_1, n_x \rangle > 0$, donc la limite est 0. On a ainsi montré que l'application rationnelle φ se prolonge en une application à valeurs dans la variété torique $\mathbb{C}\Delta$.

2.3 Structure réelle d'une variété torique

La conjugaison complexe $z^n \mapsto \overline{z}^n$ est une involution anti-holomorphe de $N \otimes \mathbb{C}^*$ dans lui-même. Il est possible de la prolonger à la variété torique $\mathbb{C}\Delta$. Cela en fait une variété réelle, dont le lieu fixe est dénoté $\mathbb{R}\Delta$. Dès lors, une courbe réelle paramétrée est une application rationnelle

$$\varphi : \mathbb{C}C \to N \otimes \mathbb{C}^*,$$

où $\mathbb{C}C$ est une surface de Riemann munie d'une involution anti-holomorphe σ, de telle sorte que φ est équivariante vis-à-vis des involutions : $\varphi(\sigma(x)) = \overline{\varphi(x)}$. Dans la pratique, cela veut dire que les fonctions rationnelles coordonnées sont choisies à coefficients réels. Dans le cas des courbes rationnelles, où l'on munit $\mathbb{C}P^1$ de sa structure réelle canonique, l'application rationnelle

$$\varphi : t \in \mathbb{C}P^1 \longmapsto \chi \prod_{i=1}^{m} (t-\alpha_i)^{n_i},$$

est réelle si χ est à valeurs dans $\mathbb{R}^*$, les scalaires α_i sont réels, ou bien viennent par paire avec leur conjugué qui a le même vecteur poids. En nommant α_i les points réels, et $\beta_j, \overline{\beta_j}$ les points complexes, elles obtiennent alors une paramétrisation de la forme suivante :

$$\varphi : t \in \mathbb{C}P^1 \longmapsto \chi \prod_{i=1}^{r} (t-\alpha_i)^{n_i} \prod_{j=1}^{s} (t^2 - 2t\Re\beta_j + |\beta_j|^2)^{n'_j}.$$

En d'autres termes, cela signifie que chacune des fractions rationnelles coordonnées sont à coefficients réels.

2.4 Amibes et Coamibes

Les variétés toriques sont munies des applications naturelles du logarithme et de l'argument coordonnée par coordonnée, provenant des applications correspondantes pour $\mathbb{C}^*$. Dans l'écriture $N \otimes \mathbb{C}^*$ indépendante du choix des coordonnées, celles-ci prennent la forme suivante :

$$\mathrm{Log} : z^n \in N \otimes \mathbb{C}^* \longmapsto (\mathrm{Log}|z|)n \in N \otimes \mathbb{R} = N_{\mathbb{R}},$$

$$\arg : z^n \in N \otimes \mathbb{C}^* \longmapsto (\arg z)n \in N \otimes (\mathbb{R}/\pi\mathbb{Z}).$$

L'argument est pris ici modulo 2π, mais il est aussi possible de le prendre modulo π. L'image d'une courbe algébrique par le logarithme s'appelle l'*amibe*. Cette dernière est au coeur de nombreux résultats, et a été étudiée par M. Passare et H. Rullgård [PR+04], ainsi que Mikhalkin [MR01]. L'image par l'application des arguments s'appelle la *coamibe*. Bien que tout aussi naturelle, elle a été moins étudiée. Le lecteur intéressé pourra se reporter à [Joh13], [FJ14] ou [FJ15].

Chapter 3

Tropical geometry and tropical curves

Without pretending to make a complete introduction to the wide topic of tropical geometry, we give a presentation of the basic notions of tropical curves. For introduction to tropical geometry, one can refer to [BS14] for a short one, [Bru+15] for a finer one, or [MS15] for an even finer one with different point of view in the constructions. In the first section, we present the general definitions relating tropical curves. Secondly, we recall some basics about tropical enumerative geometry and the moduli spaces of rational tropical curves. Last, we describe the *tropicalization* process, that allows one to get tropical curves out of families of classical curves.

Sommaire

3.1 Tropical curves

3.1.1 Abstract tropical curves

Let $\overline{\Gamma}$ be a finite connected graph without bivalent vertices. Let $\overline{\Gamma}^0_\infty$ be the set of 1-valent vertices of $\overline{\Gamma}$, and $\Gamma = \overline{\Gamma} \backslash \overline{\Gamma}^0_\infty$. If m denotes the cardinal of $\overline{\Gamma}^0_\infty$, its elements are labeled with integers from $[\![1;m]\!]$. We denote by Γ^0 the set of vertices of Γ, and by Γ^1 the set of edges of Γ. The non-compact edges resulting from the eviction of 1-valent vertices are called *unbounded ends*. The set of unbounded ends is denoted by Γ^1_∞, while its complement, the set of bounded edges, is denoted by Γ^1_b. Notice that Γ^1_∞ is also labeled by $[\![1;m]\!]$. Let $l : \gamma \in \Gamma^1_b \mapsto |\gamma| \in \mathbb{R}^*_+ =]0;+\infty[$ be a function, called length function. It endows Γ with the structure of a metric graph by decreting that a bounded edge γ is isometric to $[0;|\gamma|]$, and an unbounded end is isometric to $[0;+\infty[$.

Definition 3.1.1. Such a metric graph Γ is called an *abstract tropical curve.*

An isomorphism between two abstract tropical curves Γ and Γ' is an isometry $\Gamma \to \Gamma'$. In particular, an automorphism of Γ does not necessarily respect the labeling of the unbounded ends since it only respects the metric. Therefore, an automorphism of Γ induces a permutation of the set $I = [\![1;m]\!]$ of unbounded ends.

Definition 3.1.2. Let Γ be an abstract tropical curve. A *real structure* on Γ is an involutive isometry $\sigma : \Gamma \to \Gamma$. A *real abstract tropical curve* is an abstract tropical curve enhanced with a real structure.

Since a real structure $\sigma : \Gamma \to \Gamma$ has to preserve the metric, for any bounded edge γ, one has $|\gamma| = |\sigma(\gamma)|$. The real structure also induces an involution on the set of ends $I = [\![1;m]\!]$ of Γ. The fixed ends are called *real ends* and the pairs of exchanged ends are called the *conjugated ends*, or *complex ends.* The fixed locus of σ is denoted by $\mathrm{Fix}(\sigma)$.

Example 3.1.3.
- The trivial real structure $\sigma = \mathrm{id}_\Gamma$ is the most common example, useful despite its simplicity.
- If Γ is an abstract tropical curve and $e, e' \in \Gamma^1_\infty$ are two unbounded ends adjacent to the same vertex w, another example is given by permuting the two unbounded ends e and e', and leaving the rest of the graph invariant.

◊

In Chapter 6, we also use the following definition.

Definition 3.1.4. Let Γ be a tropical curve. A *ribbon structure* on Γ is the data of a cyclic order on the adjacent edges of each vertex of the curve.

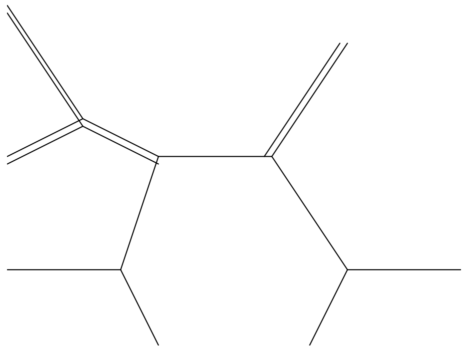

Figure 3.1 – Abstract real tropical curve with its real structure depicted by doubling the exchanged edges.

3.1.2 Parametrized tropical curves

We now define parametrized tropical curves in an affine space. Let N and M be two p-dimensional lattices dual from one another, and $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Sometimes, p is assumed to be equal to 2. Such a situation is also referred as *the planar case*. We now define parametrized tropical curves in $N_{\mathbb{R}}$.

Definition 3.1.5. A *parametrized tropical curve* in $N_{\mathbb{R}} \simeq \mathbb{R}^p$ is a pair (Γ, h), where Γ is an abstract tropical curve and $h : \Gamma \to N_{\mathbb{R}}$ is a map satisfying the following requirements:

- For every edge $E \in \Gamma^1$, the map $h|_E$ is affine. If we choose an orientation of E, the value of the differential of h taken at any interior point of E, evaluated on a tangent vector of length 1, is called the slope of h alongside E. This slope must lie in N.
- We have the so called *balancing condition*: at each vertex $V \in \Gamma^0$, if E is an edge containing V, and u_E is the slope of h along E when E is oriented outside V, then

$$\sum_{E:\partial E \ni V} u_E = 0 \in N.$$

Two parametrized curves $h : \Gamma \to N_{\mathbb{R}}$ and $h' : \Gamma' \to N_{\mathbb{R}}$ are isomorphic if there exists an isomorphism of abstract tropical curves $\varphi : \Gamma \to \Gamma'$ such that $h = h' \circ \varphi$.

Definition 3.1.6. A *real parametrized tropical curve* is a triplet (Γ, σ, h), where (Γ, h) is a parametrized tropical curve, σ is a real structure on Γ, and h is σ-invariant: $h \circ \sigma = h$.

Remark 3.1.7. In particular, two vertices that are exchanged by σ have the same image under h, and two edges that are exchanged by σ have the same slope and the same image. Such edges are called *double edges*. If they are unbounded, we call them a *double end*. Thus, the image $h(\Gamma) \subset N_{\mathbb{R}}$ may not be sufficient to recover the curve Γ used in the parametrization, and its real structure. For instance, a double end and a simple end with twice the slope have the same image. ♦

Remark 3.1.8. We could assume that $M = N = \mathbb{Z}^p$, but the distinction is now useful since the lattice M is a set of functions on the space $N_\mathbb{R}$ where the tropical curves live, while N is the space of the slopes of the edges of a tropical curve. Moreover, notice that we deal with tropical curves in the affine space $N_\mathbb{R}$, identified with its tangent space at 0. ♦

Let $h : \Gamma \to N_\mathbb{R}$ be a parametrized tropical curve. If $e \in \Gamma^1_\infty$ is an unbounded end of Γ, let $n_e \in N$ be the slope of h alongside e, oriented out of its unique adjacent vertex, *i.e.* toward infinity. The multiset

$$\Delta = \{n_e \in N | e \in \Gamma^1_\infty\} \subset N,$$

is called the *degree* of the parametrized curve. It is a multiset since an element may appear several times. Using the balancing condition, one can show that $\sum_{n_e \in \Delta} n_e = 0$.

We allow some unbounded ends to have slope zero. The image of such an end is contracted by h and consists in a single point, which is also the image of the unique adjacent vertex. Such unbounded ends are thus rather called *marked points*.

Remark 3.1.9. By a slight abuse, we say that two parametrized tropical curves have the same degree if their degrees differ only by marked points, meaning that the only vector that may have a different multiplicity on both degrees is 0. ♦

Example 3.1.10. Some degrees have a specific name. For instance, assume M and N have rank 2 and let (e_1, e_2) be a basis of N. We say that a curve is of degree d if $\Delta = \Delta_d = \{(-e_1)^d, (-e_2)^d, (e_1 + e_2)^d\}$. ♢

Definition 3.1.11.
- Let Γ be an abstract tropical curve. The *genus* of Γ is its first Betti number $b_1(\Gamma)$.
- A curve is *rational* if it is of genus 0.
- A parametrized tropical curve (Γ, h) is *rational* if Γ is rational.

Remark 3.1.12. A parametrized tropical curve is then rational if the graph that parametrizes it is a tree. ♦

3.1.3 Plane tropical curves

In this subsection, we also assume that M and N have rank 2, restricting ourselves to the planar case. In that case, classical curves have a description by polynomial equations, since they coincide with hypersurfaces. This is also the case for tropical curves. To define a plane tropical curve, we consider a *tropical polynomial*: for $x \in N_\mathbb{R}$, put

$$P(x) = \max_{u \in P_\Delta} (a_u + \langle u, x \rangle),$$

where $P_\Delta \subset M$ is the set of integer points of a convex lattice polygon, and $a_u \in \mathbb{R} \cup \{-\infty\}$ are the coefficients of the polynomial, different from $-\infty$ if $m \in M$ is a corner of P_Δ. The polygon P_Δ is called the *Newton polygon* of the plane tropical curve. If we choose a basis of M and N, the tropical polynomial P takes the following form:

$$P(x,y) = \max_{(i,j)\in P_\Delta} (a_{ij} + ix + jy).$$

The tropical polynomial P is then a piecewise affine convex function which is the maximum of a finite number of affine functions. We assume that P_Δ contains more than one point, otherwise P is an affine function. The tropical polynomial P induces a subdivision of P_Δ with the following rule: $u, u' \in P_\Delta$ are connected by an edge if $\{x \in N_\mathbb{R} : P(x) = a_u + \langle u, x\rangle = a_{u'} + \langle u', x\rangle\} \neq \emptyset$. The *corner locus* C of P, *i.e.* the set where at least two of the affine functions realize the maximum, is a rectilinear graph in $N_\mathbb{R}$. Equivalently, this is the set of points where P is not differentiable. The subdivision of P_Δ induced by P is *dual* to the corner locus C in the sense that there exists natural bijections between the following pairs of sets: edges of C and edges of the subdivision, vertices of C and polygons of the subdivision, components where P is smooth equal to one of the affine functions and vertices of the subdivision.

Definition 3.1.13. The *plane tropical curve* C associated to a tropical polynomial P is the corner locus of P, enhanced with the following weights on the edges of C: the weight of an edge is the lattice length of the dual edge in the subdivision of P_Δ. The polygon P_Δ is called the *degree* of the curve C.

Plane tropical curves can be characterized as finite weighted graphs (weights on the edges) with unbounded ends in $N_\mathbb{R}$, such that the edges are affine with slope in N, and the vertices satisfy the following balancing condition: for each edge E of weight w_E adjacent to a vertex V, let u_E be a primitive lattice vector directing E oriented outward from V, then, we have

$$\sum_{E \ni V} w_E u_E = 0.$$

For more details on plane tropical curves, see [BS14].

One can show that the image of a parametrized tropical curve is indeed a plane tropical curve with this definition. Moreover, the relation between the degree Δ of the parametrized tropical curve and the degree P_Δ of the plane curve is as follows. Let ω be the determinant form of N, a generator of $\Lambda^2 N$. It induces an isomorphism $\hat{\omega} : n \in N \mapsto \iota_n\omega = \omega(n, -) \in M$. There is a unique way to put on one another the vectors of Δ such that the convex-hull gives a convex polygon with positively oriented boundary consisting in the vectors of Δ. The image of this polygon by $\hat{\omega}$ is P_Δ.

Conversely, a plane tropical curve can always be parametrized by an abstract tropical curve if its Newton polygon is non-degenerate, thus leading to a parametrized tropical curve. If the Newton polygon is degenerate, the plane tropical curve is a union of parallel lines. Moreover, if C is a plane curve parametrized by $h : \Gamma \to N_\mathbb{R}$, the weight of an edge E of C can be recovered as the sum of the lattice lengths of the slopes of h on the edges $\gamma \in \Gamma$ which project onto E. However, there are often many ways of choosing a parametrization of a plane tropical curve, by non-isomorphic tropical curves. In fact, even the degree $P_\Delta \subset M$ does not uniquely determine the degree $\Delta \subset N$ of a parametrized curve parametrizing C: *e.g.* if C has un unbounded end of weight 2, a parametrizing graph Γ could have either an end e with $h|_e$ having slope of lattice length 2, or two ends of primitive slope with the same image by h.

We now define the usual concepts associated to plane classical curves in the case of plane tropical curves, starting with reducible curves.

Definition 3.1.14. - A plane tropical curve is *reducible* if it can be represented as the union of two distinct plane tropical curves.

- A plane tropical curve is *irreducible* if it is not reducible.

Going on with the definition of the genus, one needs to be careful since it depends on the chosen parametrization.

Definition 3.1.15. - The *genus* of a plane tropical curve is the smallest genus among its possible parametrizations.

- A plane tropical curve is *rational* if it is irreducible and can be parametrized by a rational tropical curve.

One can show that if C is a rational plane tropical curve with unbounded ends of weight 1, it admits a unique rational parametrization. More generally, we have the following statement.

Proposition 3.1.16
[Mik05] Let C be a rational plane tropical curve, and let u_e be a directing primitive lattice vector for each unbounded end e, oriented toward infinity. Let w_e be the weight of e. Then C is the image of a unique rational parametrized tropical curve of degree $\Delta = \{w_e u_e\}_e$.

3.1.4 Real parametrizations of a plane tropical curve

In this subsection, we extend Proposition 3.1.16 by describing the possible real rational parametrizations of a rational plane tropical curve, with unbounded ends of weights 1 or 2.

Let C be a rational plane tropical curve with unbounded ends of weight 1 or 2. Let $u_1, \ldots, u_r, 2v_1, \ldots, 2v_s$ be the weighted directing vectors of the unbounded ends of C, with vectors u_i, v_j being primitive vectors in N. We assume that $r \geqslant 1$. Let $h : \Gamma \to N_\mathbb{R}$ be the unique rational parametrization of C given by Proposition 3.1.16, which is of degree $\{u_i, 2v_j\}_{i,j}$. We now describe the parametrizations of C by real parametrized rational curves of degree $\{u_i, v_j^2\}_{i,j}$, which means that now all vectors are primitive, and each unbounded end of weight 2 is replaced with two ends of weight 1.

We define the subgraph Γ_{even} of Γ as the minimal subgraph satisfying both the following requirements:

- Every unbounded end of Γ of even weight (*i.e.* mapped to an end of C directed by $2v_j$ for some j) is in Γ_{even}.
- If V is a vertex of Γ and all edges adjacent to V but one are in Γ_{even}, then the remaining adjacent edge also is in Γ_{even}. Following [Shu06b], such a vertex is called an *extendable vertex.*

Remark 3.1.17. The subgraph Γ_{even} is the maximal graph on which we can "cut Γ in two" in order to obtain a new graph Γ', used to parametrize C, keeping a rational parametrization. Notice that on all the edges of Γ_{even}, the map h has an even slope. ♦

As C admits at least one odd unbounded end, each connected component $(\Gamma_{\text{even}})_i$ of Γ_{even} contains a unique *stem*, which is a non-extendable vertex. We orient the edges of $(\Gamma_{\text{even}})_i$ away from the stem. Then we say that a subset of points $\mathcal{R}_i \subset (\Gamma_{\text{even}})_i$ is *admissible* if no point of $\mathcal{R}_i$ is joint to another by an oriented path, and for each unbounded end e in $(\Gamma_{\text{even}})_i$, there is at least (and thus exactly one) point of $\mathcal{R}_i$ on the shortest path between the stem and e. Let $\mathcal{R} = \bigcup_i \mathcal{R}_i$. We then define a real abstract tropical curve $(\Gamma(\mathcal{R}), \sigma)$ with a map $h_{\mathcal{R}} : \Gamma(\mathcal{R}) \to N_{\mathbb{R}}$ that factors through $\Gamma(\mathcal{R}) \to \Gamma \to N_{\mathbb{R}}$ and makes it a real parametrized tropical curve.

Let $\Gamma_{\text{fix}}(\mathcal{R})$ be the closure of the union of the connected components of $\Gamma - \mathcal{R}$ not containing any even end. The abstract tropical curve $\Gamma(\mathcal{R})$ is obtained as the disjoint union of two copies of Γ, glued along $\Gamma_{\text{fix}}(\mathcal{R})$:

$$\begin{tikzcd} \Gamma_{\text{fix}}(\mathcal{R}) \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \Gamma \arrow[r] & \Gamma(\mathcal{R}) \end{tikzcd} .$$

In other terms, $\Gamma(\mathcal{R}) = \Gamma \coprod_{\Gamma_{\text{fix}}(\mathcal{R})} \Gamma$. It means that we have doubled the components of $\Gamma - \mathcal{R}$ containing the even ends. We denote by $\pi : \Gamma(\mathcal{R}) \to \Gamma$ the map obtained by gluing the identity maps of Γ. The complement of $\Gamma_{\text{fix}}(\mathcal{R})$ in Γ is called the *splitting graph.* It is a subset of Γ_{even}. The splitting graph is maximal if its closure is equal to Γ_{even}. The length function on $\Gamma(\mathcal{R})$ is defined as follows: we consider points of $\mathcal{R}$ as vertices of Γ, then, the length of an edge γ of $\Gamma(\mathcal{R})$ is the length of its image $\pi(\gamma)$ if it is an edge of $\Gamma_{\text{fix}}(\mathcal{R})$ and twice the length of $\pi(\gamma)$ otherwise. The involution σ is the automorphism of $\Gamma(\mathcal{R})$ that exchanges the two antecedents whenever there are two. The parametrized map $h_{\mathcal{R}} : \Gamma(\mathcal{R}) \to N_{\mathbb{R}}$ is the composition of π and h.

Remark 3.1.18. The map π really looks like a tropical cover, as defined in [CJM10] and [BM15]. However, it is not always the case. This is normal since the purpose of the notion of tropical cover is to mimick ramified covers between complex curves. The map π here plays the role of the quotient map by a real involution, which is not a ramified cover. ♦

Let γ be an edge of $\Gamma(\mathcal{R})$, and $n \in N$ be the slope of $\pi(\gamma)$. Then, one can easily check that the choice of length on $\Gamma(\mathcal{R})$ ensures that $h_{\mathcal{R}}$ has slope n if $\gamma \in \mathrm{Fix}(\sigma)$ and $\frac{n}{2}$ otherwise. However, as the edges of Γ_{even} have an even slope, it is still an element of N. One can check that the balancing condition is still satisfied. Therefore, $(\Gamma(\mathcal{R}), h_{\mathcal{R}}, \sigma)$ is a real parametrized tropical curve, of image C, and of degree $\{u_i, v_j^2\}_{i,j}$.

Proposition 3.1.19
Let C be a rational plane tropical curve of degree $P_\Delta \subset M$ having unbounded ends of weight 1 or 2. Let $\Delta \subset N$ be the degree associated to P_Δ consisting only of primitive lattice vectors. let $h : \Gamma \to N_\mathbb{R}$ be the unique rational parametrization of C given by Proposition 3.1.16. Using previous notations, every real rational parametrized curve of degree Δ having the image C is one of the curves $\Gamma(\mathcal{R})$.

Proof. The curves $(\Gamma(\mathcal{R}), h)$ provide real rational parametrizations of C. Conversely, if $h : (\Gamma', \sigma) \to N_\mathbb{R}$ is a real rational parametrization of C of degree $\{u_i, v_j^2\}$, then we have quotient curve Γ'/σ defined as follows. As a topological space, Γ/σ is the quotient by σ. The edge lengths are the same for edges in $\mathrm{Fix}(\sigma)$, and the edge length is divided by two for a pair of exchanged edges. Since h is σ-invariant, we have a quotient map $\tilde{h} : \Gamma'/\sigma \to N_\mathbb{R}$ and one can check that the above choice of edge length makes it into a parametrized tropical curve. The assumption on the weights of the unbounded ends of C ensures that the conjugated ends of Γ' are mapped to the even unbounded ends of C. Their weight is doubled when passing to the quotient. Thus, we get a rational parametrization of C of degree $\{u_i, 2v_j\}_{i,j}$. Therefore, it is isomorphic to Γ. Let $\pi : \Gamma \to \Gamma/\sigma$ be the quotient map.

The primitivity assumption on the degree ensures that near infinity, the points of the even unbounded ends of Γ have two antecedents by π. The other ends only have one. Let $\mathcal{R}$ be the boundary of $\mathrm{Fix}(\sigma)$.

— First of all, $\mathrm{Fix}(\sigma)$ is connected: if $p, q \in \mathrm{Fix}(\sigma)$, there is a unique shortest path in Γ between p and q, this path is then σ-invariant, thus in $\mathrm{Fix}(\sigma)$.

— Let Ξ be a connected component of $\Gamma \backslash \mathrm{Fix}(\sigma)$, the boundary of Ξ contains exactly one point of $\mathcal{R}$: at least one since $\Xi \neq \Gamma$, and at most one, otherwise the path between these points of $\mathcal{R}$ would lie in Ξ, and we have proven that such a path lies in $\mathrm{Fix}(\sigma)$. Thus, Ξ only contains even ends, and the construction of Γ_{even} ensures that the point of $\mathcal{R}$ on the boundary of Ξ is in Γ_{even}.

— Finally, we have proven that Γ is composed of $\mathrm{Fix}(\sigma)$, which is connected and has boundary $\mathcal{R}$, and components Ξ that are attached to $\mathrm{Fix}(\sigma)$ at those vertices. Thus, the configuration $\mathcal{R}$ is admissible: there is at least one point of $\mathcal{R}$ on the shortest path between the stem and an even end since the stem is in $\mathrm{Fix}(\sigma)$ and the end is not, and there is at most one since $\mathrm{Fix}(\sigma)$ is connected.

Finally, the set $\mathcal{R}$ being admissible, the graph Γ' is recovered as the curve $\Gamma(\mathcal{R})$. □

3.2 The moduli of tropical curves and tropical enumerative geometry

In this section we recall the definition of the moduli space of rational tropical curves, as described by A. Gathmann, M. Kerber and H. Markwig in [GKM09]. This space enables easy definitions and computations for the enumerative geometry of tropical curves, which is in our interest.

3.2.1 Moduli space of rational abstract tropical curves

Let Γ be an abstract tropical curve with m unbounded ends. We say that two tropical curves have the same combinatorial type if their underlying graphs, *i.e.* the tropical curves without their respective metrics, are the same.

Definition 3.2.1. The *combinatorial type* of a tropical curve is the homeomorphism type of its underlying labeled graph Γ, *i.e.* the labeled graph Γ without the metric.

Conversely, to give a graph a tropical structure, one just needs to specify the lengths of the bounded edges. If the curve is trivalent, meaning that every vertex is adjacent to exactly 3 edges, and has m unbounded ends, there are $m-3$ bounded edges, otherwise the number of bounded edges is $m-3-\mathrm{ov}(\Gamma)$, where $\mathrm{ov}(\Gamma)$ is the *overvalence* of the graph. The overvalence is given by $\sum_V (\mathrm{val}(V)-3)$, where V runs over the vertices of Γ, and $\mathrm{val}(V)$ denotes the valence of the vertex. Therefore, if we choose a labeling of the bounded edges, the set of curves having the same combinatorial type is in canonical bijection with $\mathbb{R}_{>0}^{m-3-\mathrm{ov}(\Gamma)}$, where the coordinates are the lengths of the bounded edges. If Γ is an abstract tropical curve, we denote by $\mathrm{Comb}(\Gamma)$ the set of curves having the same combinatorial type as Γ.

For a given combinatorial type $\mathrm{Comb}(\Gamma)$, the boundary of $\mathbb{R}_{\geqslant 0}^{m-3-\mathrm{ov}(\Gamma)}$ corresponds to curves for which the length of an edge is zero, and therefore corresponds to a graph having a different combinatorial type. This graph is obtained by deleting the edge with zero length and merging its extremities. We can thus glue together all the cones of the finitely many combinatorial types and obtain the *moduli space* $\mathcal{M}_{0,m}$ *of rational tropical curves with* m *unbounded ends.* Using the result from [SS04], it is a simplicial fan of pure dimension $m-3$ in a vector space of dimension $\binom{m}{2}-m$, and the top-dimensional cones correspond to trivalent curves. The combinatorial types of codimension 1 are called *walls.* For a more complete description, see [GKM09].

Definition 3.2.2. The *moduli space of rational tropical curves with* m *unbounded ends* is the simplicial fan $\mathcal{M}_{0,m}$.

3.2.2 Moduli space of parametrized rational tropical curves

We now turn our focus on the space of rational parametrized tropical curves. Given a rational abstract tropical curve Γ, if we specify the slope of every unbounded end, and the position of a vertex, we can define uniquely a parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$. Therefore, if $\Delta \subset N$ denotes the fixed set of slopes of the unbounded ends, the *moduli space* $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$ *of parametrized rational tropical curves of degree* Δ is $\mathcal{M}_{0,m} \times N_{\mathbb{R}}$, where the $N_{\mathbb{R}}$ factor corresponds to the position of the finite vertex adjacent to the first unbounded end.

Definition 3.2.3. The moduli space $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$ of parametrized rational tropical curves of degree Δ in $N_{\mathbb{R}}$ is naturally identified with $\mathcal{M}_{0,m} \times N_{\mathbb{R}}$.

Remark 3.2.4. Notice that we have two types of unbounded ends: the unbounded ends with a non-zero slope, which are true unbounded ends, and the unbounded ends which have a zero slope, which are called *marked points*. Such ends are sent to a unique point in $N_{\mathbb{R}}$. ♦

On this moduli space, for each unbounded end i of Γ, we have a well-defined evaluation map that associates to each parametrized curve (Γ, h) the position of the unbounded end, in the quotient of $N_{\mathbb{R}}$ by its direction. Let n_i be the slope of h along the unbounded end e_i of index $i \in [\![1; m]\!]$. The evaluation map is

$$\begin{array}{rrcl} \mathrm{ev}_i : & \mathcal{M}_0(\Delta, N_{\mathbb{R}}) & \longrightarrow & N_{\mathbb{R}}/\langle n_i \rangle \\ & (\Gamma, h) & \longmapsto & h(p) \text{ where } p \in e_i \end{array}.$$

If the slope n_i is 0, the image by the evaluation map is some point inside $N_{\mathbb{R}}$. For a marked point, we thus have

$$\begin{array}{rrcl} \mathrm{ev}_i : & \mathcal{M}_0(\Delta, N_{\mathbb{R}}) & \longrightarrow & N_{\mathbb{R}} \\ & (\Gamma, h) & \longmapsto & h(p) \end{array}.$$

Gathering the image of all unbounded ends, we get the total evaluation map:

$$\mathrm{ev} : \mathcal{M}_0(\Delta, N_{\mathbb{R}}) \longrightarrow \prod_{i=1}^{m} N_{\mathbb{R}}/\langle n_i \rangle.$$

This map can be used to impose constraints and find the parametrized tropical curves satisfying some additional property. For instance, imposing that some marked point belongs to a chosen tropical curve, or some unbounded end belongs to a chosen hyperplane. This is the domain of tropical enumerative geometry.

3.2.3 Tropical enumerative geometry and complex multiplicities

We now focus on a specific family of tropical enumerative problems, by looking at rational tropical curves that meet a bunch of constraints. To do that, we consider parametrized rational tropical curves of degree Δ in $N_{\mathbb{R}}$. Their moduli space is $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$. We allow vectors of Δ to be 0, so that the unbounded ends associated to these vectors are marked points on the curves. For each unbounded end i, let L_i be a primitive sublattice of $N/\langle n_i\rangle$, and let l_i be its corank. We denote by $L_i^{\mathbb{R}} = L_i \otimes \mathbb{R}$. Notice that if i corresponds to a marked point, L_i is a sublattice of N. Let $\mathcal{L}_i$ be a generic affine space in $N_{\mathbb{R}}/\langle n_i\rangle$ with slope $L_i^{\mathbb{R}}$. This amounts to the choice of a point λ_i in $N_{\mathbb{R}}/\left(\langle n_i\rangle \oplus L_i^{\mathbb{R}}\right)$. We consider the composition f of the evaluation map with the quotient maps by the $L_i^{\mathbb{R}}$:

$$f : \mathcal{M}_0(\Delta, N_{\mathbb{R}}) \to \prod_i N_{\mathbb{R}}/\left(\langle n_i\rangle \oplus L_i^{\mathbb{R}}\right).$$

The map f is called the *composed evaluation map.* Recall tat M and N have rank p. Assume that $\sum_i l_i = |\Delta| + p - 3$, so that the spaces have the same dimension. We can look for the parametrized tropical curves such that $f(\Gamma, h) = (\lambda_i)$, *i.e.* such that $\mathrm{ev}_i(\Gamma, h) \in \mathcal{L}_i$. In other terms, we look for the tropical curves of degree Δ that meet each one of the affine subspaces $\mathcal{L}_i$.

Remark 3.2.5. The linear constraints $\mathcal{L}_i$ could easily be replaced by tropical cycles Ξ_i of the same dimension, but we do not want to bother here with the definition of tropical cycles, so we restrict to these constraints. For more details, see [AR10]. ⧫

As the spaces have the same dimension, we expect a finite number of solutions if the affine spaces $\mathcal{L}_i$, *i.e.* the λ_i, are chosen generically. This is indeed the case.

Proposition 3.2.6
If the affine subspaces L_i are chosen generically, there is a finite number of parametrized tropical curves (Γ, h) such that $\mathrm{ev}_i(\Gamma, h) \in L_i$. Moreover, these curves are trivalent.

Remark 3.2.7. Another way to formulate the problem is to consider the image of $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$ by the evaluation map ev, living inside $\prod_i N_{\mathbb{R}}/\langle n_i\rangle$, and intersect it with $\prod_i \mathcal{L}_i \subset \prod_i N_i/\langle n_i\rangle$. These two spaces have complementary dimensions, thus, we expect a finite number of intersection points, which correspond to the desired solutions. This approach is the one of tropical intersection theory, and can be seen in [AR10] and [GKM09]. ⧫

Proof. There is a finite number of cones in the moduli space $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$. Moreover, on each cone, *i.e.* each combinatorial type, f is linear. If the restriction of f to this cone is injective, the combinatorial type contributes at most one solution.

The point (λ_i) can always be chosen outside the image of the non top-dimensional cones, since these images are included in proper subspaces of $\prod_i N_{\mathbb{R}}/(\langle n_i\rangle \oplus L_i^{\mathbb{R}})$. For a combinatorial type of top-dimension, as by assumption its dimension is equal to the dimension of $\prod_i N_{\mathbb{R}}/(\langle n_i\rangle \oplus L_i^{\mathbb{R}})$, if f is not injective, it is not surjective either, and its image is thus contained in a proper subspace. Finally, a generic choice of (λ_i) is chosen outside the image of the non top-dimensional cones and the cones where f is not injective.

For such a choice of (λ_i), there is a finite number of solutions, and the corresponding curves are trivalent since they belong to top dimensional cones of $\mathcal{M}_0(\Delta, N_\mathbb{R})$. □

Now that we have a finite number of solutions, we can count these solutions with a suitable multiplicity so that we obtain a result independent of the choice of the affine subspaces $\mathcal{L}_i$. One such choice is provided both by intersection theory [GKM09], and by the complex multiplicities that intervene in correspondence theorem [Mik05]; [NS06]; [GM07a]. It is the determinant of the composed evaluation map f. Let $h : \Gamma \to N_\mathbb{R}$ be a parametrized rational tropical curve, with Γ a trivalent graph. Then, the restriction of f to the cone $\text{Comb}(\Gamma) \times N_\mathbb{R} \simeq \mathbb{R}^{m-3}_{>0} \times N_\mathbb{R}$ is the restriction of a linear map between two vector spaces of the same dimension. Therefore, we can define the *complex multiplicity* of (Γ, h) to be the determinant of this linear map if we specify the basis that we choose on each side of the map. On the domain, we choose the canonical basis of $\mathbb{R}^{m-3}$, and a basis of N. On the image, we choose a basis of each lattice $N/(\langle n_i \rangle \oplus L_i^\mathbb{R})$. The determinant is then well-defined up to sign.

Definition 3.2.8. The complex multiplicity of (Γ, h) is

$$m_\Gamma^\mathbb{C} = |\det f|_{\text{Comb}(\Gamma)}|,$$

that is the determinant of the restriction of f to the orthant corresponding to the combinatorial type of Γ, when domain and codomain are endowed with the specified basis.

Now, let (λ_i) be chosen generically. We set

$$N_\Delta(\mathcal{L}_1, \dots, \mathcal{L}_m) = \sum_{f(\Gamma,h)=(\lambda_i)} m_\Gamma^\mathbb{C}.$$

Notice that if the subspaces $\mathcal{L}_i$ are chosen generically, there is no curve with zero multiplicity that contributes to the sum, otherwise f is not injective nor surjective on the cone corresponding to the combinatorial type of the curve, and thus (λ_i) has been chosen out of its image.

Proposition 3.2.9
The value of $N_\Delta(\mathcal{L}_1, \dots, \mathcal{L}_m)$ only depends on the slopes L_i of the affine subspaces and not their specific choice as long as it is generic.

This invariant is thus denoted by $N_\Delta(L_1, \dots, L_m)$.

Proof. As many proofs of tropical invariance, the proof goes by the study of local invariance at the walls of the tropical moduli space. The proof is similar to Proposition 4.4 in [GM08]. We proceed in two steps: first, we show that the sum of the determinants of the composed evaluation maps around a wall is zero, and then we show that the sign of these determinants characterizes the existence of solutions, thus proving the local invariance.

- We consider the wall associated to a quadrivalent vertex in the tropical curve. Let the adjacent edges be denoted by the indices $1, 2, 3, 4$. The three adjacent combinatorial types are determined by the splitting of the quadrivalent vertex into two trivalent vertices. These possibilities are denoted by $12//34$, $13//24$ and $14//23$. The cone in the moduli space $\mathcal{M}_0(N_{\mathbb{R}}, \Delta)$ corresponding to each combinatorial type is the quadrant of the vector space $N_{\mathbb{R}} \times \mathbb{R}^{m-4} \times \mathbb{R}$, consisting of the points with positive coordinates on the $\mathbb{R}$ entries, where the $N_{\mathbb{R}}$ factor corresponds to the vertex V adjacent to the edge 1, the $\mathbb{R}^{m-4}$ has canonical basis indexed by the edges of the curve with the quadrivalent vertex, which are common to all curves in the adjacent combinatorial types, and the $\mathbb{R}$ factor corresponds to the length of the edge resulting from the splitting of the quadrivalent vertex.

Let v_j be a directing vector of the edge j, oriented outward the quadrivalent vertex. For each marked point or unbounded end i, associated to a constraint L_i, let $m_{i1}, \dots, m_{ir_i}$ be linear forms defining the sublattice L_i: $L_i = \bigcap_{j=1}^{r_i} \ker m_{ij}$. Then, for the combinatorial type $12//34$, the matrix of the composed evaluation map f takes the following form:

$12//34$	$N_{\mathbb{R}}$	$\mathbb{R}^{m-8}$	1	2	3	4	$\mathbb{R}$
behind 1	m_{ij}	0 or $\langle m_{ij}, v\rangle$	$\langle m_{ij}, v_1\rangle$	0	0	0	0
behind 2	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	$\langle m_{ij}, v_2\rangle$	0	0	0
behind 3	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	$\langle m_{ij}, v_3\rangle$	0	$\langle m_{ij}, v_1 + v_2\rangle$
behind 4	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	0	$\langle m_{ij}, v_4\rangle$	$\langle m_{ij}, v_1 + v_2\rangle$

.

The columns are separated according to the decomposition of the moduli space as $N_{\mathbb{R}} \times \mathbb{R}^{m-4} \times \mathbb{R}$. Moreover, we separate the coordinates corresponding to the lengths of the edges $1, 2, 3, 4$, assuming they are bounded edges. The rows are separated according to whether which of the four edges $1, 2, 3, 4$ is on the shortest path between the vertex V and the unbounded end or marked point i. For each unbounded end or marked point i, we evaluate the linear forms $(m_{ij})_j$. For each edge e directed by v, the evaluation for the unbounded end or marked point i is $\langle m_{ij}, v\rangle$ if the edge e is part of the shortest path between V and unbounded end or marked point i, otherwise it is 0. This fact allows us to complete the middle entries of the matrix. The same rule apply for the entries of the last columns. The matrices for the combinatorial types $13//24$ and $14//23$ are respectively

$13//24$	$N_{\mathbb{R}}$	$\mathbb{R}^{m-8}$	1	2	3	4	$\mathbb{R}$
behind 1	m_{ij}	0 or $\langle m_{ij}, v\rangle$	$\langle m_{ij}, v_1\rangle$	0	0	0	0
behind 2	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	$\langle m_{ij}, v_2\rangle$	0	0	$\langle m_{ij}, v_1 + v_3\rangle$
behind 3	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	$\langle m_{ij}, v_3\rangle$	0	0
behind 4	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	0	$\langle m_{ij}, v_4\rangle$	$\langle m_{ij}, v_1 + v_3\rangle$

,

14//23	$N_{\mathbb{R}}$	$\mathbb{R}^{m-8}$	1	2	3	4	$\mathbb{R}$
behind 1	m_{ij}	0 or $\langle m_{ij}, v\rangle$	$\langle m_{ij}, v_1\rangle$	0	0	0	0
behind 2	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	$\langle m_{ij}, v_2\rangle$	0	0	$\langle m_{ij}, v_1+v_4\rangle$
behind 3	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	$\langle m_{ij}, v_3\rangle$	0	$\langle m_{ij}, v_1+v_4\rangle$
behind 4	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	0	$\langle m_{ij}, v_4\rangle$	0

.

We make the sum of the three determinants for the three adjacent combinatorial types, and use the linearity with respect to the last column, since all the other columns are equal. We get

	$N_{\mathbb{R}}$	$\mathbb{R}^{m-8}$	1	2	3	4	$\mathbb{R}$
behind 1	m_{ij}	0 or $\langle m_{ij}, v\rangle$	$\langle m_{ij}, v_1\rangle$	0	0	0	0
behind 2	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	$\langle m_{ij}, v_2\rangle$	0	0	$\langle m_{ij}, 2v_1+v_3+v_4\rangle$
behind 3	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	$\langle m_{ij}, v_3\rangle$	0	$\langle m_{ij}, 2v_1+v_2+v_4\rangle$
behind 4	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	0	$\langle m_{ij}, v_4\rangle$	$\langle m_{ij}, 2v_1+v_2+v_3\rangle$

.

Using a combination of the columns corresponding to $N_{\mathbb{R}}$ applied to v_1, and the balancing condition $v_1+v_2+v_3+v_4=0$, we get

	$N_{\mathbb{R}}$	$\mathbb{R}^{m-8}$	1	2	3	4	$\mathbb{R}$
behind 1	m_{ij}	0 or $\langle m_{ij}, v\rangle$	$\langle m_{ij}, v_1\rangle$	0	0	0	$\langle m_{ij}, v_1\rangle$
behind 2	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	$\langle m_{ij}, v_2\rangle$	0	0	$\langle m_{ij}, v_2\rangle$
behind 3	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	$\langle m_{ij}, v_3\rangle$	0	$\langle m_{ij}, v_3\rangle$
behind 4	m_{ij}	0 or $\langle m_{ij}, v\rangle$	0	0	0	$\langle m_{ij}, v_4\rangle$	$\langle m_{ij}, v_4\rangle$

.

Now, we see that the last column is the sum of the columns indexed $1,2,3,4$. Thus, the sum of the determinants is 0. If some of the edges $1,2,3,4$ was unbounded, the columns with the corresponding indices would not appear, but for the linear forms m_{ij} that would be evaluated on the corresponding unbounded end, one would already have $\langle m_{ij}, v_i\rangle = 0$ and the result is unchanged. Finally, one has

$$\det A_{12//34} + \det A_{13//24} + \det A_{14//23} = 0.$$

- We now use the previous statement to prove the invariance of the count. We denote by $\mathcal{L}$ the tuple $(\mathcal{L}_1, \dots, \mathcal{L}_m)$. Let $\mathcal{L}(t)$ be a generic path between two generic configurations $\mathcal{L}(0)$ and $\mathcal{L}(1)$, *i.e.* a path in $\prod_i N_{\mathbb{R}}/(\langle n_i\rangle \oplus L_i^{\mathbb{R}})$. Outside a finite set of values of t, the configuration $\mathcal{L}(t)$ is generic and $N_\Delta(\mathcal{L}(t))$ is given by a sum over some combinatorial types with non-zero multiplicity. More precisely, on each combinatorial type with non-zero multiplicity, the composed evaluation map f is a linear map associated with a matrix A. If the coordinates on the combinatorial type are denoted by (V,l), the equation can be formally solved: $(V,l) = A^{-1}\mathcal{L}$, and this provide a true solution of $f(\Gamma, h) = \mathcal{L}$ if the coordinates of l are non-negative.

As the multiplicity only depends on the combinatorial type, the value of $N_\Delta(\mathcal{L}(t))$ is locally constant at generic $\mathcal{L}(t)$, thus, we only need to show invariance around the special values t where $\mathcal{L}(t)$ is not generic. Let t^* be such a value. At least one of the curves of $f^{-1}(t^*)$ has a quadrivalent vertex, and it deforms into the adjacent combinatorial types when t moves slightly around t^*. Let $A_{12//34}$, $A_{13//24}$ and $A_{14//23}$ be the matrices of f on the three adjacent combinatorial types. On each combinatorial type, we can solve uniquely $A_\star(V,l) = \mathcal{L}(t)$: $(V,l) = A_\star^{-1}\lambda(t)$, where $\star$ is one of the three adjacent combinatorial types, and we get a true solution if all the coordinates of l are non-negative. This is the case for all the edges except the edge that appears with the splitting of the quadrivalent vertex. Using Cramer's rule to solve $A_\star(V,l) = \mathcal{L}(t)$, the length of the new edge is equal to $\frac{\det \tilde{A}_\star}{\det A_\star}$, where $\tilde{A}_\star$ is the matrix $A_\star$ with the last column (the one that corresponds to the length of the new edge) being replaced with $\mathcal{L} \in \prod_i N_\mathbb{R}/(\langle n_i\rangle \oplus L_i^\mathbb{R})$. As the matrices $A_{12//34}$, $A_{13//24}$ and $A_{14//23}$ only differ in their last column, the numerators are all equal, and the sign of the length of the new edge is thus determined by the sign of $\det A_\star$. Finally, the sign of the determinant determines which combinatorial type provide a true solution, and the local invariance follows from the relation

$$\det A_{12//34} + \det A_{13//24} + \det A_{14//23} = 0.$$

□

Remark 3.2.10. The invariance also results from general results of tropical intersection theory [AR10]. Had we worked with more general tropical cycles Ξ_i, the proof with intersection theory would also work, but in that case, we would have more walls to study. These walls correspond to the cases where the parametrized tropical curves do not intersect the cycles Ξ_i in their top-dimensional faces. The invariance would then result from the balancing condition for the cycles Ξ_i. ♦

3.2.4 Computation of the multiplicity

We now try to compute the multiplicity $m_\Gamma^\mathbb{C}$ of a parametrized tropical curve, defined as the determinant of the composed evaluation map. More precisely, the multiplicity can be computed in the hereby described fancy way. To keep a general setting, we consider tropical curves (Γ, h) that are not necessarily trivalent. Nevertheless, for such a curve and an evaluation map

$$f : \mathcal{M}_0(\Delta, N_\mathbb{R}) \to \prod_i N_\mathbb{R}/(\langle n_i\rangle \oplus L_i^\mathbb{R}),$$

we can define a complex multiplicity as the determinant of f provided that the cone of the combinatorial type of Γ has the same dimension as the target space. We assume it is the case.

For each primitive sublattice $L \subset N$ of corank l, there is an orthogonal dual primitive sublattice $L^\perp \subset M$ of rank l, and an associated Plücker vector $\rho \in \Lambda^l M$ defined up to sign.

This polyvector is defined as follows: let $m_1, \dots, m_l$ be a basis of $L^\perp$. Then, one has

$$\rho = m_1 \wedge \cdots \wedge m_l \in \Lambda^l M.$$

This polyvector does not depend on the chosen basis up to sign.

Given a parametrized rational tropical curve $h : \Gamma \to N_\mathbb{R}$, one has a polyvector ρ_i associated to each of the unbounded ends of the curve.

- If i is a true unbounded end, ρ_i is the Plücker vector associated to the primitive sublattice spanned by n_i and L_i.
- If i corresponds to a marked point, ρ_i is the Plücker vector associated to L_i.

We choose one vertex of Γ to be the sink of the curve and orient every edge toward it. We then cut the tropical curve, which is a tree, with the following rule: let V be a vertex different from the sink, with incoming unbounded edges directed by $n_1, \dots, n_s$, and respective polyvectors $\rho_1, \dots, \rho_s$, and a unique outgoing bounded edge, thus directed by $n_1 + \cdots + n_s$. The polyvector associated to the outgoing edge of V is

$$\rho = \iota_{n_1+\cdots+n_s}(\rho_1 \wedge \cdots \wedge \rho_s).$$

Recall that ι_n, for $n \in N$, denotes the interior product by n. In our case, all the vertices are trivalent, so that there are only two incoming edges.

Geometrically, the polyvector associated to an edge and to an unbounded end is a multiple of the Plücker vector associated to the space described by the edge when it moves. This means the following:

- For a true unbounded end i associated to a constraint $\mathcal{L}_i$, the space described by the unbounded end when $\mathrm{ev}_i(\Gamma, h)$ is in $\mathcal{L}_i$ is an affine space with direction $\langle n_i \rangle \oplus L_i^\mathbb{R}$.
- At a vertex V with incoming edges having respective polyvectors $\rho_1, \dots, \rho_s$. Assume by induction that the incoming edges move in an affine space directed by a vector subspace whose Plücker vector is ρ_i respectively. Then, the vertex moves in the intersection of all these subspaces. Therefore, the affine subspace has Plücker vector $\rho_1 \wedge \cdots \wedge \rho_s$.
- Finally, for the outgoing edge of V, it moves in an affine space equal to the affine space where V lives, enlarged by the direction of the edge: $n_1+\cdots+n_s$. Hence, the polyvector is obtained by making the interior product with $n_1 + \cdots + n_s$.

At the sink, let $\rho_1, \dots, \rho_s$ be the polyvectors associated to the incoming adjacent edges. Because of the assumption on dimensions, we have

$$\rho_1 \wedge \cdots \wedge \rho_s \in \Lambda^p M \simeq \mathbb{Z}.$$

Thus, it is an integer multiple of a generator of $\Lambda^p M$. The absolute value of the constant, obtained by evaluating on a basis of N, is the desired determinant.

Lemma 3.2.11

The value of the multiple does not depend on the chosen sink.

Proof. We prove that the obtained value does not change if we replace the sink V by one of its neighbors. Let V be a vertex, W one of its neighbors, and E the edge between them, directed by n. Let $\rho_1, \dots, \rho_s$ be the polyvectors of the edges adjacent to V different from E, and $\rho'_1, \dots, \rho'_{s'}$ the polyvectors associated to the edges adjacent to W different from E. The computation leads to the two following results:

— If V is the sink, we get
$$\rho_1 \wedge \cdots \wedge \rho_s \wedge \iota_n(\rho'_1 \wedge \cdots \wedge \rho'_{s'}).$$

— If W is the sink, we get
$$\iota_n(\rho_1 \wedge \cdots \wedge \rho_s) \wedge \rho'_1 \wedge \cdots \wedge \rho'_{s'}.$$

Therefore, the equality (up to sign) comes from the fact that $\rho_1 \wedge \cdots \wedge \rho_s \wedge \rho'_1 \wedge \cdots \wedge \rho'_{s'}$ is 0 since it is in $\Lambda^{p+1} M = \{0\}$, and that ι_n is a derivation on $\Lambda^\bullet M$. □

Theorem 3.2.12
The value obtained by the preceding algorithm is equal to the complex multiplicity $m_\Gamma^{\mathbb{C}}$.

Proof. We make an induction on the number of vertices of the curve Γ, and cut the branches one by one.

- If the curve has just one vertex, let $\rho_i = m_{i1} \wedge \cdots \wedge m_{ir_i}$ be the polyvectors associated to the lattices L_i for the s unbounded edges of the curve. The evaluation matrix of f, denoted by $[f]$, has the following form:
$$[f] = \begin{pmatrix} m_{11} \\ \vdots \\ m_{1r_1} \vdots \\ m_{sr_s} \end{pmatrix}.$$
Therefore, we have
$$\det f = m_{11} \wedge \cdots \wedge m_{1r_1} \wedge \cdots \wedge m_{sr_s} = \rho_1 \wedge \cdots \wedge \rho_s.$$

- If Γ has more than one vertex, let V be a vertex adjacent to only unbounded ends associated to polyvectors $\rho_1, \dots, \rho_s$, and one unique neighbor vertex W. We keep the same notations $\rho_i = m_{i1} \wedge \cdots \wedge m_{ir_i}$. We choose a basis of the cone associated to the combinatorial type of Γ consisting of the canonical basis of $\mathbb{R}_{\geqslant 0}^{m-3-\mathrm{ov}(\Gamma)}$, and the $N_\mathbb{R}$ factor corresponding to the position of W. Then, the matrix of f has the following form:

$$[f] = \begin{pmatrix} * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ & m_{11} & & 0 & \cdots & 0 & \langle m_{11}, n\rangle \\ & \vdots & & \vdots & \ddots & \vdots & \vdots \\ & m_{sr_s} & & 0 & \cdots & 0 & \langle m_{sr_s}, n\rangle \end{pmatrix},$$

where the first columns correspond to the evaluation of the vertex W, the last column to the evaluation corresponding to the edge between V and W. We make a development with respect to the last column. We get the following result:

$$\det f = \sum_{i=1}^{s}\sum_{j=1}^{r_i}(-1)^{\bullet}\langle m_{ij}, n\rangle \begin{vmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \\ & m_{11} & & 0 & \cdots & 0 \\ & \vdots & & \vdots & \ddots & \vdots \\ & \hat{m}_{ij} & & 0 & \cdots & 0 \\ & \vdots & & \vdots & \ddots & \vdots \\ & m_{sr_s} & & 0 & \cdots & 0 \end{vmatrix}.$$

Each determinant in the sum is the determinant of the evaluation matrix f for a tropical curve where the vertex V is deleted and the edge between V and W replaced by an unbounded end, associated with a constraint having polyvector $(-1)^{\bullet}\langle m_{ij}, n\rangle m_{11} \wedge \cdots \wedge \hat{m}_{ij} \wedge \cdots \wedge m_{sr_s}$. The sum of these vectors is precisely $\iota_n(\rho_1 \wedge \cdots \wedge \rho_s)$. Hence, the result follows by induction.

□

If we restrict to the planar case, *i.e.* $p = 2$, and assume that all sublattices L_i are $\{0\}$, the multiplicities of the enumerative problem considered in subsection 3.2.3 take a very simple form, as a product over the vertices of the curve. Let ω denote the determinant, which is a generator of $\Lambda^2 M$. For a trivalent parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$, the multiplicity of a vertex V is the integer $m_V = |\omega(a_V, b_V)|$, for a_V, b_V the slope of h on two edges adjacent to V. This multiplicity does not depend on the chosen adjacent edges.

Proposition 3.2.13
If M and N have rank 2, the complex multiplicity of a trivalent parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$ satisfies

$$m_\Gamma^{\mathbb{C}} = \prod_V m_V,$$

where the product is over the trivalent vertices of Γ.

Proof. Let $\omega \in \Lambda^2 M$ be the determinant. As each L_i is 0, the polyvector associated to any true unbounded end i is $\iota_{n_i}\omega$, while the polyvector associated to any marked point i is ω. The result then follows from the following identities on the cutting algorithm:

$$\iota_{n_1+n_2}(\iota_{n_1}\omega \wedge \iota_{n_2}\omega) = \omega(n_1, n_2)\iota_{n_1+n_2}\omega,$$

$$\iota_{n_1+n_2}(\iota_{n_1}\omega \wedge 1) = \omega(n_1, n_2)1,$$
$$\iota_n(\omega \wedge 1) = \iota_n\omega.$$

□

The complex multiplicity has a natural definition, and provides an invariance for the count of the solutions. They are not the only possible choice. One of our goals is to find other multiplicities that provide an invariant. Such an example is provided by the refined multiplicity of Block-Göttsche [BG16], as defined below. However, the complex multiplicity keeps a strong role since its nullity prevents the existence of solutions, and its invariance governs the repartition of the solutions around a wall of the moduli space.

Definition 3.2.14. The *refined multiplicity* of a simple nodal tropical curve is

$$m^q_\Gamma = \prod_V [m^{\mathbb{C}}_V]_q,$$

where $[a]_q = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}}$ is the q-analog of a.

This refined multiplicity is sometimes called the Block-Göttsche multiplicity and intervenes in the definition of the invariant $N^{\partial,\mathrm{trop}}_\Delta$ in section 5.1. Notice that the multiplicity is the same for every curve inside a given combinatorial type. We give generalization of this refined multiplicity in higher dimension in Chapter 6.

3.2.5 Moment of an unbounded end and tropical Menelaus theorem

Assume that M and N have rank 2, and let ω be a generator of $\Lambda^2 M$, *i.e.* a non-degenerate 2-form on N. It defines to a volume form on $N_\mathbb{R} \simeq \mathbb{R}^2$. Let $e \in \Gamma^1_\infty$ be an unbounded end oriented toward infinity, directed by n_e. Then the moment of e is the scalar

$$\mu_e = \omega(n_e, p) \in \mathbb{R},$$

where $p \in e$ is any point on the edge e. Remember that we identify the affine space $N_\mathbb{R}$ with its tangent space at 0, allowing us to plug in ω a tangent vector n_e and a point p. We similarly define the moment of a bounded edge if we specify its orientation. The moment of a bounded edge is reversed when its orientation is reversed.

Intuitively, the moment of an unbounded end is just a way of measuring its position alongside a transversal axis. Thus, fixing the moment of an unbounded end amounts to imposing on the curve that it goes through some point at infinity. In a way, this allows us to do toric geometry in a compactification of $N_\mathbb{R}$ but staying in $N_\mathbb{R}$. It provides a coordinate on the components of the toric boundary without even having to introduce the concept of toric boundary in the tropical world. Following this observation, the moment has also a definition in complex toric geometry, where it corresponds to the coordinate of the intersection point of the curve with the toric divisor. Let

$$\begin{array}{rccc} \varphi : & \mathbb{C}P^1 & \dashrightarrow & \mathrm{Hom}(M, \mathbb{C}^*) \simeq (\mathbb{C}^*)^2 \\ & t & \mapsto & \chi \prod_1^r (t-\alpha_j)^{n_j}. \end{array}$$

be a parametrized rational curve. Given a dual basis (e_1, e_2) of N, and its dual basis (e_1^*, e_2^*) of M, the parametrized curve given in coordinates is as follows. Let $n_i = a_i e_1 + b_i e_2$, $a = \chi(e_1^*)$ and $b = \chi(e_2^*)$, then

$$\varphi(t) = \left(a \prod_1^r (t-\alpha_i)^{a_i}, b \prod_1^r (t-\alpha_i)^{b_i} \right) \in (\mathbb{C}^*)^2.$$

This is a curve of degree $\Delta = (n_j) \subset N$. The degree Δ defines a fan Σ_Δ and a toric surface $\mathbb{C}\Delta$ to which the map φ naturally extends. The toric divisors D_k of $\mathbb{C}\Delta$ are in bijection with the rays of the fan, which are directed by the vectors n_j. Several vectors n_j may direct the same ray. Moreover, the map φ extends to the points α_j by sending α_j to a point on the toric divisor D_k corresponding to the ray directed by n_j. A coordinate on D is a primitive monomial $\chi^m \in M$ in the lattice of characters such that $\langle m, n_j \rangle = 0$. This latter equality ensures that the monomial χ^m extends on the divisor D_k. If n_j is primitive, $\iota_{n_j}\omega \in M$ is such a monomial, and then the complex moment is the evaluation of the monomial at the corresponding point on the divisor:

$$\mu_j = \left(\varphi^* \chi^{\iota_{n_j}\omega}\right)(\alpha_j).$$

The Weil reciprocity law gives us the following relation between the moments:

$$\prod_{i=1}^m \mu_i = (-1)^m.$$

We could also prove the relation using Viète formula. In the tropical world we have an analog called the *tropical Menelaus theorem*, which gives a relation between the moments of the unbounded ends of a parametrized tropical curve.

Proposition 3.2.15 (Tropical Menelaus Theorem [Mik17])
For a parametrized tropical curve of degree Δ, we have

$$\sum_{n_e \in \Delta} \mu_e = 0.$$

In the tropical case as well as in the complex case, a configuration of m points on the toric divisors is said to satisfy the *Menelaus condition* if this relation is satisfied.

If we do not assume the lattices to have rank 2 anymore, we can still choose a 2-form ω in $\Lambda^2 M$ and set the same definition. By composing the evaluation map with $\iota_n\omega = \omega(n,-)$ for any unbounded end directed by n, we get the *moment map*:

$$\begin{array}{rccc} \mathrm{mom}: & \mathcal{M}_0(\Delta, N_\mathbb{R}) & \longrightarrow & \mathbb{R}^{|\Delta|} \\ & (\Gamma, h) & \longmapsto & \mu = (\mu_i)_{1\leqslant i\leqslant m} \end{array}.$$

For the marked points, *i.e.* the unbounded ends with zero slope, we do not take the moment into account since it is equal to 0. By the tropical Menelaus theorem, the moment μ_1 is equal to the opposite of the sum of the other moments, hence one could omit it in the map.

Notice that the moment map is the composed evaluation map if we choose the lattices L_i to be $\ker\omega(n_i,-)$. Furthermore, in the planar case, both spaces have the same dimension $|\Delta|-1$. Thus, if Γ is a trivalent curve, the moment map mom on $\mathrm{Comb}(\Gamma)\times N_\mathbb{R}$ defines a *complex multiplicity* of the curve, which factors into the following product over the vertices of Γ:

$$m_\Gamma^\mathbb{C} = \prod_V m_V^\mathbb{C}.$$

This multiplicity is the one that appears in the correspondence theorem of Mikhalkin [Mik05]. Thus, in the planar case, we have a well-defined invariant given by the moment map. This invariant is denoted by $N_\Delta^{\partial,\mathrm{trop}}(1)$. This notation emphasizes the fact that the count of solutions with the refined multipliciy of Block-Göttsche also leads to an invariant, as proven in chapter 5. This refined invariant, which is a polynomial, is denoted by $N_\Delta^{\partial,\mathrm{trop}}$.

3.3 Tropicalization

We briefly recall how to obtain an abstract tropical curve and a parametrized tropical curve from a non-archimedean parametrized curve given by a rational map $f:(C,\mathbf{q})\to \mathrm{Hom}(M,\mathbb{C}((t))^*)$, where $(C,\mathbf{q})$ is a curve with marked points. For more details, see [Tyo12].

3.3.1 Tropicalization of a marked curve

Let $(C,\mathbf{q})$ be a smooth marked curve over $\mathbb{C}((t))$. Let $\mathcal{C}^{(t)}\to \mathrm{Spec}\mathbb{C}[[t]]$ be the stable model of $(C,\mathbf{q})$, defined over $\mathbb{C}[[t]]$. The marked points q_i provide sections $\mathrm{Spec}\mathbb{C}[[t]]\to\mathcal{C}^{(t)}$. We have the special fiber $\mathcal{C}^{(0)}$, which is a stable nodal curve, meaning that each irreducible component of genus zero has at least three marked points or nodes, and each irreducible component of genus 1 has at least one marked point or a node. Let $\overline{\Gamma}$ be the dual graph in the following sense: we have one finite vertex per irreducible component of the special fiber, one infinite vertex per marked point, an infinite vertex is joined to the finite vertex of the component where the point specializes, and two finite vertices are joined by an edge if they share a node. We make $\overline{\Gamma}$ into an abstract tropical curve by declaring the length of such an edge to be l if the node is locally given by $xy = t^l$ in an etale neighborhood of the node.

Remark 3.3.1. Intuitively, if our curve is rational, $\mathcal{C}^{(t)}$ is just $\mathbb{C}P^1$ with points depending on a small complex parameter t on it, *i.e.* points given by locally convergent Laurent series in $\mathbb{C}((t))$. If we take naively the special fiber $t = 0$, some marked points may collide, *i.e.* specialize on the same point, other may go to infinity, ... Taking the stable model means that we prevent that. For instance, assume a bunch of points specialize to 0. This means that they are given by formal series of the form $t^k x(t)$ with $k > 0$. We then blow-up this point and get two copies of $\mathbb{C}P^1$ sharing a node. All the points previously specializing to 0 now specialize to at least two different points on the exceptional divisor. The length of the edge between the two copies is the smallest k for all the points specializing on it. Concretely, the blow-up amounts to changing the coordinate z on $\mathbb{C}P^1$ by $t^{-k}z$. We then repeat as long as necessary. ♦

If the curve $(C, \mathbf{q})$ is a real curve, with a real configuration of points $\mathbf{q}$, the involution restricted to the special fiber induces a real structure on Γ.

3.3.2 Tropicalization of a parametrized curve

Now assume given a rational map $f : (C, \mathbf{q}) \dashrightarrow \mathrm{Hom}(M, \mathbb{C}((t))^*)$. There is a tropical curve Γ associated to $(C, \mathbf{q})$. The rational map f extends to a rational map on the stable model $\mathcal{C}^{(t)}$ of $(C, \mathbf{q})$. In order to make Γ into a parametrized tropical curve, we define a map $h : \Gamma \to N_{\mathbb{R}}$ in the following way:

— If $w \in \Gamma^0$ is a vertex dual to a component C_w of $\mathcal{C}^{(0)}$, then $h(w)$ is the element of N defined as follows:
$$h(w)(m) = \mathrm{ord}_{C_w}(f^*\chi^m),$$
where ord_{C_w} stands for the multiplicity of C_w in the divisor of $f^*\chi^m$.

— Then h maps a bounded edge to the line segment linking its extremities.

— If q_i is a marked point, then the slope of the associated unbounded end is $\mathrm{ord}_{q_i}(f^*\chi^m)$, where ord_{q_i} stands for the multiplicity of q_i in the divisor of $f^*\chi^m$.

Remark 3.3.2. The slope of the unbounded end associated to a given marked point q_i is given both by the order of vanishing of $f^*\chi^m$ at q_i, and by the multiplicity of the section defined by q_i in the divisor of $f^*\chi^m$ in the stable model $\mathcal{C}^{(t)}$. This is normal since the marked points provide divisors in $\mathcal{C}^{(t)}$ which are transverse to the special fiber. Concretely, in the rational case, if y is a coordinate on C and f is given by
$$f : y \longmapsto \chi \prod_{i=1}^{r} (y - y(q_i))^{n_i} \in \mathrm{Hom}(M, \mathbb{C}((t))^*),$$
with $\chi \in \mathrm{Hom}(M, \mathbb{C}((t))^*)$, then the slope of the edge associated to the marked point q_i is n_i. ♦

Remark 3.3.3. The special fiber is given by the equation $t = 0$. Therefore $h(w)(m) = \mathrm{ord}_{C_w}(f^*\chi^m)$ is the valuation in t of the function evaluated at the generic point of C_w. Concretely, in the rational case, let y be a coordinate on C specializing to a coordinate on C_w, which is a copy of $\mathbb{C}P^1$, such that no point specializes to ∞. Assume f is given by

$$f : y \longmapsto \chi \prod_{i=1}^{r} (y - y(q_i))^{n_i} \in \mathrm{Hom}(M, \mathbb{C}((t))^*),$$

where $\chi \in \mathrm{Hom}(M, \mathbb{C}((t))^*)$. Then $h(w)(m) = \mathrm{val}(\chi(m))$. ♦

The fact that (Γ, h) is indeed a parametrized tropical curve is proved in [Tyo12]. One essentially needs to check the balancing condition, and the fact that if γ is an edge with extremities v and w, the slope of $h(\gamma)$ lies in N, and the length of $h(w) - h(v)$ coincide with the length $|\gamma|$ in Γ. Finally, we can refine the tropicalization in the following way that is useful to compute quantum indices: on each C_w the rational map f specializes to give a parametrized complex rational curve $f_w : C_w \simeq \mathbb{C}P^1 \dashrightarrow \mathrm{Hom}(M, \mathbb{C}^*)$. Concretely, this is the curve we would obtain by taking the naive limit of f in a coordinate specializing to a coordinate of C_w. Therefore, we have a tropical curve and a complex curve associated to every vertex.

Remark 3.3.4. All our curves are taken with coefficients in $\mathbb{C}((t))$, which is not algebraically closed, and has a discrete valuation. Thus, every tropicalization data has coefficients in $\mathbb{Z}$. Instead we could take the algebraic closure, which is the field of Puiseux series $\mathbb{C}\{\{t\}\} = \bigcup_{k\geqslant 1} \mathbb{C}((t^{\frac{1}{k}}))$, but as we are using only a finite number of coefficients, all belong to $\mathbb{C}((t^{\frac{1}{k}}))$ for some k, and by taking $u = t^{\frac{1}{k}}$ we reduce it to the previous case. Therefore, we can assume that everything is defined in $\mathbb{C}((t))$, up to a change of base. ♦

3.3.3 Tropicalization of a plane curve

We finish by describing the tropicalization of a plane curve. This tropicalization is more elementary than the tropicalization of a parametrized curve. Moreover, the tropicalization of a parametrized curve gives a parametrization of the tropicalization of its image plane curve. Let C be a plane curve, defined by a polynomial $P_t \in \mathbb{C}((t))[M]$ with coefficients in $\mathbb{C}((t))$. We look for the points of the curve over the Puiseux series, *i.e.* in $N \otimes \mathbb{C}\{\{t\}\}^*$. In a basis of M, the polynomial is given in coordinates by

$$P_t(x, y) = \sum_{(i,j)\in P_\Delta} a_{i,j}(t) x^i y^j .$$

We assume that the coefficients in the corners of P_Δ are non-zero. Then, we have the associated tropical polynomial

$$\mathrm{Trop}(P_t)(x, y) = \max_{(i,j)\in P_\Delta} \left(\mathrm{val}(a_{i,j}(t)) + ix + jy\right),$$

along with a valuation map, also called *tropicalization map*:

$$\mathrm{Val} : \chi \in \mathrm{Hom}(M, \mathbb{C}\{\{t\}\}^*) \longmapsto \mathrm{val} \circ \chi \in \mathrm{Hom}(M, \mathbb{R}) = N_{\mathbb{R}}.$$

In coordinates, Val is given by the coordinatewise valuation:

$$\mathrm{Val} : (x, y) \in (\mathbb{C}\{\{t\}\}^*)^2 \longmapsto (\mathrm{val}(x), \mathrm{val}(y)) \in \mathbb{R}^2.$$

The Kapranov theorem [BS14] then ensures that the closure of the image of the vanishing locus of P_t in $(\mathbb{C}\{\{t\}\}^*)^2$ under the valuation map is equal to the tropical curve defined by $\mathrm{Trop}(P_t)$.

Theorem 3.3.5 (Kapranov)
Let C_{trop} be the tropical curve defined by $Trop(P_t)$. Then, one has

$$\overline{Val(C)} = C_{trop}.$$

Let $\alpha_{i,j} = \mathrm{val}(a_{i,j}(t))$, and $a_{i,j}(t) = t^{\alpha_{i,j}} a^0_{i,j}(t)$. The function $(i,j) \mapsto \alpha_{i,j}$ induces a convex subdivision of P_Δ, which is dual to C_{trop}. As in the tropicalization of a parametrized curve, one can recover complex curves, by specializing the polynomial P_t to one of the polygons of the subdivision. Let ϖ be one of the polygons of the subdivision of P_Δ. Then, the curve associated to ϖ is given by $P_\varpi(x, y) = \sum_{(i,j)\in\varpi} a^0_{i,j}(0) x^i y^j = 0$, defined over $\mathbb{C}$.

One can show that if $\varphi : C \to N \otimes \mathbb{C}\{\{t\}\}^*$ is a parametrized curve tropicalizing to $h : \Gamma \to N_{\mathbb{R}}$, then the image $h(\Gamma)$ and the tropicalization of the image $\overline{\mathrm{Val}(\varphi(C))}$ are the same. Moreover, the local parametrized curves $f_w : C_w \dashrightarrow N \otimes \mathbb{C}^*$ resulting from the tropicalization as parametrized curve, are precisely the irreducible components of the curves defined by $P_\varpi = 0$.

Chapter 4

Computation of some refined invariants in toric surfaces

In this chapter we adress the computation of the refined toric enumerative invariants introduced by Mikhalkin in [Mik17]. In his approach, he counts oriented real rational curves in a toric surface, and the refinement is provided by the value of a so-called *quantum index.* The definition of the quantum index is recalled in section 4.1.

Mikhalkin then considers an enumerative problem and counts its solutions according to the value of their quantum index. The enumerative problem is as follows. Let $\Delta \subset N$ be a degree, and $\mathbb{C}\Delta$ be the associated toric surface. Let $\mathcal{P}$ be a symmetric configuration of real and complex conjugated points located on the toric boundary, with some assumption on their number (see section 4.2 for more details). Any oriented rational curve of degree Δ passing through the point configuration has a well-defined quantum index, which coincide with the logarithmic area if the complex points are purely imaginary. Thus, we make a signed count of the oriented rational curves that pass through the point configuration, according to the value of their quantum index. Mikhalkin proved that the result only depends on the number of purely imaginary points on each toric divisor.

Furthermore, Mikhalkin proved that, if all the points are real, the hereby defined invariants coincide up to constant factor with the tropical invariants $N_\Delta^{\partial,\mathrm{trop}}$, previously defined, but studied and computed in chapter 5. In this chapter, we recover this result and generalize it to the case where there are some pairs of complex conjugated points which are located on common toric divisor.

In the first section, we recall the definition of the quantum index of Mikhalkin, and give a way to compute it for every real rational curve. This is an improvement since Mikhalkin restricts himself to the case of *toric type I* real curves (see Definition 6.1.16). In the rational case, such curves only have real intersection points. Then, we recall the enumerative problem considered by Mikhalkin in [Mik17], leading to the definition of his refined invariants. This enumerative problem is naturally associated to the tropical enumerative problem from section 5.1, meaning that it can be solved using a suitable correspondence theorem. Last, we give a proof of a correspondence theorem, inspired by I. Tyomkin [Tyo17], that allows us to compute Mikhalkin's refined invariants in some cases, including the one of complex points on a common divisor.

Sommaire

4.1 Quantum indices of real rational curves

We start this section by recalling the theorem about quantum indices by Mikhalkin [Mik17], restricting ourselves to the case of rational curves. We then compute the quantum indices in some specific cases and provide a general result for the computation of the quantum index of an oriented rational curve.

4.1.1 The quantum index of a real rational curve

Let

$$\varphi : t \in \mathbb{C} \longmapsto \chi \prod_1^r (t-\alpha_i)^{n_i} \prod_1^s (t-\beta_j)^{n_j} (t-\overline{\beta_j})^{n_j} \in N \otimes \mathbb{C}^*$$

be a parametrized real rational curve, with $\alpha_i \in \mathbb{R}$, $\beta_j \in \mathbb{C}\backslash\mathbb{R}$ some scalars, and $\chi \in N \otimes \mathbb{C}^*$ a co-character. Recall that the moment of the parametrized curve $(\mathbb{C}P^1, \varphi)$ at a complex point $\beta_{j_0} \in \mathbb{C} \subset \mathbb{C}P^1$ is the quantity

$$\varphi^* \chi^{\iota_{n_{j_0}}\omega}|_{\beta_{j_0}} = \chi(\iota_{n_{j_0}}\omega) \prod_1^r (\beta_{j_0}-\alpha_i)^{\omega(n_{j_0},n_i)} \prod_1^s (\beta_{j_0}-\beta_j)^{\omega(n_{j_0},n_j)} (\beta_{j_0}-\overline{\beta_j})^{\omega(n_{j_0},n_j)} \in \mathbb{C}^*.$$

Recall that ω denotes a generator of $\Lambda^2 N$, *i.e.* the determinant.

Definition 4.1.1. In the above notations, we say that the rational curve has real or purely imaginary intersection points if $\varphi^*\chi^{\iota_{n_{j_0}}\omega}|_{\beta_{j_0}} \in i\mathbb{R}$ for every j_0.

Remark 4.1.2. At the real points α_i, the moment is real since the function φ is real, that is why we only check non-real points for the purely imaginary value. Geometrically, it means that the coordinates of the intersection points of the curve with the toric boundary are either real or purely imaginary. In both cases their square is real. ♦

Recall that we have the logarithmic map

$$\mathrm{Log} : n \otimes z \in N \otimes \mathbb{C}^* \longmapsto n \otimes \mathrm{Log}|z| \in N_{\mathbb{R}}.$$

In a basis of N, it is the logarithm of the absolute value coordinate by coordinate. Similarly we define the argument map, taken modulo π rather than 2π:

$$2\arg : n \otimes z \in N \otimes \mathbb{C}^* \longmapsto n \otimes 2\arg(z) \in N \otimes \mathbb{R}/\pi\mathbb{Z}.$$

As the parametrized real rational curve $\varphi : \mathbb{C}P^1 \to N \otimes \mathbb{C}^*$ is of type I, let S be a connected component of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$, inducing a complex orientation of $\mathbb{R}P^1$. By pulling back the volume form ω on $N_{\mathbb{R}}$ to $N \otimes \mathbb{C}^*$, we can define the logarithmic area of S:

$$\mathcal{A}_{\mathrm{Log}}(S) = \int_{\varphi(S)} \mathrm{Log}^*\omega.$$

Respectively, the 2-form ω defines a 2-form ω_θ on $N \otimes \mathbb{R}/\pi\mathbb{Z}$. We can pull it back to $N \otimes \mathbb{C}^*$ and define the area of the co-amoeba of S:

$$\mathcal{A}_{\arg}(S) = \int_{\varphi(S)} (2\arg)^*\omega.$$

Assume ω is given in coordinates by $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2$. Then we have coordinates $z_1 = e^{x_1 + i\theta_1}$ and $z_2 = e^{x_2 + i\theta_2}$ on $N \otimes \mathbb{C}^*$, where $x_i \in \mathbb{R}$ and $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$. We consider the following meromorphic 2-form:

$$\begin{aligned}\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} &= (\mathrm{d}x_1 + i\mathrm{d}\theta_1) \wedge (\mathrm{d}x_2 + i\mathrm{d}\theta_2) \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 + i(\cdots).\end{aligned}$$

Notice that $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ and $\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$ are the respective pull-backs of $\omega \otimes \mathbb{R}$ by Log and $\omega \otimes \mathbb{R}/\pi\mathbb{Z}$ by $2\arg$, thus their integrals on S are precisely $\mathcal{A}_{\mathrm{Log}}(S)$ and $\mathcal{A}_{\arg}(S)$. Due to the vanishing of the meromorphic 2-form on S, one has $\mathcal{A}_{\mathrm{Log}}(S) = \mathcal{A}_{\arg}(S)$.

Theorem 4.1.3 (Mikhalkin[Mik17])
Let $\varphi : \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}^*$ be a real parametrized rational curve with real or purely imaginary intersection points, enhanced with the choice of a connected component S of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$,

inducing a complex orientation of $\mathbb{R}P^1$. Then there exists a half-integer $k(S,\varphi)$, called the quantum index of the oriented curve (S,φ), such that

$$\mathcal{A}_{\arg}(S) = \mathcal{A}_{\mathrm{Log}}(S) = k(S,\varphi)\pi^2.$$

Remark 4.1.4. It is straightforward to generalize the theorem and definitions for a general oriented type I real curve which has real or purely intersection points with the toric boundary, but the computation of the quantum index can be more complicated. ♦

Remark 4.1.5. For curves whose intersection points with the toric boundary are not purely imaginary, the quantum index is defined as a suitable shift of the logarithmic area. However, this extended definition is not needed for our case, where we assume the complex points to be purely imaginary. ♦

4.1.2 The quantum index near the tropical limit

In [Mik17], Mikhalkin proved the following result, that computes the quantum index of curves in a family near the tropical limit.

Proposition 4.1.6
[Mik17] Let $C^{(t)} = \left(f_t : \mathbb{C}P^1 \to \mathrm{Hom}(M,\mathbb{C}^*)\right)$ be a family of type I real parametrized rational curves, having real or purely imaginary intersection points, enhanced with a family of connected components of the complex locus $S^{(t)}$, inducing complex orientations of the curves. We assume that the family tropicalizes, in the sense of 3.3, to a parametrized real tropical curve $h : \Gamma \to N_{\mathbb{R}}$, such that the components $S^{(t)}$ specialize to components S_w of C_w for every vertex $w \in \mathrm{Fix}(\sigma)$, thus inducing complex orientations of the curves C_w. Then, for t large enough,

$$k(S^{(t)}, f_t) = \sum_w k(S_w, f_w),$$

where the sum is indexed over the fixed vertices of Γ.

Remark 4.1.7. In particular, and this is useful in the proof of the correspondence theorem, for one to know the quantum index of curves near the tropical limit, one only needs to know the quantum indices of the curves associated to the vertices of the tropical curve, and the way they are glued together along the edges. This means that the quantum index may be computed in the patchworking construction. ♦

If one was only interested in the computation of Mikhalkin's refined invariants, using the tropical geometry approach, the computation of the quantum index near the tropical limit would allow one to reduce the computation to the following cases:

- a real rational curve with three real intersection points with the toric boundary,
- a real rational curve with four intersection points with the toric boundary: two real and two complex conjugated,
- a real rational curve with five intersection points with the toric boundary: one real and two pairs of complex conjugated ones.

Moreover, these computations, carried out in the next subsection, can be reduced to the computation of the log-area (area of the amoeba) for a line, a parabola and a conic, using the following statement. It turns out that these computation are also enough to compute the quantum index of an oriented rational curve in the general case.

We prove that the quantum index is well-behaved under the monomial maps, which are covering maps from the complex torus to itself.

Lemma 4.1.8
Let $\varphi : \mathbb{C}C \dashrightarrow \mathrm{Hom}(M, \mathbb{C}^*)$ be a type I real curve with a choice of a connected component $S \subset \mathbb{C}C \backslash \mathbb{R}C$, inducing a complex orientation, and let $\alpha : \mathrm{Hom}(M, \mathbb{C}^*) \to \mathrm{Hom}(M', \mathbb{C}^*)$ be a monomial map, associated to a morphism $A^T : M' \to M$. We consider the composition

$$\psi : \mathbb{C}C \xrightarrow{\varphi} \mathrm{Hom}(M, \mathbb{C}^*) \xrightarrow{\alpha} \mathrm{Hom}(M', \mathbb{C}^*).$$

Let ω and ω' be the volume forms on respectively N and N', dual lattices of M and M', so that we have $A^*\omega' = (\det A)\omega$. Then, we have

$$\int_{\psi(S)} \mathrm{Log}^*\omega = \det A \int_{\varphi(S)} \mathrm{Log}^*\omega \text{ and } \int_{\psi(S)} (2\arg)^*\omega_\theta = \det A \int_{\varphi(S)} (2\arg)^*\omega_\theta.$$

Remark 4.1.9. The proposition deals with the computation of log-area in the general case of a real curve. This log-area is a quantum index only if the curve has real or purely imaginary intersection points with the toric boundary. We use this proposition to reduce the computation of a quantum index to a log-area of a curve which does not necessarily have a quantum index, but whose log-area is easier to compute. ♦

Remark 4.1.10. Notice that the different notations $\mathrm{Hom}(M, \mathbb{C}^*)$ and $\mathrm{Hom}(M', \mathbb{C}^*)$ prevent any mistakes in the direction of the various involved maps. ♦

Proof. Let N and N' be the dual lattices of M and M', so that we have linear maps

$$A^T : M' \to M,$$

$$A : N \to N'.$$

Then, we have the following commutative diagrams:

$$\begin{tikzcd} \mathbb{C}C \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ N \otimes \mathbb{C}^* \arrow[r, "\alpha"] \arrow[d, "\mathrm{Log}"] & N' \otimes \mathbb{C}^* \arrow[d, "\mathrm{Log}"] \\ N_{\mathbb{R}} \arrow[r, "A"] & N'_{\mathbb{R}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{C}C \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ N \otimes \mathbb{C}^* \arrow[r, "\alpha"] \arrow[d, "\arg"] & N' \otimes \mathbb{C}^* \arrow[d, "\arg"] \\ N \otimes \mathbb{R}/2\pi\mathbb{Z} \arrow[r, "A"] & N' \otimes \mathbb{R}/2\pi\mathbb{Z} \end{tikzcd} .$$

We denote by ω, ω' the volume forms of the lattices N and N' used to compute the log-areas. Then, we have

$$\begin{aligned}\int_{\psi(S)} \mathrm{Log}^*\omega' &= \int_S (\mathrm{Log}\circ\psi)^*\omega' \\ &= \int_S (A\circ\mathrm{Log}\circ\varphi)^*\omega' \\ &= \int_S (\mathrm{Log}\circ\varphi)^*(A^*\omega') \\ &= \det A\int_S (\mathrm{Log}\circ\varphi)^*\omega \text{ since } A^*\omega' = (\det A)\omega, \\ &= \det A\int_{\varphi(S)} \mathrm{Log}^*\omega.\end{aligned}$$

The proof is completely similar for the argument maps. □

4.1.3 Local computations

In this section, we compute the quantum indices of some specific rational curves. This includes a complex curve, a real line, a parabola whose intersection points with the abscissa axis are complex conjugated, and a conic whose intersection points with two of the axis are complex conjugated, and which is tangent to the last axis.

4.1.3.1 Log-area of a complex curve

We begin by proving that the log-area of a complex curve is zero. This proves that the non-fixed vertices of the tropical curve have no contribution to the quantum index, and thus justifies the fact that the quantum index near the tropical limit is obtained as a sum over the fixed vertices, and not the pairs of exchanged vertices. The following statement is not specific to rational curves or real curves.

Lemma 4.1.11
Let $\varphi : \mathbb{C}C \dashrightarrow N\otimes\mathbb{C}^*$ be a complex parametrized curve, with $\mathbb{C}C$ a smooth Riemann surface. Then
$$\int_{\mathbb{C}C} \mathrm{Log}^*\omega = \int_{\mathbb{C}C} (2\arg)^*\omega_\theta = 0.$$

Proof. The two integrals are known to be equal by the vanishing of the meromorphic 2-form given in coordinates by $\frac{\mathrm{d}z_1}{z_1}\wedge\frac{\mathrm{d}z_2}{z_2}$. Let $\mathbb{C}C^o$ be the open set of $\mathbb{C}C$ where φ is defined. We consider the map $\mathrm{Log}\circ\varphi : \mathbb{C}C^o \to N_\mathbb{R}$. This is a proper map between smooth oriented manifolds. Therefore, it has a well-defined degree, which corresponds both to the number of antecedents counted with signs over a generic point, and to the linear map $\mathbb{R} = H^2_c(N_\mathbb{R}) \overset{(\mathrm{Log}\circ\varphi)^*}{\longrightarrow} H^2_c(\mathbb{C}C^o) = \mathbb{R}$ between compactly supported cohomology groups. Since the map is not surjective, its degree is zero. Hence, if $\tilde{\omega}$ is a compactly supported 2-form on $N_{\mathbb{R}}$, then $\int_{\mathbb{C}C^o}(\mathrm{Log}\circ\varphi)^*\tilde{\omega}=0$. Thus, by writing ω as a (infinite) sum of compactly supported 2-forms using partitions of unity, we get the result. □

4.1.3.2 Log-area of a real line

Using a monomial map and Lemma 6.3.6, the computation of the log-area of a real line allows one to compute the quantum index of every oriented type I real curve having three intersection points with the toric boundary. This was dealt with in [Mik17].

Lemma 4.1.12
Let $\Delta \subset N$ be a family of three vectors of total sum 0, and let $P_\Delta \subset M$ be the associated triangle, of lattice area m_Δ. Let $\varphi : \mathbb{C}P^1 \dashrightarrow N\otimes\mathbb{C}^*$ be a real parametrized rational curve of degree Δ, thus having a unique real intersection point with maximal tangency with each toric divisor. Then, the quantum index of the curve is $\pm\frac{m_\Delta}{2}$ according to the choice of complex orientation.

Proof. The assumption implies that the curve is the image of a real line by a monomial map of determinant m_Δ. Hence, its quantum index is the log-area of a real line, equal to $\pm\frac{1}{2}$, times the determinant of the monomial map which is the lattice area of the triangle. □

Remark 4.1.13. The log-area of a line can be also easily computed by hand, either using the log point of view, or using the argument point of view. ♦

4.1.3.3 Log-area of a parabola

We now consider the case of a rational curve having two real punctures, and two conjugated ones. In a suitable choice of coordinates, the curve has a degree of the following form. In a basis (e_1,e_2) of N, for $m_1,m_2,m_3\in\mathbb{N}^*$, let us take

$$\Delta(m_1,m_2,m_3)=\{(m_1,2m_2);(0,m_3-m_2)^2;(-m_1,-2m_3)\}.$$

The degree of a planar curve which is parametrized by a curve of degree $\Delta(m_1,m_2,m_3)$ is the lattice polygon in M given by

$$P_\Delta(m_1,m_2,m_3)=\mathrm{Conv}\left((0,m_1),(2m_2,0),(2m_3,0)\right).$$

This polygon is drawn in Figure 4.1. Up to an automorphism of the lattice, every triangle in M having a side of even length is one of the polygons $P_\Delta(m_1,m_2,m_3)$. Let E_i be the side opposite to the i-th vertex in $P_\Delta(m_1,m_2,m_3)$, *i.e.*

$$E_1=[(2m_2,0),(2m_3,0)]\,,\ E_2=[(2m_3,0),(0,m_1)]\,,\ E_3=[(2m_2,0),(0,m_1)]\,.$$

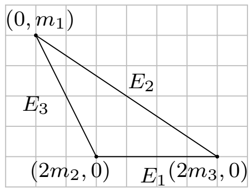

Figure 4.1 – The polygon $P_\Delta(m_1, m_2, m_3)$.

We denote by $\mathbb{C}E_i$ the associated toric divisor inside the toric surface $\mathbb{C}\Delta(m_1, m_2, m_3)$.

Such a curve has a parametrization of the form

$$\psi(t) = \left(a(t-c)^{m_1}, b(t-c)^{2m_2}(t^2+1)^{m_3-m_2}\right) \in (\mathbb{C}^*)^2,$$

where c is some real number corresponding to the coordinate of the intersection point with $\mathbb{C}E_3$, and $a, b \in \mathbb{R}^*$. The intersection point with $\mathbb{C}E_2$ corresponds to the coordinate t taking the infinite value. Under a suitable monomial map, the curve is the image of a parabola having an equation of the form $y = \lambda x^2 + \mu x + \nu$, but we rather take coordinates where the equation is of the form $y = \lambda x + \mu + \frac{\nu}{x}$. Thus, we are left to compute the log-area of a curve whose parametrization is the following one:

$$\varphi(t) = \left(t - c, \frac{t^2+1}{t-c}\right).$$

Lemma 4.1.14

Let $\mathbb{H}$ denotes the Poincaré half-plane $\{\Im t > 0\}$, i.e. the complex numbers with a positive imaginary part, inducing a complex orientation of $\mathbb{R}P^1$. Then, the log-area of $(\mathbb{H}, \varphi)$ satisfies

$$\int_{\varphi(\mathbb{H})} \mathrm{Log}^*\omega = \int_{\varphi(\mathbb{H})} (2\arg)^*\omega_\theta = 2\pi \arctan(c).$$

Proof. We compute the area of the coamoeba. According to [FJ15], the coamoeba with its order map is as on Figure 4.2. The order map has value 1 on the blue triangles, and -1 on the red ones. The center point has coordinates $(0,0)$, the square has side length 2π. The two vertical lines have respective abscissa $\pm\arg(i-c) = \pm\mathrm{arccot}(-c)$. This is the whole co-amoeba, which might fold itself, and we want to compute the area of half the co-amoeba, *i.e.* the part corresponding to $\mathbb{H}$. As $\arg(t-c) \in]0;\pi[$ if and only if $t \in \mathbb{H}$, we obtain $\arg\varphi(\mathbb{H})$ by restricting to the right half-square. Therefore, the area is obtained by taking the blue area minus the red area (because it comes with a minus sign). Let $l = \mathrm{arccot}(-c)$ the abscissa of the right vertical line. Then, we have

$$\mathcal{A}_{\arg} = l^2 - (\pi - l)^2 = 2\pi l - \pi^2 = 2\pi\left(\mathrm{arccot}(-c) - \frac{\pi}{2}\right).$$

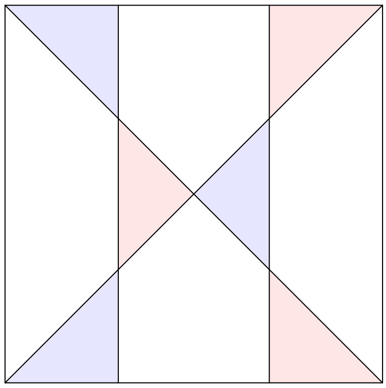

Figure 4.2 – Co-amoeba of φ_k with order map: -1 for red (triangles with left vertical side), $+1$ for blue (triangles with right vertical side).

This result might be simplified by noticing that:

$$\operatorname{arccot}(-c) - \frac{\pi}{2} = \arctan(c).$$

□

Remark 4.1.15. It is easy to find the order map of the co-amoeba of a curve, which corresponds to the number of antecedents counted with sign. This function is constant on the complement of the *shell*, which is a union of geodesics in the torus, directed by the vectors of the degree Δ of the curve. Moreover, the value of the order map changes by one when passing through one of the geodesics of the shell. This defines the order map up to a shift. The shift is fixed by the fact that the whole signed area is zero, as proved in Lemma 4.1.11. One might then expect that adding the areas of some components of the complement of the shell would give the area of half the co-amoeba, *i.e.* $\arg\varphi(\mathbb{H})$ instead of $\arg\varphi(\mathbb{C}C)$. In general it is not the case: the order maps takes into account the antecedents on both connected components of $\mathbb{C}C\backslash\mathbb{R}C$, and it is not possible to draw them apart. However it is possible here since one of the monomials provides a coordinate on the curve, here x. Thus, we get $\arg\varphi(\mathbb{H})$ by restricting to half the argument torus, here the right-half square. ♦

To get the quantum index of an oriented real type I rational curve of degree $\Delta(m_1, m_2, m_3)$, one needs to multiply by the determinant of the monomial map, whose value is $m_1(m_3 - m_2)$.

4.1.3.4 Log-area of a conic

Last, we consider the case of a conic having complex intersection points with the x and y-axis in $\mathbb{C}P^2$, and which is tangent to the z-axis. This computation is needed for the quantum index of rational curve with five punctures : two pairs of conjugated ones and one real one.

Such a conic has a parametrization of the following form:

$$\varphi : t \in \mathbb{C} \longmapsto (a(t^2+1), b((t-r)^2+s^2)),$$

where t is a coordinate chosen such that the coordinate of the intersection point with the x-axis are $\pm i$, and with the z-axis is∞. The coordinates of the intersection point with the y-axis are $r \pm is$, with $s > 0$. The logarithmic map is then

$$t = x+iy \in \mathbb{H} \longmapsto \left(\frac{1}{2}\log\left(\left((x^2-y^2+1\right)^2+4y^2x^2\right), \frac{1}{2}\log\left(\left((x-r)^2-y^2+s^2\right)^2+4y^2(x-r)^2\right)\right).$$

To compute the log-area, one needs to integrate the determinant of its differential on $\mathbb{H}$. The Jacobian matrix is equal to

$$\begin{pmatrix} \frac{2x(x^2+y^2+1)}{(x^2-y^2+1)^2+4x^2y^2} & \frac{2y(y^2-x^2-1)}{(x^2-y^2+1)^2+4x^2y^2} \\ \frac{2(x-r)((x-r)^2+y^2+s^2)}{((x-r)^2-y^2+s^2)^2+4(x-r)^2y^2} & \frac{2y(y^2-(x-r)^2-1)}{((x-r)^2-y^2+s^2)^2+4(x-r)^2y^2} \end{pmatrix},$$

and its determinant is the following rational function:

$$f(x,y) = \frac{4xy(x^2+y^2+1)(y^2-(x-r)^2-1) - 4(x-r)y(y^2-x^2-1)((x-r)^2+y^2+s^2)}{((x^2-y^2+1)^2+4x^2y^2)\left(((x-r)^2-y^2+s^2)^2+4(x-r)^2y^2\right)}.$$

The denominator is easily factored since it comes from the parametrization of φ, which is factored: the first term factors in

$$((x+i(y+1))\,((x-i(y+1))\,((x+i(y-1))\,((x-i(y-1))\,,$$

and the second factors in

$$((x-r+i(y+s))\,((x-r-i(y+s))\,((x-r+i(y-s))\,((x-r-i(y-s))\,.$$

Lemma 4.1.16
The log-area of $(\mathbb{H},\varphi)$ *satisfies*

$$\mathcal{A}_{\text{Log}}(\mathbb{H},\varphi) = 4\pi\arctan\left(\frac{r}{s+1}\right).$$

Proof. It is a painful computation. One needs to compute the integral of the rational function f on $\mathbb{R}\times]0;+\infty[$. The integral over $x\in\mathbb{R}$ can be taken care of using the residue formula, which is possible since the denominator is factored. The resulting integral over $]0;+\infty[$ is again the integral of a rational function. The decomposition in simple elements is doable but long since there are 4 residues to compute, and one needs to make a disjunction according to $s<1$ or $s>1$, and the position of y relative to s and 1 to know which ones are in the upper half-plane $\mathbb{H}$. Using a computer, one gets the result. □

Remark 4.1.17. For a homogeneized version, one could use that if the coordinates of the intersection points with the x and y-axis are $r \pm is$ and $r' \pm is'$ in a coordinate such that the intersection point with the z-axis has value ∞, and s, s' are chosen positive, the log-area is

$$\mathcal{A}_{\mathrm{Log}}(\mathbb{H}, \varphi) = 4\pi \arctan\left(\frac{r'-r}{s'+s}\right).$$

♦

Finally, using Lemma 4.1.8, this allows the computation of the quantum index of an oriented type I real rational curve having five intersection points with the toric divisors: one real and two pairs of complex conjugated ones.

4.1.4 The quantum index of an oriented real rational curve

We are now ready to compute the log-area, and thus the quantum index, of an oriented real rational curve in the general setting, provided it has at least one real point. Let

$$\varphi : t \longmapsto \chi \prod_{i=1}^{r-1}(t-\alpha_i)^{n_i} \prod_{j=1}^{s}(t^2 - 2t\Re\beta_j + |\beta_j|^2)^{n'_j} \in N \otimes \mathbb{C}^*,$$

be such a curve, with α_r be the infinite point, and β_j chosen in the upper half-plane $\mathbb{H}$. One has the following theorem.

Theorem 4.1.18
The log-area of $(\mathbb{H}, \varphi)$ *takes the following value:*

$$\mathcal{A}_{\mathrm{Log}}(\mathbb{H}, \varphi) = \begin{array}{cl} & \sum_{i<i'} \omega(n_i, n_{i'})\pi^2 \\ + & \sum_{i,j} 2\pi\omega(n_i, n'_j) \arctan\left(\frac{\alpha_i - \Re\beta_j}{\Im\beta_j}\right) \\ + & \sum_{j<j'} 4\pi\omega(n'_j, n'_{j'}) \arctan\left(\frac{\Re\beta_{j'} - \Re\beta_j}{\Im\beta_{j'} + \Im\beta_j}\right) \end{array}.$$

Proof. We use a version of Lemma 4.1.8 in a higher dimensional setting. The map $\varphi : \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}^*$ can be factored through the following monomial map $\alpha : (\mathbb{C}^*)^{r+s-1} \to N \otimes \mathbb{C}^*$, associated to the lattice map $\mathbb{Z}^{r+s-1} \to N$ that sends the first $r-1$ basis vectors e_i to n_i, and the last s basis vectors e'_j to n'_j. We denote this lattice map by $A : \mathbb{Z}^{r+s-1} \to N$, and by α the associated monomial map $A \otimes \mathbb{C}^*$. Let

$$\psi : t \longmapsto \rho \prod_{i=1}^{r}(t-\alpha_i)^{e_i} \prod_{j=1}^{s}(t^2 - 2t\Re\beta_j + |\beta_j|^2)^{e'_j} \in (\mathbb{C}^*)^{r+s-1},$$

with ρ a co-character such that $\alpha(\rho)=\chi$, so that $\varphi=\alpha\circ\psi$. Then, we have

$$\begin{aligned}\mathcal{A}_{\mathrm{Log}}(\mathbb{H},\varphi) &= \int_{\varphi(\mathbb{H})} \mathrm{Log}^*\omega \\ &= \int_{\alpha\circ\psi(\mathbb{H})} \mathrm{Log}^*\omega \\ &= \int_{\psi(\mathbb{H})} \alpha^*\mathrm{Log}^*\omega \\ &= \int_{\psi(\mathbb{H})} (\mathrm{Log}\circ\alpha)^*\omega \\ &= \int_{\psi(\mathbb{H})} (A\circ\mathrm{Log})^*\omega \\ &= \int_{\psi(\mathbb{H})} \mathrm{Log}^* A^*\omega.\end{aligned}$$

The pull-back 2-form $A^*\omega$ on $\mathbb{Z}^{r+s-1}$ decomposes in the dual basis as the sum

$$A^*\omega = \sum_{i<i'} \omega(n_i,n_{i'})e_i^*\wedge e_{i'}^* + \sum_{i<j}\omega(n_i,n'_j)e_i^*\wedge e_j^{'*} + \sum_{j<j'}\omega(n'_j,n'_{j'})e_j^{'*}\wedge e_{j'}^{'*}.$$

Each of the 2-form $e_i^*\wedge e_{i'}^*$ (resp. $e_i^*\wedge e_j^{'*}$, $e_j^{'*}\wedge e_{j'}^{'*}$) intervening in the decomposition is the pull-back of the canonical 2-form on $\mathbb{Z}^2$ by the projection $x\mapsto(x_i,x_{i'})$ (resp. (x_i,x'_j), $(x'_j,x'_{j'})$), and its integral can be computed as the integral of the log-area of a line (resp. parabola, conic) by using once again the technique from Lemma 4.1.8. Using the previous computations of the log-area of a line, parabola and conic, one gets the result. □

Remark 4.1.19. In particular, if the moments of the points are real or purely imaginary, the log-area is a half-integer multiple of π^2, although the formula does not emphasizes it. ♦

4.2 Refined curve counting in a toric surface

Let $\Delta\subset N$ be a family of m primitive lattice vectors, with total sum 0. As described in the introduction, there is an associated lattice polygon P_Δ having m lattice points on its boundary. The toric surface obtained from Δ is denoted by $\mathbb{C}\Delta$. Let $E_1,\dots,E_p$ denote the sides of the polygon P_Δ and let $n_1,\dots,n_p\in N$ be their normal primitive vectors. Let $s_i\leqslant\frac{l(E_i)}{2}$ be an integer, $r_i=l(E_i)-2s_i$, so that we have $\sum_1^p r_i+2s_i=m$. We denote by s the tuple $(s_1,\dots,s_p)$, and $|s|$ the sum $\sum_{i=1}^p s_i$. Let

$$\Delta(s)=\{n_1^{r_1},(2n_1)^{s_1},n_2^{r_2},(2n_2)^{s_2},\dots,n_p^{r_p},(2n_p)^{s_p}\}.$$

Let $\mathcal{P}$ be a configuration of m points on the toric boundary $\partial\mathbb{C}\Delta$ such that:

- each toric divisor associated to a side E_i of P_Δ contains exactly r_i real points and s_i pairs of complex conjugated points,

- the configuration satisfies the Menelaus condition.

Let $\mathcal{S}(\mathcal{P})$ be the set of oriented real rational curves such that for every $p \in \mathcal{P}$, the curve passes through p or $-p$. Such a curve is said to *pass through the symmetric configuration* $\mathcal{P}$. As the curves are oriented, each real curve is counted twice: once with each of its orientations. Notice that if a curve passes through one of the points of a pair of non-real points, it also passes through its conjugate since the curve is real. We denote by $\mathcal{S}_k(\mathcal{P})$ the subset of $\mathcal{S}(\mathcal{P})$ formed by oriented curves with quantum index k.

Let $\varphi : \mathbb{C}P^1 \to \mathbb{C}\Delta$ be an oriented real parametrized rational curve, denoted by (S, φ). The logarithmic Gauss map sends a point $p \in \mathbb{R}P^1$ to the tangent direction to $\mathrm{Log}\varphi(\mathbb{R}P^1)$ inside $N_\mathbb{R}$. We get a map

$$\gamma : \mathbb{R}P^1 \to \mathbb{P}^1(N_\mathbb{R}).$$

The first space $\mathbb{R}P^1$ is oriented since the curve is oriented, while $\mathbb{P}^1(N_\mathbb{R})$ is oriented by ω. The degree of this map is denoted by $\mathrm{Rot}_{\mathrm{Log}}(S, \varphi) \in \mathbb{Z}$. If the curve has transverse intersections with the divisors, it has the same parity as the number of boundary points m. We then set

$$\sigma(S, \varphi) = (-1)^{\frac{m - \mathrm{Rot}_{\mathrm{Log}}(S,\varphi)}{2}} \in \{\pm 1\}.$$

Now let

$$R_{\Delta,k}(\mathcal{P}) = \sum_{(S,\varphi) \in \mathcal{S}_k(\mathcal{P})} \sigma(S, \varphi),$$

and

$$R_\Delta(\mathcal{P}) = \frac{1}{4} \sum_k R_{\Delta,k}(\mathcal{P}) q^k \in \mathbb{Z}[q^{\pm\frac{1}{2}}].$$

The coefficient $\frac{1}{4}$ is here to account for the deck transformation: if $\{f(x, y) = 0\}$ is a curve in $\mathcal{S}(\mathcal{P})$, then $\{f(x, -y) = 0\}$, $\{f(-x, y) = 0\}$, $\{f(-x, -y) = 0\}$ are in $\mathcal{S}(\mathcal{P})$ too.

Remark 4.2.1. The shift by m to the logarithmic rotation number is only to keep track of its residue mod 2, while we are interested in its residue mod 4, one could also choose another convention. For instance the logarithmic rotation number of a maximal curve. ♦

Theorem 4.2.2
(Mikhalkin [Mik17]) As long as $r = \sum_1^p r_i \geqslant 1$, the value of $R_\Delta(\mathcal{P})$ is independent of the configuration $\mathcal{P}$ as long as it is generic. It only depends on Δ and s.

The obtained polynomial, independent of $\mathcal{P}$, is denoted by $R_{\Delta,s}$.

Remark 4.2.3. Although this theorem is not stated in these terms in [Mik17] because the only case considered is the one of purely imaginary points where the quantum index coincides with the log-area, the proof does not use this specific assumption, and thus applies also in this setting. ♦

Remark 4.2.4. Here, generic means that the configuration of points $\mathcal{P}$ needs to be a regular value of the evaluation map that associates to a parametrized curve the coordinates of its intersection points with the toric boundary. ♦

4.3 Realization and correspondence theorem in the real case

In this section we prove a correspondence theorem, by refining the realization theorem of Tyomkin [Tyo17] in the case of real curves. The proof follows the same steps as in [Tyo17] and presents similar calculations. We start by giving a bunch of notations which might seem a little heavy, but are useful to deal with real tropical curves having a non-trivial real structure.

4.3.1 Notations

Let Γ be a real rational abstract tropical curve with m ends. We denote by σ the involution on Γ, and Γ/σ the quotient graph. When needed, we denote by $\pi : \Gamma \to \Gamma/\sigma$ the quotient map. Let I denote the set of ends of Γ, endowed with an action of the involution, embodied in the following decomposition:

$$I = \{x_1, \dots, x_r, z_1^\pm, \dots, z_s^\pm\},$$

where the ends x_i are fixed ends (real ends) and $z_i^\pm$ are exchanged with one another (complex ends). Following this notation, we denote the set of ends of Γ/σ by:

$$I/\sigma = \{x_1, \dots, x_r, z_1, \dots, z_s\}.$$

We assume that $r \geqslant 1$, and orient the edges of both Γ and Γ/σ away from x_r, which makes them rooted trees. This orientation induces a partial order $\prec$ on the curve. For w and w' vertices or ends, we have $w \prec w'$ if and only if the shortest path from w to w' agrees with the orientation of the graph. We endow the set I/σ with a total order, different from $\prec$, for which x_r is the smallest element.

If $w \in \Gamma^0$ is a vertex of Γ, let I_w^∞ be the set of ends of Γ which are greater than w for the order $\prec$. We take a similar notation $(I/\sigma)_w^\infty$ for $w \in (\Gamma/\sigma)^0$. Notice that if $w \in \mathrm{Fix}(\sigma)$, then I_w^∞ is stable by σ, and if $w \notin \mathrm{Fix}(\sigma)$, then at most one element of each pair $\{z_j^\pm\}$ belongs to I_w^∞.

If $\gamma \in \Gamma^1$ is a bounded edge of γ (same for Γ/σ), let $\mathfrak{t}(\gamma)$ and $\mathfrak{h}(\gamma)$ be the tail and the head of γ. Notice that as the order goes from a real end to every other ends, including all complex ends, $\mathfrak{h}(\gamma) \notin \mathrm{Fix}(\sigma)$ if and only if $\gamma \notin \mathrm{Fix}(\sigma)$.

If $\gamma^\sigma \in (\Gamma/\sigma)^1$ is an edge of Γ/σ, let $\iota(\gamma^\sigma)$ be the smallest element of $(I/\sigma)^\infty_{\mathfrak{h}(\gamma^\sigma)}$. It is the smallest end among those accessible by $\mathfrak{h}(\gamma^\sigma)$. The order on I/σ along with this map ι induces an order on the edges of Γ/σ having the same tail. We thus can speak about the smallest and biggest edge leaving a vertex $\pi(w)$ of Γ/σ. We can then lift these local orders on Γ/σ to Γ:

- If $w \in \mathrm{Fix}(\sigma)$, then we have three cases for the lift of an edge γ^σ such that $\mathfrak{t}(\gamma^\sigma) = \pi(w)$:
  - $\star$ (RR) the edge $\gamma^\sigma = \{\gamma\}$ lifts to a fixed edge γ of Γ, and $\iota(\gamma^\sigma) = \{x_j\}$ is a real marking. (The (RR) stands for "Real edge Real marking".) We then set $\iota(\gamma) = x_j$.
  - $\star$ (RC) the edge $\gamma^\sigma = \{\gamma\}$ lifts to a fixed edge γ of Γ but $\iota(\gamma^\sigma) = \{z_j^\pm\}$ is a complex marking. It means that the curve Γ splits at some point on the path from w to $\iota(\gamma^\sigma)$, but not right away in γ. (The (RC) stands for "Real edge Complex marking".)
  - $\star$ (CC) the edge $\gamma^\sigma = \{\gamma^+, \gamma^-\}$ lifts to a pair of exchanged edges in Γ. If $\iota(\gamma^\sigma) = \{z_j^\pm\}$, we have up to a relabeling of $\gamma^\pm$ that z_j^+ (resp. z_j^-) is accessible via γ^+ (resp. γ^-). (The (CC) stands for "Complex edge Complex marking".)
- If $w \notin \mathrm{Fix}(\sigma)$, then $\pi : \{\gamma \in \Gamma^1 : \mathfrak{t}(\gamma) = w\} \to \{\gamma^\sigma : \mathfrak{t}(\gamma^\sigma) = \pi(w)\}$ is a bijection, therefore we have a total order on $\{\gamma \in \Gamma^1 : \mathfrak{t}(\gamma) = w\}$, and for every edge γ such that $\mathfrak{t}(\gamma) = w$ we have a unique complex end $\iota(\gamma) \in \iota(\gamma^\sigma)$ accessible by w. Notice that in this case, every edge γ^σ such that $\mathfrak{t}(\gamma^\sigma) = \pi(w)$ is of type (CC). Moreover, we also have an induced order between the edges emanating from w.

We denote by $v_\mathbb{R}$ (resp. $v_\mathbb{C}$) the number of fixed vertices (resp. pairs of exchanged vertices), and by $e_\mathbb{R}$ (resp. $e_\mathbb{C}$) the number of fixed bounded edges (resp. pairs of exchanged bounded edges).

4.3.2 Space of rational curves with given tropicalization

Let Γ be an abstract tropical curve with m ends. Let $(C^{(t)}, x_1, \dots, x_r, z_1^\pm, \dots, z_s^\pm)$ be a real smooth rational curve with a real configuration $(\mathbf{x}, \mathbf{z}^\pm) = (x_1, \dots, x_r, z_1^\pm, \dots, z_s^\pm)$ of marked points, tropicalizing on Γ. We assume that $C^{(0)}$ is the special fiber of the stable model of $C^{(t)}$, so that Γ is the dual graph of $C^{(0)}$. We assume that $r \geqslant 1$ and we also denote by σ the real structure on $C^{(t)}$.

Remark 4.3.1. By taking a coordinate, the marked curve $(C^{(t)}, \mathbf{x}, \mathbf{z}^\pm)$ can be seen as $\mathbb{P}^1(\mathbb{C}((t))) \simeq \mathbb{C}((t)) \cup \{\infty\}$, the projective line over the field of Laurent series, along with $r + 2s$ Laurent series which are the marked points, taken up to a change of coordinate in $GL_2\big(\mathbb{R}((t))\big)$. The first r Laurent series are in $\mathbb{R}((t))$, and the last s are taken in $\mathbb{C}((t))\backslash\mathbb{R}((t))$ along with their conjugate. ♦

We associate to each vertex $w \in \Gamma^0$ a coordinate y_w on $C^{(t)}$, taking into account the real structure, *i.e.* the coordinate y_w is real if $w \in \mathrm{Fix}(\sigma)$ and $y_{\sigma(w)} = \overline{y_w \circ \sigma}$ otherwise. Moreover, the coordinate y_w specializes to a coordinate on the irreducible component of $C^{(0)}$ associated to w.

- If $w \in \mathrm{Fix}(\sigma)$, the set I_w^∞ is stable by σ, but has no order induced by I/σ since for each complex marking, both are accessible. Still, let γ_a^σ and γ_b^σ be the smallest and biggest edges emanating from $\pi(w)$ in Γ/σ. We make a disjunction according to the type $(RR), (RC), (CC)$ of each edge:

(RR/RR) If γ_a^σ and γ_b^σ are both of type (RR), they lift to edges γ_a and γ_b of Γ, which have well-defined real marking $x_a = \iota(\gamma_a)$ and $x_b = \iota(\gamma_b)$. Then we take y_w such that $y_w(x_r) = \infty$, $y_w(x_a) = 0$, $y_w(x_b) = 1$. This is a real coordinate since y_w coincide with $\overline{y_w \circ \sigma}$ at three points.

(RR/RC) If γ_a^σ is of type (RR) and γ_b^σ of type (RC), then they lift up to edges $\gamma_a, \gamma_b \in \text{Fix}(\sigma)$, and we have $\iota(\gamma_a) = x_a$, and $\iota(\gamma_b) = z_b^\pm$. Then $\mathfrak{Re} z_b^\pm$ is a well-defined real Laurent series whose specialization on the component associated to w is different from the one of x_a. Then we can take y_w such that $y_w(x_r) = \infty$, $y_w(x_a) = 0$, $y_w(\mathfrak{Re} z_b^\pm) = 1$.

(RC/RR) We do the same with $\mathfrak{Re} z_a^\pm$ and x_b.

(RC/RC) If γ_a^σ and γ_b^σ are both of type (RC) we do the same with $\mathfrak{Re} z_a^\pm$ and $\mathfrak{Re} z_b^\pm$.

$(CC/-)$ If γ_a^σ is of type (CC), then γ_a^σ lifts to a pair of exchanged edges $\{\gamma_a^\pm\}$ both emanating from w. They both have a well-defined $\iota(\gamma_a^\pm) = z_a^\pm$. Then we take y_w such that $y_w(x_r) = \infty$, $y_w(z_a^\pm) = \pm i$, which also is a real coordinate.

$(-/CC)$ If γ_a^σ is of type (RR) or (RC) and γ_b^σ is of type (CC), we do the same with $z_b^\pm$.

- If $w \notin \text{Fix}(\sigma)$, then I_w^∞ consists only of complex markings, all edges emanating from w are of type (CC) and we have a well-defined $\iota(\gamma)$ for each of them. Let a and b be the smallest and biggest elements in I_w^∞. We take y_w such that $y_w(x_r) = \infty$, $y_w(a) = 0$, $y_w(b) = 1$. This choice ensures that $y_{\sigma(w)} = \overline{y_w \circ \sigma}$.

The functions y_w are all coordinates on C sending x_r to ∞, therefore we can pass from one to another by a real affine function which we now describe.

Proposition 4.3.2
Let $\gamma \in \Gamma^1$ be a bounded edge.

- *If $\gamma \notin \text{Fix}(\sigma)$, let z_a^ε and z_b^η be the smallest and biggest elements in $I_{\mathfrak{h}(\gamma)}^\infty$, then*

$$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)}{y_{\mathfrak{t}(\gamma)}(z_b^\eta) - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)} \text{ and } |\gamma| = val(y_{\mathfrak{t}(\gamma)}(z_b^\eta) - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)).$$

- *If $\gamma \in \text{Fix}(\sigma)$, we make a disjunction according to the type of $\mathfrak{h}(\gamma)$:*

(RR/RR) $y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(x_a)}{y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a)}$ *and* $|\gamma| = val(y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a))$.

(RR/RC) $y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(x_a)}{\mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a)}$ *and* $|\gamma| = val(\mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a))$.

(RC/RR) $y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(x_a)}{y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm)}$ *and* $|\gamma| = val(y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm))$.

(RC/RC) $y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm)}{\mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm)}$ *and* $|\gamma| = val(\mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm))$.

$(CC/-)$ $y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re} y_{\mathfrak{t}(\gamma)}(z_a^\pm)}{\mathfrak{Im} y_{\mathfrak{t}(\gamma)}(z_a^+)}$ *and* $|\gamma| = val\left(\mathfrak{Im} y_{\mathfrak{t}(\gamma)}(z_a^+)\right)$.

$(-/CC)$ *same with a switched by b.*

Proof. In each case we check that the right-hand term, which is a coordinate since it is obtained by an affine change from another coordinate, coincides with $y_{\mathfrak{h}(\gamma)}$ at the three points used to define it. As it is so, they are equal. The equality with the length of γ is the definition of the latter since $y_{\mathfrak{h}(\gamma)}$ and $y_{\mathfrak{t}(\gamma)}$ are coordinates on the irreducible component associated with $\mathfrak{h}(\gamma)$ and $\mathfrak{t}(\gamma)$. $\square$

For every edge we now define $\alpha_\gamma \in \mathbb{C}[[t]]^\times$ and $\beta_\gamma \in \mathbb{C}[[t]]$ which will be the coordinates on the space of real marked curves tropicalizing on Γ. Once again, the definition goes through the distinction of the type of edges emanating from $\mathfrak{h}(\gamma)$. Let $\gamma \in \Gamma_b^1$ be a bounded edge:

$$\begin{aligned}
(RR/RR)\ \alpha_\gamma &= t^{-|\gamma|}\left(y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a)\right),\\
(RR/RC)\ \alpha_\gamma &= t^{-|\gamma|}\left(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a)\right),\\
(RC/RR)\ \alpha_\gamma &= t^{-|\gamma|}\left(y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)\right),\\
(RC/RC)\ \alpha_\gamma &= t^{-|\gamma|}\left(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)\right),\\
(CC/-)\ \alpha_\gamma &= t^{-|\gamma|}\mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_a^+),\\
(-/CC)\ \alpha_\gamma &= t^{-|\gamma|}\mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_b^+).
\end{aligned}$$

Let $\gamma \in \Gamma^1$ be a non-necessarily bounded edge:

- ⋆ If $\gamma \notin \mathrm{Fix}(\sigma)$ is of type (CC) then $\beta_\gamma = y_{\mathfrak{t}(\gamma)}(z^\varepsilon_{\iota(\gamma)})$ where $z^\varepsilon_{\iota(\gamma)}$ is the lift of $z_{\iota(\gamma)}$ accessible by γ.
- ⋆ If $\gamma \in \mathrm{Fix}(\sigma)$ is of type (RR) then $\beta_\gamma = y_{\mathfrak{t}(\gamma)}(x_{\iota(\gamma)})$.
- ⋆ If $\gamma \in \mathrm{Fix}(\sigma)$ is of type (RC) then $\beta_\gamma = \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z^\pm_{\iota(\gamma)})$.

We now can define the function

$$\Psi_\gamma(y) = \beta_\gamma + t^{|\gamma|}\alpha_\gamma y,$$

which allows an easy description of the relations between the y_w.

Proposition 4.3.3
The Laurent series α_γ and β_γ satisfy the following properties.

(i) If γ is an edge, one has $\alpha_\gamma \in \mathbb{C}[[t]]^\times$. Moreover, if $\gamma \neq \gamma'$ are two different edges with the same tail $\mathfrak{t}(\gamma) = \mathfrak{t}(\gamma')$, then $\beta_\gamma - \beta_{\gamma'} \in \mathbb{C}[[t]]^\times$.

(ii) They are real: $\alpha_{\sigma(\gamma)} = \overline{\alpha_\gamma}$, $\beta_{\sigma(\gamma)} = \overline{\beta_\gamma}$. In particular $\alpha_\gamma, \beta_\gamma \in \mathbb{R}[[t]]$ if $\gamma \in \mathrm{Fix}(\sigma)$.

(iii) For each edge γ, one has $y_{\mathfrak{t}(\gamma)} = \Psi_\gamma(y_{\mathfrak{h}(\gamma)})$. In particular, for any marked point q, one has

$$y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(q) = t^{|\gamma|}\alpha_\gamma\left(y_{\mathfrak{h}(\gamma)} - y_{\mathfrak{h}(\gamma)}(q)\right).$$

Moreover Ψ_γ is real in the sense that $\Psi_{\sigma(\gamma)}(y) = \overline{\Psi_\gamma(\overline{y})}$.

(iv) Let w, w' be two vertices, and $\gamma_1, \dots, \gamma_d$ the geodesic path from w to w'. If the orientation of γ_i in Γ agree with its orientaton in the geodesic path, let $\varepsilon_i = +1$, otherwise let $\varepsilon_i = -1$. Then

$$y_w = \Psi_{\gamma_1}^{\varepsilon_1} \circ \cdots \circ \Psi_{\gamma_d}^{\varepsilon_d}(y_{w'}),$$

and in particular for any marked point q:

$$y_w - y_w(q) = t^{\sum_1^d \varepsilon_i |\gamma_i|} \left(\prod_1^d \alpha_{\gamma_i}^{\varepsilon_i} \right) (y_{w'} - y_{w'}(q)) \, .$$

(v) *Let q be a marked point associated with an unbounded end $e \in \Gamma^1_\infty$, w be a vertex, and $\gamma_1, \dots, \gamma_d, e$ be the geodesic path from w to q. Then*

$$y_w(q) = \Psi_{\gamma_1}^{\varepsilon_1} \circ \cdots \circ \Psi_{\gamma_d}^{\varepsilon_d}(\beta_e),$$

and in particular for every marked point q associated with the unbounded end e, if v_r is the vertex adjacent to the unbounded end associated to x_r,

$$y_{v_r}(q) = \beta_{\gamma_1} + t^{|\gamma_1|}\alpha_{\gamma_1} \left(\cdots (\beta_{\gamma_d} + t^{|\gamma_d|}\alpha_{\gamma_d}\beta_e \right) .$$

(vi) *For every marked point q_i and vertex $w \in \Gamma^0$, val$(y_w(q)) \geqslant 0 \Leftrightarrow q$ is accessible by w.*

(vii) *For every edge $\gamma \in \Gamma^1$ and every marked point $q \in I^\infty_{\mathfrak{t}(\gamma)} \backslash I^\infty_{\mathfrak{h}(\gamma)}$, we have val$(y_{\mathfrak{t}(\gamma)}(q) - \beta_\gamma) = 0$.*

Proof. It suffices to check every statement:

- (i) and (ii) follow from the definition of α and β,
- (iii) comes from the definition of Ψ_γ and from the fact that α and β are real,
- (iv) and (v) are just iterations from (iii),
- (vi) comes from (v) and,(vii) follows from (i) and (v).

□

Remark 4.3.4. Formally speaking, this proposition is the direct translation of Proposition 4.3 in [Tyo17], in the setting of curves with a real structure. Although the formulas seem quite repulsive at the first look, the meaning of each object must be clear. The formal series α_γ and β_γ allow one to recover the coordinates of the marked points, following the formula (v) of Proposition 4.3.3. The formal series α_γ are the *"phase length"* of the edge γ, in contrast to $t^{|\gamma|}$ which could be called *"valuation length"*, while the formal series β_γ are the directions one needs to follow at each w in order to get to the points of $I^\infty_{\mathfrak{h}(\gamma)}$. In other terms, α and β provide the necessary coefficients to find the coordinates of the marked points. One could say that the abstract tropical curve Γ only remembers the valuation information, while the formal series α and β encode the phase information. ♦

Let $v = v_r$ be the vertex adjacent to the end x_r, the uplet

$$(y_v(x_1), \dots, y_v(x_{r-1}), y_v(z_1^+), \dots, y_v(z_s^+)) \in \mathbb{R}((t))^{r-1} \times \mathbb{C}((t))^s$$

provides a system of coordinates on the moduli space of real rational marked curves, that we can restrict on the moduli space of curves tropicalizing on Γ. Notice that the choice of y_v fixes the value of some members of the uplet. The definition of α, β along with a quick induction ensures that they can be written in terms of $\big(y_v(q)\big)_q$. Conversely, the formula from Proposition 4.3.3(v) allows to recover $\big(y_v(q)\big)_q$ from α and β. Therefore, they also provide a system of coordinates. Moreover, Proposition 4.3.3 describes the set of possible values of α, β, since the formula from Proposition 4.3.3(v) gives the values of the points to choose on $\mathbb{P}^1(\mathbb{C}((t)))$, in order to make it into a marked curve with the right tropicalization and the right formal series α, β.

We denote by $\mathcal{A}$ the space $(\mathbb{R}[[t]]^\times)^{e_\mathbb{R}} \times (\mathbb{C}[[t]]^\times)^{e_\mathbb{C}}$ of possible values of α. We denote by $\mathcal{B}$ the space of possible values of β satisfying the conditions of Proposition 4.3.3.

4.3.3 Space of morphisms with given tropicalization

Let $h:\Gamma\to N_\mathbb{R}$ be a real rational parametrized tropical curve of degree Δ. In this subsection we give an explicit description of morphisms $f:(C,\mathbf{x},\mathbf{z}^\pm)\to \mathrm{Hom}(M,\mathbb{C}((t))^*)$ of degree Δ tropicalizing to it, and for which $(C,\mathbf{x},\mathbf{z}^\pm)$ is a smooth connected marked rational curve.

Let $f:C\to\mathrm{Hom}(M,\mathbb{C}((t))^*)$ be a real morphism that tropicalizes to h. By assumption, in the coordinate y_w, the morphism f takes the following form:

$$f(y_w) = t^{h(w)}\chi_w \prod_{i\in I_w^\infty}(y_w - y_w(q_i))^{n_i}\prod_{i\notin I_w^\infty}\left(\frac{y_w}{y_w(q_i)}-1\right)^{n_i} \in \mathrm{Hom}(M,\mathbb{C}((t))^*).$$

Remark 4.3.5. Notice that $t^{h(w)}$ denotes the morphism $m\mapsto t^{h(w)(m)}\in\mathbb{C}((t))^*$. The choice of normalization in the product ensures that $\chi_w: M\to\mathbb{C}((t))^*$ has value in $\mathbb{C}[[t]]^\times$. Finally, both products are indexed by the ends of Γ, and although the writing does not emphasize this aspect, there are real ends and complex ends. Furthermore, there is a constant term (-1) in the second product, corresponding to the root x_r, for which $y_w(x_r)=\infty$ for any w. ♦

We now relate the expression of f in two different vertices. We assume that they are connected by an edge γ. Let

$$\phi_\gamma = \prod_{i\in I^\infty_{\mathfrak{t}(\gamma)}\setminus I^\infty_{\mathfrak{h}(\gamma)}}\left(y_{\mathfrak{t}(\gamma)}(q_i)-\beta_\gamma\right)^{n_i}\prod_{i\notin I^\infty_{\mathfrak{t}(\gamma)}}\left(1-\frac{\beta_\gamma}{y_{\mathfrak{t}(\gamma)}(q_i)}\right)^{n_i}\in\mathrm{Hom}(M,\mathbb{C}((t))^*.$$

Notice that ϕ_γ depends only on the value of α,β, and is thus a function $\phi_\gamma(\alpha,\beta)$.

Proposition 4.3.6
We have the following properties of χ_w and ϕ_γ:

(i) The χ_w are real: $\chi_{\sigma(w)}=\overline{\chi_w}$. In particular, if $w\in\mathrm{Fix}(\sigma)$, χ_w takes values in $\mathbb{R}[[t]]^\times$.

(ii) The co-characters ϕ_γ are real: $\phi_{\sigma(\gamma)}=\overline{\phi_\gamma}$. In particular, if $\gamma\in\mathrm{Fix}(\sigma)$ is a fixed edge, the co-character ϕ_γ takes values in $\mathbb{R}[[t]]^\times$.

(iii) For any edge γ, let n_γ denote the slope of h on γ. One has

$$\phi_\gamma \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \alpha_\gamma^{n_\gamma} = 1 \in \mathrm{Hom}(M, \mathbb{C}((t))^*).$$

Proof. The first two points are immediate to check and follow from the definition of ϕ_γ along with the fact that f is real. For the last point, we start by making the quotient of the two expressions of f in the coordinates $y_{\mathfrak{t}(\gamma)}$ and $y_{\mathfrak{h}(\gamma)}$, and use Proposition 4.3.3 that ensures that $y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(q_i) = t^{|\gamma|}\alpha_\gamma(y_{\mathfrak{h}(\gamma)} - y_{\mathfrak{h}(\gamma)}(q_i))$. Hence,

$$\frac{t^{h(\mathfrak{t}(\gamma))}}{t^{h(\mathfrak{h}(\gamma))}} \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \frac{\prod_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} y_{\mathfrak{h}(\gamma)}(q_i)^{n_i}}{\prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} y_{\mathfrak{t}(\gamma)}(q_i)^{n_i}} \cdot \left(t^{|\gamma|}\alpha_\gamma\right)^{\sum_i n_i} = 1 \in \mathrm{Hom}(M, \mathbb{C}((t))^*).$$

Since $h(\mathfrak{h}(\gamma)) - h(\mathfrak{t}(\gamma)) = |\gamma| n_\gamma$, and $\sum_i n_i = 0$ by balancing condition, we get

$$t^{-|\gamma| n_\gamma} \frac{\prod_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} y_{\mathfrak{h}(\gamma)}(q_i)^{n_i}}{\prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} y_{\mathfrak{t}(\gamma)}(q_i)^{n_i}} \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} = 1 \in \mathrm{Hom}(M, \mathbb{C}((t))^*).$$

Finally, using that $y_{\mathfrak{h}(\gamma)}(q_i) = \frac{y_{\mathfrak{t}(\gamma)}(q_i) - \beta_\gamma}{t^{|\gamma|}\alpha_\gamma}$, we get that

$$\frac{\prod_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} y_{\mathfrak{h}(\gamma)}(q_i)^{n_i}}{\prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} y_{\mathfrak{t}(\gamma)}(q_i)^{n_i}} = (t^{|\gamma|}\alpha_\gamma)^{-\sum_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} n_i} \frac{\prod_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} (y_{\mathfrak{t}(\gamma)}(q_i) - \beta_\gamma)^{n_i}}{\prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} y_{\mathfrak{t}(\gamma)}(q_i)^{n_i}}.$$

Since by adding the balancing condition at the vertices not accessible via γ we get $\sum_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} n_i = -n_\gamma$, and as $\complement I^\infty_{\mathfrak{t}(\gamma)} \subset \complement I^\infty_{\mathfrak{h}(\gamma)}$, we get that

$$\frac{\prod_{i \notin I^\infty_{\mathfrak{h}(\gamma)}} y_{\mathfrak{h}(\gamma)}(q_i)^{n_i}}{\prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} y_{\mathfrak{t}(\gamma)}(q_i)^{n_i}} = t^{|\gamma| n_\gamma} \cdot \alpha_\gamma^{n_\gamma} \cdot \phi_\gamma,$$

which results in the desired formula. □

Remark 4.3.7. There is a slight misnomer in the proof: because $y_w(x_r) = \infty$, the intermediate steps of computation are not well-defined. However, the computation remains true, either by allowing a finite value to $y_w(x_r)$ and then making it ∞, or by putting it apart in the computation, which complicates the explanation. ♦

Conversely, if we are given a real family $\chi_w : M \to \mathbb{C}[[t]]^\times$ such that Proposition 4.3.6 holds for any γ, then the maps defined by the formulas

$$f(y_w) = t^{h(w)} \chi_w \prod_{i \in I^\infty_w} (y_w - y_w(q_i))^{n_i} \prod_{i \notin I^\infty_w} \left(\frac{y_w}{y_w(q_i)} - 1\right)^{n_i}$$

agree and define a real morphism f tropicalizing to $h : \Gamma \to N_\mathbb{R}$.

If G is an abelian group, let $N_G = N \otimes G$. Let $\mathcal{X} = N^{v_\mathbb{R}}_{\mathbb{R}[[t]]^\times} \times N^{v_\mathbb{C}}_{\mathbb{C}[[t]]^\times}$ be the space where the tuple χ is chosen. Then, the space of morphisms $f : (C, \mathbf{x}, \mathbf{z}^\pm) \to N \otimes \mathbb{C}((t))^*$ tropicalizing to $h : \Gamma \to N_\mathbb{R}$ is the subset of $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$ given by the following equations:

$$\forall \gamma \in \Gamma_b^1 \ : \ \phi_\gamma(\alpha, \beta) \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \alpha_\gamma^{n_\gamma} = 1 \in \mathrm{Hom}(M, \mathbb{C}((t))^*).$$

The tuples α, β deal with the tropicalization of the curve, and the tuple χ with the tropicalization of the morphism.

4.3.4 Evaluation map

We now use the previous description to write down the conditions that a curve having fixed moments must satisfy. For each $n_j \in \Delta$, let $m_j = \iota_{n_j}\omega$ be the monomial used to measure the moment of the corresponding end q_j.

Let q_j be a real or complex marked point, and v_j be the adjacent vertex of the associated unbounded end e_j. In the coordinate y_{v_j}, the expression of the moment of the marked point q_j takes the following form:

$$f^*\chi^{m_j}|_{q_j} = t^{h(v_j)(m_j)}\chi_{v_j}(m_j) \prod_{i \in I^\infty_{v_j}} \left(\beta_{e_j} - y_{v_j}(q_i)\right)^{\omega(n_j, n_i)} \prod_{i \notin I^\infty_{v_j}} \left(\frac{\beta_{e_j}}{y_{v_j}(q_i)} - 1\right)^{\omega(n_j, n_i)}.$$

We then put

$$\varphi_j = \prod_{i \in I^\infty_{v_j}} \left(\beta_{e_j} - y_{v_j}(q_i)\right)^{\omega(n_j, n_i)} \prod_{i \notin I^\infty_{v_j}} \left(\frac{\beta_{e_j}}{y_{v_j}(q_i)} - 1\right)^{\omega(n_j, n_i)} \in \mathbb{C}[[t]]^\times,$$

which, according to Proposition 4.3.3, is an invertible formal series only depending on α, β. The series is invertible since the only terms of the product having positive valuation are taken with a zero exponent.

Proposition 4.3.8
The formal series φ_j are real: $\varphi_{\sigma(j)} = \overline{\varphi_j}$ for every end e_j, and in particular, if x_j is a real marked point, then $\varphi_j \in \mathbb{R}[[t]]^\times$. Moreover, φ is a function of α, β, i.e. it does not depend on χ.

Proof. It follows from the fact that all the quantities that intervene in the definition of φ_j are real. The second part is obvious. □

Thus, with this new notation, the evaluation map takes the following form:

$$f^*\chi^{m_j}|_{q_j} = t^{h(v_j)(m_j)}\chi_{v_j}(m_j)\varphi_j.$$

4.3.5 Correspondence theorem

Now we look at the following map:

$$\begin{array}{rccl} \Theta : & \mathcal{X}\times\mathcal{A}\times\mathcal{B} & \longrightarrow & \left(N_{\mathbb{R}[[t]]^\times}^{e_\mathbb{R}}\times N_{\mathbb{C}[[t]]^\times}^{e_\mathbb{C}}\right)\times\mathbb{R}[[t]]^{\times r-1}\times\mathbb{C}[[t]]^{\times s} \\ & (\chi,\alpha,\beta) & \longmapsto & \left(\left(\phi_\gamma\cdot\frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}}\cdot\alpha_\gamma^{n_\gamma}\right)_{\gamma\in\Gamma_b^1},\left(\chi_{v_j}(m_j)\varphi_j\right)_j\right) \end{array}.$$

This map is the same as in [Tyo17] but for a curve endowed with a non-trivial real involution. That is why we take every real vertex or real edge with real coefficients, and only one of every pair of complex vertices or complex edges with complex coefficients.

One can check that the dimensions of the source and target spaces are the same:

- Since N has rank 2, the target space has dimension $2e_\mathbb{R}+4e_\mathbb{C}+r-1+2s$.
- The space $\mathcal{A}$ has dimension $e_\mathbb{R}+2e_\mathbb{C}$ since one chooses a real formal series for each real edge, and a complex one for each pair of conjugated edges.
- The space $\mathcal{X}$ has dimension $2(v_\mathbb{R}+2v_\mathbb{C})$ since N has rank 2.
- The dimension of $\mathcal{B}$ is precisely $\mathrm{ov}(\Gamma)$ since we choose a real or complex coefficient for each edge γ for which $\mathfrak{t}(\gamma)$ is not trivalent.

We thus have to check that

$$2e_\mathbb{R}+4e_\mathbb{C}+r+2s-1 = e_\mathbb{R}+2e_\mathbb{C}+2(v_\mathbb{R}+2v_\mathbb{C})+\mathrm{ov}(\Gamma),$$

which is equivalent to

$$\mathrm{ov}(\Gamma)-(r+2s) = e_\mathbb{R}+2e_\mathbb{C}-2(v_\mathbb{R}+2v_\mathbb{C})-1.$$

Since Γ is a tree, the Euler characteristic gives the following relation:

$$1+e_\mathbb{R}+2e_\mathbb{C} = v_\mathbb{R}+2v_\mathbb{C}.$$

The count of the valencies of the vertices gives also:

$$r+2s+2e_\mathbb{R}+4e_\mathbb{C} = 3(v_\mathbb{R}+2v_\mathbb{C})+\mathrm{ov}(\Gamma).$$

These relations lead to

$$\begin{aligned} \mathrm{ov}(\Gamma)-(r+2s) &= 2e_\mathbb{R}+4e_\mathbb{C}-3v_\mathbb{R}-6v_\mathbb{C} \\ &= e_\mathbb{R}+2e_\mathbb{C}-2(v_\mathbb{R}+2v_\mathbb{C})-1. \end{aligned}$$

Thus, the dimensions are equal.

Let $\zeta \in \mathbb{R}((t))^{*r-1} \times i\mathbb{R}((t))^{*s}$ be a generic family of moments, defining a real configuration of points $\mathcal{P}_0$, and thus a symmetric configuration of points $\mathcal{P}$ in $\mathbb{C}\Delta$ when considering possible changes of signs. Let $\mu_j = \mathrm{val}\zeta_j \in \mathbb{R}$ be their respective "tropical" moments, which we also assume to be generic. We denote by $\zeta_j^\Gamma = \zeta_j t^{-\mu_j} \in \mathbb{C}[[t]]^\times$ the moments normalized to have a zero valuation. Any classical curve passing through the symmetric configuration $\mathcal{P}$ tropicalizes to a real parametrized tropical curve $h : \Gamma \to N_\mathbb{R}$ satisfying $\mathrm{ev}(\Gamma) = \mu$, and specializes to a tuple (χ, α, β) in $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$, satisfying $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$. Moreover, the plane tropical curve image $h(\Gamma)$ has a unique parametrization as a parametrized tropical curve of degree $\Delta(s)$. Notice that the space $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$ depends on the choice of the parametrized tropical curve (Γ, h).

Conversely, for each real parametrized tropical curve (Γ, h) with $\mathrm{ev}(\Gamma) = \mu$, we need to find the classical curves tropicalizing to (Γ, h) and passing through the symmetric configuration $\mathcal{P}$. Such a curve corresponds to a point in the moduli space $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$. Finding the curves passing through the symmetric configuration $\mathcal{P}$ and tropicalizing on Γ thus amounts to solve for (χ, α, β) the equation $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$, for any possible sign of ζ^Γ. Recall that the complex moments are purely imaginary.

Given a parametrized tropical curve $h_0 : \Gamma_0 \to N_\mathbb{R}$ of degree $\Delta(s)$ with $\mathrm{ev}(\Gamma_0) = \mu$, and a real parametrized curve $h : \Gamma \to N_\mathbb{R}$ having the same image $C_{\mathrm{trop}} = h(\Gamma) = h(\Gamma_0)$, we say that $(\chi_0, \alpha_0, \beta_0)$ is a first order solution if $\Theta(\chi_0, \alpha_0, \beta_0) = (1, \zeta^\Gamma) \bmod t$. Notice that, as μ is generic, the image tropical curve C_{trop} is a nodal curve. The real rational curves with image C_{trop} are described in Lemma 3.1.19.

Theorem 4.3.9
For each real parametrized tropical curve (Γ, h) of degree Δ, obtained from a parametrized tropical curve of degree $\Delta(s)$ passing through μ, and each first order solution $(\chi_0, \alpha_0, \beta_0)$, such that the Jacobian of Θ at $(\chi_0, \alpha_0, \beta_0)$ is invertible at first order, there is a unique lift of $(\chi_0, \alpha_0, \beta_0)$ to a true solution (χ, α, β) in $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$.

Remark 4.3.10. In the next section, we prove that Theorem 4.3.9 applies under the assumption that all pairs of complex conjugated points sit on the same toric divisor. The Theorem does not apply in the general setting for the following reason: the Jacobian matrix may not be invertible at first order anymore. Thus, in order to find lifts of first order solutions, one would need to find solutions at a higher order, that requires more refined computations. ♦

Proof. As one only needs to solve $\Theta(\chi, \alpha, \beta)$ in $\mathcal{X} \times \mathcal{A} \times \mathcal{B}$, the Theorem is an immediate application of the Hensel's Lemma in several variables. □

4.4 Statement of result and proof

4.4.1 Statement of result and plan for the proof

Using Theorem 4.3.9, we now can relate $R_{\Delta,s}$ and $N_{\Delta(s)}^{\partial,\mathrm{trop}}$ in the case $s=(s_1,0,\dots)$. From now on, we assume that only s_1 might be non-zero.

Moreover, from now on, the complex points in the configuration are purely imaginary. To do such an assumption, one needs to prove the existence of regular values of the evaluation maps that sends a parametrized curve to the coordinates of its intersection points with the boundary. This transversality condition is needed in the proof of invariance in [Mik17]. The existence of regular purely imaginary values is proven close to the tropical limit along with the correspondence theorem, when we check the invertibility of the Jacobian matrix. As the complex points are now purely imaginary, the quantum index is equal to the log-area, and the refined count only needs to be multiplied by 2^{s_1} in order to take into account that the opposite pair of a purely imaginary pair of conjugated points is the same pair.

Let $\mathcal{P}_t$ be a symmetric configuration of points depending on a parameter t, chosen as in section 4.2, but with purely imaginary points. This means that one is given a collection of series $\pm\zeta_i(t)\in\mathbb{R}((t))^*\cup i\mathbb{R}((t))^*$ corresponding to the coordinates of the points $\pm p_i(t)$ of $\mathcal{P}_t$ on the toric divisors. Let μ be the tropicalization of the point configuration, *i.e.* for a pair of points $\pm p_i(t)$, we have $\mu_i=\mathrm{val}\ \zeta_i(t)$. The correspondence theorem, proven in the previous section, provides for t large enough a correspondence between the curves of $\mathcal{S}(\mathcal{P}_t)$, which are real parametrized curves of degree Δ, and the parametrized tropical curves (Γ_0,h_0) of degree $\Delta(s)$ such that $\mathrm{mom}(\Gamma_0,h_0)=\mu$. This is done by enhancing (h_0,Γ_0) to a real parametrized tropical curve of degree Δ admitting first order solutions, and showing that one can lift every first order solution to a true solution. By counting the first order solutions, we assign a multiplicity to each curve of $\mathrm{mom}^{-1}(\mu)$, so that the count of $\mathrm{mom}^{-1}(\mu)$ with these multiplicities gives the invariant $R_{\Delta,s}$. This multiplicity happens to be proportional to the refined multiplicity of Block-Göttsche, thus leading to the relation stated in Theorem 4.4.1.

Theorem 4.4.1
One has

$$R_{\Delta,(s_1,0,\dots,0)}=2^{s_1}\frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{m-2-s_1}}{(q-q^{-1})^{s_1}}N_{\Delta(s)}^{\partial,trop}=2^{s_1}\frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{m-2-2s}}{(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{s}}N_{\Delta(s)}^{\partial,trop}$$

Being given a parametrized tropical curve $h_0:\Gamma_0\to N_\mathbb{R}$ of degree $\Delta(s)$ with $\mathrm{mom}(\Gamma_0,h_0)=\mu$, the task of computing its multiplicity amounts to two things. The first is to find the parametrized real tropical curves of degree Δ having the same image. The second task consists in finding the first order solutions to $\Theta(\chi,\alpha,\beta)=(1,\zeta^\Gamma)$. Being given a parametrized tropical curve $h_0 : \Gamma_0 \to N_{\mathbb{R}}$ of degree $\Delta(s)$ such that $\mathrm{mom}(\Gamma_0, h_0) = \mu$, here is the list of results we need to show before getting to the proof of Theorem 4.4.1:

- Show that the real parametrized tropical curves with image $h_0(\Gamma_0)$ having a first order lift have no flat vertex and thus find the real parametrized tropical curves $h : \Gamma \to N_{\mathbb{R}}$ admitting first order solutions. This is done in subsection 4.4.2.
- Show that for each first order lifts, the Jacobian of Θ is invertible at first order, so that Theorem 4.3.9 applies. This is done in subsection 4.4.3.
- Before getting to the count of first order solutions, solve the enumerative problem for parabola, done in subsection 4.4.4.
- Count the first order solutions and finally prove Theorem 4.4.1. See subsection 4.4.5.

We also give a sketch for an alternative proof in subsection 4.4.6 using the approach of Mikhalkin [Mik05] or Shustin [Shu06a]; [Shu06b].

4.4.2 The look for tropical solutions

Let $h_0 : \Gamma_0 \to N_{\mathbb{R}}$ be a parametrized rational tropical curve of degree $\Delta(s)$ such that $\mathrm{mom}(\Gamma_0, h_0) = \mu$. Let $C_{\mathrm{trop}} = h_0(\Gamma_0)$ be its image, which is a plane tropical curve. We need to find the real parametrized tropical curves (Γ, h) of degree Δ parametrizing the plane curve C_{trop} and admitting a first order solution. The different possible real structures are described in Proposition 3.1.19. As all the pairs of complex points sit on the same toric divisor, the graph Γ_{even} only consists of the even unbounded ends. The only real structures described by Proposition 3.1.19 are the presence of quadrivalent vertices at the unique vertex of these unbounded ends, or a flat vertex. This latter case is forbidden by the following lemma.

Lemma 4.4.2
The parametrized real tropical curves with a trivalent flat vertex cannot be the tropicalization of a family of parametrized real rational curves passing through $\mathcal{P}_t$.

Proof. Assume that $h : \Gamma \to N_{\mathbb{R}}$ has a trivalent flat vertex w, in direction n_{j_0}, with two outgoing unbounded ends exchanged by the involution. Let $f_t : \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}((t))^*$ be a parametrized real rational curve tropicalizing to (Γ, h). Then, in the real coordinate y such that the two conjugated points have coordinate $\pm i$ and some real point has coordinate ∞, the morphism takes the following form:

$$f(y) = \chi_w t^{h(w)} (y^2+1)^{n_{j_0}} \prod_j \left(\frac{y}{y(q_j)} - 1 \right)^{n_j} \in \mathrm{Hom}(M, \mathbb{C}((t))^*),$$

where q_j are the other boundary points of the curve, and $\chi_w \in N \otimes \mathbb{R}((t))^*$. The moment at $\pm i$ is obtained by evaluating at $\iota_{n_{j_0}} \omega \in M$, then at $\pm i$. At the first order, the moments of the exchanged unbounded ends are real: the big product takes value 1, the coefficient $\chi_w(\iota_{n_{j_0}} \omega)$ is real, and $(y^2+1)^{n_{j_0}}$ is evaluated with 0 exponent. This is absurd since it is supposed to be purely imaginary. Hence, we cannot have any flat vertex. $\square$

Lemma 4.4.3
Among the real parametrized tropical curves $h : \Gamma \to N_{\mathbb{R}}$ of degree Δ with image C_{trop}, at most one may be the tropicalization of a family of parametrized real rational curves passing through $\mathcal{P}_t$. Moreover, this real tropical curve is the tropical curve obtained from Γ_0 with the maximal splitting graph.

Proof. There is an infinite number of parametrized curves with image C_{trop}, obtained by splitting the graph of even edges and described in Proposition 3.1.19. All the unbounded ends of C_{trop} associated to the complex markings are double edges near infinity since they correspond to two distinct marked points. Thus they belong to Γ_{even}. Since all the complex markings are on the same divisor, they have the same direction and they cannot meet at a common vertex. Therefore, there are no extendable vertex and the graph Γ_{even} only consists of the even unbounded ends. The only possibility is that the double ends separates itself at a trivalent flat vertex, sent somewhere on the unbounded end of C_{trop}. However, this is forbidden by the previous lemma. Therefore, all the double ends maximally split and there is a unique possibility. □

We have proven that for C_{trop}, there is a unique real parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$ of degree Δ with image C_{trop} that can be the tropicalization of a family of parametrized real rational curves.

4.4.3 Invertibility of the Jacobian

We now prove that for each real parametrized tropical curve obtained in subsection 4.4.2, Theorem 4.3.9 applies and one can lift any first order solution of $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$ to a true solution.

Lemma 4.4.4
Let $h : \Gamma \to N_{\mathbb{R}}$ be a real parametrized tropical curve given by Lemma 4.4.3 and $(\chi_0, \alpha_0, \beta_0)$ any first order solution to the equation $\Theta = (1, \zeta^\Gamma)$. The Jacobian of Θ is invertible at $(\chi_0, \alpha_0, \beta_0)$.

Proof. Let $h : \Gamma \to N_{\mathbb{R}}$ be one of the real parametrized tropical curves given by 4.4.3, meaning that $h(\Gamma)$ is a plane tropical curve such that its parametrization $h_0 : \Gamma_0 \to N_{\mathbb{R}}$ as a curve of degree $\Delta(s)$ satisfies $\text{mom}(\Gamma_0) = \mu$, and that (Γ, h) has no flat vertex. Let $(\chi_0, \alpha_0, \beta_0)$ be a first order solution. We show by induction that the Jacobian matrix is invertible. For simplicity, and because of the multiplicative nature of the map Θ, we use logarithmic coordinates for every variable except β. It means that we look at $\log \Theta$, depending on the new variables $(\log \chi, \log \alpha, \beta)$. Each time, the logarithm is taken coordinate by coordinate.

Notice that if γ is an edge, $\log \alpha_\gamma$ and β_γ are scalars, while if w is a real vertex, $\log \chi_w : M \to \mathbb{R}$ is an element of $N_{\mathbb{R}}$.

To compute the Jacobian relative to coordinates $(\log\chi, \log\alpha, \beta)$ at $t=0$, we can first put $t=0$. Thus, we get for every bounded edge $\gamma \in \Gamma_b^1$:

$$\phi_\gamma|_{t=0} = \prod_{\substack{\gamma' \neq \gamma \\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} (\beta_{\gamma'}|_{t=0} - \beta_\gamma|_{t=0})^{n_{\gamma'}} \in N \otimes \mathbb{R}^*,$$

and for every unbounded end e_j:

$$\varphi_j|_{t=0} = \pm \prod_{\substack{\gamma' \neq e_j \\ \mathfrak{t}(\gamma')=v_j}} (\beta_{e_j}|_{t=0} - \beta_{\gamma'}|_{t=0})^{\omega(n_j, n_{\gamma'})} \in \mathbb{C}^*.$$

Notice that at the first order, ϕ_γ depends only on β and not α. For convenience of notation, all the following computations are taken at the first order, and we drop "$|_{t=0}$" out of the notation.

The description of Γ implies that it has only real vertices and real bounded edges. For a bounded edge $\gamma \in \Gamma_b^1$, let $N_\gamma = \phi_\gamma \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \alpha_\gamma^{n_\gamma} \in N_{\mathbb{R}[[t]]^\times}$. Similarly, let $X_j = \chi_{v_j}(m_j)\varphi_j \in \mathbb{R}[[t]]^\times$ for a real end x_j, and $Z_j = \chi_{v_j}(m_j)\varphi_j \in \mathbb{C}[[t]]^\times$ for a complex end $z_j^\pm$. The variables N_γ, X_j and Z_j index the lines of the Jacobian matrix $\frac{\partial \log\Theta}{\partial(\log\chi, \log\alpha, \beta)}$. We now compute these lines.

- For a general bounded edge $\gamma \in \Gamma_b^1$, one has at the first order

$$\log N_\gamma = \sum_{\substack{\gamma' \neq \gamma \\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} n_{\gamma'} \log(\beta_{\gamma'} - \beta_\gamma) \ + \ \log\chi_{\mathfrak{t}(\gamma)} - \log\chi_{\mathfrak{h}(\gamma)} + n_\gamma \log\alpha_\gamma \in \mathrm{Hom}(M, \mathbb{R}).$$

  Therefore, one has the following partial Jacobian matrices,

$$\frac{\partial \log N_\gamma}{\partial \log\chi_{\mathfrak{t}(\gamma)}} = I_2, \ \frac{\partial \log N_\gamma}{\partial \log\chi_{\mathfrak{h}(\gamma)}} = -I_2, \ \frac{\partial \log N_\gamma}{\partial \log\alpha_\gamma} = n_\gamma.$$

  Concerning the β variables, the only cases when N_γ depends on a $\beta_{\gamma'}$ parameter at first order is when the tail $\mathfrak{t}(\gamma)$ of γ is one of the quadrivalent vertices. In that case, the intervening coordinate is precisely β_γ. Then one has

$$\frac{\partial \log N_\gamma}{\partial \beta_\gamma} = \frac{2\beta_\gamma}{\beta_\gamma^2 + 1} n_{\gamma'},$$

  where $n_{\gamma'} \in N$ is the vector directing the complex ends adjacents to $\mathfrak{t}(\gamma)$.
- For a real end x_j, corresponding to an end e_j, directed by n_j and adjacent to a vertex v_j, one has at the first order

$$\log X_j = \sum_{\substack{\gamma' \neq e_j \\ \mathfrak{t}(\gamma')=v_j}} \omega(n_j, n_{\gamma'}) \log(\beta_{e_j} - \beta_{\gamma'}) \ + \ \log\chi_{v_j}(m_j) \in \mathbb{R}.$$

The Jacobian relative to $\log \chi \in N \otimes \mathbb{R}$ is the Jacobian of the evaluation at m_j, which is linear with respect to $\log \chi$. Thus, one has

$$\frac{\partial \log X_j}{\partial \log \chi_{v_j}} = m_j = \iota_{n_j}\omega.$$

Now, the only case where $\log X_j$ depends on a β parameter is when v_j is a quadrivalent vertex. In that case, the parameter is β_{e_j}. Let $n_{\gamma'}$ be the vector directing the complex ends adjacent to v_j. Then, one has

$$\frac{\partial \log X_j}{\partial \beta_{e_j}} = \omega(n_j, n_{\gamma'})\frac{2\beta_{e_j}}{\beta_{e_j}^2+1}.$$

- For the case of Z_j, the computations are similar, only this time the target space is $\mathbb{C}$ instead of $\mathbb{R}$. The unbounded end e_j is adjacent to a quadrivalent vertex. Let γ be the real edge such that $\mathfrak{t}(\gamma) = v_j$. This edge might be unbounded. Then, one has

$$\log Z_j = \sum_{\substack{\gamma' \neq e_j \\ \mathfrak{t}(\gamma')=v_j}} \omega(n_j, n_{\gamma'}) \log(\beta_{e_j} - \beta_{\gamma'}) \ + \ \log \chi_{v_j}(m_j) \in \mathbb{C}.$$

Thus, we once again have

$$\frac{\partial \log Z_j}{\partial \log \chi_{v_j}} = m_j = \iota_{n_j}\omega \in M_{\mathbb{R}} \subset M_{\mathbb{C}},$$

and this time

$$\frac{\partial \log Z_j}{\partial \beta_\gamma} = \omega(n_\gamma, n_j)\frac{\partial \log(i-\beta_\gamma)}{\partial \beta_\gamma} = \frac{\omega(n_\gamma, n_j)}{\beta_\gamma^2+1}\begin{pmatrix}\beta_\gamma \\ 1\end{pmatrix} = \frac{\omega(n_\gamma, n_j)}{\beta_\gamma^2+1}(\beta_\gamma + i) \in \mathbb{C}.$$

Now that all the terms of the Jacobian matrix $\frac{\partial \log \Theta}{\partial(\log\chi, \log\alpha, \beta)}$ are known, we make an induction on the number of vertices to prove that it is invertible. The initialization is done within the induction step, by removing the column indexed by α_γ, and the remaining rows and columns which are not drawn on the array. Let V be a vertex of Γ which is adjacent to two real unbounded ends, or one real and two complex unbounded ends, as depicted on Figure 4.3.

- Let γ be the edge with $\mathfrak{h}(\gamma) = V$. We assume that γ is a bounded edge. In the first case, the matrix has the following form

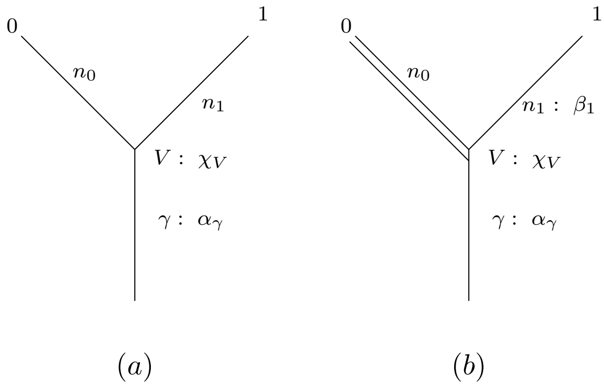

Figure 4.3 – Vertices adjacent to two real ends (a) or a real end and two complex ends (b).

$$
\mathrm{Jac}\Theta = \begin{array}{c|ccc|cc|c|}
 & & & & \chi_V & & \alpha_\gamma \\
\cline{5-7}
 & & & & 0 & 0 & 0 \\
 & & * & & \vdots & \vdots & \vdots \\
 & & & & 0 & 0 & 0 \\
\cline{2-7}
\multirow{2}{*}{N_γ} & * & \cdots & * & -1 & 0 & \multirow{2}{*}{n_0+n_1} \\
 & * & \cdots & * & 0 & -1 & \\
\cline{2-7}
C_0 & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_0} & 0 \\
\cline{2-7}
C_1 & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_1} & 0 \\
\cline{2-7}
\end{array}.
$$

By developing with respect to the last two rows, since (m_0, m_1) are free, we are left with the following determinant:

$$
\begin{array}{c|ccc|c|}
 & & & & \alpha_\gamma \\
\cline{5-5}
 & & & & 0 \\
 & & * & & \vdots \\
 & & & & 0 \\
\cline{2-5}
\multirow{2}{*}{N_γ} & * & \cdots & * & \multirow{2}{*}{n_γ} \\
 & * & \cdots & * & \\
\cline{2-5}
\end{array}.
$$

If γ was an unbounded end, we would be left with the empty matrix and we would have proven invertibleness. Otherwise, the last two rows correspond to a copy of $N_\mathbb{R}$, and are thus given by two elements of $M_\mathbb{R}$, the dual of $N_\mathbb{R}$. Up to a change of basis, one can assume that one of these elements of $M_\mathbb{R}$ is $\omega(n_\gamma, -)$, which takes 0 value on n_γ. Thus, by making a development with respect to the column, we are reduced to the determinant matrix where the bounded edge γ is replaced with an unbounded real end, directed by n_γ, and N_γ is replaced by the evaluation of the moment of this new unbounded end. Thus, the matrix is invertible by induction.

- If the vertex V is adjacent to two complex unbounded ends (directed by n_0), and to a real unbounded end (directed by n_1), let β_1 be the β coordinate associated to the real end. Let γ be the edge with $\mathfrak{h}(\gamma) = V$, thus directed by $2n_0 + n_1$. The determinant takes the following form:

$$\mathrm{Jac}\Theta = \begin{array}{c|ccc|cc|c|c|}
\multicolumn{4}{c}{} & \multicolumn{2}{c}{\chi_V} & \multicolumn{1}{c}{\alpha_\gamma} & \multicolumn{1}{c}{\beta_1} \\
\cline{5-8}
 & & & & 0 & 0 & 0 & 0 \\
 & & * & & \vdots & \vdots & \vdots & \vdots \\
 & & & & 0 & 0 & 0 & 0 \\
\cline{2-8}
\multirow{2}{*}{N_γ} & * & \cdots & * & -1 & 0 & \multirow{2}{*}{n_γ} & 0 \\
 & * & \cdots & * & 0 & -1 & & 0 \\
\cline{2-8}
\multirow{2}{*}{C_0} & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_0} & 0 & \beta_1 \\
 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
\cline{2-8}
C_1 & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_1} & 0 & 2\beta_1 \\
\cline{2-8}
\end{array}.$$

Notice that we dropped out the constant factor $\frac{\omega(2n_0,n_1)}{\beta_1^2+1}$ in the last column. Similarly to the previous case, one can make a development with respect to the penultimate row, and then the second resulting last rows. We recover the same determinant as in the case of a trivalent real vertex, which is also the empty determinant if γ is an unbounded end. Hence, the conclusion follows, reducing once again to a graph with one vertex less.

□

We have now proven that for our choice of s, Theorem 4.3.9 applies for any first order solution, which we now need to count.

4.4.4 Local resolution at the quadrivalent vertices

In this subsection, we solve the enumerative problem for curves of degree $\Delta(m_1, m_2, m_3)$, tracing back to subsubsection 4.1.3.3. This resolution is necessary for the counting of first order sollution at quadrivalent vertices, *i.e.* that have two incoming adjacent fixed edges, and two exchanged adjacent unbounded edges.

Continuing in the setting of subsubsection 4.1.3.3, we deal with real rational curves in the toric surface $\mathbb{C}\Delta(m_1, m_2, m_3)$. We choose a real point on $\mathbb{C}E_3$ and two purely imaginary conjugated points on $\mathbb{C}E_1$, and look for real rational curves of degree $\Delta(m_1, m_2, m_3)$, maximally tangent to each toric divisor at the given points. The Menelaus theorem ensures that there exists a unique point on $\mathbb{C}E_2$ such that each curve passing through the three chosen points also pass through the point on $\mathbb{C}E_2$. Such a curve has a parametrization of the form

$$\varphi(t) = \big(a(t-c)^{m_1}, b(t-c)^{2m_2}(t^2+1)^{m_3-m_2}\big) \in (\mathbb{C}^*)^2,$$

where c is some real number corresponding to the coordinate of the intersection point with $\mathbb{C}E_3$, and $a, b \in \mathbb{R}^*$. The intersection point with $\mathbb{C}E_2$ corresponds to the coordinate t taking the infinite value. The condition to pass through the specific points are given by the following equations:

$$a(i-c)^{m_1} = i\lambda \in i\mathbb{R}^* \text{ and } \frac{b^{\frac{m_1}{\delta}}}{a^{\frac{2m_2}{d}}}(c^2+1)^{\frac{m_1}{\delta}(m_3-m_2)} = \mu \in \mathbb{R}^*,$$

where $\delta = m_1 \wedge (2m_2)$ is the integer length of E_3. The first equation solves for c and a, and the second equation solves for b with a unique solution if $\frac{m_1}{\delta}$ is odd, and 0 or 2 solution according to the sign of μ when it is even. Let us do this resolution. The first equation implies that $(i-c)^{m_1} \in i\mathbb{R}$ and thus we can write it $i - c = re^{i\pi\frac{2k+1}{2m_1}}$ with $k \in \mathbb{Z}$ and $r \in \mathbb{R}$. Therefore, we have $i - re^{i\pi\frac{2k+1}{2m_1}} = c \in \mathbb{R}$. Hence,

$$\begin{aligned} \mathfrak{Im}\left(i - re^{i\pi\frac{2k+1}{2m_1}}\right) = 1 - r\sin\left(\pi\frac{2k+1}{2m_1}\right) = 0 \quad &\Rightarrow \quad r = \frac{1}{\sin\left(\pi\frac{2k+1}{2m_1}\right)} \\ &\Rightarrow \quad c = r\cos\left(\pi\frac{2k+1}{2m_1}\right) = \cot\left(\pi\frac{2k+1}{2m_1}\right) = c_k. \end{aligned}$$

We have proven that c can only take a finite number of values $c_k = \cot\left(\pi\frac{2k+1}{2m_1}\right)$, for $k \in [\![0; m_1-1]\!]$. For each value of c_k we find a unique a, and then solve for b eventually. Thus, we have proven that up to the action of the real torus $(\mathbb{R}^*)^2$, every real curve having purely imaginary intersection with $\mathbb{C}E_1$, and real intersection with both $\mathbb{C}E_2$ and $\mathbb{C}E_3$ is one of the curves

$$\psi_k : t \longmapsto \left((t-c_k)^{m_1}, (t-c_k)^{2m_2}(t^2+1)^{m_3-m_2}\right).$$

These parametrized curves are the respective images of the curves

$$\varphi_k : t \longmapsto \left(t - c_k, \frac{t^2+1}{t-c_k}\right),$$

by the monomial map $\alpha : (z,w) \mapsto (z^{m_1}, z^{m_3+m_2}w^{m_3-m_2})$. Their quantum indices have been computed in subsubsection 4.1.3.3.

Lemma 4.4.5
For any $k \in [\![0; m_1-1]\!]$, the log-area of $(\mathbb{H}, \varphi_k)$ is

$$\int_{\varphi_k(\mathbb{H})} Log^*\omega = \int_{\varphi_k(\mathbb{H})} \arg^*\omega_\theta = \left(\frac{2k+1}{m_1} - 1\right)\pi^2.$$

In particular, the quantum index of ψ_k is

$$k(\mathbb{H}, \psi_k) = (m_3 - m_2)(2k+1-m_1).$$

Proof. This follows immediatly from subsubsection 4.1.3.3. □

In particular, given two purely imaginary points on $\mathbb{C}E_1$ and one real point on $\mathbb{C}E_3$, we have proven that

- If m_1 is odd, there exists precisely m_1 curves maximally tangent to the divisors and passing through the chosen points. Moreover, according to the two choices of orientation for each of them, there are two curves of each quantum index $(m_3 - m_2)(2k + 1 - m_1)$, for $k \in [\![0; m_1 - 1]\!]$, *i.e.* all the even multiples of $m_3 - m_2$ of absolute value $< m_1$. This set is stable by one of the deck transformations. Thus, we get $2m_1$ oriented curves, two of each quantum index. If we also consider curves passing through the symmetric real point, we get $4m_1$ real oriented curves.

- If m_1 is even, we might still be in the previous case (if $\frac{m_1}{\delta}$ is odd), or there might be $2m_1$ or 0 solutions according to the sign of μ (when $\frac{m_1}{\delta}$ is even). Thus, there are $4m_1$ or 0 oriented curves passing through the points.

We have thus proven the following lemma.

Lemma 4.4.6
In $\mathbb{C}\Delta(m_1, m_2, m_3)$, being given a pair of purely imaginary points on $\mathbb{C}E_1$ and a pair of opposite real points on $\mathbb{C}E_3$, there are $4m_1$ oriented real rational curves of degree $\Delta(m_1, m_2, m_3)$ passing through the symmetric real configuration, and their refined count is

$$4 \sum_{0 \leqslant k < m_1} q^{(m_3-m_2)(2k+1-m_1)} = 4\frac{q^{m_1(m_3-m_2)} - q^{-m_1(m_3-m_2)}}{q^{m_3-m_2} - q^{-(m_3-m_2)}}.$$

Proof. We already have proven that there are $4m_1$ oriented real rational curve satisfying the conditions and computed their quantum index. Moreover, one can easily see that their logarithmic rotation number is 0. The result follows. □

4.4.5 The search for the first order solutions and proof of the Theorem

Using the previous subsections, we are now ready to count the first order solutions to $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$ for a fixed real parametrized rational curve (Γ, h) obtained from a parametrized rational curve (Γ_0, h_0) with $\mathrm{mom}(\Gamma_0, h_0) = \mu$.

Proposition 4.4.7
Let $\mathcal{P}_t$ be a symmetric real configuration of points as previously chosen, tropicalizing to a family of moments μ. Let $h_0 : \Gamma_0 \to N_\mathbb{R}$ be a parametrized tropical curve of degree $\Delta(s)$ having moments μ, and let $h : \Gamma \to N_\mathbb{R}$ be the associated real parametrized tropical curve without flat vertex such that $\mathrm{mom}(\Gamma) = \mu$. *Vertices of Γ and Γ_0 are canonically identified. Let $W_1, \dots, W_s$ be the quadrivalent vertices of Γ, adjacent to the complex unbounded ends, let m_{W_i} denotes their complex multiplicity as a trivalent vertex of Γ_0. Then there are precisely $2^{m-2s} \prod m_{W_i}$ oriented real curves passing through the symmetric configuration $\mathcal{P}_t$ and tropicalizing to (Γ, h). Their refined count according to the quantum index and sign σ is given by*

$$m'_\Gamma = 4 \prod_1^s \frac{q^{\frac{m_{W_i}}{2}} - q^{-\frac{m_{W_i}}{2}}}{q - q^{-1}} \prod_{V \neq W_i} (q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}}).$$

Before proving this proposition, we can now prove Theorem 4.4.1.

Proof of Theorem 4.4.1. This is a consequence of Theorem 4.3.9 that states that each of the first order solutions given by Proposition 4.4.7 lifts to a unique solution, and an easy computation between the multiplicities: we obtain $R_{\Delta,s}$ by counting curves with multiplicities $\frac{1}{4}m'_\Gamma$. The multiplicity $\frac{1}{4}m'_\Gamma$ is obtained from m^q_Γ by clearing the denominators of the $m-2-s$ vertices and dividing by the terms of the s quadrivalent vertices:

$$\frac{1}{4}m'_\Gamma = \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2-s}}{(q-q^{-1})^s} m^q_\Gamma.$$

Therefore, one has

$$R_{\Delta,s} = 2^s \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2-s}}{(q-q^{-1})^s} N^{\partial,\text{trop}}_{\Delta(s)}.$$

Using the identity $q - q^{-1} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$, we get that

$$R_{\Delta,s} = 2^s \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2-2s}}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^s} N^{\partial,\text{trop}}_{\Delta(s)}.$$

□

In [GS19] L. Göttsche and F. Schroeter proposed a refined way to count so-called *refined Broccoli curves* having fixed ends, and passing through a fixed configuration of "real and complex" points. In the case where there are only marked ends and no marked points, this count coincides with the count of plane tropical curves passing through the configuration with usual Block-Göttsche multiplicities from [GS14] up to a multiplication by a constant term depending on the degree and easily computed. More precisely, provided there are no marked points, the refined Broccoli multiplicity is just the refined multiplicity from [GS14] enhanced by a product over the ends of weight higher than 2, coinciding with this aformentionned constant term. In our case, since the only multiple edges are marked and of weight 2 (only one real unbounded end is unmarked and of weight one), this factor is $\frac{q+q^{-1}}{q^{1/2}+q^{-1/2}}$ for each of the s ends. If we denote by $BG_{\Delta(s)}(q)$ the refined invariant obtained in [GS19], then we have the relation

$$R_{\Delta,s}(q) = 2^s \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2-2s}}{(q+q^{-1})^s} BG_{\Delta(s)}(q).$$

We now prove Proposition 4.4.7.

Proof of Proposition 4.4.7. The proof is made with an induction on the number of vertices of the curve Γ. Exceptionally, to suit the induction, a vector n_j of N directing an unbounded end e_j may not be primitive. In that case, we denote by $m_j = \omega\left(\frac{n_j}{l(n_j)}, -\right)$ the dual vector, but still of lattice length 1. Let V be a vertex adjacent to two real ends, or one real end and two complex ends. Let γ be the edge, maybe unbounded, with $\mathfrak{h}(\gamma) = V$.

- If V is a real vertex adjacent to two real unbounded ends indexed by 0 and 1, then we have to solve for $\chi_V : M \to \mathbb{R}^*$ the following system:

$$\begin{cases} \chi_V(m_0) = \pm\zeta_0^\Gamma \in \mathbb{R}^* \\ \chi_V(m_1) = \pm\zeta_1^\Gamma \in \mathbb{R}^* \end{cases}.$$

Recall that the vectors n_0 and n_1 might not be primitive, but m_0 and m_1 are. This system leads to 4 solutions: if (e_1^*, e_2^*) is a basis of M, the absolute value of $\chi_V(e_1^*)$ and $\chi_V(e_2^*)$ is uniquely determined, while the sign may be chosen arbitrarily. Notice that some choices of signs for the right-hand side of the system may provide several solutions, while other provide none. Let m_γ be the primitive vector dual to n_γ. Let $\tilde{m}_\gamma$ be such that $(m_\gamma, \tilde{m}_\gamma)$ is a basis of M. These 4 solutions separate themselves into two groups of 2, according to the sign of $\chi_V(m_\gamma)$. If γ is unbounded, this closes the proof.

Now, if γ is bounded, we have the equation

$$\phi_\gamma \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_V} \cdot \alpha_\gamma^{n_\gamma} = 1 \in N \otimes \mathbb{R}^*.$$

We evaluate at m_γ, leading to

$$\phi_\gamma(m_\gamma)\chi_{\mathfrak{t}(\gamma)}(m_\gamma) = \chi_V(m_\gamma) \in \mathbb{R}^*.$$

Recall that according to the choice of signs of $\pm\zeta_j$, the sign of $\chi_V(m_\gamma)$ may change. Let replace the bounded edge γ by an unbounded end with direction n_γ, leading to a new parametrized tropical curve (Γ', h'). The above equation is the equation associated to this new unbounded end in Γ', in the corresponding system $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$. Thus, we can proceed by induction. Let $4R$ denote the refined signed count of oriented curves lifting Γ'. These $4R$ curves separate themselves into four groups of R according to the value of the signs the function $\phi_\gamma \cdot \chi_{\mathfrak{t}(\gamma)}$ takes on the basis $(m_\gamma, \tilde{m}_\gamma)$.

Last, we need to solve for α_γ. By evaluating at $\tilde{m}_\gamma$, we get

$$\alpha_\gamma^{\langle n_\gamma, \tilde{m}_\gamma \rangle} = \frac{\chi_V(\tilde{m}_\gamma)}{\phi_\gamma(\tilde{m}_\gamma) \cdot \chi_{\mathfrak{t}(\gamma)}(\tilde{m}_\gamma)}.$$

The solving as well as the number of solutions depends on the sign of the right-hand side.

- ⋆ If n_γ has odd integer length, then we solve uniquely for α_γ for each possible sign of $\chi_V(\tilde{m}_\gamma)$. The sign of $\chi_V(m_\gamma)$ is already determined since we have $\phi_\gamma(m_\gamma)\chi_{\mathfrak{t}(\gamma)}(m_\gamma) = \chi_V(m_\gamma)$. Thus, each of the oriented curves in each of the groups have two possible solutions for χ_V. This corresponds to the gluing of two possible curves, one increasing the logarithmic rotation number by one, the other decreasing it by one. In each case, the orientation of the curve propagates, and the signed count becomes

$$4 \times (q^{m_V/2} - q^{-m_V/2})R.$$

- ⋆ If n_γ has even integer length, then the sign of $\chi_V(m_\gamma)$ is still determined, and the sign of $\chi_V(\tilde{m}_\gamma)$ is forced in order to have at least one solution for α_γ. In that case, we have two. The two possible choices of α_γ correspond to the two ways of gluing a curve over V to one of the $4R$ curves, when such a gluing is possible. One of these choices decreases the logarithmic rotation number by one while the other increases it by one. The signed count then becomes again

$$4 \times (q^{m_V/2} - q^{-m_V/2})R.$$

- If V is adjacent to two complex ends and a real end, respectively indexed by 0 and 1, we then have to solve for β_1 and χ_V the system

$$\begin{cases} \chi_V(m_1)(\beta_1^2+1)^{\langle n_0, m_1\rangle} = \zeta_0^\Gamma \in i\mathbb{R}^* \\ \chi_V(m_0)(i-\beta_1)^{\langle n_1, m_0\rangle} = \pm\zeta_1^\Gamma \in \mathbb{R}^* \end{cases}.$$

This system was already solved in section 4.1.3. Assume that the degree $\{n_0, 2n_1, -n_0-2n_1\}$ is equivalent to $\Delta(m_0,m_1,m_2)$ as in section 4.1.3, ignoring the temporary conflict of notation m_i. (They are integers in $\Delta(m_0,m_1,m_2)$ and co-characters in $\chi_V(m_i)$) The logarithmic rotation number of the solutions is equal to 0, so each solution is counted with a positive sign. The refined count of the solutions from section 4.1.3 is equal to 0, 1 or 2 times the following sum, which covers all the possible values of the quantum index:

$$\sum_{k=0}^{m_1-1} q^{(m_3-m_2)(2k+1-m_1)} = \frac{q^{(m_3-m_2)m_1 - q^{-(m_3-m_2)m_1}}}{q^{m_3-m_2} - q^{-(m_3-m_2)}} = \frac{q^{\frac{m_W}{2}} - q^{-\frac{m_W}{2}}}{q-q^{-1}},$$

since $m_3 - m_2 = 1$, and the complex multiplicity m_W satisfies $m_W = 2m_1$, which is the multiplicity of the vertex. Accounting for both possible orientations, and both choices of signs for the ζ_i, this closes the proof if γ is an unbounded end. Otherwise, we then use the equation

$$\phi_\gamma \cdot \frac{\chi_{\mathrm{t}(\gamma)}}{\chi_V} \cdot \alpha_\gamma^{n_\gamma} = 1 \in N \otimes \mathbb{R}^*,$$

just as in the previous step. We solve for the other unknowns inductively, then for α_γ. The same disjunction provides the new signed count

$$4\frac{q^{m_V/2} - q^{-m_V/2}}{q-q^{-1}}R.$$

□

4.4.6 Alternative proof

So far, we have proven a correspondence theorem using the approach of Tyomkin in [Tyo12]. One could also carry a proof of Theorem 4.4.1 using the approach of Mikhalkin [Mik05]; [Mik17], or Shustin [Shu06b]. Both adopt a description of plane curves by polynomial equations rather than a parametrization.

Briefly, given a real parametrized tropical curve, this method consists in finding a collection of plane curves indexed by the vertices of the tropical curves, and a way of gluing them along the edges of the curve. We refer the reader to [Mik05]; [Mik17]; [Shu06b] for details and proofs. Here, we recover the multiplicity m'_Γ by counting the possible families of curves indexed by the vertices, and the number of gluings. Using the aforementioned approach to the correspondence theorem, this computation reproves Theorem 4.4.1 through a new proof of Proposition 4.4.7.

Alternative Proof of Proposition 4.4.7. We make an induction on the number of vertices. Thus, we initialize with curves Γ having a unique vertex, trivalent or quadrivalent.

Following [Mik05], to compute the multiplicity, one needs to count (in a suitable way) the local curves over the vertices of Γ, and the number of ways to glue them together. In this proof, we do not assume the vectors of Δ to be primitive.

- If there is only one trivalent vertex V in Γ, then we are looking for curves maximally tangent to the toric divisors and passing through two pairs of opposite real points. There are 4 such curves, which are exchanged by the action of the deck transformation group $\{\pm 1\}^2$ on the associated toric surface. Notice that if we specify the points in the pairs the number of curves may vary, but if we consider both points in the pair, the number remains the same. These 4 curves lead to 8 oriented curves. Half of them have logarithmic Gauss degree 1 and the other half has degree -1, leading to signs $\sigma(\mathbb{R}C) = 1$ or -1. Therefore the signed contribution is $4(q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})$.
- If there is only one quadrivalent vertex W, we look for curves passing through one pair of conjugated imaginary points on one divisor, and a pair of opposite real points. Assume that the degree of the vertex is $\Delta(m_1, m_2, m_3)$. Then, as the unbounded non-real ends are of weight 2, we have $m_3 - m_2 = 1$, and the complex multiplicity is $m_W = 2m_1$. We have seen that there are always $2m_1$ curves passing through the configuration: either m_1 for each of the real points in the pair, or $2m_1$ and 0. Therefore, there are $4m_1$ oriented curves going through the pair. Moreover, their quantum index is known. The logarithmic rotation number $\mathrm{Rot}_{\mathrm{Log}}$ can be computed thanks to the same monomial map that allowed us to compute their quantum index: if A denotes the matrix of the monomial map, then
$$\mathrm{Rot}_{\mathrm{Log}}(\psi_k) = \det A \times \mathrm{Rot}_{\mathrm{Log}}(\varphi_k).$$

  As the logarithmic rotation number of φ_k is 0, all the curves have logarithmic rotation number zero and therefore $\sigma(\overrightarrow{C}) = 1$. When accounting for both orientations, the desired count is
$$4\sum_{k=0}^{m_1-1} q^{(m_3-m_2)(2k+1-m_1)} = 4\frac{q^{(m_3-m_2)m_1 - q^{-(m_3-m_2)m_1}}}{q^{m_3-m_2} - q^{-(m_3-m_2)}} = 4\frac{q^{\frac{m_W}{2}} - q^{-\frac{m_W}{2}}}{q - q^{-1}},$$

since $m_3 - m_2 = 1$, and the complex multiplicity m_W satisfies $m_W = 2m_1$, which is the lattice area of the dual triangle.

Now that the initialization is done, assume that Γ has more than one vertex. Let V be a vertex adjacent to two unbounded real ends, or one real end and two complex ends. The last edge adjacent to V, which is bounded, is denoted by γ. Let Γ' be the parametrized tropical curve obtained by deleting this vertex and replacing the edge γ heading to V by an unbounded end with same direction. The Menelaus rule allows us to define a pair of real opposite moments associated to the new unbounded end. This moment is defined by the condition that the symmetric configuration composed by the pairs of points of the edges adjacent to V satisfies the Menelaus condition. We get a new symmetric configuration of points $\mathcal{P}'_t$, indexed by the ends of Γ'.

Let $4R$ be the refined count of oriented curves tropicalizing on Γ', passing through the symmetric configuration $\mathcal{P}'_t$. We now have to glue together the oriented curves above Γ', and the curves over the vertex V. According to [Mik05], such a gluing is possible only if the *phases* agree, and if the edge has weight two, there are two ways to do it. Recall that the *phase* of the curve C_V above V with respect to the edge γ is defined as follows: the edge γ corresponds to an intersection point p of C_V with the toric boundary, the phase is the (set of) quadrants of $(\mathbb{R}^*)^2$ in which the curve sits in a neighborhood of this intersection point. This means the following: the set of quadrants is identified with $\mathbb{F}_2^2$, and

- If the vector n_γ, which directs γ has odd lattice length, then the curve passes through the corresponding toric divisor and changes of quadrant at p. The phase is the element of $\mathbb{F}_2^2/\langle n_\gamma\rangle$ that corresponds to these quadrants.
- If the vector n_γ has even lattice length, the curve stays on the same side of the toric divisor, thus in the same quadrant. In that case, the phase is an element of $\mathbb{F}_2^2$.

In each case we inquire for the refined count over the global tropical curve Γ. We make a disjunction over the type of V, which can either be a trivalent one, or a quadrivalent one (*i.e.* one of the vertices W_i), and γ can be an odd or an even edge. According to [Mik05], if the edge is even, we have two ways of gluing the curves together when the phases agree. We use the action of the deck transformation group $\{\pm 1\}^2$.

- Assume that V is trivalent, and the bounded edge adjacent to V is odd. Then, there are two real opposite points, corresponding to the two distinct phases that the edge can have, where the gluing can happen, and they are exchanged by the deck transformation group $\{\pm 1\}^2$. Therefore, over Γ', there are $2R$ oriented curves for each of the phases (both add up to the total $4R$ oriented curves). There are 4 curves above the vertex V, two over each real phase. If a pair with compatible phases is chosen, we have a unique way of gluing. Moreover, for any possible gluing of an oriented curve and a curve, the orientation of the oriented curve extends to the new global curve. The curves above V thus get an orientation. The two oriented curves obtained this way have opposite quantum indices, and one increases by one the logarithmic rotation number while the other decreases it by one. Finally, the total contribution is

$$(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})2R+(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})2R=4(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R.$$

- Assume that V is trivalent, and γ is an even edge. We have 4 possible phases, exchanged by the deck transformation group. For each of these phases, there are R oriented curves over Γ', and just one curve over V. Each time, there are two ways of gluing the curves: one that increases the logarithmic rotation number, and one that decreases it. We thus get

$$(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R+(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R+(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R+(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R=4(q^{\frac{m_V}{2}}-q^{-\frac{m_V}{2}})R.$$

- Assume that $V=W$ is a quadrivalent vertex, and that γ is an odd edge. Assume that the dual triangle is equivalent to $\Delta(m_1,m_2,m_3)$. There are two possible phases. Over Γ', there are $2R$ oriented curves for each of the phases, while there are m_1 curves over W for each of the phases. In each case the gluing is unique and we get

$$\left(\frac{q^{\frac{m_W}{2}}-q^{-\frac{m_W}{2}}}{q-q^{-1}}\right)2R+\left(\frac{q^{\frac{m_W}{2}}-q^{-\frac{m_W}{2}}}{q-q^{-1}}\right)2R=4\left(\frac{q^{\frac{m_W}{2}}-q^{-\frac{m_W}{2}}}{q-q^{-1}}\right)R.$$

- Finally, if $V=W$ is a quadrivalent vertex and γ is an even edge, there are four phases to consider, each one with R oriented curves over Γ'. This time m_1 is even, and the distribution of the phases for the curves above the vertex might be a little trickier. We have seen that if the boundary points are fixed, there are either $2m_1$ curves above W for one of the points and zero for the other, or m_1 for each of them. In each case this does not give the complete phase at the intersection point. Anyway, for each of these curves there are R oriented curves that we can glue, and this can happen in two ways. Thus, we still get

$$2\times 2\left(\frac{q^{\frac{m_W}{2}}-q^{-\frac{m_W}{2}}}{q-q^{-1}}\right)\times R=4\left(\frac{q^{\frac{m_W}{2}}-q^{-\frac{m_W}{2}}}{q-q^{-1}}\right)R.$$

Finally, we recover the formula for m'_Γ. $\square$

Chapter 5

Recursive formula for tropical refined invariants

This chapter is devoted to the computation of the refined tropical invariants. Briefly, the invariants considered here are obtained by counting rational plane tropical curves satisfying boundary constraints using the refined multiplicity from Block-Göttsche. This multiplicity was proposed by F. Block and L. Göttsche in [BG16] to refine the complex multiplicity of plane tropical curves into a polynomial one. Recall the Definition 3.2.14 of this refined multiplicity:

Definition. The refined multiplicity of a simple nodal tropical curve $h:\Gamma\to N_\mathbb{R}\simeq\mathbb{R}^2$ is

$$m_\Gamma^q=\prod_V [m_V^\mathbb{C}]_q,$$

where $[a]_q=\frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}}$ is the q-analog of a, $m_V^\mathbb{C}$ is the complex multiplicity of a trivalent vertex (the determinant of the slopes of h on two among the three adjacent edges), and the product is indexed by the vertices of Γ, which are trivalent.

Just as the complex and the real multiplicities, the refined multiplicity is a product over the trivalent vertices of the tropical curve. Moreover, it specializes to the complex multiplicity of the tropical curve at $q=1$, and the real multiplicity at $q=-1$. Therefore, the refined count of tropical curves evaluated at ±1 gives an invariant in many situations. For instance, the refined count of genus g and degree $\Delta\subset N$ tropical curves passing through a general configuration of $|\Delta|-1+g$ points in $N_\mathbb{R}$ and evaluated at ±1 is independent of the choice of the point configuration. Furthermore, I. Itenberg and G. Mikhalkin proved in [IM13] that the refined count of the tropical curves solution to this specific tropical enumerative problem is indeed an invariant, and not only its evaluation at ±1.

Since, refinements of tropical enumerative invariants have been found in many situations. Without giving an exhaustive list, here are a few examples:

- refined invariants that interpolate between descendants invariants for rational curves and Welschinger invariants with pairs of complex conjugated points have been given by L. Blechman and E. Shustin in [BS19],
- refined invariants for counting Broccoli curves have been given by L. Göttsche and F. Schroeter in [GS19], and generalized by F. Schroeter and E. Shustin for elliptic curves in [SS18],

- refinement for rational curves in higher dimension in the context of scattering diagrams have been given by S-A. Filippini and J. Stoppa [FS15], also by T. Mandel [Man15], following the ideas of M. Gross, R. Padharipande and B. Siebert in [GPS+10].

While these refined invariants bear similarities with other known invariants such as refinements of Donaldson-Thomas invariants considered by M. Kontsevich and Y. Soibelman [KS08], their meaning remains quite mysterious. In particular, many of the aforementioned papers dealing ith refinements prove tropical invariance statements without giving an interpretation of the refined invariants in classical geometry, often because such an interpretation is unknown. Conjecturally, the refinement of Block and Göttsche coincides with the refinement of Severi degrees by the χ_{-y}-genus, as proposed by L. Göttsche and V. Shende in [GS14]. Since, other interpretations have been given. For instance, P. Bousseau [Bou19] related refined invariants to some Gromov-Witten invariants involving λ-classes, and G. Mikhalkin [Mik17] proved that in toric surfaces, the refined tropical invariants compute some signed number of real rational curves passing through a configuration of real points on the toric boundary, according to the value of some quantum index, see Chapter 4 for more details.

In this chapter, we first give a proper definition of the refined invariants $N_\Delta^{\partial,\mathrm{trop}}$ and provide a proof of their invariance, slightly different from the proof of [IM13], and which generalizes to higher dimension. Such a proof is needed in Chapter 6. Then, we give and prove a recursive formula that leads to an algorithm for their computation. Finally, we give a few values of these invariants, enlighting the functioning of the formula, and close the chapter with some recursive formulas dealing with the algebraic invariants related to $N_\Delta^{\partial,\mathrm{trop}}$ through the use of correspondence theorems.

Sommaire

5.1 Definition of the invariants and recursive formula

5.1.1 Tropical enumerative problem

We now turn our focus into the tropical enumerative problem that provides the refined tropical invariants $N_\Delta^{\partial,\mathrm{trop}}$ used in [Mik17]. This family of enumerative problems depends on the choice of the degree Δ, and the recursive formula proven in the chapter gives a relation between all these invariants.

Let $\Delta = \{v_1, \dots, v_m\} \subset N$ be a tropical degree. We do not assume the vectors v_i to be primitive since the recursive formula almost always makes appear degrees with non-primitive vectors. For each $v_i \in \Delta$ we choose some scalar $\mu_i \in \mathbb{R}$. Because of the tropical Menelaus theorem, the sum of the moments of all the unbounded ends of a tropical curve is zero, so a necessary condition for the scalars μ_i to be the moments of some tropical curve is $\sum_1^m \mu_i = 0$. We now look at the set $\mathcal{S}(\mu)$ of parametrized rational tropical curves of degree Δ that have $\mu = (\mu_i)$ as family of moments, *i.e.* $\mathrm{mom}(\Gamma) = \mu$. The invariance amounts to prove that the count of parametrized rational curves with appropriate multiplicity does not depend on μ. This tropical enumerative problem is called the Δ-problem.

Notice that due to the linear character of the moment map restricted to any combinatorial type, each type contributes at most one solution unless the map is non-injective. Let $\mathrm{Comb}(\Gamma)$ be a top-dimensional combinatorial type, for which the moment map is not injective. Since $\mathrm{Comb}(\Gamma)$ and $\mathbb{R}^{m-1}$ have the same dimension, the moment map is not surjective either. Similarly, the restriction of the evaluation map to non top-dimensional combinatorial types fails to be surjective for dimensional reasons. A family of moments μ is said to be generic if it is chosen outside the image of the moment map restricted to non top-dimensional combinatorial types, and top-dimensional types with non-injective moment map. Thus, if the configuration of moments μ is chosen generically, the set of solutions $\mathcal{S}(\mu)$ is finite, and the rational curves of $\mathcal{S}(\mu)$ solution to the problem are simple, and they have a well-defined refined tropical multiplicity m_Γ^q. Then, we set

$$N_\Delta^{\partial,\mathrm{trop}}(q,\mu) = \sum_{\Gamma\in\mathcal{S}(\mu)} m_\Gamma^q.$$

Remark 5.1.1. Let $\mathrm{Comb}(\Gamma)$ be a combinatorial type, and A the linear map associated to the restriction $\mathrm{mom}_{\mathrm{Comb}(\Gamma)}$ of mom to $\mathrm{Comb}(\Gamma)$. Let (l,V) denotes the coordinates on $\mathrm{Comb}(\Gamma)$ given by the length of the edges and the position of a specified vertex. To see if $\mathrm{Comb}(\Gamma)$ contributes a solution, one just needs to solve the system $A(l,V) = \mu$, leading to a formal solution $(l,V) = A^{-1}(\mu) \in \mathbb{R}^{m-3} \times N_\mathbb{R}$, and check that all its first $m-3$ coordinates, which correspond to the length of the bounded edges, are non-negative. ♦

Theorem 5.1.2
The value of $N_\Delta^{\partial,trop}(q,\mu)$ *does not depend on* μ *as long as* μ *is generic.*

A proof can be found in the next subsection.

5.1.2 Proof of tropical invariance

The proof of invariance goes in the same way as many tropical proofs of invariance by showing that we have a local invariance of the count around the walls of the tropical moduli space.

Proof of Theorem 5.1.2. We choose two generic configurations $\mu(0)$ and $\mu(1)$, and choose a generic path $\mu(t)$ between them. Due to the genericity, we know that the set F of values of t where $\mu(t)$ meets the non-generic configurations is finite, and $N_\Delta^{\partial,\mathrm{trop}}(q,\mu(t))$ is constant on the connected components of the complement of this exceptional set F. We now need to check that the value is constant around these special values.

Let t^* be such a special value. Thanks to the genericity of the path, it means that at least one of the curves of $\mathcal{S}(\mu(t^*))$ has a unique four-valent vertex V. There are three ways to deform this curve into a trivalent one by choosing a splitting of the quadrivalent vertex, meaning there are three maximal cones adjacent to the wall. In some cases, one of the deformation leads to a flat vertex, *i.e.* a non-injective combinatorial type. Let E_1, E_2, E_3, E_4 be the adjacent edges directed by a_1, a_2, a_3, a_4, with ingoing orientations. Their index i is taken in $\mathbb{Z}/4\mathbb{Z}$. The splittings are denoted by $12//34$, $13//24$ and $14//23$ according to the pairing of vertices. Around a wall, one curve solution may be divided into two solutions, or the other way around two solutions may merge into one solution.

We first assume that there are no parallel edges among the edges E_i. Let us prove that up to a relabeling we can assume :

- for each i we have $\omega(a_i, a_{i+1}) > 0$,
- we have $\omega(a_2, a_3) > \omega(a_1, a_2)$.

The first point essentially consists in finding some cyclic counterclockwise order on the outgoing vectors a_i. Let us take such a cyclic order and prove that it satisfies these conditions : if we had $\omega(a_4, a_1) < 0$ (same for $\omega(a_1, a_2) < 0$ and other values), then because of the counterclockwise cyclic order all the vectors a_i are in a half-plane and their sum would not be zero, which is absurd. So we have $\omega(a_i, a_{i+1}) > 0$. For the second point, the assumption that there are no parallel vectors ensures that $\omega(a_i, a_{i+1} + a_{i-1}) \neq 0$ for any i, thus there are no consecutive equal values. Hence we can assume that $\omega(a_2, a_3) > \omega(a_1, a_2)$ up to a cyclic shift of the indices.

Let us notice that

$$\omega(a_4, a_1) = \omega(-a_1 - a_2 - a_3, a_1) = \omega(a_1, a_2 + a_3) > 0,$$

$$\text{and } \omega(a_3, a_4) = \omega(a_3, -a_1 - a_2 - a_3) = \omega(a_1 + a_2, a_3) > 0.$$

To prove the local invariance, we need to know the repartition of the combinatorial types around the wall, that is, the adjacent combinatorial types providing a solution when $\mu(t)$

moves slightly. Using the correspondence theorem of Mikhalkin [Mik05] or the tropical proof of invariance of the count with complex multiplicities given by A. Gathmann-H. Markwig in [GM07b], this repartition is known to match the equality given between complex multiplicities $m_\Gamma = \prod_V m_V^{\mathbb{C}} > 0$. All the vertices in the respective products for the three adjacent combinatorial types are the same, except the two vertices resulting from the splitting of the quadri-valent vertex. The desired relation is then

$$\underset{\text{for } 12//34}{\omega(a_1,a_2)\omega(a_1+a_2,a_3)} \quad + \quad \underset{\text{for } 13//24}{\omega(a_1,a_3)\omega(a_2,a_1+a_3)} \quad + \quad \underset{\text{for } 14//23}{\omega(a_2,a_3)\omega(a_2+a_3,a_1)} \quad = \quad 0,$$

and the repartition of combinatorial types around the wall is given by the sign of each term. It means up to sign that one is positive and is on one side of the wall, and the two other ones are negative, on the other side of the wall. Hence, we just need to study the signs of each term to know which curve is on which side. We know that $\omega(a_1,a_2)$ and $\omega(a_1+a_2,a_3)$ are positive, therefore their product, which is the term of 12//34, is also positive. We know that $\omega(a_2,a_3)$ is positive, but $\omega(a_2+a_3,a_1)$ is negative, therefore their product is negative and 14//23 is on the other side of the wall. It means that the combinatorial types 12//34 and 14//23 are on opposite sides of the wall. We need to determine on which side the type 13//24 is, and that is given by the sign of the middle term. As by assumption $\omega(a_2,a_1+a_3) = \omega(a_2,a_3) - \omega(a_1,a_2) > 0$, it is determined by the sign of $\omega(a_1,a_3)$.

— If $\omega(a_1,a_3) > 0$, then 12//34 and 13//24 are on the same side, and the invariance for refined multiplicities is dealt with the identity

$$\begin{aligned} &(q^{\omega(a_2,a_3)} - q^{-\omega(a_2,a_3)})(q^{\omega(a_1,a_2+a_3)} - q^{-\omega(a_1,a_2+a_3)}) \\ =&(q^{\omega(a_1,a_2)} - q^{-\omega(a_1,a_2)})(q^{\omega(a_1+a_2,a_3)} - q^{-\omega(a_1+a_2,a_3)}) \\ +&(q^{\omega(a_1,a_3)} - q^{-\omega(a_1,a_3)})(q^{\omega(a_2,a_1+a_3)} - q^{-\omega(a_2,a_1+a_3)}), \end{aligned}$$

— and if $\omega(a_1,a_3) < 0$, then 14//23 and 13//24 are on the same side and then the invariance for refined multiplicities is true since

$$\begin{aligned} &(q^{\omega(a_2,a_3)} - q^{-\omega(a_2,a_3)})(q^{\omega(a_1,a_2+a_3)} - q^{-\omega(a_1,a_2+a_3)}) \\ +&(q^{\omega(a_3,a_1)} - q^{-\omega(a_3,a_1)})(q^{\omega(a_2,a_1+a_3)} - q^{-\omega(a_2,a_1+a_3)}) \\ =&(q^{\omega(a_1,a_2)} - q^{-\omega(a_1,a_2)})(q^{\omega(a_1+a_2,a_3)} - q^{-\omega(a_1+a_2,a_3)}). \end{aligned}$$

If we have some edges parallel among the vectors a_i, either two consecutives vectors are parallel, and then the invariance is straightforward, since there are only two adjacent combinatorial types with equal non-zero multiplicity, or we can choose a cyclic labeling such that a_1 and a_3 are parallel. We then have $\omega(a_1,a_3) = 0$. It means that one of the determinant multiplicities is zero, which is normal since the associated combinatorial type would have a flat vertex. Thus,

$$\underset{\text{for } 12//34}{\omega(a_1,a_2)\omega(a_1+a_2,a_3)} + \underset{\text{for } 14//23}{\omega(a_2,a_3)\omega(a_2+a_3,a_1)} = 0.$$

It means that the two terms are of opposite sign. Assume the first one is positive, and thus $\omega(a_1,a_2)$ and $\omega(a_1+a_2,a_3)=\omega(a_2,a_3)$ have the same sign. The refined multiplicity is then

$$(q^{\omega(a_1,a_2)}-q^{-\omega(a_1,a_2)})(q^{\omega(a_2,a_3)}-q^{-\omega(a_2,a_3)}).$$

The second term being negative, it means that $\omega(a_2,a_3)$ and $\omega(a_1,a_2+a_3)=\omega(a_1,a_2)$ have the same sign. The refined multiplicity is the the same and we have the desired local invariance. It is the same if the first term is negative. □

Remark 5.1.3. The technicalities in the proof are just needed to find the repartition of the combinatorial types around the wall. This repartition could also be found by looking at the subdivisions of the Newton polygon, which are dual to the tropical curves. The quadrilateral dual to the quadrivalent vertex has three subdivisions (resp. two in the case of a flat vertex) matching the three splittings of the vertex: one using the big diagonal, one using the small diagonal, and one by using a parallelogram. Then, we can show that the repartition is given by putting the subdivision using the big diagonal alone on one side of the wall. See [IM13]. ♦

5.1.3 Recursive formula

Before stating the formula we need to introduce some notations. Let $\Delta=\{v_1,v_2,\dots,v_m\}\subset N\simeq\mathbb{Z}^2$ be a degree for plane curves, *i.e.* a family of vectors whose sum is zero. Let $\mu\in\mathbb{R}^m$ be a family of moments satisfying the tropical Menelaus condition, and let Γ be a parametrized tropical curve in $\mathcal{S}(\mu)$. Notice that for a generic choice of μ, such tropical curves are trivalent. As Γ is a tree, there is a unique shortest path between the edges directed by v_1 and v_m, which we call a *chord*. The chord has a natural orientation from the end v_1 to the end v_m. Once we remove the chord, the curve Γ disconnects into several components $\Gamma_1,\dots,\Gamma_p$, indexed in the order in which they meet the chord. Let E_i be the edge in Γ_i adjacent to the chord, and V_i the vertex in which both meet. All these notations along with the ones which are about to follow are depicted on Figture 5.1.

There are two possibilities for Γ_i :

- either E_i is an unbounded edge, directed by some $v_i\in\Delta$, and we set $u_i=-v_i$,
- or E_i is bounded and Γ_i contains more than one unbounded edge of Γ. We then denote by $\widetilde{\Delta_i}\subset\Delta$ the set of directing vectors of unbounded ends of Γ that belong to Γ_i. Let $u_i=-\sum_{v\in\widetilde{\Delta_i}}v$ be the directing vector of E_i going toward the chord, and $\Delta_i=\widetilde{\Delta_i}\sqcup\{u_i\}$ be the degree of the curve Γ_i' obtained by letting E_i going to infinity instead of stopping when meeting the chord in V_i.

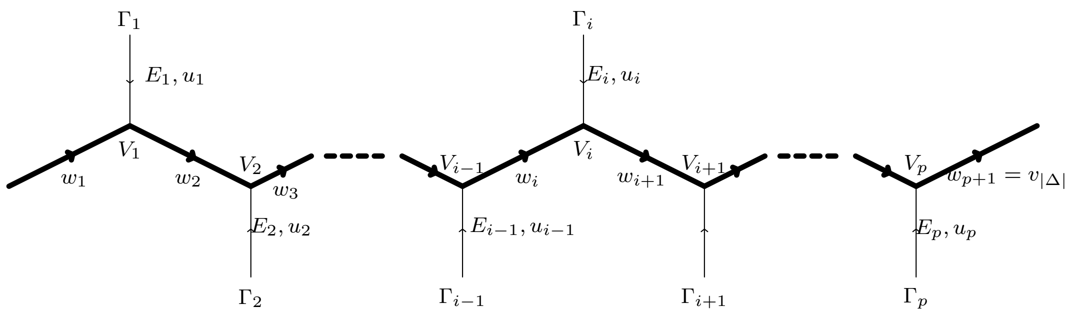

Figure 5.1 – skeleton of the curve Γ with every notation. The chord is in fat.

Finally, let w_i be the vector directing the edge of the chord between V_{i-1} and V_i. This means that $w_1 = -v_1$ and $w_{i+1} = w_i + u_i$. Let also $\sigma_i = \omega(w_i, w_{i+1})$ be the signed multiplicity of the vertex V_i. We can now derive a recursive formula from this description.

Theorem 5.1.4
With the above notation, we have

$$N_\Delta^{\partial,trop}(q) = \sum_* \prod_{i=1}^p [\sigma_i]_q N_{\Delta_i}^{\partial,trop}(q),$$

where the sum $*$ is over the ordered partitions of $\Delta - \{v_1, v_m\}$ into

$$\Delta - \{v_1, v_m\} = \bigsqcup_{i=1}^p \widetilde{\Delta_i},\ u_i = -\sum_{v\in\widetilde{\Delta_i}} v,$$

such that

$$\begin{cases} \sigma_i > 0 \Rightarrow |\widetilde{\Delta_i}| = 1 \\ \omega(\sigma_i u_i, \sigma_{i+1} u_{i+1}) \geqslant 0, \end{cases}$$

and up to a reordering of consecutive indices i having respective colinear vectors u_i.

Remark 5.1.5. The term *"ordered partition"* means that the set is subdivided into several subsets, but we keep track of their order by labeling them. Ordered partitions in p subsets are thus in bijection with the surjections to $[\![1;p]\!]$. The reordering means that orders that differ by a sequence of permutations of consecutive indices i and $i+1$ such that u_i and u_{i+1} are colinear, are counted only once. ♦

5.2 Proof of the recursive formula

To prove the recursive formula, we find a way to describe the curves solution to the problem for a specific value of μ. The idea is to choose an idealistic configuration of constraints and then a 1-parameter family $\mu(t)$ of moments getting closer and closer to this idealistic but unreachable configuration. Such a 1-parameter family $\mu(t)$ is called a deformation. Then, we describe the specific combinatorial types that continue to provide a solution through the deformation process toward this ideal configuration.

Remark 5.2.1. The same idea drives the tropical proof of the Caporaso-Harris formula in [GM07a] : one deforms the constraints by making one of the marked points going to infinity on the left. The only combinatorial types that "survive" the deformation (see definition below) are those that either have the corresponding marked point on a horizontal edge of the curve, or that split into a *floor* containing the marked point and a curve of lower degree, joined to the floor by horizontal edges. We implement this ideology in our setting to provide a way of computing the invariants $N_\Delta^{\partial,\mathrm{trop}}$. ♦

First, recall the moment map mom : $\mathcal{M}_0(\Delta, N_\mathbb{R}) \to \mathbb{R}^{m-1}$. If $\mu \in \mathbb{R}^{m-1}$ is a family of moments for the last $m-1$ ends of the curves, we say that a parametrized tropical curve is solution to the Δ-problem with value μ if $\mathrm{mom}(\Gamma) = \mu$. In some cases we see μ as a function $\Delta - \{v_1\} \to \mathbb{R}$ that assigns to any unbounded end its moment. If $\widetilde{\Xi} \subset \Delta - \{v_1\}$ is a subset, then

$$\Xi = \{-\sum_{v\in\widetilde{\Xi}} v\} \sqcup \widetilde{\Xi}$$

is still a tropical degree and this notation allows us to consider the Ξ-problem with value $\mu|_{\widetilde{\Xi}}$.

Definition 5.2.2. We call a *deformation vector* a lattice vector $\delta \in \mathbb{Z}^{m-1} \subset \mathbb{R}^{m-1}$. The associated *deformation* of an element $\mu \in \mathbb{R}^{m-1}$ is the half-line $\mu + \mathbb{R}_{\geqslant 0}\delta$, parametrized by $t \mapsto \mu + t\delta$.

Definition 5.2.3. Let Γ be a tropical curve with non-zero multiplicity. On the orthant $\mathrm{Comb}(\Gamma) \times N_\mathbb{R}$ of curves having the same combinatorial type, the moment map is linear with matrix A in the canonical basis :

$$A = \mathrm{mom}|_{\mathrm{Comb}(\Gamma)\times N_\mathbb{R}} : \mathbb{R}_{\geqslant 0}^{m-3} \times N_\mathbb{R} \to \mathbb{R}^{m-1}.$$

If δ is a deformation vector, we say that the combinatorial type of Γ *survives* the deformation if the first $m-3$ coordinates of $A^{-1}\delta$ are non-negative.

Remark 5.2.4. As the multiplicity of the curve is non-zero, the matrix A is invertible. Hence, any small deformation of the image can be pullback by the moment map to a small deformation of the curve, which means a variation of the length of the bounded edges and maybe a translation. The assumption that the first coordinates are non-negative means that going along the deformation δ, the length of the edges are non-decreasing, and the half-line $\mathrm{mom}|_{\mathrm{Comb}(\Gamma)\times N_\mathbb{R}}^{-1}(\mu) + \mathbb{R}_{\geqslant 0}A^{-1}\delta$ does not meet the boundary of the orthant, place where the deformation of the curve cannot go on since some edge has length going to zero and so a quadri-valent vertex appears. ♦

Let Γ be a tropical curve solution to the Δ-problem with value μ, with non-zero-multiplicity, and δ a deformation vector. Let f be some affine function defined on the orthant of the combinatorial type of Γ, with linear part $\overline{f}$. Then we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \overline{f}(A^{-1}\delta),$$

for the variation of f along the deformation. The functions of interest are the position of a vertex V, the length of a bounded edge E, and the moment of some edge E.

Example 5.2.5. If $f = V : \mathrm{Comb}(\Gamma) \times N_{\mathbb{R}} \to N_{\mathbb{R}}$ is the position of a vertex of Γ, then $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the direction in which V moves when the curve is deformed by making $\mathrm{mom}(\Gamma)$ go in direction δ. $\lozenge$

From now on, we consider the deformation vector $\delta = (0, \dots, 0, -1)$, which means that the moment of the edge directed by v_m goes to $-\infty$, and thus the moment of the edge directed by v_1 goes to $+\infty$. We look for combinatorial types that survive this deformation.

Proposition 5.2.6
For t large enough, the only combinatorial types that contribute a solution to $N_\Delta^{\partial,trop}(q, \mu + t\delta)$ are surviving combinatorial types.

Proof. Each combinatorial type $\mathrm{Comb}(\Gamma)$ of tropical curves provides a formal solution to the problem, meaning that we can solve $\mathrm{mom}|_{\mathrm{Comb}(\Gamma)\times N_{\mathbb{R}}}(l, V)) = \mu$ formally and find the lengths of the edges, but some of them might be negative. The formal solution is a true solution if the length of each edge is non-negative. If $\mathrm{Comb}(\Gamma)$ is not a surviving combinatorial type, the length of some edge strictly decreases when t increases, and it becomes negative if t is large enough, therefore the combinatorial type no longer provides a solution. $\square$

Remark 5.2.7. As the length of some edges might be constant through the deformation, the survival property is not enough to ensure a combinatorial type ultimately provide a true solution. More precisely, among the combinatorial types differing from one another by a permutation of consecutive indices i having colinear directing vectors u_i, exactly one ultimately provides a true solution. This is the place where the reordering appears. $\blacklozenge$

Using the balancing condition, we see that the moment μ_{E_i} of E_i is constant equal to minus the sum of coordinates of $\mu|_{\widetilde{\Delta_i}}$. This means that the edge E_i is contained in a fixed line. The vertices of Γ_i are fixed because the moment of two of their incident edges are constant. Thus, if we change the moment of v_1 and v_m, the only way the edges E_i can move is by varying their lengths while the edges in each Γ_i different from E_i are fixed. Moreover, only their extremity V_i in which Γ_i meets the chord can move, and these vertices move on the lines that respectively contain E_i.

For the combinatorial type of Γ to survive the deformation, we need to check that neither the length of the edges of the chord nor the length of the bounded edges E_i go to 0. Recall $\sigma_i = \omega(w_i, w_{i+1})$ the signed multiplicity of the vertex V_i. If $\sigma_i < 0$, that means that at V_i, the chord turns right, as we can see on Figure 5.2, and if $\sigma_i > 0$, it means that the chord turns

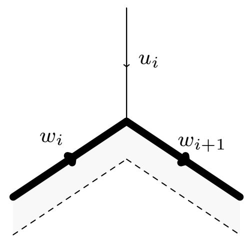

Figure 5.2 – Vertex that turns right, i.e. $\sigma_i < 0$.

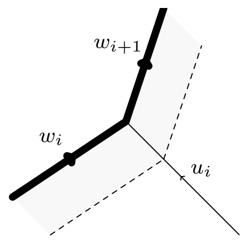

Figure 5.3 – Vertex that turns left, i.e. $\sigma_i > 0$.

left, as we can see on Figure 5.3.

Let $\tau_i = \omega(w_i, V_i) = \omega(w_i, V_{i-1})$ be the moment of the edge directed by w_i. The balancing condition ensures that

$$\tau_{i+1} = \tau_i + \mu_{E_i},$$

hence all the moments τ_i only differ from one another by a constant, and thus all go to $-\infty$ through the deformation process, since τ_{p+1} is the moment of the edge directed by v_m that goes to $-\infty$ with velocity -1 by assumption. We then have $\frac{\mathrm{d}\tau_i}{\mathrm{d}t} = -1$.

We now write down some equations whose derivation allows us to obtain the variations of the lengths of the bounded edges of the curve. We separate the case of edges which are adjacent to the chord from the edges which are part of the chord. We denote by a dependence on t the fact that the functions (position of a vertex, moment of an edge, length of an edge) are taken on the curve of $\mathrm{Comb}(\Gamma) \times N_{\mathbb{R}}$ whose evaluation is $\mu + t\delta$, in case the orthant provides a true solution. As $t = 0$ provides a true solution, the formal solutions are true solutions at least for small values of t. The formal solution is true for any t if the combinatorial type is a surviving one.

— First, let P_i be a fixed point on E_i, which is the other extremity if E_i is bounded and

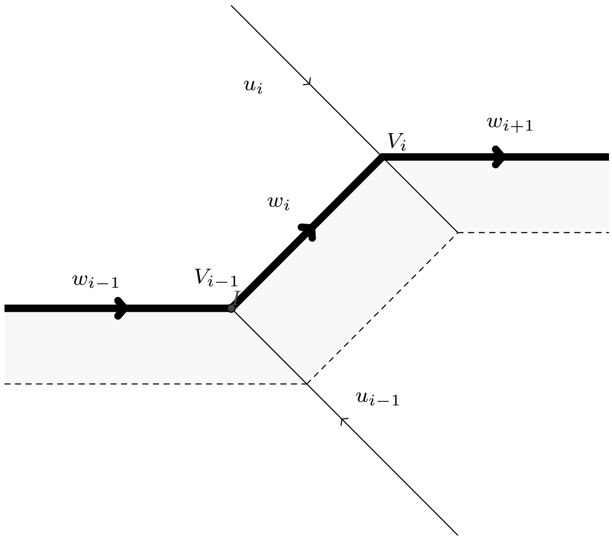

Figure 5.4 – Deformation of an edge of the chord.

any point otherwise. We have

$$V_i(t) = P_i + l_i(t)u_i = P_i + l_i(t)(w_{i+1} - w_i),$$

where $l_i(t)$ is the length of the edge between V_i and P_i. Hence,

$$\begin{aligned} \tau_i = \omega(w_i, V_i(t)) &= \omega(w_i, P_i) + l_i(t)\omega(w_i, w_{i+1} - w_i) \\ &= \omega(w_i, P_i) + l_i(t)\sigma_i. \end{aligned}$$

Thus, by derivating, we get $-1 = \frac{\mathrm{d}\tau_i}{\mathrm{d}t} = \sigma_i \frac{\mathrm{d}l_i}{\mathrm{d}t}$. This means that :

- If at V_i the chord turns left ($\sigma_i > 0$), then l_i decreases as t goes to $+\infty$, the vertex V_i goes up the edge E_i and will meet a vertex if there is one, which is the case if and only if E_i is bounded.
- If at V_i the chord turns right ($\sigma_i < 0$), then l_i increases as t goes to $+\infty$.

— We consider the edge between the vertices V_{i-1} and V_i. Let λ_i be its length so that we have

$$V_i(t) - V_{i-1}(t) = \lambda_i(t)w_i.$$

By derivating with respect to t and using previous notations, we get

$$\frac{\mathrm{d}l_i}{\mathrm{d}t}(w_{i+1} - w_i) - \frac{\mathrm{d}l_{i-1}}{\mathrm{d}t}(w_i - w_{i-1}) = \frac{\mathrm{d}\lambda_i}{\mathrm{d}t}w_i,$$

which is equivalent to

$$-\frac{w_{i+1}-w_i}{\sigma_i}+\frac{w_i-w_{i-1}}{\sigma_{i-1}}=\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}w_i$$

since $\frac{\mathrm{d}l_i}{\mathrm{d}t}=-\frac{1}{\sigma_i}$. Multiplying by $\sigma_i\sigma_{i-1}$ we get

$$\sigma_{i-1}(w_{i+1}-w_i)-\sigma_i(w_i-w_{i-1})=-\sigma_i\sigma_{i-1}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}w_i.$$

At this point we can check that $\sigma_{i-1}w_{i+1}+\sigma_i w_{i-1}$ is indeed colinear to w_i :

$$\begin{aligned}\omega(w_i,\sigma_{i-1}w_{i+1}+\sigma_i w_{i-1}) &= \sigma_{i-1}\omega(w_i,w_{i+1})+\sigma_i\omega(w_i,w_{i-1})\\ &= \sigma_{i-1}\sigma_i-\sigma_i\sigma_{i-1}\\ &= 0.\end{aligned}$$

In order to check the sign of $\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}$, we can evaluate any linear form on this vector equation, for instance $\omega(w_{i-1},-)$, which gives us

$$\sigma_{i-1}^2\sigma_i\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}=\sigma_{i-1}^2+\sigma_i\sigma_{i-1}-\sigma_{i-1}\omega(w_{i-1},w_{i+1}).$$

By noticing that

$$\begin{aligned}\omega(u_{i-1},u_i) &= \omega(w_i-w_{i-1},w_{i+1}-w_i)\\ &= \sigma_i+\sigma_{i-1}-\omega(w_{i-1},w_{i+1}),\end{aligned}$$

after dividing by σ_{i-1}, we are left with

$$\sigma_{i-1}\sigma_i\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}=\omega(u_{i-1},u_i).$$

Hence, $\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}$ is non-negative if and only if $\omega(\sigma_{i-1}u_{i-1},\sigma_i u_i)$ is non-negative.

We now can describe the conditions for a combinatorial type to survive our deformation.

Proposition 5.2.8
Let Γ be a parametrized tropical curve. In the above notations, the combinatorial type of Γ survives the deformation $t\to+\infty$ if and only if

- *the edge E_i is an unbounded edge whenever $\sigma_i>0$;*
- *for each i, we have $\omega(\sigma_{i-1}u_{i-1},\sigma_i u_i)\geqslant 0$.*

Proof. The statement follows from the previous description: a combinatorial type survives the deformation if and only if the lengths of the bounded edges are non-decreasing along the deformation. All the cases have previously been studied:

- The length of the bounded edges of the chord is non-decreasing, hence for each i we have $\omega(\sigma_{i-1}u_{i-1}, \sigma_i u_i) \geqslant 0$.
- The bounded edges inside some Γ_i but different from E_i are constant.
- The edges E_i have a non-decreasing length unless $\sigma_i > 0$, and then we need the edge to be unbounded.

□

We can now prove the recursive formula.

Proof of Theorem 5.1.4. Let $\mu \in \mathbb{R}^{m-1}$ be any value. Thanks to the previous proposition, up to a change of μ by $\mu + t\delta$ for a very large t, we can assume that the combinatorial types of the solutions to the Δ-problem with the value μ are surviving combinatorial types for our deformation. However, as noticed, the subtlety is that not all surviving combinatorial types provide a true solution. Nevertheless, each of the curves Γ_i' is solution to the Δ_i-problem with value $\mu|_{\widetilde{\Delta_i}}$. We use this to construct the solutions.

Let Γ be a solution to the Δ-problem with value $\mu + t\delta$ for t large enough. By assumption it has a surviving combinatorial type. Moreover, the Γ_i' provide solutions to the underlyings Δ_i-problems with values $\mu|_{\widetilde{\Delta_i}}$, and the multiplicity of Γ factors in the following way :

$$m_\Gamma^q = \prod_1^p [\sigma_i]_q m_{\Gamma_i}^q.$$

Conversely, any combinatorial type can be described by the ordered partition $\Delta - \{v_1, v_m\} = \bigsqcup \Delta_i$ along with the combinatorial type of the curves Γ_i. Let (Γ_i) be a family of solutions to the respective Δ_i-problems with respective values $\mu|_{\widetilde{\Delta_i}}$, and we try to glue them into a global solution, for t large enough. The gluing is given by the order of the partition in which we glue $\Gamma_1, \dots, \Gamma_p$ on the chord linking the unbounded end 1 to the unbounded end m. We have a formal solution obtained by resolving the length of the edges on the combinatorial type, and need to check that they indeed provide a true solution.

The lengths of the bounded edges inside the graphs Γ_i are constant. The only orders that have non-decreasing lengths for the edges of the chords and the edges E_i when t goes to $+\infty$ are the orders considered in the formula. If the length of some of these edges is negative but increases through the deformation, it becomes positive for t large enough. Conversely, if the order is not one of the considered, some edge length decreases along the deformation process and is ultimately negative, so the combinatorial type ceases to provide a solution. Finally, if the length of some edge of the chord is constant through the deformation, meaning that consecutive incidents vectors to the chord are colinear, there is a unique order between them that matches the order of their moments, *i.e.* the order in which a transversal oriented line would meet them, and provides positive lengths for these edges, hence the consideration of the order on $\Gamma_1,\dots,\Gamma_p$ up to a reordering of consecutive colinear vectors u_i.

Finally, by putting together the contribution of the different possibilities of Γ_i, we get

$$\sum_{\Gamma_i\in\mathcal{S}(\mu|_{\widetilde{\Delta_i}})}\prod_1^p[\sigma_i]_q m^q_{\Gamma_i}=\prod_1^p[\sigma_i]_q N^{\partial,\mathrm{trop}}_{\Delta_i},$$

and thus the desired formula. □

Provided that the deformation is big enough, the moments of all the unbounded edges except v_1 and v_m are really small regarding these two specific moments. It means that when we look at a solution to the Δ-problem with our value of μ, all the edges adjacent to the chord seem to go through the origin of $\mathbb{R}^2$ (Although they do not, but they are not far from it if we look at them from far far away) while the chord goes around the origin, changing its direction when meeting an adjacent edge in the right order.

This decomposition can be compared with the usual floor decomposition of tropical curves with an h-transverse degree coming from the tropical Caporaso-Harris formula of [GM07a]. However, here the floors have a more complicated shape. For instance, even for degree d curves, and the two edges whose moments vary are directed by $(-1,0)$ and $(0,-1)$, the chord may not be a usual floor and can make a loop around the origin as we can see on Figure 5.5. The figure shows a quartic curve, with two unbounded ends going to infinity. The movement of these ends is depicted with an arrow. The region colored in grey is the zone through which the curve travels through the deformation. We have a similar situation for a cubic depicted on Figure 5.6.

Finally, the recursive formula involves every $N^{\partial,\mathrm{trop}}_{\Delta}$, associated with different toric surfaces, contrarily to the usual Caporaso-Harris formula for curves in $\mathbb{C}P^2$, which restricts to specific degrees. Furthermore, the formula applies for curves of any degree, while the Caporaso-Harris formula restricts to h-transverse polygons.

5.3 Computations

We now provide a few values $N^{\partial,\mathrm{trop}}_{\Delta}$ for various families $\Delta\subset N$, computed using the recursive formula. Because of the exponential complexity of the algorithm, we only manage to make computations for small degrees. Concerning curves of degree d in $\mathbb{C}P^2$, computations can be done by hand up to degree 5 or 6. Degree 7 seems to be out of reach without computer assistance.

Let $d\geqslant 1$ be an integer, and let $\lambda\vdash d$ be a partition of d. We denote by $N^{\partial,\mathrm{trop}}_d(\lambda)$ the polynomial $N^{\partial,\mathrm{trop}}_{\Delta}$ when $\Delta=\{(-e_1)^d,(e_1+e_2)^d,-\lambda_1 e_2,-\lambda_2 e_2,\dots\}$. These are the degrees

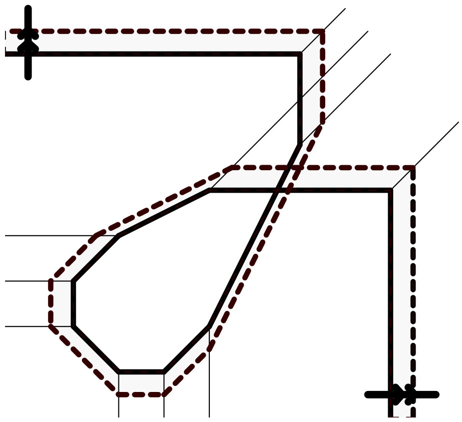

Figure 5.5 – Image of a quartic curve during the deformation.

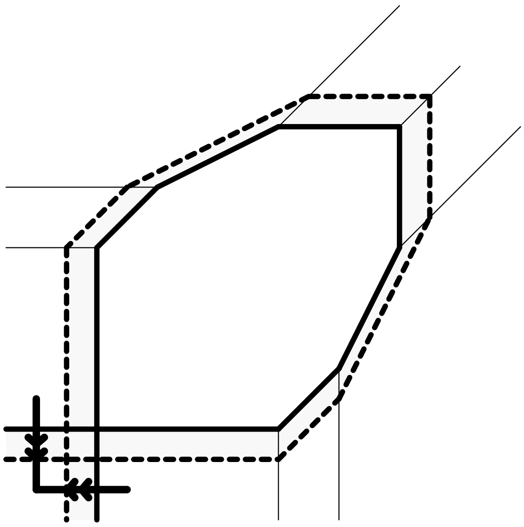

Figure 5.6 – Image of a cubic curve during the deformation.

that appear in the proof of the Caporaso-Harris formula in [GM07a].

Proposition 5.3.1
We have :

- $N_1^{\partial,trop}(1) = N_2^{\partial,trop}(1^2) = 1$,
- $N_2^{\partial,trop}(2) = q^{1/2} + q^{-1/2}$,
- $N_3^{\partial,trop}(1^3) = q + 7 + q^{-1}$,
  $N_3^{\partial,trop}(2,1) = q^{3/2} + 6q^{1/2} + 6q^{-1/2} + q^{-3/2}$,
  $N_3^{\partial,trop}(3) = q^2 + 5q + 6 + 5q^{-1} + q^{-2}$,
- $N_4^{\partial,trop}(1^4) = q^3 + 10q^2 + 55q + 172 + 55q^{-1} + 10q^{-2} + q^{-3}$,
  $N_4^{\partial,trop}(2,1^2) = q^{7/2} + 9q^{5/2} + 45q^{3/2} + 133q^{1/2} + 133q^{-1/2} + 45q^{-3/2} + 9q^{-5/2} + q^{-7/2}$,
  $N_4^{\partial,trop}(3,1) = q^4 + 8q^3 + 36q^2 + 96q + 117 + 96q^{-1} + 36q^{-2} + 8q^{-3} + q^{-4}$,
  $N_4^{\partial,trop}(4) = q^{9/2} + 7q^{7/2} + 28q^{5/2} + 68q^{3/2} + 88q^{1/2} + 88q^{-1/2} + 68q^{-3/2} + 28q^{-5/2} + 7q^{-7/2} + q^{-9/2}$,
  $N_4^{\partial,trop}(2^2) = q^4 + 8q^3 + 36q^2 + 104q + 150 + 104q^{-1} + 36q^{-2} + 8q^{-3} + q^{-4}$,
- $N_5^{\partial,trop}(1^5) = q^6 + 13q^5 + 91q^4 + 455q^3 + 1695q^2 + 5023q + 11185 + 5023q^{-1} + 1695q^{-2} + 455q^{-3} + 91q^{-4} + 13q^{-5} + q^{-6}$.

The proof is a straightforward computation. For each degree Δ one chooses two specific vectors and makes the associated unbounded edges going to infinity, reducing the computation of $N_\Delta^{\partial,\mathrm{trop}}$ to the computations of invariants with families of smaller size. We here show some of the computations for degree d curves, choosing unbounded ends directed by $(-1,0)$ and $(1,1)$. We explain only the main features, and draw the tropical curves resulting from the deformation. The shape of the tropical curves illustrates some of the involved phenomena.

Very low degrees. The values of $N_1^{\partial,\mathrm{trop}}(1), N_2^{\partial,\mathrm{trop}}(1^2)$ and $N_2^{\partial,\mathrm{trop}}(2)$ are easy to find since for each choice of boundary conditions, there is only one tropical curve matching the constraints. The only curve for $N_2(2)$ has a vertex of complex multiplicity 2, leading to the value $N_2^{\partial,\mathrm{trop}}(2) = q^{1/2} + q^{-1/2}$.

For curves of degree 3, the choice of unbounded edges going to infinity leads to tropical curves having a floor decomposition in the sense of [GM07a], and the computation is reduced to the value of $N_1^{\partial,\mathrm{trop}}$, $N_2^{\partial,\mathrm{trop}}(1^2)$ and $N_2^{\partial,\mathrm{trop}}(2)$, which we already know. The appearance of a floor means that the chord degenerates into a toric divisor, which is the coordinate axis $y = 0$ of $\mathbb{C}P^2$ in our case. The computation leads to the value $q + 7 + q^{-1}$.

Curves of degree 4 and 5. For curves of degree 4, we still get a contribution of the curves having a floor in the sense of [GM07a]. Their contribution is

$$N_4^{\partial,\mathrm{trop}}(1^4) = q^3 + 10q^2 + 55q + 172 + 55q^{-1} + 10q^{-2} + q^{-3}.$$

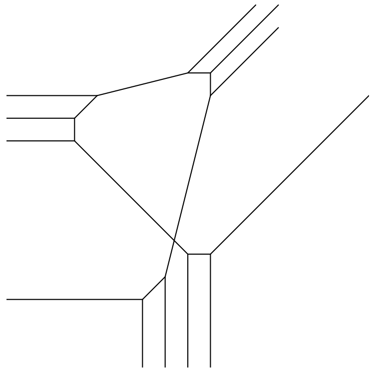

Figure 5.7 – Quartic curve with a loop

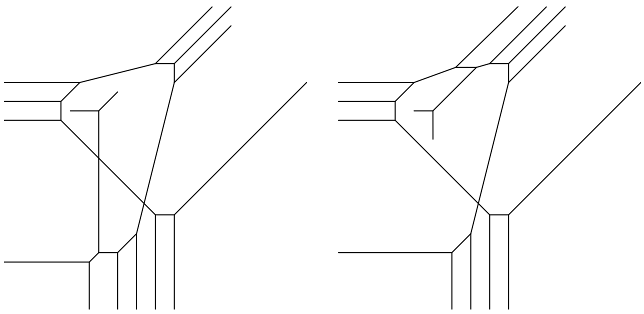

Figure 5.8 – Quintics degenerating to a loop and a line

However, it would be wrong to assume that all the contributing curves are of this form. There are in fact 6 additional curves having the shape of Figure 5.7, where the chord makes a loop around the origin. This means that the chord degenerates to the union of all toric divisors, rather than going to only one, as it would happen if it degenerated on a floor. The six curves come from the six possible repartitions of the bottom unbounded ends in two groups of two.

For curves of degree 5, the chord can yet again degenerate into a floor, or just as in the degree 4, degenerate into a loop. Only this time, contrarily to the degree 4 case, there might still be unbounded ends leftover, leading to a line attached to the loop, as we can see on Figure 5.8.

Degree 6 and higher. Up to degree 5, thanks to the particular choice of unbounded edges going to infinity, the computations were reduced only to values of the form $N_d^{\partial,\mathrm{trop}}(\lambda)$ for some λ, as in the classical Caporaso-Harris formula. Once again, it would be wrong to assume that these values are sufficient to compute $N_d^{\partial,\mathrm{trop}}(\lambda)$, in the sense that some smart choice of formula reduces the computation of $N_d^{\partial,\mathrm{trop}}(\lambda)$ to the computation of some $N_l^{\partial,\mathrm{trop}}(\mu)$ for $l < d$. Starting from $d = 6$, the curves resulting from the eviction of the chord (*i.e.* the curves Γ_i) may not be of degree l, meaning that the degree of the plane curve is not a standard triangle of size l, as we can see on Figure 5.9: the remaining curves might be of degree 1 or 2 as it

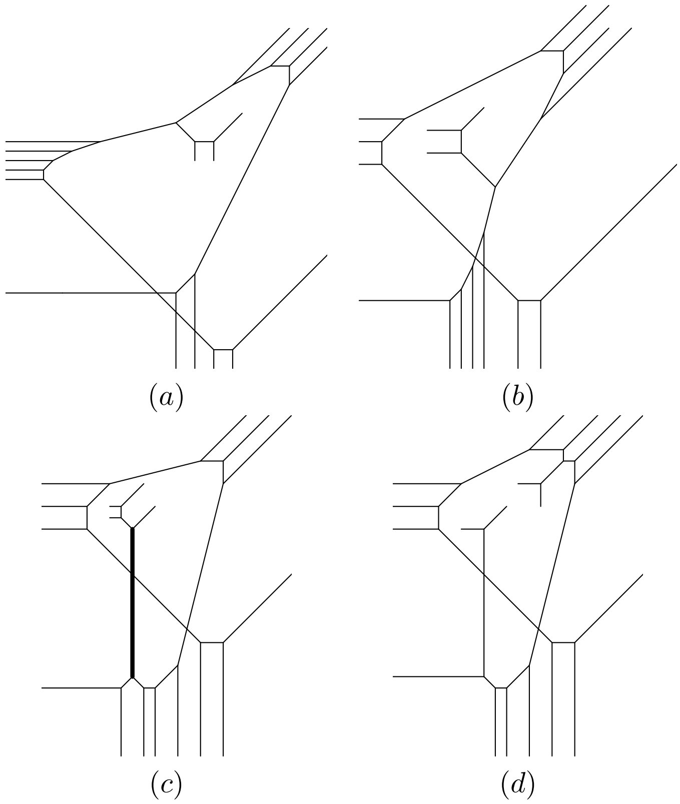

Figure 5.9 – Different examples of sextics with a loop : in (a) and (b) the degree of the remaining curve is not a standard triangle, in (c) it is a degree 2 curve, in (d) it is two lines.

is the case in (c) and (d), but can also have a more complicated shape, as we can see on (a) and (b).

For degree 7, the situation becomes even more complicated since the growing number of unbounded edges increases the number of possible degrees for the curves Γ_i, and the chord can now make two loops. The number of loops that the chord can make goes higher with the degree.

5.4 Recursive formulas for algebraic invariants

The use of correspondence theorems allows one to get relations between the tropical refined invariants $N_\Delta^{\partial,\mathrm{trop}}(q)$ and some known algebraic invariants. This approach leads to recursive formulas between these same algebraic invariants using the formula from Theorem 5.1.4.

— The first corollary deals with the number of complex rational curves of degree Δ passing through a fixed generic configuration of $|\Delta|$ points located on the toric boundary and subject to the Menelaus condition. Indeed, such a number can be computed using the correspondence theorem and the usual complex multiplicities, to which the refined multiplicity specializes at 1.

Let $\Delta \subset N$ be a degree for complex curves in $N \otimes \mathbb{C}^*$, and let $\mathcal{P} \subset \partial\mathbb{C}\Delta$ be a generic configuration of points subject to the Menelaus condition. The number of complex rational curves of degree Δ passing through $\mathcal{P}$ and not containing any toric divisor as an irreducible component is denoted by $N_\Delta^\partial(\mathcal{P})$. This number can be proved to be independent of the choice of $\mathcal{P}$.

Corollary 5.4.1
One has

$$N_\Delta^\partial = \sum_* \prod_{i=1}^p |\sigma_i| N_{\Delta_i}^\partial.$$

Proof. The value of N_Δ^∂ can be computed close to the tropical limit using the correspondence theorem of [Mik05] and the complex multiplicities of the tropical curves. The complex multiplicity satisfies $m_\Gamma^\mathbb{C} = m_\Gamma^h(1)$. Thus, one also has $N_\Delta^\partial = N_\Delta^{\partial,\text{trop}}(1)$ and the recursive formula follows. □

— The search for a recursive formula between refined tropical invariants was motivated by their relation to the refined invariants of Mikhalkin in [Mik17]. Unsurprisingly, the recursive relation thus translates to a recursive formula between the refined invariants R_Δ from section 4.2.

Recall that in case $\Delta \subset N$ consists only of primitive vectors, the invariant R_Δ counts real oriented rational curves passing through a symmetric configuration of real points located on the toric boundary of the associated toric surface. This count is independent of the chosen point configuration. We enlarge the notation to the case where Δ contains non-primitive vectors, but the result of the count is only invariant close to the tropical limit.

Corollary 5.4.2
One has

$$R_\Delta = \sum_* \prod_{i=1}^p (q^{|\sigma_i|} - q^{-|\sigma_i|}) R_{\Delta_i}.$$

Proof. The formula directly comes from Theorem 7 in [Mik17], which relates the value of R_Δ to the value of $N_\Delta^{\partial,\text{trop}}$:

$$R_\Delta = (q^{1/2} - q^{-1/2})^{|\Delta|-2} N_\Delta^{\partial,\text{trop}}(q).$$

□

— Finally, using Theorem 4.4.1, the refined tropical invariants are also related to the refined invariant from [Mik17] in case the symmetric configuration has pairs of complex points located on a common toric divisor.

Chapter 6

Curves in higher dimension

We now try to generalize the refined count of real rational curves introduced by Mikhalkin in [Mik17] to a higher dimensional setting. Here, we define a quantum class whose definition generalizes the definition of Mikhalkin for curves in a general toric variety. We provide a way to compute it for real rational curves and *toric type I* curves. This computation allows us to define a higher dimensional analog of Harnack curves. Finally, we give a classical enumerative problem and its tropical counterpart, for which we prove several results of invariance.

Sommaire

6.1 Quantum class of type I real curve

6.1.1 Setting, notations, and definition

We consider curves in a toric variety, which this time is not assumed to be a surface anymore. Let M and N be two lattices, dual from each other. We denote by m their common rank. Let $\Delta \subset N$ be a multiset of total sum 0. This allows us to consider tropical curves in $N_\mathbb{R} = N \otimes \mathbb{R}$, as well as classical curves in $N \otimes \mathbb{C}^*$, both of degree Δ. The degree Δ defines a fan Σ_Δ in $N_\mathbb{R} = N \otimes \mathbb{R}$, consisting of the rays directed by the vectors of Δ. From this fan, one can obtain a non-compact toric variety $\mathbb{C}\Delta$, whose toric divisors are in bijection with the rays of the fan Σ_Δ. The variety is smooth, but non-compact since the fan Σ_Δ has not $N_\mathbb{R}$ as its support. Nevertheless, it is possible to extend Σ_Δ in a complete fan, therefore compactifying the toric variety $\mathbb{C}\Delta$.

The standard conjugation coordinate by coordinate, *i.e.* the map $\mathrm{id}\otimes\mathrm{conj}$ from $N\otimes\mathbb{C}^*$ to itself, extends to an anti-holomorphic involution of $\mathbb{C}\Delta$. This involution, also called conjugation, makes $\mathbb{C}\Delta$ into a real variety. The fixed locus, also called the real locus, is denoted by $\mathbb{R}\Delta$. A curve is a real curve if it is invariant by conjugation.

The square map from $N\otimes\mathbb{C}^*$ to itself also extends to a morphism Sq from $\mathbb{C}\Delta$ into itself. Similarly to the planar case, we say that a point is real or purely imaginary if its image under the square map is in the fixed locus $\mathbb{R}\Delta$. This can be seen in coordinates: let $p\in\mathbb{C}\Delta$ be a point. Then p is real or purely imaginary if every monomial χ^m defined at p, for $m\in M$, takes a real or purely imaginary value. Recall that the condition for χ^m being defined at p is empty if p is not on a toric divisor, and if p sits on a toric divisor D associated to a ray of Σ_Δ directed by $n\in N$, the condition of χ^m being defined at p is equivalent to $\langle m,n\rangle=0$.

Definition 6.1.1. A real curve $\mathbb{C}C\subset\mathbb{C}\Delta$ has real or purely imaginary intersection points with the toric boundary if $\mathrm{Sq}(\mathbb{C}C)\cap\partial\mathbb{C}\Delta\subset\mathbb{R}\Delta$. Such a curve is said to have *ropip* (real or purely imaginary intersection points).

We now consider a real parametrized curve $\varphi:\mathbb{C}C\to\mathbb{C}\Delta$, of type I and degree Δ, with ropip. Let S be a connected component of $\mathbb{C}C\backslash\mathbb{R}C$. We denote by S^o the open half-curve S minus the intersection points between the curve and the toric boundary. We then consider the image of S under the following composition of maps:

$$S^o \xrightarrow{\varphi} N\otimes\mathbb{C}^* \xrightarrow{2\arg} N\otimes(\mathbb{R}/\pi\mathbb{Z}) \to (N\otimes(\mathbb{R}/\pi\mathbb{Z}))/\{\pm\mathrm{id}\}.$$

- The first map is the parametrization, that sends S^o into the complex torus $N\otimes\mathbb{C}^*$.
- The second map is the argument map coordinate by coordinate, taken mod π instead of 2π. For this map, the real points are sent to 0, and purely imaginary points to non-zero 2-torsion points of $N\otimes(\mathbb{R}/\pi\mathbb{Z})$, *i.e.* points whose coordinates are 0 or $\frac{\pi}{2}$. The target space is called the *argument torus*, and is topologically a torus of dimension $m=\mathrm{rk}N$.
- The action of the conjugation on $N\otimes\mathbb{C}^*$ induces an action on the argument torus $N\otimes(\mathbb{R}/\pi\mathbb{Z})$. The action is $-\mathrm{id}$. The last map is the quotient by this action. We denote its target space $(N\otimes(\mathbb{R}/\pi\mathbb{Z}))/\{\pm\mathrm{id}\}$ by K_N.

Remark 6.1.2. Notice that the choice of the other component $\overline{S}$ of $\mathbb{C}C\backslash\mathbb{R}C$ would lead to the same image in K_N. ♦

The homology groups of K_N can be computed using a cellular decomposition. We denote by K_m the space $(\mathbb{R}/\pi\mathbb{Z})^m/\{\pm\mathrm{id}\}$, *i.e.* $K_{\mathbb{Z}^m}$.

Example 6.1.3.
- If N has rank 2, we have a homeomorphism between K_N and K_2, which is a *pillowcase*. The morphism induced by the complex conjugation on $N\otimes(\mathbb{R}/\pi\mathbb{Z})$, *i.e.*

$-\mathrm{id}$, has 4 fixed points, which are the 4 singular points of the orbifold. Topologically, K_2 is homeomorphic to the sphere S^2.

- If N has rank 4, K_N is homeomorphic to the Kummer surface. There are 16 singularities, corresponding to the 16 fixed points of $-\mathrm{id}$. It is simply connected. Its homology groups are
$$H^2(K_4,\mathbb{Z}) \simeq \mathbb{Z}^6 \text{ and } H_2(K_4,\mathbb{Z}) \simeq \mathbb{Z}^6 \oplus (\mathbb{Z}/2\mathbb{Z})^5.$$
- In the general case, K_n has 2^n singular points.

$\Diamond$

We claim that due to ropip hypothesis, the image of S^o under these maps is a cycle in K_N, *i.e.* it has no boundary. Therefore, S defines a homology class in $H_2(K_N,\mathbb{Z})$, the second homology group of the quotient of the argument torus. Moreover, this homology class allows one to compute the area of the coamoeba for any choice of constant 2-form on $N_\mathbb{R}$.

Proposition 6.1.4
The image of S^o in K_N has no boundary, and thus defines a homology class $h(S)$. Moreover, one has $h(\overline{S}) = -h(S)$.

Proof. We consider $\hat{S}$, which is the real oriented blow-up of $S \cup \mathbb{R}C$ at the boundary points of C. This means the following: the surface $\hat{S}$ is a compact surface with boundary, and the boundary consists in several components:

- one circle for each complex boundary point, *i.e.* each point of $S - S^o$,
- one circle for each connected component of $\mathbb{R}C$. Moreover, these components are divided in segments which correspond to real boundary points and connected components of $\mathbb{R}C - (\mathbb{R}C \cap \partial\mathbb{R}\Delta)$.

This blow-up enables us to extend the map 2arg, previously undefined at the boundary point, on the blown-up surface $\hat{S}$. The map is already defined on $\mathbb{R}C - (\mathbb{R}C \cap \partial\mathbb{R}\Delta)$.

- Let $p \in S - S^o$ be a complex boundary point, with $\varphi(p)$ sitting on a toric divisor D, and let $n \in N$ be the corresponding vector in Δ. This means that n is the linear form $m \in M \mapsto \mathrm{val}_p(\varphi^*\chi^m) \in \mathbb{Z}$. We take $U \subset \mathbb{C}$ a complex chart of S with coordinate z such that the coordinate of p is 0. Let χ^m be a monomial. By definition of n we can write $\varphi^*\chi^m(z) = z^{\langle m,n\rangle}\psi(z)$ where $\psi(0) \neq 0$. Now we have for $\varepsilon > 0$,
$$\begin{aligned} 2\mathrm{arg}\ \varphi^*\chi^m(\varepsilon e^{i\theta}) &= 2\mathrm{arg}\big(\varepsilon^{\langle m,n\rangle} e^{i\langle m,n\rangle\theta}\psi(\varepsilon e^{i\theta})\big) \\ &= \langle m,n\rangle\theta + 2\mathrm{arg}\ \psi(\varepsilon e^{i\theta}) \quad \underset{\varepsilon\to 0}{\longrightarrow} \langle m,n\rangle\theta + 2\mathrm{arg}\psi(0). \end{aligned}$$
Therefore, as the quantity has a limit when $\varepsilon \to 0$ with fixed θ, the map may be extended to the oriented blow-up of S at p. And we see that the image of the circle resulting from the blow-up is a geodesic going in the direction n, when θ goes from 0 to 2π.
- The same computation works for the real boundary points, except that this time θ will only go from 0 to π. It still closes itself since we look the arguments in $\mathbb{R}/\pi\mathbb{Z}$ rather than $\mathbb{R}/2\pi\mathbb{Z}$.

We now use the ropip assumption. This means that the geodesics pass through one of the fixed points of the involution. However, their orientation is changed by the involution, so that the boundary disappears when going to the quotient K_N, because each geodesics folds itself. Therefore, the image of $\hat{S}$ inside K_N has no boundary and defines a cycle.

The statement $h(\overline{S}) = -h(S)$ easily follows from the fact that the conjugation induces $-\mathrm{id}$ on $H_2(\mathbb{C}P^1)$. □

Remark 6.1.5. Notice that in the case of a point $p \in S - S^o$, the argument map extends itself to the classic blow-up and not only the oriented blow-up. However, the non-oriented blow-up of S is not oriented anymore, and does not allow us to define a homology class.

This remark shows that already in the argument torus, the image of S has no boundary in a neighborhood of the geodesic image of p, but the neighborhood of the geodesic is a Moebius strip, and thus is non-orientable. When considering the oriented blow-up, the geodesic appears twice in the boundary: as θ goes from 0 to 2π, the geodesic is traveled twice, but in the same direction. One needs to go to the quotient K_N for the boundary to fold itself and cancel. ♦

The ropip assumption is the keypoint to ensure that the image cycle has no boundary in K_N, and thus S^o realizes a homology class in the group $H_2(K_N, \mathbb{Z})$. If the assumption is not satisfied, the boundary circles of complex boundary points may fail to pass through a fixed point of the involution $-\mathrm{id}$ and cancel themselves when passing to the quotient.

Definition 6.1.6. The homology class in $H_2(K_N, \mathbb{Z})$ defined by S is called *quantum class* and denoted by $h(S, \varphi)$, or just $h(S)$ if there is no possible confusion.

6.1.2 First properties

Due to its definition as a homology class, the quantum class is well-behaved under monomial maps between toric varieties. Let $A : N' \to N$ be a map between two lattices, and $\alpha : N' \otimes \mathbb{C}^* \to N \otimes \mathbb{C}^*$ be the monomial map between the associated complex torus. Let $\varphi : \mathbb{C}C \dashrightarrow N' \otimes \mathbb{C}^*$ be a type I real parametrized curve with ropip. Then, the parametrized curve $\alpha \circ \varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ has also ropip. Moreover, one has the following property.

Proposition 6.1.7
One has

$$h(S, \alpha \circ \varphi) = \tilde{\alpha}_* h(S, \varphi) \in H_2(K_N, \mathbb{Z}).$$

Proof. One has the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, "2\arg\circ\varphi"] \arrow[dr] & N' \otimes (\mathbb{R}/\pi\mathbb{Z}) \arrow[r, "p'"] \arrow[d, "\alpha"] & K_{N'} \arrow[d, dashed, "\exists\tilde{\alpha}"] \\ & N \otimes (\mathbb{R}/\pi\mathbb{Z}) \arrow[r, "p"] & K_N \end{tikzcd} .$$

The map between the real tori is induced by the lattice map A. It is in fact equal to $A\otimes\mathrm{id}$, but also denoted by α. The composition $p\circ\alpha$ takes the same value on two opposite elements, since p commutes to $-\mathrm{id}$, therefore, it can be factored through p', leading to a map $\tilde{\alpha} : K_{N'} \to K_N$. Thus, one gets the desired property by taking the induced maps at the homology level. □

Finally, due to the additive properties of the homology class, the quantum class can be computed close to the tropical limit, similarly to the computation of the quantum index of [Mik17] close to the tropical limit.

Proposition 6.1.8
Let $f_t : \mathbb{C}C^{(t)} \dashrightarrow N \otimes \mathbb{C}((t))^*$ be a family of type I real curves with ropip. We assume that the family admits a tropical limit $h : \Gamma \to N_\mathbb{R}$, with a curve $f_w : \mathbb{C}C_w \dashrightarrow N \otimes \mathbb{C}^*$ assigned to each vertex of Γ. We assume that each curve $\mathbb{C}C^{(t)}$ is endowed with a choice of a connected component $S^{(t)}$ of $\mathbb{C}C^{(t)}\backslash\mathbb{R}C^{(t)}$ such that $S^{(t)}$ converges. Then:

- *for each fixed vertex w of Γ, the component $S^{(t)}$ specializes to a component S_w of $\mathbb{C}C_w\backslash\mathbb{R}C_w$,*
- *for each pair $\{w, \sigma(w)\}$ of exchanged vertices of Γ, the component $S^{(t)}$ specializes to one of the curves $\mathbb{C}C_w$ or $\mathbb{C}C_{\sigma(w)}$ corresponding to the vertices, we assume this curve to be $\mathbb{C}C_w$,*
- *each curve $f_w : \mathbb{C}C_w \dashrightarrow N \otimes \mathbb{C}^*$ has ropip,*
- *close to the tropical limit, one has*

$$h(S^{(t)}, f_t) = \sum_{w\in\mathrm{Fix}(\sigma)} h(S_w, f_w) + \sum_{w\notin\mathrm{Fix}(\sigma)} h(\mathbb{C}C_w, f_w).$$

Proof. The first two points are immediate, and the third point is a consequence of the balancing condition for the phases of the curves assigned to the vertices. The last point is obtained as in [Mik17]. □

Remark 6.1.9. We recover the result of Mikhalkin in [Mik17]. However, the non-fixed vertices of the tropical curve Γ may have a contribution through some torsion element. ♦

6.1.3 Area of the amoeba and the coamoeba

We are now concerned with giving an interpretation to the previously defined quantum class. The quantum class allows one to measure the area of the amoeba of a real type I curve with ropip. Let ω be a 2-form on $N_\mathbb{R}$ and let $\mathbb{C}C \xrightarrow{\varphi} \mathbb{C}\Delta$ be a type I real curve. For now, we do not assume the curve to be ropip. Let S be a connected component of $\mathbb{C}C\backslash\mathbb{R}C$. By definition, the area of the amoeba measured by ω is

$$\mathcal{A}^{\omega}_{\mathrm{Log}}(S) = \int_S \varphi^*\mathrm{Log}^*\omega = \int_{\varphi(S)} \mathrm{Log}^*\omega.$$

The area is a signed area since we consider the pull-back of the form by the logarithm $\mathrm{Log} : N\otimes\mathbb{C}^* \to N_{\mathbb{R}}$. This area is computed only on S because the area $\int_{\mathbb{C}C}\varphi^*\mathrm{Log}^*\omega$ is equal to 0. The 2-form ω can be used to define as well a constant 2-form ω_θ on the argument torus $N\otimes(\mathbb{R}/\pi\mathbb{Z})$, that also allows a measurement of the area of the coamoeba:

$$\mathcal{A}^{\omega}_{\mathrm{arg}}(S) = \int_{\varphi(S)} (2\arg)^*\omega_\theta.$$

We also compute the area on a half-curve for the same reasons. These two areas are in fact equal.

Proposition 6.1.10
One has $\mathcal{A}^{\omega}_{\mathrm{arg}}(S) = \mathcal{A}^{\omega}_{\mathrm{Log}}(S)$.

Proof. To see that, let $z_j = e^{x_j}e^{i\theta_j}$ be coordinates on $N\otimes\mathbb{C}^*$, so that the logarithm is $(z_j)\mapsto(x_j)$, and the argument map $(z_j)\mapsto(\theta_j)$. Then, ω and ω_θ are given in coordinates by

$$\omega = \sum_{i<j} a_{ij}\mathrm{d}x_i\wedge\mathrm{d}x_j \text{ and } \omega_\theta = \sum_{i<j} a_{ij}\mathrm{d}\theta_i\wedge\mathrm{d}\theta_j.$$

We then consider the following holomorphic 2-form

$$\begin{aligned}\varphi &= \sum a_{ij}\frac{\mathrm{d}z_i}{z_i}\wedge\frac{\mathrm{d}z_j}{z_j}\\ &= \sum a_{ij}(\mathrm{d}x_i\wedge\mathrm{d}x_j - \mathrm{d}\theta_i\wedge\mathrm{d}\theta_j) + i(\cdots)\\ &= \log^*\omega - (2\arg)^*\omega_\theta + i(\cdots),\end{aligned}$$

where $z_i = x_ie^{i\theta_i}$. This holomorphic 2-form vanishes on the curve $\mathbb{C}C$ and thus, the area of the amoeba measured by ω, which the integral of the pull-back of ω on S, is equal to the value of the integral of the pull-back of ω_θ, which is the area of the coamoeba. □

We now assume that the curve real C has ropip. The 2-form ω_θ on $N\otimes(\mathbb{R}/\pi\mathbb{Z})$ realizes a cohomology class in $H^2(N\otimes(\mathbb{R}/\pi\mathbb{Z}),\mathbb{R})$. As the 2-form is invariant by the antipodal map, it descends to a 2-form on K_N, where it realizes some cohomology class in $H^2(K_N,\mathbb{R})$. We thus can evaluate the obtained cohomology class on $h(S)$ to get a number $k_\omega(S) := \langle\omega, h(S)\rangle$ called the *quantum index*. Therefore, we see that the area of the amoeba with respect to ω is just the evaluation of cohomology class realized by ω_θ on the quantum class $h(S)$. Hence, it can only take a discrete set of values if ω is integer valued.

Proposition 6.1.11
One has

$$\mathcal{A}^{\omega}_{Log}(S) = \mathcal{A}^{\omega}_{\arg}(S) = \langle \omega, h(S) \rangle.$$

Remark 6.1.12. The torsion part of the quantum class disappears when we evaluate some cohomology class, since the coefficients are now chosen to be in $\mathbb{R}$ instead of $\mathbb{Z}$. Therefore, the quantum class carries more information than the given areas for every 2-form ω. ♦

Remark 6.1.13. The quantum index $k_\omega(S,\varphi)$ is linear in ω. ♦

Using Proposition 6.1.7, one easily proves the following statement.

Proposition 6.1.14
Let $A : N' \to N$ be a lattice map, $\alpha : N' \otimes \mathbb{C}^* \to N \otimes \mathbb{C}^*$ be the associated monomial map, ω a 2-form on $N_{\mathbb{R}}$, and $\varphi : \mathbb{C}C \dashrightarrow N' \otimes \mathbb{C}^*$ be a type I real curve with ropip and S a connected component of $\mathbb{C}C\backslash\mathbb{R}C$. Then, one has

$$k_\omega(S, \alpha \circ \varphi) = k_{A^*\omega}(S, \varphi).$$

Proof. One has

$$\begin{aligned} k_\omega(S, \alpha \circ \varphi) &= \langle \omega, h(S, \alpha \circ \varphi) \rangle \\ &= \langle \omega, \tilde{\alpha}_* h(S, \varphi) \rangle \\ &= \langle A^*\omega, h(S, \varphi) \rangle . \end{aligned}$$

□

Remark 6.1.15. Notice that since the ropip hypothesis is used only to ensure that the area is indeed a quantum index, this functoriality property might also be used more generally to compute log-area, as in Lemma 4.1.8. ♦

6.1.4 Computation of the quantum class for toric type I real curves

Let $\varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ be a type I real parametrized curve. Let S be a connected component of $\mathbb{C}C\backslash\mathbb{R}C$, inducing a complex orientation of $\mathbb{R}C$. The map φ induces a morphism

$$\varphi_* : \pi_1(S^o) \to \pi_1(N \otimes \mathbb{C}^*) = H_1(N \otimes \mathbb{C}^*, \mathbb{Z}) \simeq N.$$

Definition 6.1.16. A type I real curve $\varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ with S a connected component of $\mathbb{C}C\backslash\mathbb{R}C$ is of *toric type I* if $\varphi_*\pi_1(S^o) = \{0\} \subset H_1(N \otimes \mathbb{C}^*, \mathbb{Z})$.

Lemma 6.1.17
The intersection points of a toric type I real curve with the toric boundary are real. Therefore, the curve has ropip.

Proof. Let $p \in \mathbb{C}C$ be a point such that $\varphi(p) \in D$, where D is a toric divisor associated to a ray directed by $n \in N$. Then, a small loop around p is sent to the a multiple of $n \in N \simeq H_1(N \otimes \mathbb{C}^*, \mathbb{Z})$ by φ_*. More precisely, this multiple is the cocharacter associated to the map $m \mapsto \mathrm{val}_p \varphi^* \chi^m$. Therefore, all the intersection points of a toric type I real curve with the toric boundary are real, hence the ropip property. □

Lemma 6.1.18
Let $\varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ be a toric type I real curve. Let $\mathbb{R}C_i$ be a connected component of the real locus. Let $p_1, \dots, p_m \in \mathbb{R}C_i$ be its intersection points with the toric boundary, and let $n_j \in N$ be the co-character associated to φ at p_j. Then one has

$$\sum_j n_j = 0.$$

Proof. Let γ be a loop in S^o that is a small shift of the boundary component $\mathbb{R}C_i$. By assumption, the image of this loop by φ_* is 0. However, this loop is homologous to the boundary component in the oriented blow-up $\hat{S}$ of S introduced in the proof of Proposition 6.1.4. Moreover, this proof showed that the image of a half-loop around an intersection point between $\varphi(\mathbb{R}C_i)$ and the toric boundary realizes the class n in $H_1(N \otimes (\mathbb{R}/\pi\mathbb{Z}), \mathbb{Z}) \simeq N$. Therefore, the image of $\varphi_*(\gamma)$ is also equal to $\sum_j n_j$. Hence, $\sum_j n_j = 0$. □

As all the intersection points of a toric type I real curve with the toric boundary are real, we have a well-defined quantum class. We now compute the quantum class in $H_2(K_N, \mathbb{Q})$ of a toric type I real curve.

Proposition 6.1.19
Let $\varphi : \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ be a toric type I real curve. Let S be a connected component of $\mathbb{C}C \backslash \mathbb{R}C$, inducing a complex orientation of the real part. Let $\mathbb{R}C_1, \dots, \mathbb{R}C_r$ be the connected components of the real locus, oriented as the boundary of S. For each component $\mathbb{R}C_i$, let $p_{i1}, \dots, p_{im_i}$ be its intersection points with the toric boundary, cyclically ordered, and let n_{ij} be the co-character associated to φ at p_{ij}. Then, one has

$$h(S, \varphi) = \sum_{\mathbb{R}C_i} \sum_{1 \leqslant j < k \leqslant m_i} n_{ij} \wedge n_{ik} \in H_2(K_N, \mathbb{Q}).$$

Remark 6.1.20. Notice that, as $\sum_j n_{ij} = 0$ for each i, the second sum does not depend on which point is the first, as long as the cyclic order is respected. ♦

Proof. The assumption that the curve is of toric type I ensures that the map $\alpha : \hat{S} \to N \otimes (\mathbb{R}/\pi\mathbb{Z})$ can be lifted to the universal cover $N_\mathbb{R}$ of the argument torus. We then consider a lift $\tilde{\alpha} : \hat{S} \to N_\mathbb{R}$. The image of $\hat{S}$ under this lift is a surface with a boundary consisting of polygonal loops in bijection with the real components $\mathbb{R}C_i$ of $\mathbb{C}C$. Let ω be a constant 2-form on $N \otimes (\mathbb{R}/\pi\mathbb{Z})$. The knowing of $\int_S \varphi^*(2\arg)^*\omega$ is enough to compute the quantum class $h(S, \varphi) \in H_2(K_N, \mathbb{R})$ with real coefficients. The form ω lifts to a 2-form on $N_\mathbb{R}$. Let $p : N_\mathbb{R} \to N_(\mathbb{R}/\pi\mathbb{Z})$ be the projection, so that we have $p \circ \tilde{\alpha} = \alpha$. The area we want to

compute is equal to
$$\int_{\alpha(S)} \omega = \int_{\tilde{\alpha}(S)} p^*\omega.$$
On $N_\mathbb{R}$, the form $p^*\omega$ is exact, so the value of the integral is independent of the surface $\tilde{\alpha}(S)$ provided that the boundary is the same.

The polygonal line γ_i bounding $\tilde{\alpha}(S)$ corresponding to the component $\mathbb{R}C_i$ is formed by the vectors $(n_{ij})_j$ on top of one another, in their cyclic order. The Lemma 6.1.18 ensures that it is indeed a polygonal loop. To compute the area, we change the surface $\tilde{\alpha}(S)$ by considering instead of $\tilde{\alpha}(\hat{S})$ a triangulation of each polygonal loop γ_i, by drawing a segment between one preferred vertex of the polygonal line, and every other vertex. The area is then given by

$$\begin{aligned}
&\omega(n_{i1}, n_{i2}) \\
&+\omega(n_{i1} + n_{i2}, n_{i3}) \\
&+\omega(n_{i1} + n_{i2} + n_{i3}, n_{i4}) \\
&+\cdots \\
&=\sum_{j<k} \omega(n_{ij}, n_{ik}) \\
&=\left\langle \omega, \sum_{j<k} n_{ij} \wedge n_{ik} \right\rangle.
\end{aligned}$$

We then just need to add up the contributions of the various components $\mathbb{R}C_i$ to get the result. □

Remark 6.1.21. In particular, if $\varphi; \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}^*$ is type I real rational curve with real intersection points with the toric boundary, it is of toric type I, and its quantum class is given by
$$h(\mathbb{H}, \varphi) = \sum_{1 \leqslant i < j \leqslant m} n_i \wedge n_j.$$
♦

6.1.5 Computation of the quantum class for rational curves

We conclude this first section about quantum class by giving a way of computing the quantum class with real coefficients for a real oriented rational curve with ropip. To do so, we use the fact that for any lattice map $N' \to N$ whose image contains $\Delta \subset N$, any parametrized curve $\mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}^*$ of degree Δ might be lifted to $N' \otimes \mathbb{C}^*$.

Let $\Delta \subset N$ be a multiset of total sum 0. Let $\varphi : \mathbb{C}P^1 \to N \otimes \mathbb{C}^*$ be a real rational of degree Δ. If t is a coordinate on $\mathbb{C}P^1$, φ takes the following form:

$$\varphi : t \longmapsto \chi \prod_{i=1}^{r} (t-\alpha_i)^{n_i} \prod_{j=1}^{s} (t^2 - 2\Re(\beta_j)t + |\beta_j|^2)^{n'_j} \in N \otimes \mathbb{C}^*,$$

where χ is a co-character with real values, and α_i, β_j are the coordinates of the intersection points with the toric boundary. We can always assume that $\Im\beta_j > 0$. We assume $r \geqslant 1$, and that the point α_r does not appear in the product, *i.e.* $\alpha_r = \infty$ in the coordinate t. Let $\mathbb{H}$ be the component of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$ consisting of points with $\Im t > 0$. Then we consider the lattice $\mathbb{Z}^{r+s-1}$, with basis $(e_1, \dots, e_{r-1}, f_1, \dots, f_s)$. Let e_r be such that $\sum_1^r e_i + 2\sum_1^s f_j = 0$. We then have a map $A : \mathbb{Z}^{r+s-1} \to N$ such that $Ae_i = n_i$, and $Af_j = n'_j$. This map is associated to a monomial map $\alpha(\mathbb{C}^*)^{r+s-1} \to N \otimes \mathbb{C}^*$, and the map φ factors through α by the map defined as follows:

$$\psi : t \longmapsto \rho \prod_{i=1}^{r} (t-\alpha_i)^{e_i} \prod_{j=1}^{s} (t^2 - 2\Re(\beta_j)t + |\beta_j|^2)^{f_j} \in (\mathbb{C}^*)^{r+s-1},$$

where ρ is a co-character such that $\alpha(\rho) = \chi$. Moreover, we can pull-pack any 2-form ω on N and obtain a 2-form $A^*\omega$ on $\mathbb{Z}^{r+s-1}$. It satisfies the following relations:

$$\begin{cases} A^*\omega(e_i, e_{i'}) = \omega(n_i, n_{i'}), \\ A^*\omega(f_j, f_{j'}) = \omega(n'_j, n'_{j'}), \\ A^*\omega(e_i, f_j) = \omega(n_i, n'_j). \end{cases}$$

The Proposition 6.1.14 along with Remark 6.1.15 allows us to compute the quantum index in $N \otimes \mathbb{C}^*$ as a log-area in $(\mathbb{C}^*)^{r+s-1}$:

$$\begin{aligned} k_\omega(\mathbb{H}, \varphi) &= k_\omega(\mathbb{H}, \alpha \circ \psi) \\ &= k_{A^*\omega}(\mathbb{H}, \psi). \end{aligned}$$

Now, we use the linearity of the quantum index with respect to the 2-form, and the decomposition

$$A^*\omega = \sum_{i<i'} \omega(n_i, n_{i'}) e_i^* \wedge e_{i'}^* + \sum_{j<j'} \omega(n'_j, n'_{j'}) f_j^* \wedge f_{j'}^* + \sum_{i,j} \omega(n_i, n'_j) e_i^* \wedge f_j^*.$$

We are left with the computation of the quantum index for each of the 2-forms appearing in the decomposition. Fortunately, those forms have a big kernel, and they can be expressed as the pull-back of a 2-form on a 2-dimensional lattice, and we get back to the planar case. We are lead to compute the log-area for the curves

$$\begin{cases} \phi_{ii'} : t \longmapsto (t-\alpha_i, t-\alpha_{i'}), \\ \phi_{jj'} : t \longmapsto (t^2 - 2\Re(\beta_j) + |\beta_j|^2, t^2 - 2\Re(\beta_{j'}) + |\beta_{j'}|^2), \\ \phi_{ij} : t \longmapsto (t-\alpha_i, t^2 - 2\Re(\beta_j) + |\beta_j|^2). \end{cases}$$

All these curves are in $(\mathbb{C}^*)^2$. Let ω_0 be the canonical 2-form of the subjacent lattice of co-characters. Then, one has

$$\begin{cases} k_{\omega_0}(\mathbb{H},\phi_{ii'}) = \pi^2, \\ k_{\omega_0}(\mathbb{H},\phi_{jj'}) = 4\pi \arctan\left(\frac{\Re\beta_{j'}-\Re\beta_j}{\Im\beta_{j'}+\Im\beta_j}\right), \\ k_{\omega_0}(\mathbb{H},\phi_{ij}) = 2\pi \arctan\left(\frac{\alpha_i-\Re\beta_j}{\Im\beta_j}\right). \end{cases}$$

Thus, we get

$$\begin{aligned} k_\omega(\mathbb{H},\varphi) &= \textstyle\sum_{i<i'} \omega(n_i,n_{i'}) \\ &\quad + \textstyle\sum_{j<j'} 4\pi \arctan\left(\frac{\Re\beta_{j'}-\Re\beta_j}{\Im\beta_{j'}+\Im\beta_j}\right) \omega(n'_j,n'_{j'}) \\ &\quad + \textstyle\sum_{i<j} 2\pi \arctan\left(\frac{\alpha_i-\Re\beta_j}{\Im\beta_j}\right) \omega(n_i,n'_j). \end{aligned}$$

Finally, we have proven the following theorem:

Theorem 6.1.22
The quantum class of a real rational parametrized curve

$$\varphi : t \longmapsto \chi \prod_{i=1}^{r} (t-\alpha_i)^{n_i} \prod_{j=1}^{s} (t^2 - 2\Re(\beta_j)t + |\beta_j|^2)^{n'_j} \in N \otimes \mathbb{C}^*,$$

is equal to

$$h(\mathbb{H},\varphi) = \begin{array}{l} \sum_{i<i'} n_i \wedge n_{i'} \\ + \sum_{j<j'} 4\pi \arctan\left(\frac{\Re\beta_{j'}-\Re\beta_j}{\Im\beta_{j'}+\Im\beta_j}\right) n'_j \wedge n'_{j'} \\ + \sum_{i<j} 2\pi \arctan\left(\frac{\alpha_i-\Re\beta_j}{\Im\beta_j}\right) n_i \wedge n'_j \end{array} \in H_2(K_N,\mathbb{R}) \simeq N.$$

Proof. We have computed the value of the quantum class on every cohomology class, which is sufficient to characterize it with coefficients chosen in $\mathbb{R}$. □

6.2 Harnack curves in higher dimension

We now turn our focus on the set of homology classes realizable as the quantum class of some type I real curve with ropip.

Proposition 6.2.1
For a fixed degree $\Delta \subset N$, there is a finite number of homology classes achievable as the quantum class of some type I real curve.

Proof. We only need to show that for each 2-form ω, the set of quantum indices is finite. As any 2-form decomposes as a sum of 2-form of the form $\alpha \wedge \beta$, we can assume that $\omega = \alpha \wedge \beta$, for some linear form α, β on N.

Then, ω is the pull-back of some 2-form on $N/(\ker\alpha \cap \ker\beta)$, which is a rank 2 lattice, since the quotient has no torsion. We denote this form by ω_0, so that $\alpha \wedge \beta = p^*\omega_0$. Let p be the projection $N \to N/(\ker\alpha \cap \ker\beta)$, and $\pi : N \otimes \mathbb{C}^* \to N/(\ker\alpha \cap \ker\beta) \otimes \mathbb{C}^*$ the map between the associated complex torus, so that we have $\omega = \alpha \wedge \beta = p^*\omega_0$. Therefore, for any oriented type I real curve $\varphi : S \subset \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$, using Proposition 6.1.14, one has

$$k_{\alpha\wedge\beta}(S, \varphi) = k_{\omega_0}(S, \pi \circ \varphi).$$

According to Mikhalkin [Mik17], the quantum index of an oriented plane curve is bounded by a constant depending on the degree, so $k_{\alpha\wedge\beta}$ only takes a finite number of values if the degree is fixed. □

In the planar case, Mikhalkin proved in [Mik17] that among curves of a fixed degree, curves that maximize the quantum index are the Harnack curves, defined as follows. Let Δ be a degree, associated to a polygon P_Δ, whose sides are labeled $E_1, \dots, E_p$, in the cyclic order induced as the boundary of P_Δ.

Definition 6.2.2. A curve of degree Δ and genus g is a Harnack curve if:

— It is a M-curve: it has the maximal number of real components $g + 1$.

— There are g real components non-intersecting the toric divisors, and the last component intersects each toric divisor $\mathbb{C}E_i$ in $l(E_i)$ points, and does it in their cyclic order, meaning that the parameters of the intersection points are ordered as follows:

$$a_1^1 < \cdots < a_{l(E_1)}^1 < a_1^2 < \cdots < a_1^p < \cdots < a_{l(E_p)}^p < a_1^1.$$

This follows from the majoration of the quantum index by the degree, and results of [MR01].

Let ω be a 2-form on N. It induces a linear form on $H_2(K_N, \mathbb{Z})$. We can try to find the curves that maximize this linear form, extending the definition of Harnack curves in higher dimension.

Theorem 6.2.3
Let $\Delta \subset N$ be a degree, and ω be a generic 2-form. The toric type I real curve having a unique real component that intersects the toric divisors in an order prescribed by ω maximize the quantum index among the toric type I real curves.

Proof. If $\varphi : S \subset \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ is an oriented toric type I real curve, we have proven that its quantum index only depends on the order in which each of its real component intersects the toric divisors. We proceed in three times: first, we show that the order leads to a maximization of the quantum index only if each real component intersects cyclically the divisors, then we show that among the curves satisfying these conditions, the curves with the intersection points on the same component have a bigger quantum index, and finally we show that such curves exist.

- Let $\varphi : S \subset \mathbb{C}C \dashrightarrow N \otimes \mathbb{C}^*$ be an oriented toric type I real curve. We know the quantum index $k_\omega(S,\varphi)$ is the sum over the real components of $\mathbb{R}C$ of the cyclic sums $\sum_{i<j} \omega(v_i, v_j)$, where the vectors v_i are the vectors from Δ corresponding to the toric divisors that the real component intersects. Let $\{n_1^{a_1}, \dots, n_r^{a_r}\}$ be the multiset of these vectors for a chosen component of $\mathbb{R}C$. Using Lemma 6.1.18, we know that $\sum a_i n_i = 0$. We now write these vectors with respect to the order given by the oriented real component, emphasizing the role of n_1. We get

$$n_1^{t_1}, w_1, n_1^{t_2}, w_2, \dots, n_1^{t_p}, w_p,$$

with the following notations: the integer t_i correspond to n_1 being present several times in the sequence, and the vectors w_i are the elements of $\{n_1^{a_1}, \dots, n_r^{a_r}\}$ different from n_1. Therefore, we have $p = \sum_2^r a_i$, the integers t_i might take the value 0, and $\sum_1^p t_i = a_1$. The contribution of the real component to the quantum index is equal to

$$\sum_{1 \leqslant i<j \leqslant p} \omega(w_i, w_j) + t_i \omega(n_1, w_j) + t_j \omega(w_i, n_1).$$

As this sum is an affine function of the tuple (t_i), that belongs to the integer points of the simplex defined by $\{t_i \geqslant 0\} \cap \{\sum t_i = a_1\}$, its maximum is reached at its corner: all the integers t_i but one are zero. Therefore, we obtain a greater value if all the copies of n_1 are together. By iterating this argument, we have proven that the quantum index is maximized when the real component intersects cylically the divisors, provided that such a curve exists.

- The previous step guarantees that each real component intersects cyclically the toric divisors. Let $v_{i1}, \dots, v_{im_i}$ be the vectors associated to the divisors that the real component $\mathbb{R}C_i$ intersects. The quantum index is equal to the quantum index of a curve that has only one real component intersecting the diviors in the following order:

$$v_{11}, \dots, v_{1m_1}, v_{21}, \dots, v_{2m_2}, \dots$$

Then, we reapply the previous step to get a cyclic intersection.

- We have proven that the quantum index of a toric type I real oriented curve is always smaller that the quantum index of some hypothetical toric type I real curve having a unique real component intersecting cyclically the toric divisors. We still need to prove that such a curve exists. If we choose an order on the divisors, there exists an oriented real rational curve that meets them in the desired order. In fact, let $\alpha_1 < \cdots < \alpha_m$, then the rational curve

$$\varphi : t \longmapsto \prod_1^r (t - \alpha_i)^{n_i} \in N \otimes \mathbb{C}^*,$$

meets the divisors associated to the rays directid by the vectors n_i in the order prescribed by the scalars α_i.

Finally, if $\Delta = \{n_1^{a_1}, \dots, n_r^{a_r}\}$, the cyclic orders that maximize the quantum indices are the orders maximizing the value of $\sum_{i<j} a_{\sigma(i)} a_{\sigma(j)} \omega(n_{\sigma(i)}, n_{\sigma(j)})$ for $\sigma \in \mathcal{S}_r$. $\square$

Remark 6.2.4. We have proven that the maximal quantum indices among toric type I real curves are realized by curves with a cyclic intersection with the divisors, not that they are the only curves. There might be some problem for instance when Δ splits in two subdegrees $\Delta_1 \sqcup \Delta_2$ such that for each $n_1 \in \Delta_1$ and $n_2 \in \Delta_2$, $\omega(n_1, n_2) = 0$. ♦

Remark 6.2.5. We have proven that curves intersecting cyclically the toric divisors in some specific order depending on the chosen 2-form maximize the quantum index for toric type I real curves. In the planar case, these curves are called Harnack curves and have been the subject of many studies. Following the ideas of [Mik00] and [MR01], the next questions should be: do they indeed maximize the quantum index among all curves ? are they the only one to maximize the quantum index ? are maximizing curves M-curves ? ♦

6.3 Enumerative problem

6.3.1 Tropical moment problem

Recall that we consider tropical curves in $N_{\mathbb{R}}$, which is an m-dimensional vector space. Let ω be a 2-form on N, and $\Delta \subset N$ be a multiset of total sum 0. We consider the moduli space $\mathcal{M}_0(N_{\mathbb{R}}, \Delta)$ of rational tropical curves of degree Δ in $N_{\mathbb{R}}$. We assume that vectors of Δ do not belong to $\ker \omega$. In fact, we could even mod out by $\ker \omega$. For each unbounded end e directed by n_e, we have evaluation map, that associates to a parametrized rational tropical curve the position of the unbounded end:

$$\mathrm{ev}_e : \mathcal{M}_0(N_{\mathbb{R}}, \Delta) \to N_{\mathbb{R}}/\langle n_e \rangle.$$

This map might be composed by any linear form on $N_{\mathbb{R}}/\langle n_e \rangle$, such as $\omega(n_e, -)$, thus getting the *moment* of the unbounded end. This linear form is non-zero since n_e is assumed not to belong to $\ker \omega$. Hence, we get the following moment map:

$$\mathrm{mom} : (\Gamma, h) \in \mathcal{M}_0(N_{\mathbb{R}}, \Delta) \longmapsto (\mu_e) \in \mathbb{R}^{\Delta}.$$

The tropical Menelaus relation ensures that the image of this evaluation map lives in the hyperplane $\{\sum \mu_e = 0\}$. The domain has dimension $m + |\Delta| - 3$, and the target space has dimension $|\Delta| - 1$. Therefore, they have the same dimension only if N has rank 2. Otherwise, one needs an additional constraint of the suitable dimension to obtain an interesting enumerative problem. There are several ways to do that.

- One way is to consider the moment map, enlarged by the position of some unbounded end, assumed to be the first unbounded end:

  $$\mathrm{mom} \times \mathrm{ev}_1 : \mathcal{M}_0(N_{\mathbb{R}}, \Delta) \longrightarrow \mathbb{R}^{|\Delta|-1} \times N_{\mathbb{R}}/\langle n_1 \rangle.$$

  Then, we look for parametrized curves (Γ, h) with fixed moments such that $\mathrm{ev}_1(\Gamma, h)$ belongs to some chosen tropical 1-cycle in $N_{\mathbb{R}}/\langle n_1 \rangle$. For instance, one could choose a line D with rational slope δ. This cycle needs to be transversal to the hyperplane $\{\omega(n_1, -) =$

$\mu_1\}$. We show that the count of the solutions with suitable refined multiplicities does not depend on the choice of the moments and the choice of the cycle. We expect it to neither depend on the chosen unbounded end with an additional constraint.

- Another way, quite similar, is to consider one additional marked point on the curve:

$$\mathrm{mom} \times \mathrm{ev}_0 : \mathcal{M}_{0,1}(N_{\mathbb{R}}, \Delta) \longrightarrow \mathbb{R}^{|\Delta|-1} \times N_{\mathbb{R}},$$

  and then look for curves with fixed moments such that the additional marked point belongs to a fixed tropical 1-cycle. Once again, the count of solutions with suitable refined multiplicities is independent of the choice of μ, and expected to be independent on the choice of this cycle.

- Yet another way would be to add several conditions on various unbounded ends, or mix it up with conditions of points inside the main strata $N_{\mathbb{R}}$, such that the constraints have the right dimension, but we do not know if it gives refined invariants.

We define the *refined multiplicity* of a rational tropical curve, that appears in the curve count of the solutions to the previous enumerative problems.

Definition 6.3.1. Let $h : \Gamma \to N_{\mathbb{R}}$ be a trivalent rational parametrized tropical curve. Let v be a vertex of Γ, and a_v, b_v the slopes of h on two outgoing edges of v, chosen so that $\omega(a_v, b_v) > 0$. Then, we set

$$m^h_\Gamma = \prod_v (q^{a_v \wedge b_v} - q^{b_v \wedge a_v}) \in \mathbb{Z}[\Lambda^2 N],$$

where the products is over the vertices of Γ.

Remark 6.3.2. This refined multiplicity using a wedge product rather than the evaluation of some 2-form is inspired by Shustin [Shu18]. ♦

The h in exponent stands for homology, because under suitable hypothesis, the refinement gives the quantum classes, which are homology classes. We can easily check that the vertex multiplicity $q^{a_v \wedge b_v} - q^{b_v \wedge a_v}$ does indeed not depend on the two chosen outgoing edges a_v and b_v. This is due to the antisymmetric property of the wedge product.

6.3.2 Invariance

In this section we state and prove the invariance statements in the announced cases:

- Additional constraint on the first unbounded end (constraint at infinity),
- Additional constraint on an additional marked point (constraint in the main strata).

6.3.2.1 Constraints at infinity

Let $\mu \in \mathbb{R}^{|\Delta|-1}$ and $D \subset N_{\mathbb{R}}/\langle n_1 \rangle$ be a line with slope $\delta \in N/\langle n_1 \rangle$ chosen generically among the lines with that slope. We assume $\omega(n_1, \delta) \neq 0$.

Lemma 6.3.3
There is a finite number of parametrized tropical curves $h : \Gamma \to N_{\mathbb{R}}$ with $\mathrm{mom}(\Gamma, h) = \mu$ and $\mathrm{ev}_1(\Gamma, h) \in D$. Moreover, these curves are trivalent.

Proof. There is only a finite number of combinatorial types of parametrized curves, hence a finite number of top-dimensional cones in $\mathcal{M}_0(N_{\mathbb{R}}, \Delta)$. The dimension of these cones is equal to the dimension of $\mathbb{R}^{|\Delta|-1} \times N_{\mathbb{R}}/\langle n_1, \delta \rangle$, *i.e.* $|\Delta| + m - 3$. Therefore, the statement follows if D and μ are chosen outside the image of the cones where $\mathrm{mom} \times \mathrm{ev}_1$ is not surjective. That includes some top-dimensional cones and the not top-dimensional cones. □

Remark 6.3.4. This statement also follows from intersection theory: the image of $\mathcal{M}_0(N_{\mathbb{R}}, \Delta)$ under $\mathrm{mom} \times \mathrm{ev}_1$ is a polyhedral complex, and $\{\mu\} \times D$ is also a polyhedral complex of complementary dimension. Thus, if chosen generically, their finite number of intersection points belong to the relative interior of their top-dimensional faces. Hence, we recover the result. ♦

The complex multiplicity of the solutions, defined as the determinant of the composed evaluation map, is easily computed using the approach from 3.2.12.

Lemma 6.3.5
Let $h : \Gamma \to N_{\mathbb{R}}$ be a parametrized tropical curve with $\mathrm{mom}(\Gamma, h) = \mu$ and $\mathrm{ev}_1(\Gamma, h) \in D$. The tropical curve is trivalent. Then, its complex multiplicity is

$$m_\Gamma^{\mathbb{C}} = \left| \omega(n_1, \delta) \prod_v \omega(a_v, b_v) \right|,$$

where the product is over the vertices of Γ, and for each vertex v, a_v and b_v denote the slope of h on two edges adjacent to the vertex v.

Proof. Notice that if n and n' are co-characters, one has

$$\begin{aligned} \iota_{n+n'}(\iota_n \omega \wedge \iota_{n'} \omega) &= \iota_{n+n'} \iota_n \omega \wedge \iota_{n'} \omega - \iota_n \omega \wedge \iota_{n+n'} \iota_{n'} \omega \\ &= \omega(n, n') \iota_{n'} \omega - \omega(n', n) \iota_n \omega \\ &= \omega(n, n') \iota_{n+n'} \omega. \end{aligned}$$

Therefore, the multiplicity is equal to the product of the vertex multiplicities, times a constant term, computed on the first unbounded end. This contribution is equal to $\omega(n_1, \delta)$. □

As the complex multiplicity is a product over the trivalent vertices, we refine this multiplicity using Definition 6.3.1. For a trivalent curve solution to the enumerative problem, we set

$$N_\Delta^{\mathrm{trop},1}(\mu, D) = |\omega(n_1, \delta)| \sum_{\substack{\mathrm{mom}(\Gamma)=\mu \\ \mathrm{ev}_1(\Gamma)\in D}} m_\Gamma^h \in \mathbb{Z}[\Lambda^2 N],$$

where the 1 in exponent emphasizes the fact that the additional constraint is put on the first unbounded end.

If instead of D we fix a degree $\Theta \subset N/\langle n_1 \rangle$ and choose a generic tropical 1-cycle Ξ of degree Θ, we have a similar statement.

Lemma 6.3.6
If μ and Ξ are chosen generically, there is a finite number of parametrized tropical curves $h : \Gamma \to N_\mathbb{R}$ with $\mathrm{mom}(\Gamma, h) = \mu$ and $\mathrm{ev}_1(\Gamma, h) \in \Xi$. Moreover, these curves are trivalent and intersect Ξ at an edge.

Proof. The proof is similar to proof of Lemma 6.3.3, given by intersection theory: the polyhedral complexes $\mathrm{mom} \times \mathrm{ev}_1(\mathcal{M}_0(N_\mathbb{R}, \Delta)$ and $\{\mu\} \times \Xi$ are of complementary dimension in $\mathbb{R}^{|\Delta|-1} \times N_\mathbb{R}/\langle n_1 \rangle$. Thus, if the second polyhedral complex is chosen generically, they have a finite number of intersection points and their intersection points belong to the relative interior of top-dimensional faces. Therefore, the curves solutions are trivalent, and the intersection point with Ξ belongs to the relative interior of some edge of Ξ. $\square$

Similarly, one can compute the complex multiplicity. The result is the same if we replace the slope δ of D by the slope of Ξ. Let $h : \Gamma \to N_\mathbb{R}$ be a parametrized tropical curve such that $\mathrm{ev}_1(\Gamma, h) \in \Xi$, and $v \in N/\langle n_1 \rangle$ is the slope of Ξ at $\mathrm{ev}_1(\Gamma, h)$, we denote by $h(\Gamma) \cdot \Xi$ the integer $|\omega(n_1, v)|$. Then, the complex multiplicity satisfies

$$m_\Gamma^\mathbb{C} = (h(\Gamma) \cdot \Xi) \prod_v |\omega(a_v, b_v)|,$$

with obvious notations. We now refine the multiplicity using Definition 6.3.1.

$$N_\Delta^{\mathrm{trop},1}(\mu, \Xi) = \sum_{\substack{\mathrm{mom}(\Gamma)=\mu \\ \mathrm{ev}_1(\Gamma)\in \Xi}} (h(\Gamma) \cdot \Xi) m_\Gamma^h \in \mathbb{Z}[\Lambda^2 N].$$

Notice that this definition generalizes the previous one when the line D is considered as a tropical cycle, since in that case the intersection index $h(\Gamma) \cdot D$ is always equal to $|\omega(\delta, n_1)|$.

We can now state our first invariance statements.

Theorem 6.3.7
We have the following results of invariance:

- *The value of* $N_\Delta^{\mathrm{trop},1}(\mu, D)$ *does not depend on the generic choice of* μ *and choice of* D *among the lines of slope* δ.
- *The value of* $N_\Delta^{\mathrm{trop},1}(\mu, \Xi)$ *does not depend on the generic choice of* μ *and choice of* Ξ *among the cycles of degree* Θ.

Proof. In both cases, using Proposition 3.2.9, we know that the count of solutions with complex multiplicity leads to an invariant. This means that the repartition of the solutions around a wall matches the invariance of the count with complex multiplicities. Therefore, we need to check for each wall that the count with refined multiplicities is also invariant.

For the first invariance statement, there is only one wall to check: the wall corresponding to a quadrivalent vertex of the tropical curve. Let a_1, a_2, a_3, a_4 be the slopes of the outer edges, with index taken in $\mathbb{Z}/4\mathbb{Z}$. Up to a relabeling, we assume that $\omega(a_i, a_{i+1}) > 0$ for every i. If some value is equal to 0, the proof remains unchanged. Moreover, we assume that $\omega(a_2, a_3) > \omega(a_1, a_2)$. The invariance of the complex count amounts to the following relation:

$$\underset{\text{for } 12//34}{\omega(a_1,a_2)\omega(a_1+a_2,a_3)} \quad + \quad \underset{\text{for } 13//24}{\omega(a_1,a_3)\omega(a_2,a_1+a_3)} \quad + \quad \underset{\text{for } 14//23}{\omega(a_2,a_3)\omega(a_2+a_3,a_1)} \quad = \quad 0,$$

and the repartition of combinatorial types around the wall is given by the sign of each term. It means up to sign that one is positive and is on one side of the wall, and the two other ones are negative, on the other side of the wall. Hence, we just need to study the signs of each term to know which curve is on which side. We know that $\omega(a_1, a_2)$ and $\omega(a_1 + a_2, a_3) = \omega(a_3, a_4)$ are positive. Therefore, their product, which is the term of 12//34, is also positive. We know that $\omega(a_2, a_3)$ is positive, but $\omega(a_2 + a_3, a_1) = -\omega(a_4, a_1)$ is negative. Therefore, their product is negative and 14//23 is on the other side of the wall. It means that the combinatorial types 12//34 and 14//23 are on opposite sides of the wall. We need to determine on which side the type 13//24 is, and that is given by the sign of the middle term. As by assumption $\omega(a_2, a_1+a_3) = \omega(a_2, a_3)-\omega(a_1, a_2) > 0$, it is determined by the sign of $\omega(a_1, a_3)$.

— If $\omega(a_1, a_3) > 0$, then 12//34 and 13//24 are on the same side, and the invariance for refined multiplicities is dealt with the identity

$$\begin{aligned}
&(q^{a_2\wedge a_3)} - q^{a_3\wedge a_2})(q^{a_1\wedge(a_2+a_3)} - q^{(a_2+a_3)\wedge a_1}) \\
=&(q^{a_1\wedge a_2} - q^{a_2\wedge a_1})(q^{(a_1+a_2)\wedge a_3} - q^{a_3\wedge(a_1+a_2)}) \\
+&(q^{a_1\wedge a_3} - q^{a_3\wedge a_1})(q^{a_2\wedge(a_1+a_3)} - q^{(a_1+a_3)\wedge a_2}),
\end{aligned}$$

— and if $\omega(a_1, a_3) < 0$, then 14//23 and 13//24 are on the same side and then the invariance for refined multiplicities is true since

$$\begin{aligned}
&(q^{a_2\wedge a_3}-q^{a_3\wedge a_2})(q^{a_1\wedge(a_2+a_3)}-q^{(a_2+a_3)\wedge a_1})\\
+&(q^{a_3\wedge a_1}-q^{a_1\wedge a_3})(q^{a_2\wedge(a_1+a_3)}-q^{(a_1+a_3)\wedge a_2})\\
=&(q^{a_1\wedge a_2}-q^{a_2\wedge a_1})(q^{(a_1+a_2)\wedge a_3}-q^{a_3\wedge(a_1+a_2)}).
\end{aligned}$$

If we have some edges parallel among the vectors a_i, either two consecutives vectors are parallel, and then the invariance is straightforward, since there are only two adjacent combinatorial types with equal non-zero multiplicity, or we can choose a cyclic labeling such that a_1 and a_3 are parallel. We then have $\omega(a_1, a_3) = 0$. It means that one of the determinant multiplicities is zero, which is normal since the associated combinatorial type would have a flat vertex. Thus,

$$\underset{\text{for } 12//34}{\omega(a_1,a_2)\omega(a_1+a_2,a_3)} \quad + \quad \underset{\text{for } 14//23}{\omega(a_2,a_3)\omega(a_2+a_3,a_1)} \quad = \quad 0.$$

It means that the two terms are of opposite sign. Assume the first one is positive, and thus $\omega(a_1, a_2)$ and $\omega(a_1 + a_2, a_3) = \omega(a_2, a_3)$ have the same sign. The refined multiplicity is then

$$(q^{a_1\wedge a_2}-q^{a_2\wedge a_1})(q^{a_2\wedge a_3}-q^{a_3\wedge a_2}).$$

The second term being negative, it means that $\omega(a_2, a_3)$ and $\omega(a_1, a_2+a_3) = \omega(a_1, a_2)$ have the same sign. The refined multiplicity is the the same and we have the desired local invariance. It is the same if the first term is negative. This closes the proof of invariance in case of $N_\Delta^{\mathrm{trop},1}(\mu, D)$. In case of $N_\Delta^{\mathrm{trop},1}(\mu, \Xi)$, we have one more wall to study, which corresponds to the tropical curve passing through a vertex of Ξ. Let $v_1, \dots, v_k$ be the slopes of Ξ at this vertex. Then, the invariance of complex multiplicities amounts to the following relation:

$$\omega(n_1, v_1) + \cdots + \omega(n_1, v_k) = 0,$$

which comes from the balancing condition of Ξ. This relation is preserved in the refined multiplicities. Therefore, we keep invariance around this wall. □

Our second invariance result concerns the degree Θ of the additional cycle. We denote by $\omega(n_1, \Theta)$ the intersection index between the hyperplane $\{\omega(n_1, -) = \mu_1\}$ in $N_\mathbb{R}/\langle n_1\rangle$ and Ξ. This index is defined as follows: if Ξ and μ_1 are chosen generically, their intersection points belong to edges of Ξ, let $p_1, \dots, p_k$ be the intersection points, and $v_1, \dots, v_k$ be the respective slope of Ξ at these points. Then, the index is

$$\sum_1^k |\omega(n_1, v_i)|.$$

Using the balancing condition, one can see that this sum does only depend on the degree Θ of Ξ. Alternatively, this follows from general results of intersection theory.

Theorem 6.3.8
The value of $\frac{1}{\omega(n_1,\Theta)}N_\Delta^{\mathrm{trop},1}(\mu,\Xi)$*, already independent on the choice of* μ *and* Ξ*, does not depend on the choice of* Θ*.*

The value of $\frac{1}{\omega(n_1,\Theta)}N_\Delta^{\mathrm{trop},1}(\mu,\Xi)$, now independent of μ and Ξ along with its degree Θ, is denoted by $N_\Delta^{\mathrm{trop},1}$.

Proof. First, notice that invariant $\frac{1}{|\omega(n_1,\delta)|}N_\Delta^{\mathrm{trop},1}(\mu,D)$ does not depend on the slope δ of D: the set of solutions only depend on the intersection point of D with the hyperplane $\{\omega(n_1,-)=\mu_1\}\subset N_\mathbb{R}/\langle n_1\rangle$. The slope only appears in the multiplicity through the presence of $\omega(n_1,\delta)$. Hence, the result is invariant. We denote it by $\mathcal{N}_\Delta^1$.

Let Ξ be a generic cycle of degree Θ, μ be a generic family of moments. Let $p_1,\dots,p_k$ be the intersection points of Ξ with the hyperplane $\{\omega(n_1,-)=\mu_1\}\subset N_\mathbb{R}/\langle n_1\rangle$. Every tropical curve solution of the enumerative problem passes through one of these points. Let $v_1,\dots,v_k$ be the respective slopes of Ξ at these points. Then the set of tropical curves with moments μ and meeting Ξ at p_j are in bijection with the tropical curves having moment μ and meeting the line $D_j=p_j+\mathbb{R}v_j$ (also at p_j). The case of a line constraint ensures that the contribution of these solutions is equal to $|\omega(n_1,v_j)|N_\Delta^{\mathrm{trop},1}$. Therefore, adding the various contributions, we get $(h(\Gamma)\cdot\Xi)N_\Delta^{\mathrm{trop},1}$. Hence, the result does not depend on Θ. □

6.3.2.2 Constraint in the main strata

The next announced result concerning invariants is when we allow the tropical 1-cycle to be in $N_\mathbb{R}$ instead of $N_\mathbb{R}/\langle n_1\rangle$. For that, we need to consider an additional marked point, and thus, the following evaluation map:

$$\mathrm{mom}\times\mathrm{ev}_0:\mathcal{M}_{0,1}(N_\mathbb{R},\Delta)\longrightarrow\mathbb{R}^{|\Delta|-1}\times N_\mathbb{R},$$

that associates to a parametrized tropical curve with a marked point the moment of its unbounded ends and the position of the new marked point. Let $\Theta\subset N$ be a degree and let Ξ be a 1-dimensional tropical cycle of degree Θ. Let μ be a family of moments. We assume μ and Ξ are chosen generically.

Lemma 6.3.9
There is a finite number of parametrized tropical curves $h:\Gamma\to N_\mathbb{R}$ *such that* $\mathrm{mom}(\Gamma,h)=\mu$ *and* $\mathrm{ev}_0(\Gamma,h)\in\Xi$*. Moreover, these curves are trivalent and intersect* Ξ *along the relative interior of their respective top-dimensional faces.*

Proof. The proof is entirely similar to the proof of Lemma 6.3.6. □

Once again, the complex multiplicity can be computed using the approach from Theorem

3.2.12.

Lemma 6.3.10
Let $h : \Gamma \to N_{\mathbb{R}}$ be a trivalent tropical curve having moment μ and meeting Ξ at a point where their respective slopes are n_0 and v. Then its complex multiplicity is equal to

$$m_\Gamma^{\mathbb{C}} = \left| \omega(n_1, v) \prod_v \omega(a_v, b_v) \right| ,$$

with obvious notations.

Proof. Notice that

$$\iota_{n+n'}(\iota_n\omega \wedge \iota_{n'}\omega) = \omega(n, n')\iota_{n+n'}\omega,$$

and

$$\iota_{n+n'}(\iota_n\omega \wedge 1) = \omega(n, n') \times 1 \in \Lambda^\bullet M.$$

Therefore, we can compute the multiplicity by cutting the curve and meeting at the marked point. The marked point cuts the curve in two halves. One half has every unbounded end with a fixed moment, and the first relation allows us to cut it up to the edge where it meets the marked point, making the product of the vertex multiplicities of this first half appear. The other half tropical curve has every end but one with a fixed moment. The last unbounded end is free. Using the two previous relations, we can still cut it, and we get the product of the vertices multiplicities for the second half. The last contribution, given by the marked point, is equal to

$$\iota_{n_0}\omega \wedge \rho \wedge 1 \in \Lambda^m M,$$

where ρ denotes the polyvector of $\Lambda^{m-1}M$ corresponding to the edge of Ξ. This contribution is equal to $\omega(n_0, v)$. □

Then, every parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$ with a marked point such that $\mathrm{mom}(\Gamma, h) = \mu$ and $\mathrm{ev}_0(\Gamma, h) \in \Xi$ is trivalent, and the marked point belongs to a 1-dimensional cell of Ξ. Let n_0 and v be the respective slopes of h and Ξ at this common point. We denote by $h(\Gamma)\cdot\Xi$ the integer $|\omega(n_1, v)|$ and we refine the complex multiplicity using Definition 6.3.1:

$$N_\Delta^{\mathrm{trop},0}(\mu, \Xi) = \sum_{\substack{\mathrm{mom}(\Gamma)=\mu \\ \mathrm{ev}_0(\Gamma)\in\Xi}} (h(\Gamma) \cdot \Xi) m_\Gamma^h \in \mathbb{Z}[\Lambda^2 N].$$

Theorem 6.3.11
The value of $N_\Delta^{\mathrm{trop},0}(\mu, \Xi)$ does not depend on the choices of μ and Ξ among the cycles of degree Θ as long as these are generic.

The value, now independent of the choice of μ and Ξ, is denoted by $N_\Delta^{\mathrm{trop},0}(\Theta)$.

Proof. As in the first proofs of invariance, we know that the count with complex multiplicities leads to an invariant count. This gives the repartition of the solutions around each wall, and we need to check the invariance of refined multiplicities around these walls. Let $h : \Gamma \to N_{\mathbb{R}}$ be a tropical curve inside a wall. In our case, there are three walls to study:

- The parametetrized marked tropical curve (Γ, h) has a quadrivalent vertex. The proof is the same as in proof of Theorem 6.3.7.
- The tropical curve meets Ξ at one of its vertices. Proof is again the same as in Theorem 6.3.7.
- Last, the marked point coincides with a vertex of the tropical curve. In that case, let a_1, a_2, a_3 be the slope of h on the edges adjacent to the marked vertex. Let v be the slope of Ξ at the marked point. The complex invariance amounts to the following relation:

  $$\omega(a_1, v) + \omega(a_2, v) + \omega(a_3, v) = 0.$$

  This relation also ensures the invariance of the refined count.

□

Remark 6.3.12. As in the case of constraints at infinity, the dependence in Θ of the invariant $N_\Delta^{\mathrm{trop},0}(\Theta)$ is expected to be of the form $f(\Delta, \Theta) N_\Delta^{\mathrm{trop},0}$, for some integer function f, and $\mathcal{N}_\Delta^0 \in \mathbb{Z}[\Lambda^2 N]$ being equal to the assumed common value of $N_\Delta^{\mathrm{trop},i}$, for $i \geqslant 1$. ♦

6.3.3 Classical counterpart

We now provide a classical enumerative problem, tropicalizing to the moment problem in the sense that the classical problem can be resolved using the tropical geometry approach of the analog tropical enumerative problem and a suitable correspondence theorem, such as [NS06] or[Tyo17]. We expect the refined enumeration of tropical curves to correspond to some refined count of classical curves, with respect to their quantum class, provided such classes are well-defined for the solutions of the enumerative problem. We still consider real rational curves of degree $\Delta \subset N$, and choose a 2-form ω on N.

6.3.3.1 Real moment problem

We start with the *real problem*, analog to the problem considered by Mikhalkin in [Mik17] when all the points are real.

Recall that in the toric variety $\mathbb{C}\Delta$, the moment of a point p belonging to a toric divisor D associated to a ray of the fan Σ_Δ directed by n is the scalar $\chi^{\iota_n \omega}(p)$. The set of points in D having the same moment is the orbit of p under the action of the subgroup $(\ker \iota_n \omega) \otimes \mathbb{C}^* \subset N \otimes \mathbb{C}^*$. Thus, we have the following moment map:

$$\mathrm{mom} : \mathcal{M}_0(N \otimes \mathbb{C}^*, \Delta) \to (\mathbb{C}^*)^{|\Delta|}.$$

As in the tropical case, the Weil reciprocity law ensures that the product of the moments at the boundary points only depends on the degree Δ, with a value equal to ± 1. Also, just as in the tropical case, we need an additional constraint to obtain an enumerative problem. If we fix the moments $\mu \in (\mathbb{C}^*)^{|\Delta|}$, once again, there are several possibilities for the additional constraint:

- Let $\delta \in N$ be a co-character, we ask for the first marked point, *i.e.* the point corresponding to the first unbounded end and already belonging to a toric divisor D, to belong to an orbit $\mathcal{D}$ of D under the action of the subgroup $\langle \delta \rangle \otimes \mathbb{C}^*$.
- Instead of being an orbit, the additional constraint may be a real curve inside $\mathbb{C}\Delta$.
- One could consider an additional marked point on the curve, and ask for this new marked point to belong to a chosen real curve, curve that might be an orbit or not.
- Finally, one could mix up these constraints just as in the tropical case.

We can see that the tropicalization of these enumerative problems are indeed the tropical problems described in subsection 6.3.1. This means that the classical problems, known to provide an invariant in the complex setting, can be solved near the tropical limit through the use of a correspondence theorem such as the one from [Tyo17], in case all the constraints are torus orbits. This allows one to compute the invariants.

Theorem 6.3.13 (Tyomkin [Tyo17])
Let the constraints μ and $\mathcal{D}$ be tropically general. Then, we have a correspondence between the tropical curves solutions to the tropical problem, and the complex curves solutions to the classical problem. Furthermore, each tropical curve is the limit of $m_\Gamma^{\mathbb{C}}$ complex solution.

The complex number of solution is equal to the number of tropical curves if those curves are counted with multiplicity the absolute of the determinant of the tropical evaluation map.

We prove that the signed count, refined by the value of the quantum class, of the solutions near the tropical limit is invariant. Before stating properly this theorem, we define the sign of the solutions near the tropical limit. Recall that a real oriented structure on a tropical curve is a ribbon structure. Moreover, given a 2-form ω and a trivalent curve $h : \Gamma \to N_\mathbb{R}$ with non-zero complex multiplicity, we have a ribbon structure given by ω: if a_1, a_2, a_3 are slope of the three edges adjacent to a vertex v, the order for the ribbon structure is a_1, a_2, a_3 if $\omega(a_1, a_2) = \omega(a_2, a_3) = \omega(a_3, a_1) > 0$. It is the reversed order otherwise.

Definition 6.3.14. Let $h : \Gamma \to N_\mathbb{R}$ be a tropical curve, enhanced with a real oriented structure $\mathfrak{o}$. At each vertex, the orientation given by the ribbon structure may coincide or not with the orientation prescribed by ω. Let w be the number of vertices where the orientation differs. The sign $\sigma(\Gamma, h, \mathfrak{o})$ is set to be $(-1)^w$.

Let μ be a generic family of real moments in $(\mathbb{R}^*)^{|\Delta|}$ and $\mathcal{D}$ a generic orbit under $\langle \delta \rangle \otimes \mathbb{C}^*$ inside the toric divisor D_1. Then, we set

$$\mathcal{N}_\Delta^1(\mu, \mathcal{D}) = \sum_C \sigma(C) q^{h(C)},$$

where the sum is over the real oriented rational curves having moments μ, and passing through $\mathcal{D}$. Such curves have real intersection with the toric divisor. Thus, they have a well-defined quantum class $h(C) \in H_2(K_N, \mathbb{R})$. Moreover, they have a well-defined sign near the tropical limit, equal to the sign $\sigma(\Gamma, h, \mathfrak{o})$ of their tropical limit curve enhanced by the ribbon structure obtained as the limit of the real curve.

Theorem 6.3.15
Near the tropical limit, the value of $\mathcal{N}^1_\Delta(\mu, \mathcal{D})$ is independent of the choice of μ, $\mathcal{D}$, and equal to $N^{\text{trop},1}_\Delta$.

Proof. This is a straight-forward application of the correspondence theorem. □

Remark 6.3.16. As we are counting oriented real rational curves, this count might be seen as a count for holomorphic disks with boundary in the real locus of the toric variety, which happens to be a lagrangian. ♦

Remark 6.3.17. The definition of the signs of real oriented curves, along with the invariance of their refined count, is an open question when we are not close to the tropical limit. ♦

6.3.3.2 Imaginary constraints

Although the meaning of the previous enumerative problem might bear some symplectic interpretation, the problem can be seen purely algebraically, as we look for real oriented rational curves satisfying some algebraic constraints. In a perspective of refinement, the introduction of non real constraints in the problem complicates the setting since algebraic condition are not enough to ensure the curves are ropip, which is a necessary condition for them to have a quantum class.

In the real problem, the intersection points with the toric boundary are real. Thus, any curve with real intersection with the toric boundary is ropip, and has a well-defined quantum class, which is the case of the solution to the real problem. If we ask for a point to have purely imaginary moment, there is no guarantee that the other coordinates of the point are real or purely imaginary, as it was the case in the planar problem from [Mik17].

Maybe evaluating the quantum class on the class provided by the 2-form ω allows us to enlarge the set of curves having a quantum index, maybe by only asking for a real or purely imaginary moment. Hopefully, a real rational curve can be ensured to have a quantum index by satisfying only some algebraic constraints. Otherwise, maybe one should look at lagrangian constraints.

Bibliography

[AR10] Lars Allermann and Johannes Rau. « First steps in tropical intersection theory ». In: *Mathematische zeitschrift* 264.3 (2010), pp. 633–670 (cit. on pp. 29, 33).

[BG16] Florian Block and Lothar Göttsche. « Refined curve counting with tropical geometry ». In: *Compositio Mathematica* 152.1 (2016), pp. 115–151 (cit. on pp. 6, 9, 37, 81).

[Blo20] Thomas Blomme. « A Tropical Computation of Refined Toric Invariants II ». In: *preprint* (2020) (cit. on p. 11).

[BM15] Arne Buchholz and Hannah Markwig. « Tropical covers of curves and their moduli spaces ». In: *Communications in Contemporary Mathematics* 17.01 (2015), p. 1350045 (cit. on p. 25).

[Bou19] Pierrick Bousseau. « Tropical refined curve counting from higher genera and lambda classes ». In: *Inventiones mathematicae* 215.1 (2019), pp. 1–79 (cit. on pp. 6, 82).

[BP15] Erwan Brugallé and Nicolas Puignau. « On Welschinger invariants of symplectic 4-manifolds ». In: *Comment. Math. Helv.* 90.4 (2015), pp. 905–938 (cit. on p. 5).

[Bru] E Brugallé. « Real surgery of real symplectic manifolds and enumerative geometry ». In: *arXiv preprint arXiv:1601.05708* (), pp. 1–25 (cit. on p. 5).

[Bru+15] Erwan Brugallé et al. « Brief introduction to tropical geometry ». In: *Proceedings of the Gökova Geometry-Topology Conference 2014* (2015), pp. 1–75 (cit. on p. 19).

[Bru18] Erwan Brugallé. « On the invariance of Welschinger invariants ». In: *arXiv preprint arXiv:1811.06891* (2018) (cit. on p. 5).

[BS14] Erwan Brugallé and Kristin Shaw. « A bit of tropical geometry ». In: *The American Mathematical Monthly* 121.7 (2014), pp. 563–589 (cit. on pp. 19, 23, 42).

[BS19] Lev Blechman and Eugenii Shustin. « Refined descendant invariants of toric surfaces ». In: *Discrete & Computational Geometry* 62.1 (2019), pp. 180–208 (cit. on pp. 6, 81).

[CH98] Lucia Caporaso and Joe Harris. « Counting plane curves of any genus ». In: *Inventiones mathematicae* 131.2 (1998), pp. 435–392 (cit. on p. 2).

[Che18a] Xujia Chen. « Solomon's relations for Welshinger's invariants: examples ». In: *arXiv preprint math/1809.08938* (2018) (cit. on p. 5).

[Che18b] Xujia Chen. « Steenrod Pseudocycles, Lifted Cobordisms, and Solomon's Relations for Welschinger's Invariants ». In: *arXiv preprint arXiv:1809.08919* (2018) (cit. on p. 5).

[CJM10] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. « Tropical Hurwitz numbers ». In: *Journal of Algebraic Combinatorics* 32.2 (2010), pp. 241–265 (cit. on p. 25).

[FJ14] Jens Forsgård and Petter Johansson. « Coamoebas and line arrangements in dimension two ». In: *Mathematische Zeitschrift* 278.1-2 (2014), pp. 25–38 (cit. on p. 18).

[FJ15] Jens Forsgård and Petter Johansson. « On the order map for hypersurface coamoebas ». In: *Arkiv för Matematik* 53.1 (2015), pp. 79–104 (cit. on pp. 18, 50).

[FS15] Sara Angela Filippini and Jacopo Stoppa. « Block–Göttsche invariants from wall-crossing ». In: *Compositio Mathematica* 151.8 (2015), pp. 1543–1567 (cit. on pp. 7, 82).

[Ful93] William Fulton. *Introduction to toric varieties.* 131. Princeton University Press, 1993 (cit. on p. 15).

[GKM09] Andreas Gathmann, Michael Kerber, and Hannah Markwig. « Tropical fans and the moduli spaces of tropical curves ». In: *Compositio Mathematica* 145.1 (2009), pp. 173–195 (cit. on pp. 27, 29, 30).

[GM07a] Andreas Gathmann and Hannah Markwig. « The Caporaso–Harris formula and plane relative Gromov-Witten invariants in tropical geometry ». In: *Mathematische Annalen* 338.4 (2007), pp. 845–868 (cit. on pp. 5, 11, 30, 88, 94, 96).

[GM07b] Andreas Gathmann and Hannah Markwig. « The numbers of tropical plane curves through points in general position ». In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2007.602 (2007), pp. 155–177 (cit. on pp. 5, 85).

[GM08] Andreas Gathmann and Hannah Markwig. « Kontsevich's formula and the WDVV equations in tropical geometry ». In: *Advances in Mathematics* 217.2 (2008), pp. 537–560 (cit. on pp. 5, 30).

[GMS13] Andreas Gathmann, Hannah Markwig, and Franziska Schroeter. « Broccoli curves and the tropical invariance of Welschinger numbers ». In: *Advances in Mathematics* 240 (2013), pp. 520–574 (cit. on p. 5).

[GPS+10] Mark Gross, Rahul Pandharipande, Bernd Siebert, et al. « The tropical vertex ». In: *Duke Mathematical Journal* 153.2 (2010), pp. 297–362 (cit. on pp. 7, 82).

[GS14] Lothar Göttsche and Vivek Shende. « Refined curve counting on complex surfaces ». In: *Geometry & Topology* 18.4 (2014), pp. 2245–2307 (cit. on pp. 6, 75, 82).

[GS19] Lothar Göttsche and Franziska Schroeter. « Refined broccoli invariants ». In: *Journal of Algebraic Geometry* 28.1 (2019), pp. 1–41 (cit. on pp. 6, 75, 81).

[IKS03] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin. « Welschinger invariant and enumeration of real rational curves ». In: *International Mathematics research notices* 2003.49 (2003), pp. 2639–2653 (cit. on p. 5).

[IKS04] Ilia Itenberg, Viatcheslav Mikhailovich Kharlamov, and Evgenii Isaakovich Shustin. « Logarithmic equivalence of Welschinger and Gromov-Witten invariants ». In: *Russian Mathematical Surveys* 59.6 (2004), p. 1093 (cit. on p. 5).

[IKS09] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin. « A Caporaso-Harris type formula for Welschinger invariants of real toric del Pezzo surfaces ». In: *Commentarii Math. Helvetici* 84.1 (2009), pp. 87–126 (cit. on p. 5).

[IM13] Ilia Itenberg and Grigory Mikhalkin. « On Block–Göttsche multiplicities for planar tropical curves ». In: *International Mathematics Research Notices* 2013.23 (2013), pp. 5289–5320 (cit. on pp. 6, 81, 82, 86).

[Joh13] Petter Johansson. « The argument cycle and the coamoeba ». In: *Complex variables and elliptic equations* 58.3 (2013), pp. 373–384 (cit. on p. 18).

[KM94] Maxim Kontsevich and Yu Manin. « Gromov-Witten classes, quantum cohomology, and enumerative geometry ». In: *Communications in Mathematical Physics* 164.3 (1994), pp. 525–562 (cit. on p. 2).

[KS08] Maxim Kontsevich and Yan Soibelman. « Stability structures, motivic Donaldson-Thomas invariants and cluster transformations ». In: *arXiv preprint arXiv:0811.2435* (2008) (cit. on pp. 6, 82).

[Man15] Travis Mandel. « Scattering diagrams, theta functions, and refined tropical curve counts ». In: *arXiv preprint arXiv:1503.06183* (2015) (cit. on pp. 7, 82).

[Mik00] Grigory Mikhalkin. « Real algebraic curves, the moment map and amoebas ». In: *Annals of Mathematics* 151.1 (2000), pp. 309–326 (cit. on p. 114).

[Mik05] Grigory Mikhalkin. « Enumerative tropical algebraic geometry in $\mathbb{R}^2$ ». In: *Journal of the American Mathematical Society* 18.2 (2005), pp. 313–377 (cit. on pp. 2, 4–7, 24, 30, 39, 67, 77–79, 85, 99).

[Mik17] Grigory Mikhalkin. « Quantum indices and refined enumeration of real plane curves ». In: *Acta Mathematica* 219.1 (2017), pp. 135–180 (cit. on pp. 7–13, 38, 43–46, 49, 55, 66, 77, 78, 82, 99–101, 105, 112, 122, 124).

[MR01] Grigory Mikhalkin and Hans Rullgård. « Amoebas of maximal area ». In: *International mathematics research notices* 2001.9 (2001), pp. 441–451 (cit. on pp. 18, 112, 114).

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. Vol. 161. American Mathematical Soc., 2015 (cit. on p. 19).

[NPS18] Johannes Nicaise, Sam Payne, and Franziska Schroeter. « Tropical refined curve counting via motivic integration ». In: *Geometry & Topology* 22.6 (2018), pp. 3175–3234 (cit. on p. 6).

[NS06] Takeo Nishinou and Bernd Siebert. « Toric degenerations of toric varieties and tropical curves ». In: *Duke Mathematical Journal* 135.1 (2006), pp. 1–51 (cit. on pp. 3, 5, 30, 122).

[PR+04] Mikael Passare, Hans Rullgård, et al. « Amoebas, Monge-Ampare measures, and triangulations of the Newton polytope ». In: *Duke Mathematical Journal* 121.3 (2004), pp. 481–507 (cit. on p. 18).

[Shu06a] Eugenii Shustin. « A tropical approach to enumerative geometry ». In: *St. Petersburg Mathematical Journal* 17.2 (2006), pp. 343–375 (cit. on pp. 3, 5, 67).

[Shu06b] Eugenii Shustin. « A tropical calculation of the Welschinger invariants of real toric del Pezzo surfaces ». In: *J. Algebraic Geom. 15* 2 (2006), pp. 285–322 (cit. on pp. 4, 5, 25, 67, 77, 78).

[Shu18] Eugenii Shustin. *On refined count of rational tropical curves.* 2018. arXiv: 1812.08038 [math.AG] (cit. on p. 115).

[SS04] David Speyer and Bernd Sturmfels. « The tropical grassmannian ». In: *Advances in Geometry* 4.3 (2004), pp. 389–411 (cit. on p. 27).

[SS18] Franziska Schroeter and Eugenii Shustin. « Refined elliptic tropical enumerative invariants ». In: *Israel Journal of Mathematics* 225.2 (2018), pp. 817–869 (cit. on pp. 6, 81).

[Tyo12] Ilya Tyomkin. « Tropical geometry and correspondence theorems via toric stacks ». In: *Mathematische Annalen* 353.3 (2012), pp. 945–995 (cit. on pp. 3, 39, 41, 77).

[Tyo17] Ilya Tyomkin. « Enumeration of rational curves with cross-ratio constraints ». In: *Advances in Mathematics* 305 (2017), pp. 1356–1383 (cit. on pp. 3, 43, 56, 60, 64, 122, 123).

[Vir84] Oleg Ya Viro. « Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7 ». In: *Topology.* Springer, 1984, pp. 187–200 (cit. on p. 3).

[Wel05a] Jean-Yves Welschinger. « Enumerative invariants of stongly semipositive real symplectic six-manifolds ». In: *arXiv preprint math/0509121* (2005) (cit. on p. 4).

[Wel05b] Jean-Yves Welschinger. « Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry ». In: *Inventiones mathematicae* 162.1 (2005), pp. 195–234 (cit. on pp. 3, 4).

Résumé — La géométrie tropicale a permis le calcul de nombreux invariants de géométrie complexe (invariants de Gromov-Witten), ainsi que réelle (invariants de Welschinger) à travers l'utilisation de théorèmes de *correspondance*. Ceux-ci mettent à jour des liens profonds entre la géométrie tropicale et la géométrie dite classique. La richesse des objets tropicaux alliée à leur simplicité apparente a également permis de proposer de nouveaux invariants, dits raffinés, dont les interprétations en géométrie classique restent à ce jour encore mystérieuses, bien que plusieurs conjectures, comme celle de Göttsche-Shende, laissent présager d'une connexion profonde avec certaines quantités géométriques classiques. Une de ces interprétations est proposée par Mikhalkin en 2015, à travers le comptage de courbes rationnelles réelles dans les surfaces toriques, en fonction de la valeur d'un *"indice quantique"*. Les courbes comptées sont astreintes à passer par certains points réels ou complexes conjugués situés sur les diviseurs toriques de la surface, et le résultat s'avère ne dépendre que du nombre de points complexes. Dans le cas où les points sont réels, Mikhalkin relie l'invariant classique ainsi obtenu aux invariants tropicaux raffinés. En donnant une manière de calculer l'indice quantique d'une courbe rationnelle quelconque, nous étendons ensuite cette relation entre invariants classiques et tropicaux dans le cas où certains des points de la configuration sont imaginaires purs, et fournissons une formule récursive qui permet un calcul effectif de ces invariants raffinés tropicaux. Enfin, on propose une généralisation des invariants raffinés au cas de variétés toriques de dimension arbitraire.

Mots clés : géométrie tropicale, invariants raffinés.

Abstract — Tropical geometry enabled the computation of numerous invariants in complex geometry (Gromov-Witten invariants), as well as in real geometry (Welschinger invariants) using *correspondence* theorems. These theorems reveal a deep connection between tropical geometry and classical geometry. The richness of tropical objects coupled with their simplicity of use also enabled the definition of tropical refined invariants, whose interpretation on the classical geometry side remains quite mysterious, although several conjectures, *e.g.* the Göttsche-Shende conjecture, suggest an even deeper connection to other classical geometric quantities. One such interpretation is proposed by Mikhalkin in 2015, through the counting of real rational curves in toric surfaces, according to the value of a so-called *"quantum index"*. The refined count of curves, which have to pass through some real and complex conjugated points chosen on the toric boundary of the surface, happens to depend only on the number of complex points on each divisor. In the case where all the chosen points are real, Mikhalkin related the obtained invariant to tropical refined invariants. After giving a way of computing the quantum index of rational curves, we extend this relation between classical and tropical invariants in the case where some of the points of the configuration are purely imaginary, and we give a recursive formula that allows one to compute the involved tropical refined invariants. Finally, we propose a generalization of these refined tropical invariants in toric varieties of higher dimension.

Keywords: tropical geometry, refined invariants.