Thick points of random walk and multiplicative chaos

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted before for any degree or other qualification except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the Mathematics Degree Committee.

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Thick points of random walk and multiplicative chaos

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Abstract

This thesis concentrates on the classical problem of understanding the multifractal properties of Brownian motion and random walk. Specifically, we will be interested in the set of thick points, that is points where the trajectory goes back unusually often. The study of such points was initiated sixty years ago by Erdős and Taylor and has attracted a lot of attention, but we are able to make considerable progress on this topic by establishing a deep connection with an *a priori* unrelated area of Probability theory, called Gaussian multiplicative chaos.

Firstly, in two dimensions, we answer a question of [DPRZ01] and compute the number of thick points of planar random walk, assuming that the increments are symmetric and have a finite moment of order two. The proof provides a streamlined argument based on the connection to the Gaussian free field and works in a very general setting including isoradial graphs. In higher dimensions, we study the scaling limit of the set of thick points. In particular, we show that the rescaled number of thick points converges to a nondegenerate random variable and that the centred maximum of the local times converges to a randomly shifted Gumbel distribution.

Next, we construct the analogue of Gaussian multiplicative chaos measures for the local times of planar Brownian motion by exponentiating the square root of the local times of small circles. We also consider a flat measure supported on points whose local time is within a constant of the desired thickness level and show a simple relation between the two objects. Our results extend those of [BBK94] and in particular cover the entire L^1-phase or subcritical regime. These results allow us to obtain a nondegenerate limit for the appropriately rescaled size of thick points, thereby considerably refining estimates of [DPRZ01].

Finally, we characterise the multiplicative chaos measure $\mathcal{M}$ associated to planar Brownian motion by showing that it is the only random Borel measure satisfying a list of natural properties. These properties only serve to fix the average value of the measure and to express a spatial Markov property. As a consequence of our characterisation, we establish the scaling limit of the set of thick points of planar simple random walk, stopped at the first exit time of a domain, by showing the weak convergence towards $\mathcal{M}$ of the point measure associated to the thick points. As a corollary, we obtain the convergence of

the appropriately normalised number of thick points of random walk to a nondegenerate random variable.

Contents

Chapter 1

Introduction

A first rigorous mathematical definition of Brownian motion was given by Wiener [Wie23] in 1923. A century later, Brownian motion and its discrete counterpart, random walk, are one of the most central objects in Probability Theory. This thesis proposes to pursue the study, initiated by Erdős and Taylor in 1960 [ET60], of points that have been visited unusually often by Brownian motion in the critical dimension two. We make considerable progress on this old topic by establishing a deep connection with an *a priori* unrelated area of Probability, called Gaussian multiplicative chaos theory (GMC). This theory was introduced by Kahane [Kah85] in the eighties but regained a lot of importance in the mathematical community in the last two decades, playing a central role in the probabilistic formulation of Liouville conformal field theory and showing up in different branches of mathematics such as the study of the Riemann zeta function on the critical line and the study of large random matrices.

This introduction will start by presenting our main object of focus in the Brownian realm, that is the set of thick points, and the associated results available in the literature. We will then move on to a section that explains why a potential connection between Brownian motion and GMC might exist. This section is a cornerstone of this thesis. It guides the intuition that the local times of Brownian motion behave like the square of a logarithmically-correlated Gaussian field. Such fields have been extensively studied in the past twenty years and the next part of the introduction recalls some of their properties with an emphasis on the similarities with the Brownian results. We will finish this introduction by giving slightly more context and presenting some further results that are not contained in the current thesis.

1.1 Thick points of planar Brownian motion

1.1.1 Multiple points and intersection local times

It is a classical result from Dvoretzky, Erdős and Kakutani [DEK54] that planar Brownian motion possesses points of any given multiplicity p, i.e. points that have been visited at least p times. This makes the two-dimensional case special since in dimension three Brownian motion has double points but no triple points and in higher dimensions Brownian motion is a simple curve. For this reason, the current thesis will be mainly focused on the 2D case. Taylor [Tay66] continued the study of multiple points of planar Brownian motion and showed that for any p, the set D_p of p-multiple points has Hausdorff dimension equal to 2 a.s. Quoting Le Gall [LG87a], note that "the dimension of D_p is the same for all integers p's, while it is intuitively clear that there are much more points of multiplicity p than points of multiplicity $p+1$." In a series of papers [LG87a, LG87b, LG89], Le Gall gave a rigorous result confirming this intuition, by showing that the Hausdorff measure m_p associated to the gauge function $h_p(x) = x^2 \left(\log \frac{1}{x} \log \log \log \frac{1}{x} \right)^p$ and restricted to the set D_p can be written as a sum of finite and positive measures (see [LG87a]). Moreover, the measure $m_p(\cdot \cap D_p)$ gives a natural (infinite) measure which is supported on the points of multiplicity p. In the following paragraph, we describe another (which turns out to be the same) natural measure supported on D_p: the so-called **intersection local time**.

Denote $(B_t)_{0 \leq t \leq 1}$ a planar standard Brownian motion and let I_p be the set

$$I_p := \{ (t_1, \dots, t_p) \in [0,1]^p : 0 \leq t_1 < \dots < t_p \leq 1 \}.$$

The intersection local time α_p of B with itself is a measure on I_p formally given by

$$\alpha_p(dt_1, \dots, dt_p) = \mathbf{1}_{\left\{ B_{t_1} - B_{t_2} = 0 \right\}} \dots \mathbf{1}_{\left\{ B_{t_{p-1}} - B_{t_p} = 0 \right\}} dt_1 \dots dt_p.$$

It can be rigorously defined via an approximation procedure: one smooths out the indicator functions and then renormalises; see [LG85]. As suggests the formal definition, α_p is supported on $\{ (t_1, \dots, t_p) \in I_p : B_{t_1} = \dots = B_{t_p} \}$. Therefore, one obtains a measure supported on the set D_p by pushing forward α_p with the mapping $(t_1, \dots, t_p) \mapsto B_{t_1}$. [LG89] shows that this measure coincides (up to a multiplicative constant) with the Hausdorff measure $m_p(\cdot \cap D_p)$ from the previous paragraph.

1.1.2 Thick points

An analogous study for points having been visited unusually and infinitely many often was initiated by Bass, Burdzy and Khoshnevisan [BBK94] in 1994. These points are now known in the literature as **thick points**. The trajectory they consider is a Brownian path starting at the origin and killed upon hitting for the first time the unit circle. To quantify how often a point has been visited, [BBK94] considers the number $N_{x,\varepsilon}$ of excursions from x to the circle centred at x with radius ε. For any thickness parameter $a > 0$ (replacing the multiplicity p), define the set of a-thick points by

$$A_a := \left\{ x \in \mathbb{D} : \lim_{\varepsilon \to 0} \frac{N_{x,\varepsilon}}{|\log \varepsilon|} = a \right\}. \tag{1.1}$$

The main result of [BBK94] is:

1. For any $a \in (0, 1/2)$, a.s. there exists a measure β_a carried by A_a whose carrying Hausdorff dimension equals $2 - a$;[1]

2. For any $a > 0$, the Hausdorff dimension of A_a is at most $2 - a/e$ a.s.

The measure β_a can be thought of as the thick point analogue of the intersection local time. The restriction $a \in (0, 1/2)$ is a limitation of their proof and the goal of Chapter 3 is to extend the construction of this measure to the whole subcritical regime $a \in (0, 2)$. This construction is heavily inspired by **Gaussian multiplicative chaos** (GMC) theory. In the GMC slang, $a \in (0, 1/2)$ is a strict subset of the L^2-phase which corresponds to $a \in (0, 1)$. We will see that going beyond the L^2-phase leads to major difficulties that are at the heart of the Erdős–Taylor conjecture that we present in the next section.

We conclude this section by mentioning that the following lower bound is a consequence of Chapter 3 (see also [AHS20]): for all $a \in (0, 2)$, $\dim(A_a) \geq 2 - a$ a.s. To the best of our knowledge, the complementary upper bound is still open and has not been improved since [BBK94]. In a landmark paper, Dembo, Peres, Rosen and Zeitouni [DPRZ01] introduce a different notion of thick point for which they settle the analogous question. These thick points are this time defined using the occupation measure of small discs: for all $x \in \mathbb{D}$ and $\varepsilon > 0$, define

$$\mu_{\mathrm{occ}}(D(x, \varepsilon)) := \int_0^\tau \mathbf{1}_{\{|B_t - x| < \varepsilon\}} dt,$$

where τ is the first hitting time of the unit circle. [DPRZ01] shows that

$$\lim_{\varepsilon \to 0} \frac{\sup_{x \in \mathbb{D}} \mu_{\mathrm{occ}}(D(x, \varepsilon))}{\varepsilon^2 (\log \varepsilon)^2} = 2 \qquad \text{a.s.} \tag{1.2}$$

which leads to the natural notion of thick points:

$$\mathcal{T}_a := \left\{ x \in \mathbb{D} : \lim_{\varepsilon \to 0} \frac{\mu_{\mathrm{occ}}(D(x, \varepsilon))}{\varepsilon^2 (\log \varepsilon)^2} = a \right\}, \qquad a \in (0, 2). \tag{1.3}$$

[1] The carrying Hausdorff dimension of a measure μ is given by the infimum of $d > 0$ such that there exists a Borel set A with Hausdorff dimension d which carries μ, in the sense that $\mu(A^c) = 0$.

They establish that, for all $a \in (0,2)$, the Hausdorff dimension of $\mathcal{T}_a$ is equal to $2-a$ a.s. This paper had very strong consequences on a long-standing conjecture by Erdős and Taylor that we present now.

1.1.3 Erdős–Taylor conjecture

The analogous study of points visited unusually often by planar random walk can be tracked back at least to the famous paper [ET60] by Erdős and Taylor in 1960. In particular, they asked: "How many times does a planar simple random walk revisit the most visited site upon the first n steps?". This simple question was settled forty years later in the aforementioned article [DPRZ01] and is central for this thesis.

Consider a (discrete-time) planar simple random walk $(S_n)_{n\geq 0}$ on the square lattice $\mathbb{Z}^2$ which starts at the origin, and for any $x \in \mathbb{Z}^2$ and $n \geq 0$, let

$$\ell_x^n := \sum_{k=0}^{n} \mathbf{1}_{\{X_k = x\}}$$

be the total amount of time accumulated at the vertex x in the first n steps. ℓ_x^n is called the **local time** at x up to time n. We will kill the walk at the first exit time τ_N of the large square $V_N := \{-N, \dots, N\}^2$. For orientation, recall that most of the points of V_N will not have been visited by the walk before time τ_N and that, conditioned on the fact that a given point has been visited, its local time up to time τ_N is of order $\log N$. Erdős and Taylor [ET60] were interested in exceptional points where the local time is atypically large. They proved that

$$\frac{1}{\pi} \leq \liminf_{N\to\infty} \frac{\sup_{x\in\mathbb{Z}^2} \ell_x^{\tau_N}}{(\log N)^2} \leq \limsup_{N\to\infty} \frac{\sup_{x\in\mathbb{Z}^2} \ell_x^{\tau_N}}{(\log N)^2} \leq \frac{4}{\pi} \qquad \text{a.s.}$$

and conjectured that the upper bound is sharp. Getting rid of the factor of 4 between the lower and upper bounds leads to major complications and, as we will see later, is closely related to being able to cover the whole range $a \in (0,2)$ in [BBK94]'s theorem mentioned above, as well as being able to go beyond the so-called L^2-phase in Gaussian multiplicative chaos theory. This was achieved forty years later by Dembo, Peres, Rosen and Zeitouni [DPRZ01]. They also considered atypical points, which are the discrete analogue of the two notions of thick points defined in (1.1) and (1.3), where the local time is at least a fraction of the asymptotic maximum. For any parameter $a \in (0,2)$, define

$$\mathcal{T}_N(a) := \left\{ x \in V_N : \ell_x^{\tau_N} \geq \frac{2}{\pi} a (\log N)^2 \right\}. \tag{1.4}$$

[DPRZ01] established that $\mathcal{T}_N(a)$ asymptotically contains $N^{2-a+o(1)}$ points. In words, this shows that the "fractal dimension" of the set of a-thick points is equal to $2 - a$.

The proof of the Erdős-Taylor conjecture proposed by [DPRZ01] proceeds as follows. They first show the analogous result for Brownian motion (see (1.2)) and then, they transfer the result to the random walk setting via strong approximations (KMT-type couplings). This strategy has the drawback that it only applies to random walk with symmetric increments that have all moments finite (otherwise the coupling is not strong enough). Rosen [Ros05] and later Bass and Rosen [BR07] gave a proof based directly on random walk computations. As a result, a broader type of random walks could be treated as well, under a non-optimal assumption. One of the goals of Chapter 2 is to provide a very different approach to this question. The proof will be considerably streamlined and, as a result, solves the Erdős-Taylor conjecture under the optimal assumption that the increments have finite variance, solving a conjecture made by [DPRZ01]. The proof relies on a strong connection to a Gaussian field, called **Gaussian free field**. This connection is an instance of **isomorphism theorems** that we present in the next section. Finding a way to use such an isomorphism in our context was far from obvious and, in fact, Dembo raised this question in his Saint-Flour lecture notes [Dem05, Open problem 11].

1.2 Relation to the Gaussian free field

Isomorphism theorems form a class of results, originating with Dynkin [Dyn84b, Dyn84a] and Brydges, Fröhlich and Spencer [BFS82], relating local times of symmetric Markov processes and Gaussian fields. These isomorphism theorems regained a lot of interest in the past two decades because of their numerous applications in the study of cover time of random walk [DLP12, Din14, Din12, AB19, CLS18], and because of new isomorphisms for non-symmetric Markov processes and their links with loop soups [EK09, FR14, LJ10, LJ11, LJMR15, LJMR17, Lup20].

The isomorphism that we are about to state is due to Eisenbaum [Eis95] (which can be found in English in [MR06, Chapter 8]). For simplicity, consider the special case of a continuous-time simple random walk $(Y_t)_{t \geq 0}$ on the 2D square lattice $\mathbb{Z}^2$ which starts at the origin and which is killed at the first time τ_N that it exists $V_N = \{-N, \ldots, N\}^2$. Let us emphasise that this isomorphism is not restricted to this setting and basically covers any symmetric transient Markov process. The local times in our context are defined by

$$\ell_x^{\tau_N} := \int_0^{\tau_N} \mathbf{1}_{\{Y_t = x\}} dt, \qquad x \in V_N.$$

They will be related to the so-called discrete Gaussian free field (GFF) ϕ_N that is defined as follows. It is a centred Gaussian field $(\phi_N(x))_{x \in V_N}$ indexed by vertices of V_N whose covariance is given by the Green function

$$\mathbb{E}\left[\phi_N(x)\phi_N(y)\right] = \mathbb{E}_x\left[\ell_y^{\tau_N}\right], \tag{1.5}$$

where $\mathbb{E}_x$ denotes that the walk starts at x. As a side remark, we can see here that the symmetry of the Markov process is essential to ensure that the Green function is symmetric. Eisenbaum's isomorphism states that for any $s > 0$ and any measurable bounded function $f : \mathbb{R}^{V_N} \to \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}\left[f\left\{\left(\ell_x^{\tau_N} + \frac{1}{2}(\phi_N(x)+s)^2\right)_{x\in V_N}\right\}\right] \\ &= \mathbb{E}\left[\left(1 + \frac{\phi_N(0)}{s}\right) f\left\{\left(\frac{1}{2}(\phi_N(x)+s)^2\right)_{x\in V_N}\right\}\right]. \end{aligned}$$

On the left hand side of the above identity, the GFF ϕ_N and the walk Y are taken to be independent. The special vertex 0 appears on the right hand side because it is the starting point of the walk. $s > 0$ is usually considered as a fixed parameter. It does not have any specific interpretation, but it does in the case of other isomorphisms. For instance, in the second generalised Ray-Knight theorem (isomorphism due to [EKM+00]), the analogue of the parameter s corresponds to the total amount of time spent by the walk at the boundary point. In Section 1.5, we will present another isomorphism due to Le Jan that links the GFF to the so-called Brownian loop soup.

The interest of these isomorphisms is that it allows one to transfer computations in the random walk realm to Gaussian computations and vice versa. This will be used extensively in Chapter 2. Moreover, and maybe even more importantly, it guides the intuition. In Chapters 3 and 4, we will not be able to directly use any isomorphism, but our analysis will be heavily inspired by the analogy that the local time behaves roughly like half of the GFF squared.

1.3 Gaussian free field and Liouville measure

In view of the isomorphism theorems, points where the local time of random walk is atypically large can be thought of as being analogous to points where the Gaussian free field is atypically large. These latter points have been extensively studied in the last two decades and we present some aspects of this theory. We will start by introducing the continuum Gaussian free field. Because of the logarithmic blow-up of the correlations, its rigorous definition requires some care. We will then define the Liouville measure. This is a special instance of Gaussian multiplicative chaos measure and will be supported on thick points of the GFF. We will finally come back to the discrete setting and explain that

Liouville measure encodes the scaling limit of the set of thick points of the discrete GFF.

1.3.1 Gaussian free field - continuum

The GFF is of great importance and can be seen as being the analogue of Brownian bridge where the time interval has been replaced by a two-dimensional domain. It pops up in many different contexts. For instance, it arises as a universal scaling limit of a wide range of models such as the height function of dimer models [Ken01], the characteristic polynomial of large random matrices [HKO01, RV07, FKS16] and the Ginzburg–Landau model [Mil11, NS97] (see the review [Pow20] for more references). It also plays a central role in the mathematical construction of Liouville Conformal Field Theory (see the lecture notes [Var17] and references therein). This ubiquity can be explained by the fact that it is the only random generalised function in 2D which is both conformally invariant and satisfies a certain spatial Markov property [BPR20, BPR21].

Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain and for any $t > 0$ and $x, y \in D$, let $p_t^D(x,y)$ be the transition probability of Brownian motion killed at the boundary of D. This transition probability can be expressed as $p_t^D(x,y) = p_t(x,y)\pi_t^D(x,y)$ where $p_t(x,y) = 1/(2\pi t)\exp\left(-|x-y|^2/(2t)\right)$ is the heat kernel and $\pi_t^D(x,y)$ is the probability for a Brownian bridge from x to y of duration t to stay in D. We then define the Green function G_D with zero-boundary condition

$$G_D(x,y) = \pi \int_0^\infty p_t^D(x,y) dt, \qquad x, y \in D. \tag{1.6}$$

The Gaussian free field ϕ in D with zero-boundary conditions is formally defined as being a random Gaussian process indexed by points of D with covariance given by the Green function G_D. Because of the logarithmic blow-up of the Green function on the diagonal

$$G_D(x,y) \sim -\log|x-y| \quad \text{as} \quad |x-y| \to 0, \tag{1.7}$$

the GFF is actually not well-defined pointwise. Instead, it is a generalised function (in the sense of Schwartz) that satisfies: for any test functions f and g,

$$\mathbb{E}\left[\langle \phi, f\rangle \langle \phi, g\rangle\right] = \int_{D\times D} f(x) G_D(x,y) g(y) dx dy.$$

The GFF ϕ can be defined via its Karhunen–Loève expansion:

$$\phi := \sum_{n\geq 1} X_n f_n, \tag{1.8}$$

where $X_n, n \geq 1$, are i.i.d. standard Gaussian random variables and $\{f_n, n \geq 1\}$ is an

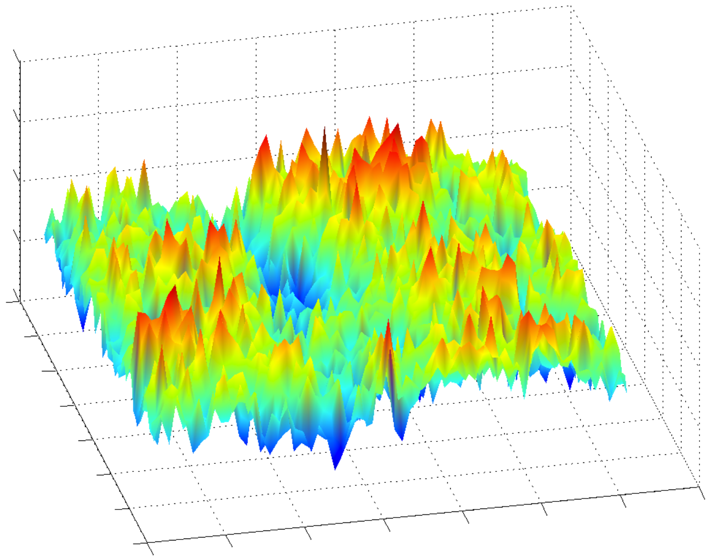


Figure 1.1: Simulation of 2D Gaussian free field made by R. Rhodes and V. Vargas.

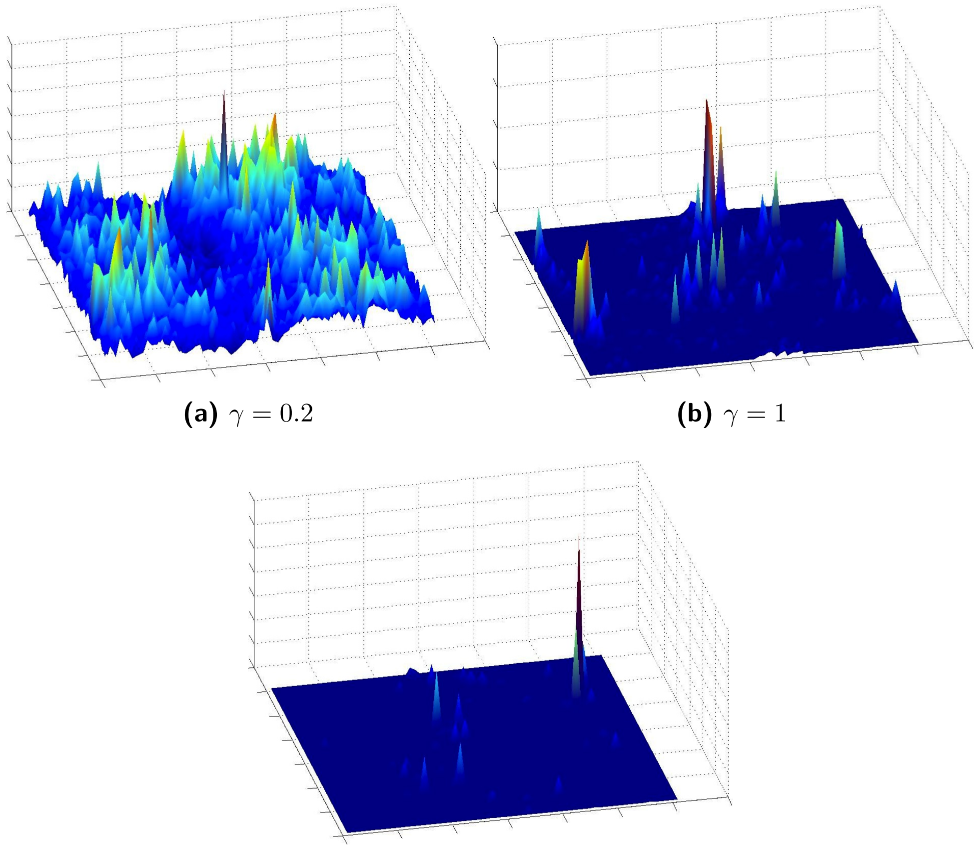
orthonormal basis of the Sobolev space H_0^1. It can be shown that, for any $\varepsilon > 0$, the series (1.8) converges a.s. in the Sobolev space $H^{-\varepsilon}$. The GFF is then defined as being the limiting random generalised function. Comprehensive introductions to the GFF can be found in [Ber16, WP20, BP21]. See Figure 1.1 for a simulation.

1.3.2 Liouville measure

Liouville measure is part of Gaussian multiplicative chaos theory which was introduced by Kahane [Kah85] and which builds and studies random measures formally defined by exponentiating γ times a log-correlated Gaussian field, such as the 2D GFF. γ is a real parameter that will be related to the thickness parameter a from Section 1.1. Exponentiating a generalised function is not a well-defined operation and defining such a measure requires some non-trivial justification. After numerous works in the last decade [RV10, DS11, RV11, Sha16, Ber17], this is by now well-understood. When the underlying log-correlated is the GFF, the associated GMC measure is very special and is called Liouville measure. This measure is deeply related to random conformal geometry. A slight variant of Liouville measure can be thought of as being the volume form of a surface chosen randomly in some canonical way (see e.g. [Gwy20] and references therein). See [Ber16, BP21] for an introduction to Liouville measure.

We now explain two natural approaches to define Liouville measure.

Convolution approximation Let $\theta : \mathbb{R}^2 \to [0, \infty)$ be a smooth mollifier such that $\int \theta = 1$. For any $\varepsilon > 0$, let $\theta_\varepsilon := \varepsilon^{-2}\theta(\cdot/\varepsilon)$ and let $\phi_\varepsilon = \phi * \theta_\varepsilon$ be an ε-approximation of



(a) $\gamma = 0.2$

(b) $\gamma = 1$

(c) $\gamma = 1.8$

Figure 1.2: Simulation of Liouville measure made by R. Rhodes and V. Vargas.

the GFF ϕ. We can now define the approximation version of the measure by

$$\mu_\gamma^\varepsilon(dx) := \varepsilon^{\gamma^2/2} e^{\gamma \phi_\varepsilon(x)} dx,$$

where $\gamma \in (0,2)$ is a parameter. Notice the normalising constant $\varepsilon^{\gamma^2/2}$ that has been chosen to exactly compensate the blow-up of the first moment. The difficult task consists in showing that μ_ε possesses a limit as $\varepsilon \to 0$. Importantly, the limiting measure is universal in the sense that it does not depend on the approximation procedure (two different mollifiers θ and θ' will lead to the same measure up to a multiplicative constant). The GMC measure μ_γ associated to the GFF is then defined to be the limiting measure. See Figure 1.2 for a simulation.

Martingale approximation Another natural approach consists in using the expansion (1.8) of the GFF and considering the approximation

$$\mu_\gamma^N(dx) := \exp\left(\gamma \sum_{n=1}^N X_n f_n(x) - \frac{\gamma^2}{2} \sum_{n=1}^N f_n(x)^2 \right) dx. \tag{1.9}$$

This approximation is particularly nice since for any Borel set $A \subset D$, $(\mu_\gamma^N(A), N \geq 1)$ is a positive martingale. In particular, we directly know that it converges almost surely as $N \to \infty$. This is true for any value of γ. What is not trivial however is that the limiting measure is nondegenerate and does not depend on the way the field was approximated. This was the approach used by Kahane [Kah85]. In particular, he showed that the regime $\gamma \in (0, 2)$ exactly corresponds to the regime where the limiting measure is not equal to zero. Moreover, we now know that the limiting measure is (up to an explicit density involving conformal radii) the measure μ_γ built using convolution approximations (see [Ber17] for instance).

1.3.3 GFF thick points

In Section 1.1, we explained that the intersection local time was the canonical measure supported on the set of multiple points of Brownian motion. Bass, Burdzy and Khoshnevisan [BBK94] initiated the construction of an analogous measure for thick points of Brownian motion. In this section, we will see that Liouville measure can be thought of as being the natural measure supported on the set of GFF thick points. We now precisely define these thick points. Because the GFF ϕ is not defined pointwise, we again need to use an approximation procedure to make sense of points where the field is atypically large. This can be done in many ways. One can again consider the convolution approximation used in Section 1.3.2 and define

$$\mathcal{T}_\gamma^{\mathrm{GFF}} := \left\{ x \in D : \lim_{\varepsilon \to 0} \frac{\phi_\varepsilon(x)}{|\log \varepsilon|} = \gamma \right\}, \quad \gamma > 0.$$

This does correspond to exceptional points since for a fixed deterministic point $x \in D$, $\phi_\varepsilon(x) = O(\sqrt{|\log \varepsilon|})$. Hu, Miller and Peres [HMP10] showed that for a slightly different notion of thick points, the Hausdorff dimension of $\mathcal{T}_\gamma^{\mathrm{GFF}}$ is equal to $2 - \gamma^2/2$ a.s. when $\gamma \in (0, 2]$ and that $\mathcal{T}_\gamma^{\mathrm{GFF}} = \varnothing$ a.s. when $\gamma > 2$. These results are closely related to the ones proven by [DPRZ01] (see (1.2)).

As mentioned earlier, Liouville measure is almost surely supported by $\mathcal{T}_\gamma^{\mathrm{GFF}}$, i.e. $\mu_\gamma(D \setminus \mathcal{T}_\gamma^{\mathrm{GFF}}) = 0$ a.s. Heuristically, this can be understood by the trade-off between, on the one hand, the fact that the larger $\phi_\varepsilon(x)$, the larger the contribution of x to μ_γ^ε and, on the other hand, the fact that the dimension of $\mathcal{T}_\gamma^{\mathrm{GFF}}$ decreases with γ.

In fact, Liouville measure is the natural measure which is supported on $\mathcal{T}_\gamma^{\mathrm{GFF}}$. This will become even more apparent in the discrete setting that we present now.

Thick points of discrete GFF The thick points of the GFF have also been extensively studied in the discrete setting. For concreteness, consider the discrete GFF $(\phi_N(x))_{x \in V_N}$

in the square $V_N = \{-N, \dots, N\}^2$ as defined in (1.5). The leading order of the maximum is known since [BDG01]. They showed that

$$\lim_{N\to\infty} \frac{\sup_{x\in V_N} \phi_N(x)}{\log N} = \frac{2\sqrt{2}}{\sqrt{\pi}} \qquad \text{in probability.}$$

Later, Daviaud [Dav06] showed that for any $\gamma \in (0,2)$, the set of γ-thick points

$$\mathcal{T}_N^{\mathrm{GFF}}(\gamma) := \left\{ x \in V_N : \phi_N(x) \geq \frac{\sqrt{2}}{\sqrt{\pi}} \gamma \log N \right\} \tag{1.10}$$

contains $N^{2-\gamma^2/2+o(1)}$ points, where $o(1) \to 0$ in probability. These results can be seen as being the GFF counterparts of the results proven by [DPRZ01] about thick points of random walk (see (1.4)). In particular, the number of thick points of $\ell_x^{\tau_N}$ and thick points of $\frac{1}{2}\phi_N(x)^2$ have the same asymptotics at the level of exponent. This simple observation is at the heart of Chapter 2.

Thanks to considerable progress in the study of logarithmically-correlated fields and Gaussian multiplicative chaos theory, Biskup and Louidor [BL19] were able to greatly refine Daviaud's estimate by establishing the scaling limit of the set (1.10) of thick points. They encode the set $\mathcal{T}_N^{\mathrm{GFF}}(\gamma)$ in the following point measure

$$\nu_\gamma^N(A) := \frac{\sqrt{\log N}}{N^{2-\gamma^2/2}} \sum_{x \in \mathcal{T}_N^{\mathrm{GFF}}(\gamma)} \mathbf{1}_{\{x/N \in A\}}, \qquad A \subset \mathbb{R}^2 \text{ Borel set}$$

and show that the sequence of random measures $(\nu_\gamma^N, N \geq 1)$ converges in distribution for the topology of weak convergence towards a multiple of Liouville measure μ_γ (the underlying domain in the continuum being the square $(-1,1)^2$). As a consequence,

$$\frac{\sqrt{\log N}}{N^{2-\gamma^2/2}} \# \mathcal{T}_N^{\mathrm{GFF}}(\gamma)$$

converges in distribution towards the total mass of μ_γ. The ultimate goal of this thesis is to establish an analogous result concerning thick points of random walk (1.4) (see Theorem 4.1).

1.4 Outline of the thesis and main results

With all the main characters introduced, we can now describe the main goals of this thesis. Chapters 2 and 3 are each based on a published article: [Jeg20b] and [Jeg20a], respectively. Chapter 4 is based on the paper [Jeg19] that has been submitted.

- Chapter 2: The first purpose of this chapter is to give a streamlined proof of the Erdős-Taylor conjecture that was first settled by [DPRZ01] (see Section 1.1.3). The strategy is to use Eisenbaum's isomorphism (see Section 1.2) to transfer the computations to the Gaussian realm. As a result, this chapter can cover a very broad range of two-dimensional random walks solving a conjecture made in [DPRZ01].

  The second part of this chapter (Section 2.4) is focused on the analogous question in dimensions three and higher. We establish the scaling limit of the set of thick point, as well as the convergence of the properly centred maximum of thick points toward a randomly shifted Gumbel distribution.

- Chapter 3: This chapter aims at building the analogue of Gaussian multiplicative chaos measure in the Brownian setting. In view of the isomorphism theorems, we will make sense of the measure formally defined as the exponential of γ times the square root of the local times of planar Brownian motion. These results cover the entire L^1-phase and extend the construction of the measure β_a by [BBK94] to the range of parameters $a \in (0, 2)$. We call this measure Brownian multiplicative chaos.

- Chapter 4: The last chapter of this thesis shows that the Brownian multiplicative chaos measure constructed in Chapter 3 encodes the scaling limit of the set of planar random walk thick points, considerably refining the estimates given by [DPRZ01]. This convergence is based on a characterisation of the law of the Brownian chaos measure. We show that it is the only measure satisfying a list of natural properties which fix the average value and provide a spatial Markov property.

  A key new idea for this characterisation is to introduce measures describing the intersection between different independent Brownian trajectories and how they interact to create thick points.

We now state the main results of this thesis.

1.4.1 Chapter 2

Consider $Y_t = S_{N_t}, t \geq 0$, a continuous time random walk on $\mathbb{Z}^2$ starting at the origin where $S_n = \sum_{i=1}^n X_i, n \geq 0$, is the jump process with i.i.d. increments $X_i \in \mathbb{Z}^2$ and $(N_t)_{t \geq 0}$ is an independent Poisson process of parameter 1. As in Section 1.1.3, we let $V_N = \{-N, \dots, N\}^2$ be the square centred at the origin of side length $2N+1$, τ_N be the first exit time of V_N and $(\ell_x^t, x \in \mathbb{Z}^2, t \geq 0)$ be the local times defined by

$$\ell_x^t := \int_0^t \mathbf{1}_{\{Y_s = x\}} ds. \tag{1.11}$$

For any thickness parameter $0 \le a \le 1$, we call $\mathcal{M}_N(a)$ the set of a-thick points

$$\mathcal{M}_N(a) := \left\{ x \in V_N : \ell_x^{\tau_N} \ge \frac{2}{\pi\sqrt{\det \mathcal{G}}} a (\log N)^2 \right\}$$

where $\mathcal{G}$ is defined below. The main two-dimensional result of Chapter 2 reads as follows:

Theorem 1.1 (Theorem 2.1)**.** *Assume that the law of the increments is symmetric (i.e.* $-X \overset{d}{=} X$*), with a finite variance and denote* $\mathcal{G} = \mathbb{E}[XX']$ *the covariance matrix of the increments. Then we have the following two a.s. limits:*

$$\lim_{N\to\infty} \frac{\max_{x\in V_N} \ell_x^{\tau_N}}{(\log N)^2} = \frac{2}{\pi\sqrt{\det \mathcal{G}}} \text{ and } \forall a \in [0,1),\ \lim_{N\to\infty} \frac{\log |\mathcal{M}_N(a)|}{\log N} = 2(1-a).$$

The proof of this results relies on Eisenbaum's isomorphism and will in fact be a particular case of a much more general theorem (Theorem 2.14).

We now turn to the high-dimensional part of Chapter 2. In this case, we will establish much more precise convergence results and we will be able to cover the critical case where the local times are close to the maximum. To ease the exposition, this chapter focuses on the continuous time simple random walk $(Y_t)_{t\ge 0}$ on the square lattice $\mathbb{Z}^d$, with $d \ge 3$. We again kill the walk at the first exit time τ_N of $\{-N, \dots, N\}^d$ and we consider the local times $\ell_x^t, x \in \mathbb{Z}^d, t \ge 0$ defined as in (1.11). Let $g := \mathbb{E}_0[\ell_0^\infty]$.

We describe thick points through a more precise encoding by considering for $a \in [0,1]$ the point measure:

$$\nu_N^a := \frac{1}{N^{2(1-a)}} \sum_{x\in V_N} \delta_{\left(x/N, \ell_x^{\tau_N} - 2ga \log N\right)}.$$

Let us emphasise that the normalisation factor is equal to 1 when $a = 1$ and that ν_N^a is viewed as a random measure on $[-1,1]^d \times \mathbb{R}$. We compare the thick points of random walk with the thick points of i.i.d. exponential random variables with mean g located at each site visited by the walk. More precisely, we denote $\mathcal{M}_N(0) := \{x \in V_N : \ell_x^{\tau_N} > 0\}$ and taking $E_x, x \in \mathbb{Z}^d$, i.i.d. exponential variables with mean g independent of $\mathcal{M}_N(0)$, we define

$$\mu_N^a := \frac{1}{N^{2(1-a)}} \sum_{x\in \mathcal{M}_N(0)} \delta_{(x/N, E_x - 2ga \log N)}. \tag{1.12}$$

We finally denote by τ the first exit time of $[-1,1]^d$ of Brownian motion starting at the origin and by μ_{occ} the occupation measure of Brownian motion starting at the origin and killed at τ. Then we have:

Theorem 1.2 (Theorem 2.4)**.** *For all* $a \in [0,1]$ *there exists a random Borel measure* ν^a *on* $[-1,1]^d \times \mathbb{R}$ *such that, with respect to the topology of vague convergence of measures on*

$[-1,1]^d \times \mathbb{R}$ *(on* $[-1,1]^d \times (0,\infty)$ *if* $a=0$*), we have:*

$$\lim_{N\to\infty} \nu_N^a = \lim_{N\to\infty} \mu_N^a = \nu^a \text{ in law.}$$

Moreover, for all $a \in [0,1)$ *the distribution of* ν^a *does not depend on* a *and*

$$\nu^a(dx, d\ell) \overset{\text{(d)}}{=} \frac{1}{g}\mu_{\text{occ}}(dx) \otimes e^{-\ell/g}\frac{d\ell}{g}.$$

At criticality, ν^1 *is a Poisson point process:*

$$\nu^1 \overset{\text{(d)}}{=} \text{PPP}\left(\frac{1}{g}\mu_{\text{occ}}(dx) \otimes e^{-\ell/g}\frac{d\ell}{g}\right).$$

This theorem has two important consequences:

Theorem 1.3 (Theorem 2.5)**.** *If we define for every* $a \in [0,1]$ *the set of* a*-thick points:*

$$\mathcal{M}_N(a) := \left\{x \in V_N : \ell_x^{\tau_N} > 2ga\log N\right\},$$

then there exist random variables M_a *such that for all* $a \in [0,1]$

$$\frac{|\mathcal{M}_N(a)|}{N^{2(1-a)}} \xrightarrow[N\to\infty]{\text{(d)}} M_a.$$

Moreover, for all $a \in [0,1)$ *the distribution of* M_a *does not depend on* a *and*

$$M_a \overset{\text{(d)}}{=} \tau/g.$$

M_1 *is a Poisson variable with parameter* τ/g*: for all* $k \geq 0$

$$\mathbb{P}(M_1 = k) = \frac{1}{k!}\mathbb{E}\left[e^{-\frac{\tau}{g}}\left(\frac{\tau}{g}\right)^k\right].$$

Theorem 1.4 (Theorem 2.6)**.** *There exists an almost surely finite random variable* L *such that*

$$\sup_{x\in V_N} \ell_x^{\tau_N} - 2g\log N \xrightarrow[N\to\infty]{\text{(d)}} L.$$

Moreover, L *is a Gumbel variable with mode* $g\log(\tau/g)$ *(location of the maximum) and scale parameter* g*, i.e. for all* $t \in \mathbb{R}$

$$\mathbb{P}(L \leq t) = \mathbb{E}\left[\exp\left(-\frac{\tau}{g}e^{-t/g}\right)\right].$$

1.4.2 Chapter 3

This chapter initiates a Gaussian multiplicative chaos theory for planar Brownian motion. Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain and let $(B_t)_{0 \le t \le \tau}$ be a planar Brownian motion killed at the boundary of D. This chapter makes sense of a measure formally defined as $e^{\gamma\sqrt{L_x}}dx$ where $\gamma \in (0,2)$ is a parameter and L_x is the "local time at x". Exactly like in the case of the 2D Gaussian free field, L_x is not well-defined pointwise (for any fixed $x \in D$, the Brownian trajectory will almost surely avoid x) and a renormalisation procedure is needed. This chapter uses local times of small circles, formally defined as:

$$L_{x,\varepsilon}(\tau) = \int_0^\tau \mathbf{1}_{\{|B_t - x| = \varepsilon\}} dt, \quad x \in D, \varepsilon > 0.$$

The main reason for choosing this approximation is that these local times exhibit a certain Markovian structure (see (3.7)). This is very much analogous to the fact that the process of circle averages of the continuum GFF, when the centre is fixed and the radius varies, has the law of a 1D Brownian motion.

The local times of circles can be defined simultaneously for all $x \in D$ and $\varepsilon > 0$ (Proposition 3.5) and we can therefore define an approximating version of the Brownian chaos measure by setting for all Borel set $A \subset D$,

$$\mu_\varepsilon^\gamma(A) := \sqrt{|\log \varepsilon|}\varepsilon^{\gamma^2/2} \int_A e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} dx.$$

To study fine properties of thick points of Brownian motion, we introduce another measure which is to be compared with (1.12): for all Borel sets $A \subset D$ and $T \subset \mathbb{R}$, define

$$\nu_\varepsilon^\gamma(A \times T) := |\log \varepsilon| \, \varepsilon^{-\gamma^2/2} \int_A \mathbf{1}_{\left\{\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} - \gamma \log \frac{1}{\varepsilon} \in T\right\}} dx.$$

We will show that:

Theorem 1.5 (Theorem 3.1)**.** *For all $\gamma \in (0,2)$, the sequences of random measures ν_ε^γ and μ_ε^γ converge as $\varepsilon \to 0$ in probability for the topology of vague convergence on $D \times (\mathbb{R} \cup \{+\infty\})$ and on D respectively towards Borel measures ν^γ and μ^γ.*

The measure ν^γ can be decomposed as a product of a measure on D and a measure on $\mathbb{R}$. Moreover, the component on D agrees with μ^γ and the component on $\mathbb{R}$ is exponential:

Theorem 1.6 (Theorem 3.2)**.** *For all $\gamma \in (0,2)$, we have $\mathbb{P}_{x_0}$-a.s.,*

$$\nu^\gamma(dx, dt) = (2\pi)^{-1/2} \mu^\gamma(dx) e^{-\gamma t} dt.$$

These results are the two-dimensional analogues of Theorem 1.2 in the continuum.

Chapter 4 will provide the discrete analogue of this result, but, as we will see, the discrete setting leads to major difficulties. Indeed, the proof of the convergences stated in Theorem 1.5 are based on the L^1 convergence of the measures evaluated at some Borel sets. A very different approach is required in the discrete since two random walks defined on lattices with different mesh sizes are a priori not defined on the same probability space (coupling them via KMT-type approximations is too rough).

We have called the measure μ^γ Brownian multiplicative chaos measure. This measure has first been built by [BBK94] for a partial range of γ ($\gamma \in (0,1)$). The paper [Jeg20a], simultaneously with [AHS20], extended the construction to the whole subcritical regime $\gamma \in (0,2)$.

1.4.3 Chapter 4

Let $(Y_t)_{t\geq 0}$ be a continuous time simple random on the two-dimensional lattice $\mathbb{Z}^2$ starting at the origin. As in Section 1.4.1, we kill the walk at the first exit time τ_N of $\{-N,\dots,N\}^2$ and we denote ℓ_x^t, $x \in \mathbb{Z}^2, t \geq 0$, the local times of the walk. Recall the definition (1.4) of the set $\mathcal{T}_N(a)$ of a-thick points. We study this set via the following measure: for all Borel sets $A \subset \mathbb{C}$,

$$\mu_N^a(A) := \frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\left\{\ell_x^{\tau_N} \geq \frac{2}{\pi} a \log^2 N\right\}}.$$

The main result of Chapter 4 is:

Theorem 1.7 (Theorem 4.1)**.** *For all $a \in (0,2)$, the sequence $\mu_N^a, N \geq 1$, converges weakly for the topology of weak convergence on $\mathbb{R}^2$. Moreover, there exists a universal constant c_0 such that the limiting measure has the same distribution as $e^{c_0 a/g}\mu^\gamma$ from Theorem 1.5, where the underlying domain is the square $[-1,1]^2$ and the starting point is the origin.*

As already alluded to in Section 1.4.2, this result requires an approach very different to the one used in the continuum. Chapter 4 will establish a characterisation of the law of Brownian multiplicative chaos (Theorem 4.5).

1.5 Other works of the author (not part of this thesis)

In this section, we will mention two extra works that we wrote concerning the Brownian chaos. Even though they are not part of this thesis, we briefly outline them to give a bit more perspective to our results.

Criticality All the results mentioned so far (except the high-dimensional part of Chapter 2) concern the subcritical regime, meaning that the thickness parameter a and the GMC parameter γ were always assumed to be *strictly* smaller than 2. The critical case concerns the study of the most extreme points whose local times are close to the maximum. This regime is much more delicate to study, partly because certain observables are not in L^1 any more.

In the Gaussian setting, this is now well understood. In the continuum, critical Gaussian multiplicative chaos has been constructed and analysed [DRSV14b, DRSV14a, JS17, JSW19, Pow18]. In the discrete, a substantial amount of articles [BZ12, BDZ11, Din13, DZ14] were devoted to studying the subleading orders of the maximum of the discrete GFF. This series or papers culminates with the convergence in law of the recentred maximum [BDZ16] (see also [Mad15] for the case of log-correlated Gaussian fields in the continuum).

In the article [Jeg21] published in PTRF, we initiate the study of the critical case in the Brownian setting by constructing a critical version of Brownian multiplicative chaos. This paper provides the first instance of critical chaos associated to a non-Gaussian log-correlated field.

Connection to Liouville quantum gravity Brownian multiplicative chaos shares striking similarities with Liouville measure. For instance, the formulas for their first moment have very similar flavours and they are both conformally covariant. On the other hand, they are far from being equal since the Brownian chaos measure is supported on the trace of a Brownian trajectory, whereas Liouville measure sees the whole domain. In a joint work [ABJL21] with É. Aïdékon, N. Berestycki and T. Lupu, we establish a concrete link between these two measures. The starting point of this work is Le Jan's isomorphism that relates the (normalised) occupation measure of critical Brownian loop soup to the (Wick) square of the GFF [LJ11].

The Brownian loop soup was introduced by Lawler and Werner [LW04] and has become a central object in planar random conformal geometry. It is an infinite collection of Brownian-like loops in a given domain; see Figure 1.3 for a simulation. It is sampled according to a Poisson point process with intensity equal to a real parameter θ times a loop measure. Sheffield and Werner [SW12] showed that, when $\theta \leq 1/2$, there are infinitely many clusters of loops, where by clusters we mean sets of loops that can be joined by a path of intersecting loops. In that case the boundaries of these clusters are closely related to Conformal Loop Ensemble (CLE). When $\theta > 1/2$, there is only one "large" cluster. We therefore see that $\theta = 1/2$ is special, and indeed, at this particular intensity, the occupation measure of Brownian loop soup is equal in distribution to half of the GFF squared [LJ11]. Making sense of such an isomorphism in the continuum requires some care (the square of a generalised function is a priori not well-defined) and a renormalisation is needed on both

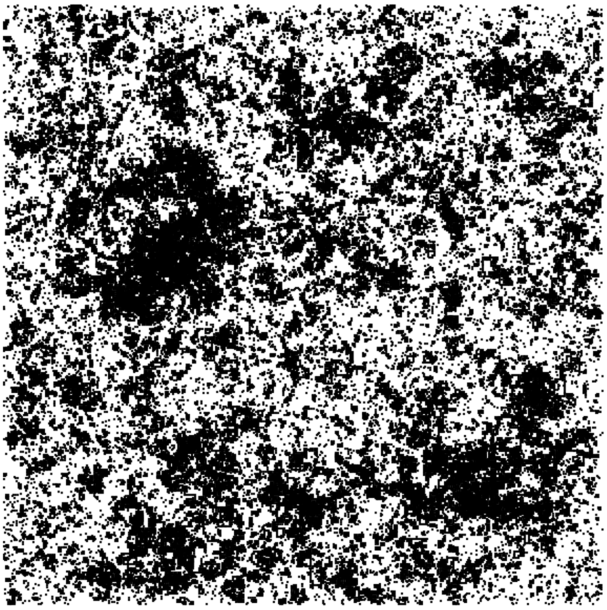


Figure 1.3: Simulation of Brownian loop soup made by S. Nacu and W. Werner.

sides of this equality. To summarise our work [ABJL21] in one sentence, we show that Brownian chaos provides the contribution of one loop to Liouville measure in Le Jan's coupling.

Chapter 2

Thick points of random walk and the Gaussian free field

We consider the thick points of random walk, i.e. points where the local time is a fraction of the maximum. In two dimensions, we answer a question of [DPRZ01] and compute the number of thick points of planar random walk, assuming that the increments are symmetric and have a finite moment of order two. The proof provides a streamlined argument based on the connection to the Gaussian free field and works in a very general setting including isoradial graphs. In higher dimensions, we study the scaling limit of the set of thick points. In particular, we show that the rescaled number of thick points converges to a nondegenerate random variable and that the centred maximum of the local times converges to a randomly shifted Gumbel distribution.

2.1 Results

For $d \geq 2$, consider a continuous time simple random walk $(Y_t)_{t \geq 0}$ on $\mathbb{Z}^d$ with rate 1. Let us denote $\mathbb{P}_x$ the law of $(Y_t)_{t \geq 0}$ starting from x and $\mathbb{E}_x$ the associated expectation. Defining $V_N = \{-N, \dots, N\}^d$, we denote τ_N the first exit time of V_N and $\left(\ell_x^t, x \in \mathbb{Z}^d, t \geq 0\right)$ the local times defined by:

$$\tau_N := \inf \{t \geq 0, Y_t \notin V_N\} \text{ and } \forall x \in V_N, \forall t \geq 0, \ell_x^t := \int_0^t \mathbf{1}_{\{Y_s = x\}} ds. \tag{2.1}$$

In 1960, Erdős and Taylor [ET60] studied the behaviour of the local time of the most frequently visited site. By translating their work to our context of continuous time random walk, they proved that

$$\begin{aligned}
&\text{if } d=2,\ \frac{1}{\pi} \leq \liminf_{N\to\infty} \frac{\sup_{x\in V_N} \ell_x^{\tau_N}}{(\log N)^2} \leq \limsup_{N\to\infty} \frac{\sup_{x\in V_N} \ell_x^{\tau_N}}{(\log N)^2} \leq \frac{4}{\pi} \qquad \mathbb{P}_0\text{−a.s.},\\
&\text{if } d\geq 3,\ \lim_{N\to\infty} \frac{\sup_{x\in V_N} \ell_x^{\tau_N}}{\log N} = 2\mathbb{E}_0\left[\ell_0^\infty\right] \qquad \mathbb{P}_0\text{−a.s.}
\end{aligned} \tag{2.2}$$

and conjectured that the limit also exists in dimension two and is equal to the upper bound. This conjecture was proved forty years later in a landmark paper [DPRZ01]. Estimates on the number of thick points, which are the points where the local times are larger than a fraction of the maximum, are also given in this paper. Briefly, their proof establishes the analogous results for the thick points of occupation measure of planar Brownian motion; taking in particular advantages of symmetries such as rotational invariance and certain exact computations on Brownian excursions. The discrete case is then deduced from the Brownian case through strong coupling/KMT arguments. This method requires all the moments of the increments to be bounded but the authors suspected that only finite second moments are needed. Later, the article [Ros05] showed that the paper [DPRZ01] can be entirely rewritten in terms of random walk giving a proof without using Brownian motion. The strategy of [Ros05] has then been refined in [BR07] to treat the case of random walks on $\mathbb{Z}^2$ with symmetric increments having finite moment of order $3+\varepsilon$. A crucial aspect of this latter article consists in controlling the jumps over discs. Such a control is achieved by developing Harnack inequalities requiring further assumptions on the walk (Condition A of [BR07]).

This paper has two purposes. Firstly, we exploit the links between the local times and the Gaussian free field (GFF) provided by Dynkin-type isomorphisms to give a simpler and more robust proof of the two-dimensional result. The proof works in a very general setting (Theorem 2.14). In particular, we weaken the assumptions of [BR07] answering the question of [DPRZ01] about walks with only finite second moments and we also treat the case of random walks on isoradial graphs. Secondly, we obtain more precise results in dimension $d \geq 3$. Namely, we show that the field $\{\ell_x^{\tau_N}, x \in V_N\}$ behaves like the field composed of i.i.d. exponential variables with mean $\mathbb{E}_0\left[\ell_0^\infty\right]$ located at each site visited by the walk. In particular, we show that the centred supremum of the local times as well as the rescaled number of thick points converge to nondegenerate random variables.

We first state two results for the planar case. Both are in fact corollaries of a more general theorem (Theorem 2.14) which will be stated later. We will then present the result in dimension $d \geq 3$.

2.1.1 Dimension two

Consider $Y_t = S_{N_t}, t \geq 0$, a continuous time random walk on $\mathbb{Z}^2$ starting from the origin where $S_n = \sum_{i=1}^n X_i, n \geq 0$, is the jump process with i.i.d. increments $X_i \in \mathbb{Z}^2$ and $(N_t)_{t\geq 0}$ is an independent Poisson process of parameter 1. As before, we consider the square V_N of side length $2N+1$, the first exit time τ_N of V_N and the local times $(\ell_x^t, x \in \mathbb{Z}^2, t \geq 0)$ defined as in (2.1). For any thickness parameter $0 \leq a \leq 1$, we call $\mathcal{M}_N(a)$ the set of a-thick points

$$\mathcal{M}_N(a) := \left\{ x \in V_N : \ell_x^{\tau_N} \geq \frac{2}{\pi\sqrt{\det \mathcal{G}}} a (\log N)^2 \right\}$$

where $\mathcal{G}$ is defined below. Then we have the following:

Theorem 2.1. *Assume that the law of the increments is symmetric (i.e. $-X \stackrel{d}{=} X$), with a finite variance and denote $\mathcal{G} = \mathbb{E}[XX']$ the covariance matrix of the increments. Then we have the following two a.s. limits:*

$$\lim_{N\to\infty} \frac{\max_{x\in V_N} \ell_x^{\tau_N}}{(\log N)^2} = \frac{2}{\pi\sqrt{\det \mathcal{G}}} \text{ and } \forall a \in [0,1), \lim_{N\to\infty} \frac{\log|\mathcal{M}_N(a)|}{\log N} = 2(1-a).$$

This theorem answers a question asked in the last section of [DPRZ01] with the additional assumption of symmetry. The assumption of symmetry is needed in our approach since otherwise we cannot define an associated GFF.

Our approach is sufficiently general that it can handle random walks with a very different flavour; for instance we discuss here the case of random walk on isoradial graphs.

We recall briefly the definitions and introduce some notation (we use the same one as [CS11]). Let $\Gamma = (V, E)$ be any connected infinite isoradial graph, with common radius 1, i.e. Γ is embedded in $\mathbb{C}$ and each face is inscribed into a circle of radius 1. Note that if $x, y \in V$ are adjacent then x and y, together with the centres of the two faces adjacent to the edge $\{x, y\}$, form a rhombus. We denote by $2\theta_{x,y}$ the interior angle of this rhombus at x (or at y). See Figure 2.1 for an example. For instance, the square (resp. triangular, hexagonal, etc) lattice is an isoradial graph with $\theta_{x,y} = \pi/4$ (resp. $\pi/6, \pi/3$, etc) for all $x \sim y$. We assume the following ellipticity condition:

$$\exists \eta \in \left(0, \frac{\pi}{4}\right), \forall x \sim y, \theta_{x,y} \in \left(\eta, \frac{\pi}{2} - \eta\right).$$

Define $\forall x \sim y \in V$ the conductance $c_{x,y} = \tan(\theta_{x,y})$ and let $(Y_t)_{t\geq 0}$ be a Markov jump process with conductances $(c_e)_{e\in E}$. Y is a continuous time walk which waits an exponential with mean $1/\sum_{y\sim x} c_{x,y}$ time in each vertex x and then jumps from x to y with probability $c_{x,y}/\sum_{z\sim x} c_{x,z}$. Take a starting point $x_0 \in V$ and denoting d_Γ the graph distance we define

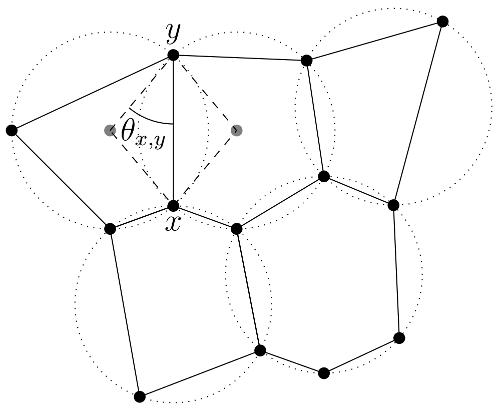


Figure 2.1: Isoradial graph and rhombic half-angle. The solid lines represent the edges of the graph. Each face is inscribed into a dotted circle of radius 1. The centres of the two faces adjacent to the edge $\{x, y\}$ are in grey.

for all $N \in \mathbb{N}$,

$$V_N := \{x \in V : d_\Gamma(x, x_0) \leq N\}$$

and as before (equation (2.1)), we consider the first exit time τ_N of V_N and the local times. We will denote $\mathbb{P}_x$ the law of the walk $(Y_t)_{t \geq 0}$ starting from $x \in V$ and $\mathbb{E}_x$ the associated expectation.

As confirmed by the theorem below, a sensible definition of a-thick points is given by

$$\mathcal{M}_N(a) := \left\{ x \in V_N : \ell_x^{\tau_N} \geq \frac{a}{\pi} (\log N)^2 \right\}.$$

Theorem 2.2. *We have the following two $\mathbb{P}_{x_0}$-a.s. limits:*

$$\lim_{N \to \infty} \frac{\max_{x \in V_N} \ell_x^{\tau_N}}{(\log N)^2} = \frac{1}{\pi} \text{ and } \forall a \in [0, 1), \lim_{N \to \infty} \frac{\log |\mathcal{M}_N(a)|}{\log N} = 2(1 - a).$$

Remark 2.3. Theorems 2.1 and 2.2 also hold when we consider the walk stopped at a deterministic time, N^2 say, rather than the first exit time τ_N of V_N, since

$$\lim_{N \to \infty} \frac{\log \tau_N}{\log N} = 2 \qquad \text{a.s.}$$

(easy to check but can also be seen from these two theorems). They also hold if we consider discrete time random walks rather than continuous time random walks. In that case, we have to multiply the discrete local times by the average time the continuous time walk stays in a given vertex before its first jump. See Remark 2.7 ending Section 2.1.2 for a short discussion about this.

Let us just confirm that Theorems 2.1 and 2.2 are coherent: in the square lattice case, the average time between successive jumps by the walk Y of Theorem 2.2 is $1/4$ rather than 1. We also mention that it is plausible that the arguments of [Ros05] can be adapted to show Theorem 2.2. However, we include it here since it is a straightforward consequence of our approach (Theorem 2.14).

2.1.2 Higher dimensions

We now come back to the setting of the beginning of Section 2.1 for $d \geq 3$ and we denote $g := \mathbb{E}_0 [\ell_0^\infty]$. In this section, the walk starts at the origin of $\mathbb{Z}^d$.

We describe thick points through a more precise encoding by considering for $a \in [0,1]$ the point measure:

$$\nu_N^a := \frac{1}{N^{2(1-a)}} \sum_{x \in V_N} \delta_{\left(x/N, \ell_x^{\tau_N} - 2ga \log N\right)}. \tag{2.3}$$

Let us emphasise that the normalisation factor is equal to 1 when $a = 1$ and that ν_N^a is viewed as a random measure on $[-1,1]^d \times \mathbb{R}$. We compare the thick points of random walk with the thick points of i.i.d. exponential random variables with mean g located at each site visited by the walk. More precisely, we denote $\mathcal{M}_N(0) := \{x \in V_N : \ell_x^{\tau_N} > 0\}$ and taking $E_x, x \in \mathbb{Z}^d$, i.i.d. exponential variables with mean g independent of $\mathcal{M}_N(0)$, we define

$$\mu_N^a := \frac{1}{N^{2(1-a)}} \sum_{x \in \mathcal{M}_N(0)} \delta_{(x/N, E_x - 2ga \log N)}.$$

We finally denote by τ the first exit time of $[-1,1]^d$ of Brownian motion starting at the origin and by μ_{occ} the occupation measure of Brownian motion starting at the origin and killed at τ. Then we have:

Theorem 2.4. *For all $a \in [0,1]$ there exists a random Borel measure ν^a on $[-1,1]^d \times \mathbb{R}$ such that, with respect to the topology of vague convergence of measures on $[-1,1]^d \times \mathbb{R}$ (on $[-1,1]^d \times (0,\infty)$ if $a = 0$), we have:*

$$\lim_{N \to \infty} \nu_N^a = \lim_{N \to \infty} \mu_N^a = \nu^a \text{ in law.}$$

Moreover, for all $a \in [0,1)$ the distribution of ν^a does not depend on a and

$$\nu^a(dx, d\ell) \overset{(\mathrm{d})}{=} \frac{1}{g} \mu_{\mathrm{occ}}(dx) \otimes e^{-\ell/g} \frac{d\ell}{g}. \tag{2.4}$$

At criticality, ν^1 is a Poisson point process:

$$\nu^1 \overset{(\mathrm{d})}{=} \mathrm{PPP}\left(\frac{1}{g} \mu_{\mathrm{occ}}(dx) \otimes e^{-\ell/g} \frac{d\ell}{g} \right). \tag{2.5}$$

We will see that this statement will imply the following two theorems:

Theorem 2.5. *If we define for every $a \in [0,1]$ the set of a-thick points:*

$$\mathcal{M}_N(a) := \left\{ x \in V_N : \ell_x^{\tau_N} > 2ga \log N \right\},$$

then there exist random variables M_a such that for all $a \in [0,1]$

$$\frac{|\mathcal{M}_N(a)|}{N^{2(1-a)}} \xrightarrow[N \to \infty]{\text{(d)}} M_a.$$

Moreover, for all $a \in [0,1)$ the distribution of M_a does not depend on a and

$$M_a \overset{\text{(d)}}{=} \tau/g. \tag{2.6}$$

M_1 is a Poisson variable with parameter τ/g: for all $k \geq 0$

$$\mathbb{P}\left(M_1 = k\right) = \frac{1}{k!} \mathbb{E}\left[e^{-\frac{\tau}{g}} \left(\frac{\tau}{g}\right)^k \right]. \tag{2.7}$$

Theorem 2.6. *There exists an almost surely finite random variable L such that*

$$\sup_{x \in V_N} \ell_x^{\tau_N} - 2g \log N \xrightarrow[N \to \infty]{\text{(d)}} L.$$

Moreover, L is a Gumbel variable with mode $g \log(\tau/g)$ (location of the maximum) and scale parameter g, i.e. for all $t \in \mathbb{R}$

$$\mathbb{P}\left(L \leq t\right) = \mathbb{E}\left[\exp\left(-\frac{\tau}{g} e^{-t/g} \right) \right].$$

To the best of our knowledge, this result is not present in the current literature. A detailed study of the local times of random walk in dimension greater than two has been done in a series of papers by Csáki, Földes, Révész, Rosen and Shi (see [CFR07b] for a survey of this work). In particular, Theorem 1 of [Rév04] and the corollary following the main theorem of [CFR06] improved the estimate of Erdős and Taylor (equation (2.2)). By translating their work to our setting of continuous time random walk (see the next remark), they showed that a.s. for all $\varepsilon > 0$, there exists $N_0 < \infty$ a.s. such that for all $N \geq N_0$,

$$-(4+\varepsilon) g \log \log N \leq \sup_{x \in V_N} \ell_x^{\tau_N} - 2g \log N \leq (2+\varepsilon) g \log \log N.$$

Let us also mention the fact that Theorem 2 of [Rév04] states that for all $\varepsilon > 0$, almost surely we have $\sup_{x \in V_N} \ell_x^{\tau_N} - 2g \log N \geq (2(d-4)/(d-2) - \varepsilon) \log \log N$ for infinitely many N. This is not in contradiction with our Theorem 2.6 because we only give the typical behaviour (i.e. at a fixed time) of $\sup_{x \in V_N} \ell_x^{\tau_N} - 2g \log N$.

Remark 2.7. We have stated our results in the case of continuous time random walk but they hold as well for discrete time random walk. As already mentioned, the statements in the planar case do not need to be changed. The reason for this is because in dimension two we were essentially comparing exponential (continuous time) or geometrical (discrete time) variables with mean $g \log N$ to $ag(\log N)^2$ for some $g > 0$ and $a \in (0,1)$. In both cases, if we divide these variables by $g \log N$ then they converge to exponential variables with parameter 1. Thus there is no difference between the continuous time case and the discrete time one. On the contrary, in higher dimensions, we are comparing exponential or geometrical variables with mean g to $ga \log N$ and these two distributions have slightly different behaviour. In the discrete time setting, our results claim that the field composed of the local times behaves like the field composed of independent geometrical variables with mean g located at each site visited by the walk. Theorems 2.4–2.6 then have to be modified accordingly.

2.2 Outline of proofs and literature overview

Section 2.3 will be dedicated to the dimension two whereas Section 2.4 will deal with the dimensions greater or equal to three. Let us first describe the two dimensional case.

We first recall the definition of the GFF on the square lattice. With the notations of Theorem 2.2 in the square lattice case, the Gaussian free field is the centred Gaussian field ϕ_N, indexed by the vertices in V_N, whose covariances are given by the Green function:

$$\mathbb{E}[\phi_N(x)\phi_N(y)] = \mathbb{E}_x \left[\ell_y^{\tau_N} \right].$$

See [Ber16], [Zei12] for introductions to the GFF. Our argument will simply relate the thick points of the random walk to those of the GFF: see [Kah85], [HMP10] in the continuum and [BDG01], [Dav06] in the discrete case.

We now explain the interest of exploiting the connection to the GFF. As usual, the proofs of Theorems 2.1 and 2.2 rely on the method of (truncated) second moment. That is, a first moment estimate on $|\mathcal{M}_N(a)|$ gives us the upper bound, while a matching upper bound on the second moment of $|\mathcal{M}_N(a)|$ would supply the lower bound. Moreover, it is necessary to first consider a truncated version of $|\mathcal{M}_N(a)|$, where we consider points that are never too thick at all scales (this is similar to the idea in [Ber17]). Computing the corresponding correlations is not easy with the random walk, but is essentially straightforward with the GFF as this is basically part of the definition. As only an upper bound on the second moment is needed, comparisons to the GFF with Dynkin-type isomorphisms go in the right direction. We will see that the Eisenbaum's version will be the most convenient to work with.

We now state this isomorphism. Consider $\Gamma = (V, E)$ a non-oriented connected infinite graph without loops, not necessary planar, equipped with symmetric conductances $(W_{xy})_{x,y \in V}$. Let E' be the edge set $E' = \{\{x,y\} : x, y \in V, W_{xy} > 0\}$. Let $\mathbb{P}_x$ be the law under which $(Y_t)_{t \geq 0}$ is a symmetric Markov jump process with conductances $(W_{yz})_{y,z \in V}$ (i.e. jump rates W_{yz} from y to z) starting at x at time 0. Y is thus a nearest neighbour random walk on (V, E') but not necessary on $\Gamma = (V, E)$. As in the isoradial case, we denote $\ell_x^t, x \in V, t \geq 0$, its local times, x_0 a starting point, V_N the ball of radius N and centre x_0 for the graph distance of Γ, τ_N the first exit time of V_N. Because Y is a *symmetric* Markov process, the following expression is symmetric in x, y:

$$\mathbb{E}_x \left[\ell_y^{\tau_N} \right] = \mathbb{E}_y \left[\ell_x^{\tau_N} \right].$$

This allows us to define a centred Gaussian field ϕ_N whose covariances are given by the previous expression. ϕ_N is called Gaussian free field and we will denote $\mathbb{P}$ its law. The following theorem establishes a relation between the local times and the GFF (see lectures notes [Ros14] for a good overview of this topic)

Theorem A (Eisenbaum's isomorphism)**.** *For all $s > 0$ and all measurable bounded function $f : \mathbb{R}^{V_N} \to \mathbb{R}$,*

$$\begin{aligned} &\mathbb{E}_{x_0} \otimes \mathbb{E} \left[f \left\{ \left(\ell_x^{\tau_N} + \frac{1}{2} (\phi_N(x) + s)^2 \right)_{x \in V_N} \right\} \right] \\ &= \mathbb{E} \left[\left(1 + \frac{\phi_N(x_0)}{s} \right) f \left\{ \left(\frac{1}{2} (\phi_N(x) + s)^2 \right)_{x \in V_N} \right\} \right]. \end{aligned}$$

Remark 2.8. We are now going to explain why we chose to use this isomorphism instead of the maybe more well-known generalised second Ray-Knight theorem. To ease the comparison, we are going to state this other isomorphism in the setting that is of interest to us. Consider the graph (V_N, E_N) with $E_N = \{\{x, y\} : x, y \in V_N, W_{xy} > 0\}$. Let $\mathbb{P}_x$ be the law under which $(Y_t)_{t \geq 0}$ is a symmetric Markov jump process with conductances $(W_e)_{e \in E_N}$ starting at x at time 0. Let $\ell_x^t, x \in V_N, t > 0$, be the associated local times and for $u > 0$, define $\tau_u := \inf\{t > 0 : \ell_{x_0}^t \geq u\}$ and $\tau_{x_0} := \inf\{t > 0 : Y_t = x_0\}$. We can now define $\mathbb{P}$ the law under which $(\psi_N(x), x \in V_N)$ is the GFF in V_N with zero-boundary condition at x_0, i.e. ψ_N is a centred Gaussian vector whose covariance matrix is given by

$$\mathbb{E}[\psi_N(x)\psi_N(y)] = \mathbb{E}_x \left[\ell_y^{\tau_{x_0}} \right].$$

The generalised second Ray-Knight theorem states that (see again the lecture notes

[Ros14]):

$$\left(\ell_x^{\tau_u} + \frac{1}{2}\psi_N(x)^2\right)_{x \in V_N} \overset{(d)}{=} \left(\frac{1}{2}\left(\psi_N(x) + \sqrt{2u}\right)^2\right)_{x \in V_N} \tag{2.8}$$

under $\mathbb{P}_{x_0} \otimes \mathbb{P}$ and $\mathbb{P}$.

It would have been possible to use this isomorphism to show Theorems 2.1 and 2.2. Compared to the Eisenbaum's isomorphism above, this has the advantage that the laws of the GFFs on the left hand side and right hand side are the same. However this has a drawback: indeed it is necessary to stop the walk where it starts, i.e. at x_0. This isomorphism then leads to a GFF ψ_N pinned at x_0. This is essentially equivalent to adding a global noise to the Dirichlet GFF ϕ_N of order $\sqrt{\log N}$ which is sufficient to ruin second moment approach. This noise would have to be removed by hand in order to apply the method of second moment. This is possible but makes the proof substantially longer.

The generalised second Ray-Knight isomorphism has been used several times to study problems related to local times (see for instance [DLP12]). We now mention two works that are maybe the most relevant to us. The isomorphism (2.8) immediatly gives the following stochastic domination:

$$\left(\sqrt{\ell_x^{\tau_u}}\right)_{x \in V_N} \prec \left(\frac{1}{\sqrt{2}}\left|\psi_N(x) + \sqrt{2u}\right|\right)_{x \in V_N}$$

under $\mathbb{P}_{x_0}$ and $\mathbb{P}$. One can actually show a stronger result and replace the absolute value on the right hand side by $\max(\cdot, 0)$ (Theorem 3.1 of [Zha14]). Abe [Abe15] exploited this and used the symmetry of the GFF to make links between what was called thin points and thick points of the random walk on the two-dimensional torus, up to a multiple of the cover time.

Let us also mention that Abe and Biskup [AB19] have announced a work in preparation which relates the thick points of random walk to the Liouville quantum gravity in dimension two. This is in the same spirit as this paper as they also rely on a connection to the GFF. However, we emphasise some important differences. First, the walk they consider is on a box and has wired boundary conditions, meaning that the walk is effectively re-randomised every time it hits the boundary of the box. Second, they consider the local time profile at a regime comparable to the cover time, so that the comparison to the GFF is perhaps more clear.

Organisation - planar case: The two-dimensional part of the paper will be organised as follows. In Section 2.3.1 we will present the general framework that we treat (Theorem 2.14). We will then show that Theorems 2.1 and 2.2 are simple corollaries. The upper bound, which is the easy part, will be briefly proved at the end of the same section. Section 2.3.2 is devoted to the lower bound. We first show that the probability to have a lot of thick points does not decay too quickly. This is the heart of our proof and makes use of the comparison to the GFF. We then bootstrap this argument to obtain the same statement with high probability, see Lemma 2.16 at the beginning of Section 2.3.2. This lemma is a key feature of our proof and allows us to use the comparison to the GFF. Indeed, since we do not require very precise estimates, we can deal with the change of measure coming from the isomorphism through very rough bounds, such as: $|\phi_N(x_0)| \leq (\log N)^2$ with high probability (see Lemma 2.17). This only introduces a poly-logarithmic multiplicative error in the estimate of the probabilities that two given points are thick, and so does not matter for the computation of the fractal dimension of the number of thick points on a polynomial scale.

If we want more accurate estimates, more ideas are required. For instance, for the simple random walk on the square lattice, the comparison between the number of thick points for the random walk and for the GFF breaks down: the two following expectations converge as N goes to infinity:

$$\lim_{N\to\infty} \frac{\log N}{N^{2(1-a)}} \mathbb{E}_0 \left[\# \left\{ x \in V_N : \ell_x^{\tau_N} \geq \frac{4a}{\pi} (\log N)^2 \right\} \right] \in (0,\infty), \tag{2.9}$$

$$\lim_{N\to\infty} \frac{\sqrt{\log N}}{N^{2(1-a)}} \mathbb{E} \left[\# \left\{ x \in V_N : \tfrac{1}{2}\phi_N(x)^2 \geq \frac{4a}{\pi} (\log N)^2 \right\} \right] \in (0,\infty). \tag{2.10}$$

In the article [BL19] the thick points of the discrete GFF ϕ_N were encoded in point measures of a similar form as the one we defined in (2.3). The authors showed the convergence of such measures. As a consequence, they went beyond the estimate (2.10) and showed that

$$\frac{\sqrt{\log N}}{N^{2(1-a)}} \# \left\{ x \in V_N : \tfrac{1}{2}\phi_N(x)^2 \geq \frac{4a}{\pi} (\log N)^2 \right\} \tag{2.11}$$

converges in law to a nondegenerate random variable.

Question 2.9. *In the case of simple random walk on the square lattice starting at the origin, does*

$$\frac{\log N}{N^{2(1-a)}} \# \left\{ x \in V_N : \ell_x^{\tau_N} \geq \frac{4a}{\pi} (\log N)^2 \right\} \tag{2.12}$$

converge to a nondegenerate random variable as N goes to infinity?

Notice that the renormalisations are different in (2.11) and in (2.12). These differences suggest scraping the GFF approach if we want optimal estimates. This is what we will do in higher dimensions.

Update: after this work was completed, this question has been solved in [Jeg20a] (Corollary 3.4), [AB19] and [Jeg19] (Corollary 4.2). The framework of [Jeg19] is the above-described setting of planar random walk stopped upon hitting the boundary of V_N for the first time, whereas [Jeg20a] works in an analogue setting for planar Brownian *motion. The article [AB19] considers a different type of walks that are run up to a time proportional to the cover time of a planar graph and that have wired boundary condition (see Remark 2.8).*

We have finished to discuss the two-dimensional case and we now describe the situation in higher dimensions. The article [DPRZ00] studied the thick points of occupation measure of Brownian motion in dimensions greater or equal to three. They obtained the leading order of the maximum and computed the Hausdorff dimension of the set of thick points. The article [CFR$^+$05b], as well as [CFR05a], [CFR06], [CFR07a], [CFR07c] (again, see [CFR07b] for a survey on this series of paper), studied the case of symmetric transient random walk on $\mathbb{Z}^d$ with finite variance. One of their results computed the leading order of the maximum of the local times too. In both [DPRZ00] and [CFR$^+$05b], a key feature of the proofs is a localisation property (Lemma 3.1 of [DPRZ00] and Lemma 2.2 of [CFR$^+$05b]) which roughly states that a thick point accumulates most of its local time in a short interval of time. This property allows them to consider independent variables and makes the situation simpler compared to the two-dimensional case.

Let us also mention the paper [CCH15] which studied the scaling limit of the discrete GFF in dimension greater or equal to three. The authors obtained a result similar to Theorem 2.4. Namely, they showed that in the limit the field behaves as independent Gaussian variables. More precisely, they defined a point process analogous to ν_N^1 (see (2.3)) which encodes the thickest points of the GFF. They showed that this point process converges to a Poisson point process. Their situation is simpler because the intensity measure is governed by the Lebesgue measure rather than the occupation measure of Brownian motion. In particular, they could use the Stein-Chen method which allowed them to consider only the first two moments.

Organisation - higher dimensions: Let us now present the main lines of our proofs and the organisation of the paper. In Section 2.4.1, Theorems 2.4, 2.5 and 2.6 will all be obtained from the joint convergence of the sequences of real-valued random variables $\nu_N^a(A_1 \times T_1), \dots, \nu_N^a(A_r \times T_r)$, for all suitable $A_i \subset [-1,1]^d$ and $T_i \subset \mathbb{R}$. We will obtain this fact by computing explicitly all the moments of these variables (Proposition 2.19). This is actually the heart of our proofs and Section 2.4.2 will be entirely dedicated to it. To compute the k-th moment of $\nu_N^a(A \times T)$, we will estimate the probability that the local times in k different points, say $x_1, \dots, x_k$, belong to $2ga \log N + T$. In the subcritical regime ($a < 1$), we will be able to assume that these points are far away from each other. In that case, Lemma 2.22 will show that we can restrict ourselves to the event that there exists a permutation σ of the set of indices $\{1, \dots, k\}$ which orders the vertices so that we have the following: the walk first hits $x_{\sigma(1)}$, accumulates a big local time in $x_{\sigma(1)}$, then hits $x_{\sigma(2)}$, accumulates a big local time in $x_{\sigma(2)}$, etc. When the walk has visited $x_{\sigma(i)}$ it does not come back to the vertices $x_{\sigma(1)}, \dots, x_{\sigma(i-1)}$. The local times can thus be treated as if they were independent.

At criticality ($a = 1$), we do not renormalise the number of thick points and we will a priori have to take into account points which are close to each other. Here, the key observation - contained in Lemma 2.24 and already present in Corollary 1.3 of [CFR+05b] - is that if two distinct points are close to each other, then the probability that they are both thick is much smaller than the probability that one of them is thick, even if they are neighbours! This is specific to the case of dimensions greater or equal to 3 and tells that the thick points do not cluster. Thus, only the points which are either equal or far away from each other will contribute to the k-th moment.

Section 2.4.3 will contain the proofs of four intermediate lemmas that are needed to prove Proposition 2.19 on the convergence of the moments of $\nu_N^a(A_1 \times T_1), \dots, \nu_N^a(A_r \times T_r)$ for suitable $A_i \subset [-1,1]^d$ and $T_i \subset \mathbb{R}$.

2.3 Dimension two

2.3.1 General framework and upper bound

We now describe the general setup for the theorem. Consider $\Gamma = (V, E)$ a non-oriented connected infinite graph without loops, not necessary planar, equipped with *symmetric* conductances $(W_{xy})_{x,y \in V}$. As before, we take $x_0 \in V$ a starting point and write d_Γ for the graph distance. We will also write

$$\forall N \in \mathbb{N}, V_N(x_0) := \{x \in V : d_\Gamma(x, x_0) \leq N\}.$$

Let $\mathbb{P}_x$ be the law under which $(Y_t)_{t \geq 0}$ is a symmetric Markov jump process with conductances $(W_{yz})_{y,z \in V}$ (i.e. jump rates W_{yz} from y to z) starting at x at time 0. Y is thus a nearest neighbour random walk on (V, E'), where $E' = \{\{x,y\} : x, y \in V, W_{xy} > 0\}$, but not necessary on Γ. We introduce the first exit time of $V_N(x_0)$ and the local times:

$$\tau_N(x_0) := \inf\{t \geq 0, Y_t \notin V_N(x_0)\} \text{ and } \forall x \in V, \forall t \geq 0, \ell_x^t := \int_0^t \mathbf{1}_{\{Y_s = x\}} ds.$$

Finally we will denote $G_N^{x_0}$ the Green function, i.e.:

$$G_N^{x_0}(x,y) := \mathbb{E}_x\left[\ell_y^{\tau_N(x_0)}\right]. \tag{2.13}$$

If there is no confusion, we will simply write V_N, τ_N and G_N instead of $V_N(x_0)$, $\tau_N(x_0)$ and $G_N^{x_0}$.

Notation: For two real-valued sequences $(u_N)_{N\geq 1}$ and $(v_N)_{N\geq 1}$ and for some parameter α, we will denote $u_N = o_\alpha(v_N)$ if

$$\forall \varepsilon > 0, \exists N_0 = N_0(\alpha,\varepsilon) > 0, \forall N \geq N_0, |u_N| \leq \varepsilon\, |v_N|\,,$$

and we will denote $u_N = O_\alpha(v_N)$ if

$$\exists C = C(\alpha) > 0, \exists N_0 = N_0(\alpha), \forall N \geq N_0, |u_N| \leq C\, |v_N|\,.$$

We now make the following assumptions on the graph Γ and on the walk Y: We start with two assumptions on the geometry of the graph Γ.

Assumption 2.10. $\#V_N(x_0) = N^{2+o(1)}$ *and for all* $x_0' \in V_N(x_0)$ *there exists a subset* $Q_N(x_0') \subset V_N(x_0')$ *with* $N^{2+o(1)}$ *points such that*

$$\forall \alpha < 2, \sum_{x,y\in Q_N(x_0')} \left(\frac{N}{d_\Gamma(x,y)\vee 1}\right)^\alpha = N^{4+o_\alpha(1)}. \tag{2.14}$$

Assumption 2.11. *For all* $\eta \in (0,1)$, $x_0' \in V_N(x_0)$, $x \in Q_N(x_0')$ *and* $R \in [1, N^{1-\eta}]$, *we can find a subset* $C_R(x) \subset Q_N(x_0')$ *which can be thought of as a circle of radius* R *centred at* x*:*

$$\forall y \in C_R(x), \log \frac{R}{d_\Gamma(x,y)} = o_\eta(\log N), \tag{2.15a}$$

$$\frac{1}{\#C_R(x)^2} \sum_{y,y'\in C_R(x)} \log\left(\frac{R}{d_\Gamma(y,y')\vee 1}\right) = o_\eta(\log N). \tag{2.15b}$$

We now assume that we have good controls on the Green function:

Assumption 2.12. *There exists* $g > 0$ *such that:*

$$\forall x \in V_N(x_0), G_N^{x_0}(x,x) \leq g \log N + o(\log N), \tag{2.16a}$$

$$\forall x_0' \in V_N(x_0), \forall x,y \in Q_N(x_0'), G_N^{x_0'}(x,y) = g \log\left(\frac{N}{d_\Gamma(x,y)\vee 1}\right) + o(\log N), \tag{2.16b}$$

$$\forall x_0' \in V_N(x_0), \forall x \in Q_N(x_0'), G_N^{x_0'}(x_0',x) \geq (1/N)^{o(1)}. \tag{2.16c}$$

Finally, we assume that the jumps are not unreasonable:

Assumption 2.13. *For all $K_N = N^{1-o(1)} \leq N$, $x_0' \in V_{N-K_N}(x_0)$ and $M > 0$,*

$$\mathbb{P}_{x_0'}\left(d_\Gamma\left(x_0', Y_{\tau_{K_N}(x_0')}\right) \geq K_N + M\right) \leq K_N N^{o(1)}/M. \tag{2.17}$$

where $\tau_{K_N}(x_0')$ is the first exit time of $V_{K_N}(x_0')$.

We now briefly discuss the above assumptions. Note that we have assumed that all the bounds do not depend on the starting point $x_0' \in V_N(x_0)$. This will be important for our Lemma 2.16. Assumption 2.11 is needed to go beyond the L^2 phase whereas Assumption 2.13 is needed to bootstrap the probability to have a lot of thick points (Lemma 2.16). This latter assumption can be weakened. We could replace $K_N N^{o(1)}/M$ by $f(K_N N^{o(1)}/M)$ with a function $t \in (0,\infty) \mapsto f(t) \in (0,\infty)$ which goes to zero quickly enough as t goes to zero. For instance, any positive power of t would do.

As confirmed by the theorem below, a sensible definition of a-thick points is given by

$$\mathcal{M}_N(a) := \left\{x \in V_N : \ell_x^{\tau_N} \geq 2ag(\log N)^2\right\}.$$

Theorem 2.14. *Assuming the above assumptions we have the following two $\mathbb{P}_{x_0}$-a.s. convergences:*

$$\lim_{N\to\infty} \frac{\max_{x\in V_N} \ell_x^{\tau_N}}{(\log N)^2} = 2g \text{ and } \forall a \in [0,1), \lim_{N\to\infty} \frac{\log|\mathcal{M}_N(a)|}{\log N} = 2(1-a).$$

We now check that Theorems 2.1 and 2.2 are consequences of this last theorem. Theorem 2.2 naturally fits into the setting of continuous time random walks defined using symmetric conductances, whereas the setting of Theorem 2.1 corresponds to the above-described general framework with Γ being the square lattice equipped with weights $W_{xy} = \mathbb{P}(X = y - x)$. These weights are symmetric thanks to the assumption $X \overset{(d)}{=} -X$. We now need to check that these two setups satisfy Assumptions 2.10 - 2.13 above.

For the isoradial case, the walk is a nearest-neighbour random walk so Assumption 2.13 is clear. The following lemma checks that all the other assumptions are fulfilled if we define

$$\forall x_0' \in V_N(x_0), Q_N(x_0') := \begin{cases} V_{N/R_N}(x_0') & \text{in the square lattice case,} \\ V_{\varepsilon N}(x_0') & \text{in the isoradial case,} \end{cases}$$

where R_N and ε are defined Lemma 2.15 below, and if we define in both cases

$$\forall x_0' \in V_N(x_0), \forall x \in Q_N(x_0'), \forall R \geq 1, C_R(x) := \{y \in Q_N(x_0') : d_\Gamma(x,y) = R\}.$$

Lemma 2.15. *1. Square Lattice. Consider a walk Y as in Theorem 2.1 and denote by*

$\mathcal{G}$ the covariance matrix of the increments. Let $x_0' \in \mathbb{Z}^2$ be a starting point. Then there exists $C > 0$ independent of x_0' such that for all $M > 0$,

$$\mathbb{P}_{x_0'}\left(d_\Gamma\left(x_0', Y_{\tau_N(x_0')}\right) \geq N + M\right) \leq CN/M. \tag{2.18}$$

Moreover for all $\eta \in (0,1)$,

$$\forall x, y \in V_N(x_0'), G_N^{x_0'}(x,y) \leq \frac{1}{\pi\sqrt{\det \mathcal{G}}} \log\left(\frac{N}{|x-y| \vee 1}\right) + o(\log N), \tag{2.19}$$

$$\forall x, y \in V_{(1-\eta)N}(x_0'), G_N^{x_0'}(x,y) \geq \frac{1}{\pi\sqrt{\det \mathcal{G}}} \log\left(\frac{N}{|x-y| \vee 1}\right) + o_\eta(\log N) \tag{2.20}$$

and there exists a sequence $R_N = N^{o(1)}$ such that

$$\forall x \in V_{N/R_N}(x_0'), G_N^{x_0'}(0, x_0') \geq N^{o(1)}. \tag{2.21}$$

2. *Isoradial Graphs. Consider a walk Y as in Theorem 2.2. Let $x_0' \in V$ be a starting point. Then for all $\eta \in (0,1)$,*

$$\forall x, y \in V_N(x_0'), G_N^{x_0'}(x,y) \leq \frac{1}{2\pi} \log\left(\frac{N}{|x-y| \vee 1}\right) + C, \tag{2.22}$$

$$\forall x, y \in V_{(1-\eta)N}(x_0'), G_N^{x_0'}(x,y) \geq \frac{1}{2\pi} \log\left(\frac{N}{|x-y| \vee 1}\right) - C(\eta) \tag{2.23}$$

for some $C, C(\eta) > 0$ independent of x_0'. Moreover, there exist $c, \varepsilon > 0$ independent of x_0' such that

$$\forall x \in V_{\varepsilon N}(x_0'), G_N^{x_0'}(x_0', x) \geq c. \tag{2.24}$$

Proof. Square lattice. We first start to prove (2.18). By translation invariance, we can assume that $x_0' = 0$. We consider the discrete time random $(S_i)_{i\geq 0}$ associated and we are going to abusively write τ_N to denote the first time the discrete time walk exits V_N. Take $\lambda > 0$ to be chosen later on. The probability we are interested in is not larger than

$$\begin{aligned}\mathbb{P}_0\left(d_\Gamma\left(S_{\tau_N - 1}, S_{\tau_N}\right) \geq M\right) &\leq \mathbb{P}_0\left(\exists i \leq \tau_N - 1, d_\Gamma(S_i, S_{i+1}) \geq M\right) \\ &\leq \mathbb{P}_0\left(\exists i \leq \lambda N^2 - 1, d_\Gamma(S_i, S_{i+1}) \geq M\right) + \mathbb{P}_0\left(\tau_N > \lambda N^2\right).\end{aligned}$$

As the increments have a finite variance, the first term on the right hand side is not larger than $C\lambda N^2/M^2$ for some $C > 0$ by the union bound. Secondly,

$$\mathbb{P}_0\left(\tau_N > \lambda N^2\right) \leq \mathbb{P}_0\left(d_\Gamma\left(0, S_{\lambda N^2}\right) \leq N\right).$$

Theorem 2.3.9 of [LL10] gives estimates on the heat kernel and in particular implies that there exists $C > 0$ such that for all $x \in \mathbb{Z}^2$, $\mathbb{P}_0(S_i = x) \leq C/i$. Hence

$$\mathbb{P}_0\left(\tau_N > \lambda N^2\right) \leq C'/\lambda.$$

We obtain (2.18) by taking $\lambda = M/N$.

Now, (2.19) and (2.20) are consequences of the estimate on the potential kernel $a(x)$ made in Theorem 4.4.6 of [LL10]:

$$a(x) = \frac{1}{\pi\sqrt{\det \mathcal{G}}} \log|x| + o(\log|x|) \text{ as } |x| \to \infty$$

which is linked to the Green function by:

$$G_N(x,y) = \sum_{z \in V_N^c} \mathbb{P}_x\left(Y_{\tau_N} = z\right) a(y-z) - a(y-x). \tag{2.25}$$

If $z \in V_N^c$ is such that $d_\Gamma(x_0, z) \leq N(\log N)^2$, then

$$\frac{1}{\pi\sqrt{\det \mathcal{G}}} \log N + o_\eta(\log N) \leq a(y-z) \leq \frac{1}{\pi\sqrt{\det \mathcal{G}}} \log N + o(\log N)$$

where the lower bound (resp. upper bound) is satisfied by all $y \in V_{(1-\eta)N}$ (resp. V_N). (2.18) implying that $\mathbb{P}_x\left(d_\Gamma\left(x_0, Y_{\tau_N}\right) \leq N(\log N)^2\right) = 1 + o(1)$, we are thus left to show that the elements z such that $d_\Gamma(x_0, z) > N(\log N)^2$ do not contribute to the sum in the equation (2.25). Thanks to (2.18), we have

$$\begin{aligned}
&\sum_{\substack{z \in \mathbb{Z}^2 \\ d_\Gamma(x_0,z) > N(\log N)^2}} \mathbb{P}_x\left(Y_{\tau_N} = z\right) \log|z| \\
&\leq \sum_{p=0}^{\infty} \mathbb{P}_x\left(2^p \leq d_\Gamma\left(x_0, Y_{\tau_N}\right)/(N(\log N)^2) < 2^{p+1}\right) \log\left(N(\log N)^2 2^{p+1}\right) \\
&\leq \frac{C}{(\log N)^2} \sum_{p=0}^{\infty} \frac{1}{2^p} \log\left(N(\log N)^2 2^{p+1}\right) \leq \frac{C'}{\log N}
\end{aligned}$$

which goes to zero as N goes to infinity. It completes the proof of (2.19) and (2.20). (2.21) is a direct consequence of (2.20).

Isoradial graphs. (2.22) and (2.23) are a direct consequences of Theorem 1.6.2 and Proposition 1.6.3 of [Law96] in the case of simple random walk on the square lattice. Kenyon extended this result to general isoradial graphs (see [Ken02] or Theorem 2.5 and Definition 2.6 of[CS11]). (2.24) follows from (2.23). □

From now on, we will work with a graph Γ and a walk Y which satisfy Assumptions

2.10 - 2.13. An upper bound on the Green function G_N is already enough to prove the upper bound of Theorem 2.14:

Proof of the upper bound of Theorem 2.14. Let $a \geq 0$ and $N \geq 1$. For every $\varepsilon > 0$ we obtain by Markov inequality:

$$\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \geq N^{2(1-a)+\varepsilon}\right) \leq N^{-2(1-a)-\varepsilon} \sum_{x \in V_N} \mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2\right).$$

But for every $x \in V_N$, under $\mathbb{P}_x$, $\ell_x^{\tau_N}$ is an exponential variable with mean $G_N(x,x)$. Hence by (2.16a),

$$\begin{aligned}\mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2\right) &= \mathbb{P}_{x_0}\left(\ell_x^{\tau_N} > 0\right)\mathbb{P}_x\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2\right) \\ &= \mathbb{P}_{x_0}\left(\ell_x^{\tau_N} > 0\right)\exp\left(-2ga(\log N)^2/G_N(x,x)\right) \\ &\leq CN^{-2a+o(1)}. \end{aligned} \tag{2.26}$$

The upper bound for the convergence in probability follows. To show that

$$\limsup_{N\to\infty} \frac{\log|\mathcal{M}_N(a)|}{\log N} \leq 2(1-a), \qquad \mathbb{P}_{x_0}\text{–a.s.},$$

we observe that, taking $N = 2^n$ in (2.26),

$$\mathbb{P}_{x_0}\left(\#\left\{x \in V_{2^{n+1}} : \ell_x^{\tau_{2^{n+1}}} \geq 2ga\left(\log 2^n\right)^2\right\} \geq (2^n)^{2(1-a)+\varepsilon}\right)$$

decays exponentially and so is summable. Moreover, if $2^n \leq N < 2^{n+1}$,

$$|\mathcal{M}_N(a)| \leq \#\left\{x \in V_{2^{n+1}} : \ell_x^{\tau_{2^{n+1}}} \geq 2ga\left(\log 2^n\right)^2\right\}.$$

Hence the Borel–Cantelli lemma implies that

$$\limsup_{N\to\infty} \frac{\log|\mathcal{M}_N(a)|}{\log N} \leq 2(1-a)+\varepsilon, \qquad \mathbb{P}_{x_0}\text{–a.s.}$$

This concludes the proof of the upper bound on $|\mathcal{M}_N(a)|$. We notice that the above reasoning also shows that for all $\varepsilon > 0$, almost surely, for all N large enough, $|\mathcal{M}_N(1+\varepsilon)| = 0$. The upper bound on $\sup_{x\in V_N} \ell_x^{\tau_N}$ then follows from

$$\left\{\sup_{x\in V_N} \ell_x^{\tau_N} \geq 2g(1+\varepsilon)(\log N)^2\right\} \subset \left\{|\mathcal{M}_N(1+\varepsilon)| \geq 1\right\}.$$

$\square$

2.3.2 Lower bound

We first start this section by establishing a lemma which simplifies a bit the problem: we only need to show that the probability to have a lot of thick points decays sub-polynomially. For all starting point $x_0' \in V_N$, define $\mathcal{M}_N(a, x_0')$ the set of a-thick points in the ball $V_N(x_0')$:

$$\mathcal{M}_N(a, x_0') = \left\{ x \in V_N(x_0') : \ell_x^{\tau_N(x_0')} \geq 2ga(\log N)^2 \right\}.$$

Lemma 2.16. *Suppose that for all starting point $x_0' \in V_N(x_0)$, for all $a \in (0,1), \varepsilon > 0$ and $N \in \mathbb{N}$,*

$$\mathbb{P}_{x_0'}\left(|\mathcal{M}_N(a, x_0')| \geq N^{2(1-a)-\varepsilon} \right) \geq p_N,$$

with $p_N = p_N(a) > 0$ decaying slower than any polynomial, i.e. $\log p_N = o_{a,\varepsilon}(\log N)$. Then for all $a \in (0,1)$,

$$\liminf_{N \to \infty} \frac{\log |\mathcal{M}_N(a)|}{\log N} \geq 2(1-a), \qquad \mathbb{P}_{x_0}-\text{a.s.}$$

Proof. A similar but weaker statement appears in [DPRZ01] and [Ros05] where they assumed that p_N was bounded away from 0. The idea is to decompose the walk in the ball $V_N(x_0)$ into several walks in smaller balls to bootstrap the probability we are interested in.

First of all, let us remark that if $p_N \in (0,1)$ decays slower than any polynomial, then so does $(\inf_{n \leq N} p_n)_{N \geq 1}$. Consequently, we can assume without loss of generality that the sequences p_N in the statement of the lemma are non increasing.

Fix $\varepsilon > 0$ and take N large and $K_N \in \mathbb{N}$ much smaller than N such that $K_N = N^{1-o(1)}$. Let us introduce the stopping times

$$\sigma(0) := 0 \text{ and } \forall i \geq 1, \sigma(i) := \inf \left\{ t > \sigma(i-1) : d_\Gamma\left(Y_t, Y_{\sigma(i-1)}\right) \geq K_N \right\}$$

and

$$i_{\max} := \max \left\{ i \geq 0, d_\Gamma\left(x_0, Y_{\sigma(i)}\right) \leq N - K_N \right\}.$$

Let $k \geq 1$. If $i_{\max}+1 \geq k$, then all the walks $\left(Y_{\sigma(i)+t}, 0 \leq t \leq \sigma(i+1) - \sigma(i)\right)$, $i = 0 \dots k-1$, are contained in the walk $(Y_t, 0 \leq t \leq \tau_N)$. So by a repeated application of Markov property, we see that for all $\delta > 0$, if N is large enough so that $a(\log N)^2 \leq (a+\delta)(\log K_N)^2$ (which is possible by assumption on K_N), we have:

$$\begin{aligned}
&\mathbb{P}_{x_0}\Big(|\mathcal{M}_N(a)| \leq N^{2(1-a)-\varepsilon} \Big) \\
&\leq \sup_{x_0' \in V_{N-K_N}(x_0)} \mathbb{P}_{x_0'}\left(|\mathcal{M}_{K_N}(a+\delta, x_0')| \leq N^{2(1-a)-\varepsilon} \right)^k + \mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right) \\
&\leq \sup_{x_0' \in V_{N-K_N}(x_0)} \mathbb{P}_{x_0'}\left(|\mathcal{M}_{K_N}(a+\delta, x_0')| \leq K_N^{(2(1-a)-\varepsilon)\sqrt{1+\delta/a}} \right)^k + \mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right).
\end{aligned}$$

If $\delta > 0$ is small enough we have $(2(1-a)-\varepsilon)\sqrt{1+\delta/a} < 2(1-a-\delta)$. Hence with $p_N = p_N(a+\delta)$

$$\begin{aligned}\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \leq N^{2(1-a)-\varepsilon}\right) &\leq (1-p_{K_N})^k + \mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right) \\ &\leq (1-p_N)^k + \mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right). \end{aligned} \tag{2.27}$$

To conclude, we have to choose K_N small enough to ensure that $i_{\max}$ is large with high probability. If the walk were a nearest neighbour random walk, we could say that $i_{\max} + 1 \geq \lfloor N/K_N \rfloor$ $\mathbb{P}_{x_0}$-a.s. Here, the jumps may be unbounded but large jumps are costly (Assumption 2.13) so we will be able to recover a lower bound fairly similar on $i_{\max}$. By the triangle inequality, we have for all $k \geq 1$

$$\begin{aligned}\mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right) &\leq \mathbb{P}_{x_0}\left(\exists i \leq k-1, d_\Gamma\left(Y_{\sigma(i)}, Y_{\sigma(i+1)}\right) \geq (N-K_N)/k\right) \\ &\leq \sum_{i=0}^{k-1} \mathbb{P}_{x_0}\left(Y_{\sigma(i)} \in V_{N-K_N}, d_\Gamma\left(Y_{\sigma(i)}, Y_{\sigma(i+1)}\right) \geq (N-K_N)/k\right) \\ &\leq k \sup_{x_0' \in V_{N-K_N}} \mathbb{P}_{x_0'}\left(d_\Gamma\left(x_0', Y_{\tau_{K_N}}\right) \geq (N-K_N)/k\right).\end{aligned}$$

Assumption 2.13 allows us to bound this last probability: there exists $(\varepsilon_N)_{N\geq 1} \subset (0,\infty)$ which converges to zero such that if $M > 0$,

$$\mathbb{P}_{x_0'}\left(d_\Gamma\left(x_0', Y_{\tau_{K_N}}\right) \geq M + K_N\right) \leq K_N N^{\varepsilon_N}/M.$$

Hence

$$\mathbb{P}_{x_0}\left(i_{\max}+1 \leq k\right) \leq \frac{k^2 K_N N^{\varepsilon_N}}{N-(k+1)K_N}.$$

Coming back to the estimate (2.27) and taking $k = (\log N)/p_N$, we have obtained

$$\begin{aligned}\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \leq N^{2(1-a)-\varepsilon}\right) &\leq (1-p_N)^{(\log N)/p_N} + \mathbb{P}_{x_0}\left(i_{\max}+1 \leq (\log N)/p_N\right) \\ &\leq \left(\sup_{0<p<1}(1-p)^{1/p}\right)^{\log N} + C\frac{(\log N)^2 K_N N^{\varepsilon_N}}{(p_N)^2(N-(1+(\log N)/p_N)K_N)}.\end{aligned}$$

We can choose

$$K_N = \frac{p_N^2}{(\log N)^4} N^{1-\varepsilon_N} = N^{1-o(1)}$$

so that the previous estimates gives

$$\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \leq N^{2(1-a)-\varepsilon}\right) \leq C/(\log N)^2.$$

We now conclude as in the proof of the upper bound of Theorem 2.14. We apply the

Borel–Cantelli lemma along the sequence $(2^p)_{p\in\mathbb{N}}$ which yields

$$\liminf_{p\to\infty} \frac{\log|\mathcal{M}_{2^p}(a)|}{\log(2^p)} \geq 2(1-a), \qquad \mathbb{P}_{x_0}-\text{a.s.}$$

This finishes the proof of the lemma because $\log(2^{p+1})/\log(2^p) \to 1$ as $p\to\infty$. □

As mentioned at the end of Section 2.2, when we will use Eisenbaum's isomorphism, we will have to bound from above expectations of the form:

$$\mathbb{E}\left[1+\frac{\phi_N(x_0)}{s};A\right] := \mathbb{E}\left[\left(1+\frac{\phi_N(x_0)}{s}\right)\mathbf{1}_A\right]$$

for some given event A. We will use the following elementary lemma which we state here only for convenience:

Lemma 2.17. *For all N large enough and for all events A,*

$$\mathbb{E}\left[\left(1+\frac{\phi_N(x_0)}{s}\right);A\right] \leq (\log N)^2\mathbb{P}(A) + N^{-\log N}.$$

Proof. Using (2.16a), we have:

$$\begin{aligned}\mathbb{E}\left[\left(1+\frac{\phi_N(x_0)}{s}\right);A\right] &\leq (\log N)^2\,\mathbb{P}(A) + \mathbb{E}\left[\left(1+\frac{\phi_N(x_0)}{s}\right)\mathbf{1}_{\{1+\phi_N(x_0)/s\geq(\log N)^2\}}\right]\\ &\leq (\log N)^2\,\mathbb{P}(A) + \exp\left(-\frac{s^2}{2g}(\log N)^3(1+o(1))\right),\end{aligned}$$

which concludes the lemma. □

We now provide our proof of the lower bound of Theorem 2.14. In the following, we write our arguments with the starting point x_0 but note that the same also works for all starting points $x_0' \in V_N(x_0)$, which is required to apply Lemma 2.16.

Proof of the lower bound of Theorem 2.14. During the entire proof we will fix some small $\eta>0$. To ease notations, we will denote $Q_N := Q_N(x_0)$. Recall that if $x\in Q_N$ and $1\leq R\leq N^{1-\eta}$, Assumption 2.11 gives the existence of a subset $C_R(x)\subset Q_N$ which can be thought of as a circle of radius R around x. We will denote M_R^x the operator corresponding to taking the mean value of a function on this circle: if f is a function defined on Q_N, then

$$M_R^x f = \frac{1}{\#C_R(x)}\sum_{y\in C_R(x)} f(y) \in \mathbb{R}.$$

We use Eisenbaum's isomorphism with some $s>0$ ($s=1$ will do). Let $\varepsilon_N = 1/\sqrt{\log N}$ and for some $b>a$ (to be chosen later on, close to a) and ϕ_N a GFF independent of the walk, we define the good events at x:

$$G_N^{b,\eta}(x,\ell^{\tau_N}) = \left\{ M_R^x \ell^{\tau_N} \leq 2gb \left(\log \frac{N}{R}\right)^2, \forall R \in (2^p)_{p\in\mathbb{N}} \cap \left\{1,\dots,N^{1-\eta}\right\}\right\},$$
$$G_N^{\eta}(x,\phi_N) = \left\{ M_R^x \left(\frac{1}{2}(\phi_N+s)^2\right) \leq \varepsilon_N \left(\log \frac{N}{R}\right)^2, \forall R \in (2^p)_{p\in\mathbb{N}} \cap \left\{1,\dots,N^{1-\eta}\right\}\right\},$$

and

$$G_N^{b,\eta}(x) = G_N^{b,\eta}(x,\ell^{\tau_N}) \cap G_N^{\eta}(x,\phi_N). \tag{2.28}$$

We require the points to be never to thick at any scales (similar to [Ber17]). We restrict ourselves to Q_N (the subset of V_N where we control the Green function G_N) by considering:

$$\widetilde{\mathcal{M}}_N(a) = \mathcal{M}_N(a) \cap Q_N$$

and we will abusively write $\left|\widetilde{\mathcal{M}}_N(a) \cap G_N^{b,\eta}\right|$ when we mean $\sum_{x\in Q_N} \mathbf{1}_{\left\{x\in\widetilde{\mathcal{M}}_N(a)\right\}} \mathbf{1}_{G_N^{b,\eta}(x)}$. The Paley–Zigmund inequality gives:

$$\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \geq \frac{1}{2}\mathbb{E}_{x_0}\otimes \mathbb{E}\left[\left|\widetilde{\mathcal{M}}_N(a)\cap G_N^{b,\eta}\right|\right]\right) \geq \frac{1}{4}\frac{\mathbb{E}_{x_0}\otimes \mathbb{E}\left[\left|\widetilde{\mathcal{M}}_N(a)\cap G_N^{b,\eta}\right|\right]^2}{\mathbb{E}_{x_0}\otimes \mathbb{E}\left[\left|\widetilde{\mathcal{M}}_N(a)\cap G_N^{b,\eta}\right|^2\right]}$$

and it remains to estimate the first and second moments on the right hand side.

First Moment Estimate Firstly, we estimate the first moment without restricting to any event. Thanks to assumptions (2.16b) and (2.16c) and because, starting from x, the law of $\ell_x^{\tau_N}$ is exponential, we have:

$$\begin{aligned}
\mathbb{E}_{x_0}\left[\left|\widetilde{\mathcal{M}}_N(a)\right|\right] &= \sum_{x\in Q_N} \mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2\right)\\
&= \sum_{x\in Q_N} \frac{G_N(x_0,x)}{G_N(x,x)}\mathbb{P}_x\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2\right)\\
&= \sum_{x\in Q_N} \frac{G_N(x_0,x)}{G_N(x,x)} \exp\left(-\frac{2ga(\log N)^2}{G_N(x,x)}\right) = N^{2-2a+o(1)}.
\end{aligned}$$

To estimate the probability $\mathbb{P}\left(G_N^{\eta}(x,\phi_N)\right)$ we will first derive a large deviation estimate for $M_R^x\left((\phi_N+s)^2\right)$. The estimate we obtain is rough and does not take into account the fact that if R is large we should expect $M_R^x\left((\phi_N+s)^2\right)$ to be close to its mean. Writing $\mathcal{N}(\mu,\sigma^2)$ a Gaussian variable with mean μ and variance σ^2, by Jensen's inequality we have

$\forall \lambda > 0$ and $\forall t \in (0, 1/(2g))$

$$\begin{aligned}\mathbb{P}\Big(M_R^x\left((\phi_N+s)^2\right) \geq \lambda \log N\Big) &\leq e^{-t\lambda}\mathbb{E}\left[\exp\left(\frac{t}{\log N}M_R^x\left((\phi_N+s)^2\right)\right)\right] \\ &\leq e^{-t\lambda}\frac{1}{\#C_R(x)}\sum_{y\in C_R(x)}\mathbb{E}\left[\exp\left(\frac{t}{\log N}(\phi_N(y)+s)^2\right)\right] \\ &\leq e^{-t\lambda}\mathbb{E}\left[\exp\left\{(tg+o(1))\mathcal{N}(o(1),1+o(1))^2\right\}\right] \leq C(t)e^{-t\lambda}\end{aligned}$$

where $0 < C(t) < \infty$ because tg is smaller than $1/2$. Hence, we have obtained: for all $t \in (0, 1/(2g))$, there exists $C(t) \in (0, \infty)$ such that

$$\forall x \in Q_N, \forall 1 \leq R \leq N^{1-\eta}, \forall \lambda > 0, \mathbb{P}\left(M_R^x\left((\phi_N+s)^2\right) \geq \lambda \log N\right) \leq C(t)e^{-t\lambda}. \tag{2.29}$$

Hence, using the above estimate with $t = 1/(4g)$ for instance, if $x \in Q_N$, the probability that the good event at x linked to ϕ_N does not hold is:

$$\begin{aligned}\mathbb{P}\left(G_N^\eta(x,\phi_N)^c\right) &\leq \sum_{\substack{R=2^p,\ p\in\mathbb{N}\\ 1\leq R\leq N^{1-\eta}}}\mathbb{P}\left(M_R^x\left(\frac{1}{2}(\phi_N+s)^2\right) > \varepsilon_N\left(\log\frac{N}{R}\right)^2\right) \\ &\leq \sum_{\substack{R=2^p,\ p\in\mathbb{N}\\ 1\leq R\leq N^{1-\eta}}}\mathbb{P}\left(M_R^x\left(\frac{1}{2}(\phi_N+s)^2\right) > \eta^2\varepsilon_N(\log N)^2\right) \\ &\leq \exp\left(-C(\eta)\varepsilon_N\log N\right) \xrightarrow[N\to\infty]{} 0\end{aligned}$$

for some $C(\eta) > 0$. By independence of ϕ_N and the local times of the random walk, we thus have

$$\mathbb{P}_{x_0}\otimes\mathbb{P}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}(x)\right) = (1-o_\eta(1))\,\mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}(x,\ell^{\tau_N})\right).$$

Now, using the Eisenbaum's isomorphism and Lemma 2.17, we can bound from above the probability $\mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}(x,\ell^{\tau_N})^c\right)$, for a given $x \in Q_N$, by the sum over $R \in \{2^p, p\in\mathbb{N}\} \cap [1, N^{1-\eta}]$ of

$$\begin{aligned}&\mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2, M_R^x\left(\ell^{\tau_N}\right) \geq 2gb\left(\log\frac{N}{R}\right)^2\right) \\ &\leq \mathbb{E}\left[\left(1+\frac{\phi_N(x_0)}{s}\right); |\phi_N(x)+s|^2 \geq 4ga(\log N)^2, M_R^x\left(|\phi_N+s|^2\right) \geq 4gb\left(\log\frac{N}{R}\right)^2\right] \\ &\leq (\log N)^2\mathbb{P}\left(|\phi_N(x)+s|^2 \geq 4ga(\log N)^2, M_R^x\left(|\phi_N+s|^2\right) \geq 4gb\left(\log\frac{N}{R}\right)^2\right) \\ &\qquad + O\left(N^{-\log N}\right).\end{aligned}$$

By taking $\delta = 2\sqrt{a/g}$, we can bound from above the probability appearing in the last equation by:

$$\begin{aligned}
&(2+o(1))\mathbb{P}\left(\phi_N(x) \geq (2\sqrt{ga}+o(1))\log N, M_R^x\left(|\phi_N+s|^2\right) \geq 4gb\left(\log\frac{N}{R}\right)^2\right)\\
&= (2+o(1))\mathbb{P}\left(e^{\delta\phi_N(x)}\mathbf{1}_{\left\{M_R^x((\phi_N+s)^2)\geq 4gb\left(\log\frac{N}{R}\right)^2\right\}} \geq N^{2\sqrt{ga}\delta+o(1)}\right)\\
&\leq N^{-4a+o(1)}\mathbb{E}\left[e^{\delta\phi_N(x)}\mathbf{1}_{\left\{M_R^x((\phi_N+s)^2)\geq 4gb\left(\log\frac{N}{R}\right)^2\right\}}\right]\\
&= N^{-4a+o(1)}e^{\frac{\delta^2}{2}\mathbb{E}\left[\phi_N(x)^2\right]}\widetilde{\mathbb{P}}\left(M_R^x((\phi_N+s)^2) \geq 4gb\left(\log\frac{N}{R}\right)^2\right)
\end{aligned}$$

where $\widetilde{\mathbb{P}}$ is the shifted probability:

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{\delta\phi_N(x)-\frac{\delta^2}{2}\mathbb{E}\left[\phi_N(x)^2\right]}.$$

By Cameron–Martin theorem, under this new probability, ϕ_N has the same covariance structure but the mean of $\phi_N(y)$ is now given by:

$$\mathrm{Cov}_{\mathbb{P}}(\phi_N(y), \delta\phi_N(x)) = (2\sqrt{ga}+o_\eta(1))\log\frac{N}{d_\Gamma(x,y)} = (2\sqrt{ga}+o_\eta(1))\log\frac{N}{R} \text{ if } y \in C_R(x).$$

As we have taken $b > a$, we can apply our tail estimate (2.29) to show that,

$$\mathbb{P}_{x_0}\left(\ell_x^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}\left(x, \ell^{\tau_N}\right)^c\right) \leq N^{-2a-t+o(1)}$$

for some small $t > 0$ which may depend on η, a and b. With the estimate on the first moment without the event $G_N^{b,\eta}$, this shows that:

$$\mathbb{E}_{x_0} \otimes \mathbb{E}\left[\left|\widetilde{\mathcal{M}}_N(a) \cap G_N^{b,\eta}\right|\right] \geq N^{2(1-a)+o(1)}.$$

Second Moment Estimate To control the second moment, we adapt the ideas of [Ber17] to our framework: let $x, y \in Q_N$ such that $d_\Gamma(x,y) \leq N^{1-\eta}$. We can find some $R \in (2^p)_{p\in\mathbb{N}}, R \leq N^{1-\eta}$ such that

$$\frac{1}{2}\left(d_\Gamma(x,y) \vee 1\right) \leq R \leq d_\Gamma(x,y) \vee 1.$$

As before, we apply the Eisenbaum isomorphism, Lemma 2.17, an exponential Markov inequality, and using the fact that by Cauchy–Schwarz $|M_R^x\phi_N| \leq \sqrt{M_R^x((\phi_N+s)^2)}+s$, we have:

$$\begin{aligned}
&\mathbb{P}_{x_0} \otimes \mathbb{P}\left(\ell_x^{\tau_N} \text{ and } \ell_y^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}(x), G_N^{b,\eta}(y)\right) \\
&\leq (2+o(1))(\log N)^2 \mathbb{P}\Bigg(\phi_N(x) \text{ and } \phi_N(y) \geq (2\sqrt{ga}+o(1))\log N, \\
&\qquad\qquad\qquad\qquad M_R^x \phi_N \leq \left(2\sqrt{gb}+o_\eta(1)\right)\log\frac{N}{R}\Bigg) + N^{-\log N} \\
&\leq N^{-4a+o(1)}\left(\frac{N}{d_\Gamma(x-y)\vee 1}\right)^{4a} \widetilde{\mathbb{P}}\left(M_R^x\phi_N \leq \left(2\sqrt{gb}+o_\eta(1)\right)\log\frac{N}{R}\right) + N^{-\log N} \qquad (2.30)
\end{aligned}$$

where $\widetilde{\mathbb{P}}$ denotes the shifted probability defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{\delta\phi_N(x)+\delta\phi_N(y)-\frac{\delta^2}{2}\mathbb{E}\left[(\phi_N(x)+\phi_N(y))^2\right]} \text{ with } \delta = 2\sqrt{\frac{a}{g}}.$$

By Cameron–Martin theorem, under the probability $\widetilde{\mathbb{P}}, \phi_N$ has the same covariance structure but the mean of $\phi_N(z)$ is now given by:

$$\mathrm{Cov}_{\mathbb{P}}(\phi_N(z), \delta\phi_N(x)+\delta\phi_N(y)) = (4\sqrt{ga}+o_\eta(1))\log\frac{N}{R} \text{ if } z \in C_R(x)$$

by our particular choice of R. Thanks to Assumptions (2.16b) and (2.15b), one can check that the variance of $M_R^x\phi_N$ is equal to $(g+o_\eta(1))\log\frac{N}{R}$. Hence

$$\begin{aligned}
&\widetilde{\mathbb{P}}\left(M_R^x\phi_N \leq \left(2\sqrt{gb}+o_\eta(1)\right)\log\frac{N}{R}\right) \\
&\leq \mathbb{P}\left(\mathcal{N}(0,1) \leq -\left(2(2\sqrt{a}-\sqrt{b})+o_\eta(1)\right)\sqrt{\log\frac{N}{R}}\right) \\
&\leq \left(\frac{N}{R}\right)^{-2(2\sqrt{a}-\sqrt{b})^2+o_\eta(1)}.
\end{aligned}$$

Again thanks to our particular choice of R, we have obtained:

$$\begin{aligned}
&\mathbb{P}_{x_0} \otimes \mathbb{P}\left(\ell_x^{\tau_N}, \ell_y^{\tau_N} \geq 2ga(\log N)^2, G_N^{b,\eta}(x), G_N^{b,\eta}(y)\right) \\
&\leq N^{-4a+o_\eta(1)}\left(\frac{N}{d_\Gamma(x,y)\vee 1}\right)^{4a-2(2\sqrt{a}-\sqrt{b})^2}.
\end{aligned}$$

As $a<1$, we can choose $b>a$ close enough to a to ensure that the exponent $4a-2(2\sqrt{a}-\sqrt{b})^2$ is less than 2. We can then sum over all $x, y \in Q_N$ such that $|x-y| \leq N^{1-\eta}$ and use assumption (2.14) to find that:

$$\mathbb{E}_{x_0} \otimes \mathbb{E}\left[\left|\widetilde{\mathcal{M}}_N(a) \cap G_N^{b,\eta}\right|^2\right] \leq N^{4(1-a)+o_\eta(1)} + \sum_{\substack{x,y \in Q_N \\ d_\Gamma(x,y) \geq N^{1-\eta}}} \mathbb{P}_{x_0}\left(\ell_x^{\tau_N}, \ell_y^{\tau_N} \geq 2ga(\log N)^2\right).$$

We eventually treat our last sum noticing that the probability in this sum is not larger than (using (2.30) without the term $\tilde{\mathbb{P}}(\cdots)$):

$$N^{-4a+o(1)} \left(\frac{N}{d_\Gamma(x,y)}\right)^{4a} \leq N^{-4a+4a\eta+o(1)}.$$

This shows that the second moment is not larger than $N^{4(1-a+a\eta)+o_\eta(1)}$. To come back to the probability we wanted to bound from below, this implies:

$$\mathbb{P}_{x_0}\left(|\mathcal{M}_N(a)| \geq N^{2(1-a)+o(1)}\right) \geq N^{-4a\eta+o_\eta(1)}.$$

As this is true for all $\eta > 0$, it means that the probability is not less than $(1/N)^{o(1)}$. We can then use Lemma 2.16 to conclude the proof of Theorem 2.14. □

2.4 Higher dimensions

2.4.1 Proofs of Theorems 2.4, 2.5 and 2.6

This section is devoted to the proofs of Theorems 2.4, 2.5 and 2.6. Let us first recall the setting and introduce some new notations. Consider a continuous time (rate 1) random walk $(Y_t)_{t\geq 0}$ on $\mathbb{Z}^d$ for $d \geq 3$ and denote $\mathbb{P}_x$ and $\mathbb{E}_x$ its law and expectation starting from x. Writing $V_N = \{-N, \ldots, N\}^d$, we consider the first exit time of V_N and the first hitting time of x:

$$\tau_N := \inf\{t \geq 0, Y_t \notin V_N\}, \forall x \in \mathbb{Z}^d, \tau_x := \inf\{t \geq 0 : Y_t = x\}. \tag{2.31}$$

We will denote G and G_N the Green function on $\mathbb{Z}^d$ and on V_N respectively: for all $x, y \in \mathbb{Z}^d$,

$$G(x,y) := \mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{Y_t=y\}} dt\right] \text{ and } G_N(x,y) := \mathbb{E}_x\left[\int_0^{\tau_N} \mathbf{1}_{\{Y_t=y\}} dt\right]. \tag{2.32}$$

Finally, we denote $g := G(0,0)$ the value of G on the diagonal and $\omega(x, dz)$ the harmonic measure on $[-1,1]^d$: for all $x \in [-1,1]^d, E \subset \partial[-1,1]^d, \omega(x,E)$ denotes the probability that a Brownian motion starting from x exits $[-1,1]^d$ through E. In the following, if $x \in \mathbb{R}^d$, we will denote $\lfloor x \rfloor$ one element of $\mathbb{Z}^d$ which is closest to x.

Let us first recall the behaviour of G_N in dimension greater or equal to 3:

Lemma 2.18. *For all $\eta \in (0,1)$, we have the following estimates:*

$$\forall x \in V_N, G_N(x,x) \leq g,$$
$$\forall x \in V_{(1-\eta)N}, G_N(x,x) \geq g + O_\eta\left(N^{2-d}\right).$$

Moreover, if $a_d = d/2\ \Gamma(d/2-1)\pi^{-d/2}$, we have for all $x \neq y \in V_N$,

$$G_N(x,y) = a_d\left(|x-y|^{2-d} - q_N(x,y)\right)$$

where $q_N(x,y) \geq O\left(|x-y|^{-d}\right)$ and for all $\tilde{x}, \tilde{y} \in (-1,1)^d$, we have the following pointwise estimate:

$$\lim_{N\to\infty} N^{d-2} q_N\left(\lfloor N\tilde{x}\rfloor, \lfloor N\tilde{y}\rfloor\right) = \int_{\partial[-1,1]^d} |\tilde{y}-\tilde{z}|^{2-d}\omega(\tilde{x}, d\tilde{z}) =: q(\tilde{x},\tilde{y}). \tag{2.33}$$

The proof of this lemma will be given in Section 2.4.3. As mentioned in Section 2.2, a key point is to show that all the moments of the number of thick points converge which is the purpose of the next proposition. Before stating it, let us introduce some notations.

Notation: If $k \geq 1$ and $q \geq 1$, we denote by $f(k \to q)$ the number of ways to partition a set with k elements into q non empty sets. As this is equal to the number of surjective functions from $\{1 \dots k\}$ to $\{1 \dots q\}$ divided by $q!$, we have

$$f(k \to q) = \frac{1}{q!}\sum_{i=1}^{q}\binom{q}{i}(-1)^{q-i}i^k. \tag{2.34}$$

If X is a topological space we will denote by $\mathcal{B}(X)$ the class of Borel sets of X.

Proposition 2.19. *Let $r \geq 1$ and for all $i = 1\dots r$, take $k_i \geq 1, A_i \in \mathcal{B}([-1,1]^d)$ such that the Lebesgue measure of $\bar{A}_i \backslash A_i^\circ$ vanishes, $T_i \in \mathcal{B}(\mathbb{R})$ with $\inf T_i > -\infty$. Moreover, we assume that the $A_i \times T_i$'s are pairwise disjoint. By denoting $k = k_1 + \cdots + k_r$ we define*

$$\begin{aligned} m(A_i \times T_i, k_i, i = 1\dots r) := &\left(\frac{a_d}{g}\right)^k \prod_{i=1}^{r}\left(\int_{T_i} e^{-t/g}\frac{dt}{g}\right)^{k_i} \\ &\times \sum_{\sigma \in \mathfrak{S}_k}\int_{A_1^{k_1}\times\cdots\times A_r^{k_r}}\prod_{i=0}^{k-1}\left(\left|y_{\sigma(i+1)} - y_{\sigma(i)}\right|^{2-d} - q\left(y_{\sigma(i)}, y_{\sigma(i+1)}\right)\right)dy_1\dots dy_k \end{aligned} \tag{2.35}$$

with the convention $y_{\sigma(0)} = 0$.

1. *Subcritical regime: let $a \in [0,1)$ and if $a = 0$ assume furthermore that $T_i \subset (0,\infty)$*

for all i. Then

$$\lim_{N\to\infty} \mathbb{E}_0 \left[\prod_{i=1}^r \left\{ \nu_N^a \left(A_i \times T_i \right) \right\}^{k_i} \right] = m(A_i \times T_i, k_i, i = 1 \dots r). \tag{2.36}$$

2. At criticality,

$$\lim_{N\to\infty} \mathbb{E}_0 \left[\prod_{i=1}^r \left\{ \nu_N^1 \left(A_i \times T_i \right) \right\}^{k_i} \right] = \sum_{\substack{1 \le q_i \le k_i \\ i=1\dots r}} \left(\prod_{i=1}^r f(k_i \to q_i) \right) m \left(A_i \times T_i, q_i, i = 1 \dots r \right). \tag{2.37}$$

The previous results also hold if we replace ν_N^a by μ_N^a.

We postpone the proof of this proposition to the next section and we now explain how we can deduce Theorems 2.4, 2.5 and 2.6 from it. We start with Theorem 2.4.

Proof of Theorem 2.4. This proof will be decomposed in three small parts. First, we will show that the previous proposition implies the joint convergence of $(\nu_N^a(A_1 \times T_1), \dots, \nu_N^a(A_r \times T_r))$ with suitable A_i's and T_i's. The second part is relatively standard and shows that it then implies the convergence in law of the sequence of random measures $\{\nu_N^a, N \geq 1\}$. The third part is dedicated to the identification of the limiting measures.

Step 1. Take $a \in [0,1]$. Let us first show that the previous proposition implies the convergence of the joint distribution $(\nu_N^a(A_1 \times T_1), \dots, \nu_N^a(A_r \times T_r))$ where the A_i's and T_i's are as in the statement of the proposition. As all their moments converge, we just need to check that the limiting moments do not grow too rapidly. Take $k_1 \dots k_r \geq 1$. We notice that for all $x \in [-1,1]^d$,

$$\begin{aligned} 0 \le \int_{[-1,1]^d} \left(|y-x|^{2-d} - q(x,y) \right) dy &\le \int_{[-1,1]^d} |y-x|^{2-d} \, dy \\ &\le \int_{[-2+x,2+x]^d} |y-x|^{2-d} \, dy = C \end{aligned}$$

for some universal constant C depending only on the dimension d. Hence there exists C' depending on d and on the T_i's such that

$$m(A_i \times T_i, k_i, i = 1 \dots r) \le C'^k k! \tag{2.38}$$

with $k = k_1 + \dots + k_r$. In particular, it implies that the moment generating function associated to those moments has a positive radius of convergence and they determine a unique law. It thus proves the claimed convergence in the subcritical regime. At criticality, we notice that for all $q \leq k$,

$$\sum_{\substack{1 \leq q_i \leq k_i \\ i=1\dots r}} \mathbf{1}_{\{q_1+\cdots+q_r=q\}} \prod_{i=1}^r f(k_i \to q_i)$$

is not larger than the number of ways to partition a set of k elements into no more than q parts which is equal to $q^k/(q!)$. Using (2.38), it implies that

$$\begin{aligned}
&\sum_{\substack{1 \leq q_i \leq k_i \\ i=1\dots r}} \left(\prod_{i=1}^r f(k_i \to q_i) \right) m\left(A_i \times T_i, q_i, i = 1 \dots r\right) \\
&\qquad \leq \sum_{q=r}^k C'^q q! \sum_{\substack{1 \leq q_i \leq k_i \\ i=1\dots r}} \mathbf{1}_{\{q_1+\cdots+q_r=q\}} \prod_{i=1}^r f(k_i \to q_i) \leq \sum_{q=r}^k C'^q q^k \leq C'^k k^{k+1} \leq \tilde{C}^k k!.
\end{aligned}$$

Again the radius of convergence of the associated moment generating function is positive and it gives the required convergence in the critical case as well. We will denote $\nu^a(A_1 \times T_1), \dots, \nu^a(A_r \times T_r)$ random variables which have the limiting distribution of $(\nu_N^a(A_1 \times T_1), \dots, \nu_N^a(A_r \times T_r))$.

Step 2. We now show the convergence of the sequence of random measures $\{\nu_N^a, N \geq 1\}$. Recalling that the underlying topology is the topology of vague convergence, it is enough to show that for all function $\phi : [-1,1]^d \times \mathbb{R} \to [0,\infty)$ which are $\mathcal{C}^\infty$ with compact support (included in $[-1,1]^d \times (0,\infty)$ if $a = 0$),

$$\langle \nu_N^a, \phi \rangle := \int_{[-1,1]^d \times \mathbb{R}} \phi(x,t) d\nu_N^a(x,t)$$

converges in distribution. It is enough to check that for all L-Lipschitz function $h : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}_0\left[h(\langle \nu_N^a, \phi \rangle)\right]$ converges. By Lemma 2.26, we can uniformly approximate ϕ by a sequence of functions $(\phi_p)_{p \geq 1}$ taking the following form:

$$\phi_p = \sum_{i=1}^p a_i^{(p)} \mathbf{1}_{A_i^{(p)} \times T_i^{(p)}}$$

where $A_i^{(p)} \in \mathcal{B}([-1,1]^d)$ with the Lebesgue measure of $\bar{A}_i^{(p)} \backslash (A_i^{(p)})^\circ$ vanishing, $T_i^{(p)} \in \mathcal{B}(\mathbb{R})$ with $\inf T_i^{(p)} > -\infty$ ($\inf T_i^{(p)} > 0$ if $a = 0$) and $a_i^{(p)} \in \mathbb{C}$. By the joint convergence proven in Step 1, for all $p \geq 1$,

$$\lim_{N \to \infty} \langle \nu_N^a, \phi_p \rangle \overset{\text{(d)}}{=} \langle \nu^a, \phi_p \rangle$$

and we can define the law (by dominated convergence theorem for instance)

$$\langle \nu^a, \phi \rangle \overset{\text{(d)}}{:=} \lim_{p \to \infty} \langle \nu^a, \phi_p \rangle .$$

We are going to show that we can exchange the two limits, i.e. that $\langle \nu_N^a, \phi \rangle$ converges in law to $\langle \nu^a, \phi \rangle$. Recalling that h is L-Lipschitz, $|\mathbb{E}_0 [h(\langle \nu_N^a, \phi \rangle)] - \mathbb{E}_0 [h(\langle \nu^a, \phi \rangle)]|$ is not larger than

$$\begin{aligned} &|\mathbb{E}_0 [h(\langle \nu_N^a, \phi_p \rangle)] - \mathbb{E}_0 [h(\langle \nu^a, \phi_p \rangle)]| + L\mathbb{E}_0 [\langle \nu_N^a, |\phi - \phi_p| \rangle] \\ &+ |\mathbb{E}_0 [h(\langle \nu^a, \phi \rangle)] - \mathbb{E}_0 [h(\langle \nu^a, \phi_p \rangle)]| . \end{aligned}$$

By the first part of the proof, the first term goes to zero as N goes to infinity. If $t_0 \in \mathbb{R}$ is such that the support of ϕ is included in $[-1,1]^d \times (t_0, \infty)$, then the second term is not larger than

$$L \left\| \phi - \phi_p \right\|_\infty \mathbb{E}_0 \left[\nu_N^a([-1,1]^d \times (t_0, \infty)) \right] \xrightarrow[N \to \infty]{} L 2^{-p} \mathbb{E}_0 \left[\nu^a([-1,1]^d \times (t_0, \infty)) \right].$$

Thus the limit of the second term goes to zero when $p \to \infty$. The third term goes to zero by definition and we have proved

$$\lim_{N \to \infty} \mathbb{E}_0 [h(\langle \nu_N^a, \phi \rangle)] = \mathbb{E}_0 [h(\langle \nu^a, \phi \rangle)] .$$

Step 3. The convergence of the sequence of random measures $\{\nu_N^a, N \geq 1\}$ has thus been proved. We are now going to identify the limit. What we did in Step 1 and Step 2 shows that the limiting distribution is entirely determined by the limiting moments from Proposition 2.19. In particular, the same conclusion holds for both $\{\nu_N^a, N \geq 1\}$ and $\{\mu_N^a, N \geq 1\}$ and this shows that these two sequences converge and have the same limiting distribution. We are now going to show that the limiting measures can be expressed in terms of the occupation measure μ_{occ} and a Poisson point process as explained in Theorem 2.4. We start with the subcritical regime ($a < 1$). Take $A_i \times T_i, i = 1 \dots r$, as in Proposition 2.19, $k_1, \dots, k_r \geq 1$ and denote $k = k_1 + \dots + k_r$. As

$$(x, y) \mapsto a_d \left(|x - y|^{2-d} - q(x, y) \right)$$

is the Green function associated to Brownian motion killed at the first exit time τ of $[-1,1]^d$ (see equation (3.15) of [Bas95] for instance), it is not hard to see that

$$\begin{aligned} &\mathbb{E}_0 \left[\prod_{i=1}^r \mu_{\mathrm{occ}}(A_i)^{k_i} \right] \\ &= \sum_{\sigma \in \mathfrak{S}_k} \int_{A_1^{k_1} \times \dots \times A_r^{k_r}} \prod_{i=0}^{k-1} a_d \left(\left| y_{\sigma(i+1)} - y_{\sigma(i)} \right| - q \left(y_{\sigma(i)}, y_{\sigma(i+1)} \right) \right) dy_1 \cdots dy_k \end{aligned}$$

with the convention $y_{\sigma(0)} = 0$. Thus

$$\mathbb{E}\left[\prod_{i=1}^{r}\left(\frac{1}{g}\mu_{\mathrm{occ}}(A_i)\int_{T_i} e^{-t_i/g}\frac{dt_i}{g}\right)^{k_i}\right] = m(A_i \times T_i, k_i, i = 1 \dots r). \tag{2.39}$$

This proves the identification (2.4) of the limiting measure in the subcritical regime. Let us now consider the critical case $a = 1$. Recalling the definition of f in (2.34) we see that the equation (2.60) of Lemma 2.25 implies that if $P_1(\lambda_1), \dots, P_r(\lambda_r)$ are independent Poisson random variables with parameters $\lambda_1, \dots, \lambda_r$,

$$\mathbb{E}\left[P_1(\lambda_1)^{k_1} \dots P_r(\lambda_r)^{k_r}\right] = \sum_{\substack{1 \leq q_i \leq k_i \\ i=1\dots r}} \left(\prod_{i=1}^{r} f(k_i \to q_i)\right) \lambda_1^{q_1} \dots \lambda_r^{q_r}.$$

Using (2.39), this now shows (2.5) and it concludes the proof. □

We now move on to the proof of Theorem 2.5.

Proof of Theorem 2.5. Take $a \in [0, 1]$. In the proof of Theorem 2.4 we showed that

$$|\mathcal{M}_N(a)| / N^{2(1-a)} = \nu_N^a([-1,1]^d \times (0,\infty))$$

converges to $\nu^a([-1,1]^d \times (0,\infty))$. The identities (2.6) and (2.7) come from (2.4) and (2.5) and from the fact that $\mu_{\mathrm{occ}}([-1,1]^d) = \tau$ a.s. □

We will finish this section by proving Theorem 2.6.

Proof of Theorem 2.6. Let $t \in \mathbb{R}$. Because the discrete random variables

$$\nu_N^1\left([-1,1]^d \times (t,\infty)\right), N \geq 1,$$

converge in law to a Poisson distribution with parameter $\tau e^{-t/g}/g$, we have

$$\begin{aligned} \lim_{N\to\infty} \mathbb{P}_0\left(\sup_{x\in V_N} \ell_x^{\tau_N} - 2g\log N \leq t\right) &= \lim_{N\to\infty} \mathbb{P}_0\left(\nu_N^1\left([-1,1]^d \times (t,\infty)\right) = 0\right) \\ &= \mathbb{E}\left[\exp\left(-\frac{\tau}{g}e^{-t/g}\right)\right]. \end{aligned}$$

This concludes the proof. □

2.4.2 Proof of Proposition 2.19

In this section, we will prove Proposition 2.19 stated in the previous section. We are first going to lay the groundwork by stating some technical lemmas which will be used in the proof of Proposition 2.19. These lemmas, except the next one, will be proven in Section 2.4.3.

We start with a well-known and easy lemma that we state for convenience. This lemma is valid for more general Markov chains.

Lemma 2.20. *For all subset $A \subset \mathbb{Z}^d$, starting from x, $\ell_x^{\tau_A}$ and $Y_{\tau_A} \mathbf{1}_{\{\tau_A < \infty\}}$ are independent.*

Proof. Consider a trajectory of the random walk Y starting at x and killed at τ_A. We can decompose it according to the excursions away from x. There is a geometric number of independent excursions. The last one is conditioned to not come back to x whereas the previous ones are i.i.d. excursions conditioned to come back to x. To conclude the proof, we notice that $Y_{\tau_A} \mathbf{1}_{\{\tau_A < \infty\}}$ depends on the last excursion whereas $\ell_x^{\tau_A}$ depends on the previous ones. □

Remark 2.21. This lemma implies in particular that conditioned on $Y_{\tau_A} \mathbf{1}_{\{\tau_A < \infty\}}$ and starting from x, $\ell_x^{\tau_A}$ is still an exponential variable with mean $\mathbb{E}_x \left[\ell_x^{\tau_A} \right]$. We also want to emphasise that this lemma is no longer true if the walk does not start at x.

Now, consider the k-th moment of $\nu_N^a (A \times T)$. To compute it, we will have to estimate the probability that in k different points, say $x_1, \dots, x_k$, the local times belong to $2ga \log N + T$. To capture the correlations of those local times, we will denote by E (to ease notation, we omit the dependence in N and $x_1, \dots, x_k$) the number of excursions between the x_i's before the time τ_N. More precisely, if we define

$$\begin{aligned} \varsigma_0 &:= \inf \{ t \geq 0 : Y_t \in \{x_1, \dots, x_k\} \}, \\ \forall p \geq 1, \varsigma_p &:= \inf \left\{ t \geq \varsigma_{p-1} : Y_t \in \{x_1, \dots, x_k\} \backslash \left\{ Y_{\varsigma_{p-1}} \right\} \right\}, \end{aligned}$$

then

$$E := \max \{ p \in \mathbb{N}, \varsigma_p \leq \tau_N \} \tag{2.40}$$

with the convention $\max \varnothing = -\infty$. The lemma below studies some properties of E. It roughly states that the typical way to visit all the points $x_1, \dots, x_k$ corresponds to $E = k-1$. It means that there exists a permutation σ of the set of indices $\{1, \dots, k\}$ so that we have the following: the walk first hits $x_{\sigma(1)}$, then hits $x_{\sigma(2)}$, etc. When the walk has visited $x_{\sigma(i)}$ it does not come back to the vertices $x_{\sigma(1)}, \dots, x_{\sigma(i-1)}$. We will denote $\mathfrak{S}_k$ the set of permutations of $\{1, \dots, k\}$.

Lemma 2.22. *There exist $C_k > 0$ and an integrable function*

$$U : \left\{ (y_1, \dots, y_k) \in \left([-1,1]^d \backslash \{0\} \right)^k : \forall i \neq j, y_i \neq y_j \right\} \to (0, \infty) \tag{2.41}$$

such that the following is true. For all $(y_1,\dots,y_k)$ *and* $(y_1',\dots,y_k')$ *where* U *is defined we have*

$$U(y_1,\dots,y_k) \leq \max_{0\leq i\neq j\leq k} \left(\frac{\left|y_i'-y_j'\right|}{\left|y_i-y_j\right|} \right)^{d-2} U(y_1',\dots,y_k') \tag{2.42}$$

with the convention $y_0=y_0'=0$. *For all* $p\geq k-1$ *and all* $x_1,\dots,x_k$ *non zero and pairwise distinct elements of* V_N,

$$\mathbb{P}_0\left(E=p,\tau_{x_i}<\tau_N\ \forall i=1\dots k\right) \leq C_k^{p+1}\left(\max_{i\neq j}|x_i-x_j|^{2-d}\right)^{p-k+1} N^{(2-d)k} U\left(\tfrac{x_1}{N},\dots,\tfrac{x_k}{N}\right). \tag{2.43}$$

Moreover, if $x_1=\lfloor Ny_1\rfloor,\dots,x_k=\lfloor Ny_k\rfloor$, *for* $y_1,\dots,y_k$ *non zero and pairwise distinct elements of* $(-1,1)^d$, *we have the following pointwise estimate:*

$$\begin{aligned}\lim_{N\to\infty} & N^{(d-2)k}\mathbb{P}_0\left(E=k-1,\tau_{x_i}<\tau_N\ \forall i=1\dots k\right) \\ &= \left(\frac{a_d}{g}\right)^k \sum_{\sigma\in\mathfrak{S}_k}\prod_{i=0}^{k-1}\left(\left|y_{\sigma(i+1)}-y_{\sigma(i)}\right|^{2-d} - q\left(y_{\sigma(i)},y_{\sigma(i+1)}\right)\right)\end{aligned} \tag{2.44}$$

with the convention $y_{\sigma(0)}=0$.

Remark 2.23. It is important for us to give a better estimate than

$$\forall p\geq k-1, \mathbb{P}_0\left(E=p,\tau_{x_i}<\tau_N\ \forall i=1\dots k\right) \leq C_k^p \max_i |x_i|^{2-d}\left(\max_{i\neq j}|x_i-x_j|^{2-d}\right)^p$$

because the function

$$(y_1,\dots,y_k)\in\prod_{i=1}^k(-1,1)^d \mapsto \max_i |y_i|^{2-d}\left(\max_{i\neq j}|y_i-y_j|^{2-d}\right)^{k-1}\in(0,\infty)$$

is not integrable if $(k-1)(d-2)\geq d$.

As mentioned in Section 2.2, in the subcritical regime we will be able to restrict ourselves to points $x_1,\dots,x_k$ which are far away from each other. At criticality we will have to deal with points which are close to each other. The following lemma shows that two distinct close points are not thick at the same time with high probability:

Lemma 2.24. *For* $x,y\in\mathbb{Z}^d$, *consider a sequence* $\left(\ell_x^{\infty,i},\ell_y^{\infty,i}\right), i\geq 1$, *of i.i.d. variables with the same law as* $\left(\ell_x^\infty,\ell_y^\infty\right)$ *under* $\mathbb{P}_x$. *If* $x\neq y$, *then for all* $p\geq 1$, *there exists* $\varepsilon_p>0$ *independent of* x *and* y *such that for all* $t\in\mathbb{R}$,

$$\mathbb{P}\left(\sum_{i=1}^p \ell_x^{\infty,i}, \sum_{i=1}^p \ell_y^{\infty,i} \geq 2g\log N + gt\right) \leq N^{-2-\varepsilon_p+o(1)}.$$

We have now all the ingredients we need to start the proof of Proposition 2.19.

Proof of Proposition 2.19. To ease notations, we will restrict ourselves to the case of the k-th moment of $\nu_N^a(A \times T)$ for $A \in \mathcal{B}([-1,1]^d)$ such that the Lebesgue measure of $\bar{A} \backslash A^\circ$ vanishes and $T \in \mathcal{B}(\mathbb{R})$ with $\inf T > -\infty$ ($\inf T > 0$ if $a = 0$). Indeed, the proof of the general case follows almost entirely along the same lines and throughout the proof we will explain which arguments need to be changed to treat the case of mixed moments

$$\mathbb{E}_0 \left[\prod_{i=1}^r \{\nu_N^a(A_i \times T_i)\}^{k_i} \right].$$

When we will refer to the general case, k will denote $k_1 + \cdots + k_r$.

In the following, we will take N large enough so that $2ga \log N + T \subset (0, \infty)$. To ease notations, we will denote

$$M_N := \nu_N^a(A \times T) \text{ and } A_N := \{x \in V_N : x/N \in A\}. \tag{2.45}$$

The k-th moment of M_N can be written as

$$\mathbb{E}_0 \left[(M_N)^k \right] = N^{-2(1-a)k} \sum_{x_1, \dots, x_k \in A_N} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2ga \log N + T \right).$$

For some $r_N = N^{o(1)}$ (to be chosen later on), we introduce the set of well-separated points

$$A_{N,k} := \left\{ (x_1, \dots, x_k) \in (A_N \backslash \{0\})^k : \min_{i \neq j} |x_i - x_j| > 2r_N \right\}.$$

The proof will be decomposed in four parts. The first one will estimate the contribution of $A_{N,k}$ to the k-th moment of M_N. This part does not need to treat the subcritical ($a < 1$) and critical ($a = 1$) cases separately. Then, the second part shows that the contribution of points $(x_1, \dots, x_k) \in (A_N)^k \backslash A_{N,k}$ to the k-th moment of M_N vanishes in the subcritical regime. The third part deals with the critical case and handles the points that are close to each other. The fourth part will briefly show the results on μ_N^a.

2.4.2.1 Contribution of points far away from each other, ν_N^a.

The goal of this part is to show that for all $a \in [0,1]$,

$$\lim_{N \to \infty} N^{-2(1-a)k} \sum_{(x_1, \dots, x_k) \in A_{N,k}} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2ga \log N + T \right) = m(A \times T, k). \tag{2.46}$$

We will write

$$M_{N,k} := N^{-2(1-a)k} \sum_{(x_1,\dots,x_k)\in A_{N,k}} \mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} \in 2ga\log N + T\right).$$

For a given $x \in V_N \backslash \partial V_N$, the Lebesgue measure of the set $\{y \in (-1,1)^d : \lfloor Ny \rfloor = x\}$ is $(1/N)^d$. Hence we can write

$$M_{N,k} = N^{(d-2+2a)k} \int_{\prod_{i=1}^k (-1,1)^d} \mathbb{P}_0\left(\ell_{\lfloor Ny_1\rfloor}^{\tau_N},\dots,\ell_{\lfloor Ny_k\rfloor}^{\tau_N} \in 2ga\log N + T\right) \times \mathbf{1}_{\{(\lfloor Ny_1\rfloor,\dots,\lfloor Ny_k\rfloor)\in A_{N,k}\}} dy_1 \dots dy_k. \tag{2.47}$$

We will first bound from above the integrand. This will provide us the domination we need in order to apply the dominated convergence theorem and we will be left to show the pointwise limit.

Let $(x_1,\dots,x_k) \in A_{N,k}$. By definition of E (equation (2.40)), if the walk visits all the x_i's before τ_N, then $E \geq k-1$. Thus

$$\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} \in 2ga\log N + T, E \leq k-2\right) = 0.$$

In this paragraph, we will use Lemma 2.22 to show that the probability

$$\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} \in 2ga\log N + T, E \geq k\right)$$

is very small. First, by denoting $t := \inf T/g$, we can bound

$$\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} \in 2ga\log N + T, E \geq k\right) \leq \mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} > 2ga\log N + gt, E \geq k\right).$$

Starting from x_1, the law of the time spent in x_1 before hitting $\partial V_N \cup \{x_2,\dots,x_k\}$ is an exponential law with mean at most g. Also, if $E = p$, the number of excursions from x_1 to $\{x_2,\dots,x_k\}$ before τ_N is not larger than p. Hence, by Lemma 2.20 conditioned on the event $\{E = p, \tau_{x_i} < \tau_N \ \forall i \leq k\}$, the joint law $(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N})$ is stochastically dominated by the law of k independent Gamma random variables with shape parameter $p+1$ and scale parameter g. Using the claim (2.61) of Lemma 2.25 about the Gamma distribution, it implies that

$$\begin{aligned}
&\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} > 2ga\log N + gt | E = p, \tau_{x_i} < \tau_N \ \forall i \leq k\right) \\
&\leq N^{-2ak} e^{-kt} \sum_{q=0}^{kp} (2a\log N + t)^q \frac{k^q}{q!}.
\end{aligned}$$

By definition of $A_{N,k}$, $\min_{i\neq j}|x_i - x_j| \geq 2r_N$. Let $U(x_1,\dots,x_k)$ be as in Lemma 2.22. Then

$$\begin{aligned}
&\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} > 2ga\log N + gt, E \geq k\right)\\
&= \sum_{p\geq k}\mathbb{P}_0\left(E=p, \tau_{x_i}<\tau_N\ \forall i\leq k\right)\\
&\qquad\qquad \times \mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} > 2ga\log N + gt | E=p, \tau_{x_i}<\tau_N\ \forall i\leq k\right)\\
&\leq N^{-(d-2+2a)k}e^{-kt}U\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right)\sum_{p\geq k}\left(C_k r_N^{\frac{2-d}{k}}\right)^p\sum_{q=0}^{kp}(2a\log N+t)^q\frac{k^q}{q!}\\
&= N^{-(d-2+2a)k}e^{-kt}U\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right)\sum_{q\geq 0}\frac{((2a\log N+t)k)^q}{q!}\sum_{p\geq\lceil q/k\rceil\vee k}\left(C_k r_N^{\frac{2-d}{k}}\right)^p\\
&\leq C_k' N^{-(d-2+2a)k}e^{-kt}U\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right)\sum_{q\geq 0}\frac{((2a\log N+t)k)^q}{q!}\left(C_k r_N^{\frac{2-d}{k}}\right)^{\lceil q/k\rceil\vee k}\\
&\leq C_k'' r_N^{\frac{2-d}{2}} N^{-(d-2+2a)k}e^{-kt}U\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right)\sum_{q\geq 0}\left\{(2a\log N+t)kC_k^{\frac{1}{2k}}r_N^{\frac{2-d}{2k^2}}\right\}^q/q! \qquad (2.48)
\end{aligned}$$

because $\left\lceil\frac{q}{k}\right\rceil\vee k \geq \frac{k}{2}+\frac{q}{2k}$ for all $q\geq 0$. If we choose $r_N = \exp\left(\sqrt{\log N}\right) = N^{o(1)}$ for instance, then $(2a\log N+t)kC_k^{1/(2k)}r_N^{(2-d)/(2k^2)}$ goes to zero and we have obtained:

$$\mathbb{P}_0\left(\ell_{x_1}^{\tau_N},\dots,\ell_{x_k}^{\tau_N} \geq 2ga\log N + gt, E\geq k\right) \leq o(1)N^{-(d-2+2a)k}e^{-kt}U\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right). \quad (2.49)$$

According to Lemma 2.22, the function $(y_1,\dots,y_k)\in(-1,1)^k \mapsto U(y_1,\dots,y_k)\in(0,\infty)$ is integrable. Moreover, the equation (2.42) of Lemma 2.22 implies that if $y_1,\dots,y_k\in(-1,1)^d$ are such that $(\lfloor Ny_1\rfloor,\dots,\lfloor Ny_k\rfloor)\in A_{N,k}$, then

$$U\left(\frac{\lfloor Ny_1\rfloor}{N},\dots,\frac{\lfloor Ny_k\rfloor}{N}\right)\leq C_{k,d}U(y_1,\dots,y_k)$$

for some $C_{k,d}>0$. Coming back to the equation (2.47) we have thus shown with the equation (2.49) that:

$$\begin{aligned}
M_{N,k} = o(1) + N^{(d-2+2a)k}\int_{\prod_{i=1}^k(-1,1)^d} & dy_1\dots dy_k \mathbf{1}_{\{(\lfloor Ny_1\rfloor,\dots,\lfloor Ny_k\rfloor)\in A_{N,k}\}} \qquad (2.50)\\
&\times\mathbb{P}_0\left(\ell_{\lfloor Ny_1\rfloor}^{\tau_N},\dots,\ell_{\lfloor Ny_k\rfloor}^{\tau_N}\in 2ga\log N + T, E=k-1\right).
\end{aligned}$$

Our last task consists in controlling the probability appearing in the equation (2.50). By Lemma 2.20, conditioning on the event $\{E=k-1, \tau_{x_i}<\tau_N\ \forall i=1\dots k\}$, the local times $\ell_{x_i}^{\tau_N}, i=1\dots k$, are independent exponential variables with mean $\mathbb{E}_{x_i}\left[\ell_{x_i}^{\tau_N\wedge\min_{j\neq i}\tau_{x_j}}\right]\leq g$.

Consequently,

$$\mathbb{P}_0(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2ga \log N + T, E = k-1) \\ \leq N^{-2ak} \left(\int_T \frac{1}{g} e^{-s/g} ds \right)^k \mathbb{P}_0 \left(E = k-1, \tau_{x_i} < \tau_N \ \forall i \leq k \right). \tag{2.51}$$

Using the first estimate of Lemma 2.22, it implies that $M_{N,k}$ is bounded and it also provides us the domination we need to use the dominated convergence theorem. We have already done everything we need for the pointwise convergence. Indeed, if $x_1 = \lfloor N y_1 \rfloor, \dots, x_k = \lfloor N y_k \rfloor$, for $y_1, \dots, y_k$ non zero and pairwise distinct elements of $(-1,1)^d$, Lemma 2.22 provides an explicit expression for the pointwise limit

$$\lim_{N \to \infty} N^{(d-2)k} \mathbb{P}_0 \left(E = k-1, \tau_{x_i} < \tau_N \ \forall i = 1 \dots k \right)$$

and a small modification of the arguments in the proof of Lemma 2.22 shows that

$$\mathbb{E}_{\lfloor N y_i \rfloor} \left[\ell_{\lfloor N y_i \rfloor}^{\tau_N \wedge \min_{j \neq i} \tau_{\lfloor N y_j \rfloor}} \right] = g + O_{y_1, \dots, y_k} \left(N^{2-d} \right).$$

Hence

$$\lim_{N \to \infty} N^{2ka} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2ga \log N + T | E = k-1, \tau_{x_i} < \tau_N \ \forall i = 1 \dots k \right) \\ = \left(\int_T e^{-s/g} \tfrac{ds}{g} \right)^k .$$

Moreover,

$$\mathbf{1}_{\{\forall i \neq j, y_i \in A^\circ \setminus \{0\}, y_i \neq y_j\}} \leq \liminf_{N \to \infty} \mathbf{1}_{\left\{ (\lfloor N y_1 \rfloor, \dots, \lfloor N y_k \rfloor) \in A_{N,k} \right\}} \\ \leq \limsup_{N \to \infty} \mathbf{1}_{\left\{ (\lfloor N y_1 \rfloor, \dots, \lfloor N y_k \rfloor) \in A_{N,k} \right\}} \leq \mathbf{1}_{\left\{ \forall i \neq j, y_i \in \bar{A} \setminus \{0\}, y_i \neq y_j \right\}}.$$

Notice the interior A° and the closure $\bar{A}$ in the previous inequalities. As we have supposed that the Lebesgue measure of $\bar{A} \setminus A^\circ$ vanishes, putting things together leads to the convergence of $M_{N,k}$ to

$$\left(\frac{a_d}{g} \right)^k \left(\int_T e^{-s/g} \frac{ds}{g} \right)^k \sum_{\sigma \in \mathfrak{S}_k} \int_{A^k} \times \prod_{i=0}^{k-1} \left(\left| y_{\sigma(i+1)} - y_{\sigma(i)} \right|^{2-d} - q(y_{\sigma(i)}, y_{\sigma(i+1)}) \right) dy_1 \dots dy_k$$

with the convention $y_{\sigma(0)} = 0$. This completes the proof of (2.46).

2.4.2.2 Subcritical regime, ν_N^a.

We now show how the previous part allows us to conclude the proof in the subcritical regime. Suppose that $a < 1$. We show that the k-th moment of M_N converges towards $m(A \times T, k)$ by induction on $k \geq 1$. Thanks to (2.46), it only remains to control the contribution of points $(x_1, \dots, x_k) \in (A_N)^k \backslash A_{N,k}$ to the k-th moment of M_N. This contribution is at most

$$\begin{aligned} C(k,d) N^{-2(1-a)k} r_N^d \sum_{x_1,\dots,x_{k-1} \in A_N} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_{k-1}}^{\tau_N} \in 2ga \log N + T \right) \\ = C(k,d) N^{-2(1-a)} r_N^d \mathbb{E}_0 \left[(M_N)^{k-1} \right] \end{aligned}$$

which goes to zero: this is clear for $k = 1$ (because $r_N = N^{o(1)}$ and $a < 1$) and comes from the induction hypothesis for $k \geq 2$. With (2.46), we have shown that

$$\mathbb{E}_0 \left[(M_N)^k \right] = m(A \times T, k) + o(1).$$

This is exactly (2.36) in the case $r = 1$. In the general case of a mixed moment, we recover the result by the exact same method.

2.4.2.3 At criticality, ν_N^a.

Let us now consider the critical case $a = 1$. Unlike in the subcritical regime, the points $(x_1, \dots, x_k) \in (A_N)^k \backslash A_{N,k}$ will contribute to $\mathbb{E}_0 \left[(M_N)^k \right]$. We first notice that the points $(x_1, \dots, x_k) \in (A_N)^k$ with one of the x_i's being equal to zero do not contribute. Indeed, by ignoring the points which are within a distance $2r_N$ to each other or to zero, which contributes at most Cr_N^d for every such point, we have:

$$\begin{aligned} &\sum_{\substack{(x_1,\dots,x_k) \in (A_N)^k \\ \exists i, x_i = 0}} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2g \log N + T \right) \\ &\qquad \leq C_k \sum_{l=0}^{k-1} \left(Cr_N^d \right)^{k-1-l} \sum_{\substack{\forall i=1\dots l, |x_i| \geq 2r_N \\ \forall i \neq j, |x_i - x_j| \geq 2r_N}} \mathbb{P}_0 \left(\ell_0^{\tau_N}, \ell_{x_1}^{\tau_N}, \dots, \ell_{x_l}^{\tau_N} \in 2g \log N + T \right). \end{aligned}$$

The last sum is over l different points and we require the local times to be large in $l+1$ different points. We can then use the same arguments as in Section 2.4.2.1 (all the points are far away from each other) to show that this last sum is at most CN^{-2}. As $r_N = N^{o(1)}$ it shows that this contribution vanishes.

We are going to estimate

$$\sum_{(x_1,\dots,x_k) \in (A_N \backslash \{0\})^k \backslash A_{N,k}} \mathbb{P}_0 \left(\ell_{x_1}^{\tau_N}, \dots, \ell_{x_k}^{\tau_N} \in 2g \log N + T \right). \tag{2.52}$$

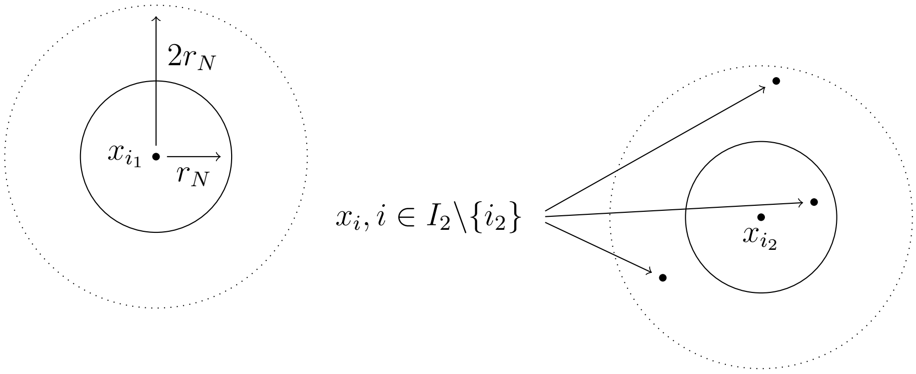


Figure 2.2: Decomposition of $(A_N \backslash \{0\})^k \backslash A_{N,k}$. The balls in solid lines do not overlap. Here $r = 2$.

If $(x_1, \dots, x_k) \in (A_N \backslash \{0\})^k \backslash A_{N,k}$, by definition of $A_{N,k}$, it means that there are at least two balls $B(x_i, r_N)$ which overlap. In the following, we will partition the set $(A_N \backslash \{0\})^k \backslash A_{N,k}$ according to the maximum number r ($r \leq k-1$) of balls which do not overlap. We will denote by x_{i_p}, $p = 1 \dots r$, the centres of such balls and we will partition the set of indices $\sqcup_{p=1}^r I_p = \{1, \dots, k\}$ such that for all $p = 1 \dots r, i \in I_p, \left| x_i - x_{i_p} \right| \leq 2r_N$. See Figure 2.2. The reader should think of the balls as small balls which are far away from each other. The choice of the partition (I_p) may be not unique. In this case, we make an arbitrary choice.

Our decomposition is thus:

$$(A_N \backslash \{0\})^k \backslash A_{N,k} = \bigcup_{r=1}^{k-1} \bigcup_{\substack{\sqcup_{p=1}^r I_p \\ = \{1, \dots, k\}}} W_{N,k,r,(I_p)}$$

where

$$W_{N,k,r,(I_p)} = \left\{ (x_1, \dots, x_k) \in (A_N \backslash \{0\})^k : \begin{array}{l} \forall p \neq q, \exists i_p \in I_p, i_q \in I_q, \left| x_{i_p} - x_{i_q} \right| > 2r_N, \\ \forall i \in I_p, \left| x_i - x_{i_p} \right| \leq 2r_N \end{array} \right\}.$$

For a given $W_{N,k,r,(I_p)}$, the contribution to the sum (2.52) of the elements $(x_1, \dots, x_k) \in W_{N,k,r,(I_p)}$ such that for all $p = 1 \dots r$, for all $i, j \in I_p, x_i = x_j$ is equal to

$$\sum_{(y_1, \dots, y_r) \in A_{N,r}} \mathbb{P}_0 \left(\ell_{y_1}^{\tau_N}, \dots, \ell_{y_r}^{\tau_N} \in 2g \log N + T \right)$$

which converges to $m(A \times T, r)$ (see (2.46)). As the number of ways to partition the set $\{1, \dots, k\}$ into r non empty sets is exactly equal to $f(k \to r)$, the claim of the proposition is equivalent to saying that the contribution of $W_{N,k,r,(I_p)}$ to the sum (2.52) comes only from these points. In other words, if we denote

$$W^{\neq}_{N,k,r,(I_p)} = \left\{ (x_1, \dots, x_k) \in W_{N,k,r,(I_p)} : \exists p = 1 \dots r, \exists i, j \in I_p, x_i \neq x_j \right\}$$

then we are going to show that

$$\sum_{(x_1,\dots,x_k) \in W^{\neq}_{N,k,r,(I_p)}} \mathbb{P}_0 \left(\ell^{\tau_N}_{x_1}, \dots, \ell^{\tau_N}_{x_k} \in 2g \log N + T \right) \xrightarrow[N \to \infty]{} 0.$$

By denoting $t := \inf T/g$, we can first bound:

$$\mathbb{P}_0 \left(\ell^{\tau_N}_{x_1}, \dots, \ell^{\tau_N}_{x_k} \in 2g \log N + T \right) \leq \mathbb{P}_0 \left(\ell^{\infty}_{x_1}, \dots, \ell^{\infty}_{x_k} > 2g \log N + gt \right).$$

If $(x_1, \dots, x_k) \in W^{\neq}_{N,k,r,(I_p)}$, then there exists $p_0 \in \{1, \dots, r\}$ and $j_{p_0} \in I_{p_0}$ such that $x_{i_{p_0}} \neq x_{j_{p_0}}$. To bound from above this last sum, for each $p \neq p_0$ we keep track of only one $x_k, k \in I_p$, by considering x_{i_p}. As for all $k \in I_p$, $\left| x_k - x_{i_p} \right| \leq 2r_N$, our estimate is increased by a multiplicative factor of order r_N^d for each point that we forget. For $p = p_0$, we keep track of both $x_{i_{p_0}}$ and $x_{j_{p_0}}$. Furthermore, $x_{j_{p_0}}$ will absorb all the $x_{i_p}, p \neq p_0$ which are within a distance $2r_N$ of $x_{j_{p_0}}$. This procedure implies that:

$$\begin{aligned} &\sum_{(x_1,\dots,x_k) \in W^{\neq}_{N,k,r,(I_p)}} \mathbb{P}_0 \left(\ell^{\infty}_{x_1}, \dots, \ell^{\infty}_{x_k} > 2g \log N + gt \right) \\ &\qquad \leq C \sum_{s=1}^{r} (r_N^d)^{k-s-1} \sum_{\substack{x_0,\dots,x_s \in A_N \\ x_0 \neq x_1, |x_0 - x_1| \leq 2r_N \\ \forall i \neq j, \{i,j\} \neq \{0,1\}, |x_i - x_j| > 2r_N}} \mathbb{P}_0 \left(\ell^{\infty}_{x_0}, \dots, \ell^{\infty}_{x_s} > 2g \log N + gt \right) \end{aligned} \tag{2.53}$$

where $C > 0$ may depend on d, k, r. We will conclude by showing that this last sum is not larger than $N^{-\varepsilon}$ for some $\varepsilon > 0$. Take $s \in \{1, \dots, r\}$ and $(x_0, x_1, \dots, x_s)$ as in the previous sum. If $s = 1$ it means that we just need to control the local times $\ell^{\infty}_{x_0}, \ell^{\infty}_{x_1}$. This has already been done in Lemma 2.24 and we are going to explain the slightly more delicate case $s \geq 2$. The idea is fairly similar to the one we used in the subcritical regime. Let us denote E the number of excursions between the sets $\{x_0, x_1\}, \{x_2\}, \dots, \{x_s\}$. First of all, let us notice that if we take $p_{\max} \geq s$, a small modification of the equation (2.48) gives:

$$\begin{aligned}
&\sum_{\substack{x_0,\ldots,x_s \in A_N \\ x_0 \neq x_1, |x_0 - x_1| \leq 2r_N \\ \forall i \neq j, \{i,j\} \neq \{0,1\}, |x_i - x_j| > 2r_N}} \mathbb{P}_0\left(\ell^\infty_{x_0}, \ldots, \ell^\infty_{x_s} > 2g\log N + gt, E \geq p_{\max}\right) \\
&\leq C(s,d)(r_N)^d \sum_{\substack{x_1,\ldots,x_s \in A_N \\ \forall i \neq j, |x_i - x_j| > 2r_N}} \mathbb{P}_0\left(\ell^\infty_{x_1}, \ldots, \ell^\infty_{x_s} > 2g\log N + gt, E \geq p_{\max}\right) \\
&\leq Ce^{-st} r_N^{(p_{\max}-s)(2-d)+d} N^{-ds} \sum_{\substack{x_1,\ldots,x_s \in A_N \\ \forall i \neq j, |x_i - x_j| > 2r_N}} U\left(\frac{x_1}{N}, \ldots, \frac{x_s}{N}\right) \leq Ce^{-st} r_N^{(p_{\max}-s)(2-d)+d}.
\end{aligned}$$

Hence if $p_{\max}$ is large enough, the negative power $(p_{\max} - s)(2-d) + d$ of r_N will kill the positive power $(k-s-1)d$ of r_N in the equation (2.53) and we are now left to control:

$$\sum_{\substack{x_0,\ldots,x_s \in A_N \\ x_0 \neq x_1, |x_0 - x_1| \leq 2r_N \\ \forall i \neq j, \{i,j\} \neq \{0,1\}, |x_i - x_j| > 2r_N}} \mathbb{P}_0\left(\ell^\infty_{x_0}, \ldots, \ell^\infty_{x_s} > 2g\log N + gt, E < p_{\max}\right).$$

Thanks to Lemmas 2.25 and 2.24 and using the notations in those lemmas, we have

$$\begin{aligned}
&\mathbb{P}_0\left(\ell^\infty_{x_0}, \ldots, \ell^\infty_{x_s} > 2g\log N + gt | E = p, \tau_{\{x_0,x_1\}}, \tau_{x_2}, \ldots, \tau_{x_s} < \infty\right) \\
&\leq \mathbb{P}\left(\Gamma(p+1, g) > 2g\log N + gt\right)^{s-1} \mathbb{P}\left(\forall \alpha = 0, 1, \sum_{i=1}^{p+1}\sum_{j=1}^{A_i} \ell^i_{x_\alpha, j} > 2g\log N + gt\right) \\
&\leq N^{-2s-\varepsilon_p}.
\end{aligned}$$

By summing (2.43) of Lemma 2.22 over all $p \geq s-1$, we also have

$$\begin{aligned}
\mathbb{P}_0\left(E = p, \tau_{\{x_0,x_1\}}, \tau_{x_2}, \ldots, \tau_{x_s} < \infty\right) &\leq 2\max_{\alpha=0,1} \mathbb{P}_0\left(\tau_{x_\alpha}, \tau_{x_2}, \ldots, \tau_{x_s} < \infty\right) \\
&\leq CN^{(2-d)s} \max_{\alpha=0,1} U\left(\frac{x_\alpha}{N}, \frac{x_2}{N}, \ldots, \frac{x_s}{N}\right).
\end{aligned}$$

We have obtained the existence of $\varepsilon > 0$ such that

$$\begin{aligned}
&\sum_{\substack{x_0,\dots,x_s \in A_N \\ x_0 \neq x_1, |x_0 - x_1| \leq 2r_N \\ \forall i \neq j, \{i,j\} \neq \{0,1\}, |x_i - x_j| \geq 2r_N}} \mathbb{P}_0 \left(\ell^\infty_{x_0}, \dots, \ell^\infty_{x_s} > 2g \log N + gt, E < p_{\max} \right) \\
&\leq N^{-ds-\varepsilon} \sum_{\substack{x_0,\dots,x_s \in A_N \\ x_0 \neq x_1, |x_0 - x_1| \leq 2r_N \\ \forall i \neq j, \{i,j\} \neq \{0,1\}, |x_i - x_j| \geq 2r_N}} \max_{\alpha=0,1} U \left(\frac{x_\alpha}{N}, \frac{x_2}{N}, \dots, \frac{x_s}{N} \right) \\
&\leq C(d)(r_N)^d N^{-ds-\varepsilon} \sum_{\substack{x_1,\dots,x_s \in A_N \\ \forall i \neq j, |x_i - x_j| \geq 2r_N}} U \left(\frac{x_1}{N}, \frac{x_2}{N}, \dots, \frac{x_s}{N} \right) \leq C(r_N)^d N^{-\varepsilon}
\end{aligned}$$

where we justify as before the last inequality thanks to the integrability of U and by (2.42). This concludes the proof of the estimates on $\mathbb{E}_0 \left[\{\nu_N^a(A \times T)\}^k \right]$ at criticality (equation (2.37) with $r = 1$).

In the general case of a mixed moment, we have to deal with points

$$\left\{ (x_1, \dots, x_k) \in (A_{1N} \backslash \{0\})^{k_1} \times \dots \times (A_{rN} \backslash \{0\})^{k_r} : \exists i \neq j, |x_i - x_j| \leq 2r_N \right\}.$$

As before, we decompose this set according to blocks of points which are close to each other. Again, only points which are equal inside a same block will contribute. As we have assumed that the $A_i \times T_i$'s are pairwise disjoint, they will not interact between each other meaning that if $1 \leq i \neq j \leq r$, if $x_i \in A_i$ and $x_j \in A_j$, either $x_i \neq x_j$ or $T_i \cap T_j = \varnothing$. Now, take $r_i \leq k_i$ for $i = 1 \dots r$. We notice that the number of ways to partition the sets $\{1, \dots, k_i\}$ into r_i non empty sets, for $i = 1 \dots r$, is equal to

$$\prod_{i=1}^r f(k_i \to r_i).$$

Thus, the contribution of points $(x_1, \dots, x_k) \in (A_{1N} \backslash \{0\})^{k_1} \times \dots \times (A_{rN} \backslash \{0\})^{k_r}$ such that for all $i = 1 \dots r$, $\{x_{k_1 + \dots + k_{i-1} + 1}, \dots, x_{k_1 + \dots + k_i}\}$ is composed of r_i well-separated points converges to

$$\left(\prod_{i=1}^r f(k_i \to r_i) \right) m(A_i \times T_i, r_i, i = 1 \dots r).$$

This shows (2.37) in the general case $r \geq 1$.

2.4.2.4 Estimates on μ_N^a.

We now briefly end the proof of Proposition 2.19 by explaining how the results for μ_N^a are obtained. Take $a \in [0,1]$, $T \in \mathcal{B}(\mathbb{R})$ and $A \subset [-1,1]^d$ such that the Lebesgue measure of $\bar{A} \backslash A^\circ$ vanishes. By definition of $f(k \to r)$ and since $(E_x)_{x \in V_N}$ are i.i.d. exponential variables with mean g independent of $\mathcal{M}_N(0)$, the normalised k-th moment $\mathbb{E}_0\left[(\mu_N^a(A\times T))^k\right]$ is equal to

$$\begin{aligned}
&\frac{1}{N^{2(1-a)k}}\mathbb{E}_0\left[\sum_{x_1,\dots,x_k\in A_N\cap\mathcal{M}_N(0)}\mathbf{1}_{\left\{E_{x_1},\dots,E_{x_k}\in 2ga\log N+T\right\}}\right]\\
&=\frac{1}{N^{2(1-a)k}}\sum_{r=1}^{k}f(k\to r)\mathbb{E}_0\left[\sum_{\substack{x_1,\dots,x_r\in A_N\cap\mathcal{M}_N(0)\\ \forall i\neq j, x_i\neq x_j}}\mathbf{1}_{\left\{E_{x_1},\dots,E_{x_r}\in 2ga\log N+T\right\}}\right]\\
&=\frac{1}{N^{2(1-a)k}}\sum_{r=1}^{k}f(k\to r)N^{-2ar}\left(\int_T e^{-s/g}\frac{ds}{g}\right)^r\mathbb{E}_0\left[\sum_{\substack{x_1,\dots,x_r\in A_N\\ \forall i\neq j, x_i\neq x_j}}\mathbf{1}_{\left\{\ell_{x_1}^{\tau_N},\dots,\ell_{x_r}^{\tau_N}>0\right\}}\right].
\end{aligned}$$

We have already shown that

$$\lim_{N\to\infty}\frac{1}{N^{2r}}\mathbb{E}_0\left[\sum_{\substack{x_1,\dots,x_r\in A_N\\ x_i\neq x_j\forall i\neq j}}\mathbf{1}_{\left\{\ell_{x_1}^{\tau_N},\dots,\ell_{x_r}^{\tau_N}>0\right\}}\right]=m(A\times(0,\infty),r)$$

so $\mathbb{E}_0\left[(\mu_N^a(A\times T))^k\right]$ converges to

$$\sum_{r=1}^{k}f(k\to r)\left(\int_T e^{-s/g}\frac{ds}{g}\right)^r m(A\times(0,\infty),r)\times\begin{cases}1 & \text{if } a=1 \text{ or } r=k\\ 0 & \text{if } a<1 \text{ and } r<k\end{cases}$$

which is exactly the stated result. The extension to the general case of a mixed moment is obtained exactly as for ν_N^a. □

2.4.3 Proof of technical lemmas

We start this section by proving Lemma 2.18 which gives estimates on the Green function G_N (defined in (2.32) as well as the Green function G on $\mathbb{Z}^d$) in dimension greater of equal to 3.

Proof of Lemma 2.18. As in dimension 2, these estimates follow from [Law96] and [LL10]: Proposition 1.5.8 in [Law96] gives

$$G_N(x,y)=G(x,y)-\sum_{z\in\partial V_N}\mathbb{P}_x\left(Y_{\tau_N}=z\right)G(z,y)\qquad(2.54)$$

and Theorem 4.3.1 in [LL10] (or Theorem 1.5.4 in [Law96] for a slightly worse estimate) gives

$$G(x,y) = a_d \left|x-y\right|^{2-d} + O\left(\left|x-y\right|^{-d}\right) \text{ as } |x-y| \to \infty. \tag{2.55}$$

Our first two estimates on the Green function on the diagonal follow since if $y \in V_{(1-\eta)N}$ for some $\eta > 0$, then for all $z \in \partial V_N$, $|z-y| \geq \eta N$. The lower bound on $q_N(x,y)$ follows as well. We are going to explain how to obtain the pointwise limit estimate (2.33). Take $\tilde{x} \neq \tilde{y} \in (-1,1)^d$. By (2.54) and (2.55), we have

$$N^{d-2} G_N\left(\lfloor N\tilde{x} \rfloor, \lfloor N\tilde{y} \rfloor\right) = a_d \left|x-y\right|^{2-d} - a_d \mathbb{E}_{\lfloor N\tilde{x} \rfloor}\left[\left|\frac{Y_{\tau_N}}{N} - \tilde{y}\right|^{2-d}\right] + O_{\tilde{x},\tilde{y}}\left(N^{2-d}\right).$$

By Donsker's invariance principle, starting from $\lfloor N\tilde{x} \rfloor$, Y_{τ_N}/N converges in law to the exit distribution of $[-1,1]^d$ of Brownian motion starting from $\tilde{x}$. We thus obtain (2.33). □

We now move on to the proof of Lemma 2.22. We consider k non zero and pairwise distinct points $x_1, \dots, x_k \in V_N$ and we recall the definitions of E and of the stopping times ς_p in (2.40).

Proof of Lemma 2.22. As mentioned just before Lemma 2.22, if $E = k-1$ and $\tau_{x_i} < \tau_N$ $\forall i = 1 \dots k$ then the stopping times $\varsigma_p, p = 0 \dots k-1$, define a permutation σ of the set of indices $\{1, \dots, k\}$ which keeps track of the order of visits of the set $\{x_1, \dots, x_k\}$. By a repeated application of Markov property, we thus have:

$$\begin{aligned} \mathbb{P}_0\left(E = k-1, \tau_{x_i} < \tau_N \ \forall i = 1 \dots k\right) = \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}_0\left(\tau_{x_{\sigma(1)}} < \tau_N \wedge \min_{j \neq 1} \tau_{x_{\sigma(j)}}\right) \\ \times \prod_{i=1}^{k-1} \mathbb{P}_{x_{\sigma(i)}}\left(\tau_{x_{\sigma(i+1)}} < \tau_N \wedge \min_{j \neq i,i+1} \tau_{x_{\sigma(j)}}\right) \mathbb{P}_{x_{\sigma(k)}}\left(\tau_N < \min_{j \neq k} \tau_{x_{\sigma(j)}}\right). \end{aligned} \tag{2.56}$$

But for all $\sigma \in \mathfrak{S}_k$ and $i = 1 \dots k-1$,

$$\mathbb{P}_{x_{\sigma(i)}}\left(\tau_{x_{\sigma(i+1)}} < \tau_N \wedge \min_{j \neq i,i+1} \tau_{x_{\sigma(j)}}\right) \leq \mathbb{P}_{x_{\sigma(i)}}\left(\tau_{x_{\sigma(i+1)}} < \tau_N\right) = \frac{G_N(x_{\sigma(i)}, x_{\sigma(i+1)})}{G_N(x_{\sigma(i+1)}, x_{\sigma(i+1)})}.$$

We bound from below the denominator $G_N(x_{\sigma(i+1)}, x_{\sigma(i+1)})$ by 1 and from above the numerator $G_N(x_{\sigma(i)}, x_{\sigma(i+1)})$ by $C\left|x_{\sigma(i)} - x_{\sigma(i+1)}\right|^{2-d}$ (see Lemma 2.18). Coming back to (2.56), this leads to

$$\mathbb{P}_0\left(E = k-1, \tau_{x_i} < \tau_N \ \forall i = 1 \dots k\right) \leq C^k \sum_{\sigma \in \mathfrak{S}_k} \prod_{i=0}^{k-1} \left|x_{\sigma(i)} - x_{\sigma(i+1)}\right|^{2-d}.$$

with the convention $x_{\sigma(0)} = 0$.

The general case $p \geq k-1$ follows from the same lines but now the order of visits of the set $\{x_1, \dots, x_k\}$ is not as simple as before. In the following, $\sigma \in \mathfrak{S}_k$ will keep track of the order of new visits of the vertices $x_1, \dots, x_k$: $x_{\sigma(1)}$ is the first vertex visited among the x_i's, $x_{\sigma(2)}$ the second one... We will focus on the transitions which explore new vertices, so we introduce the notion: $(\sigma, f) \in \mathfrak{S}_k \times \{1, \dots, k\}^{\{2,\dots,k\}}$ is said to be admissible if

$$\forall i = 2 \dots k, f(i) \in \{\sigma(1), \dots, \sigma(i-1)\}.$$

$x_{f(i)}$ will denote the vertex visited just before visiting the vertex $x_{\sigma(i)}$. Now we define

$$U(x_1, \dots, x_k) := \sum_{(\sigma,f) \text{ admissible}} \left| x_{\sigma(1)} \right|^{2-d} \prod_{i=1}^{k-1} \left| x_{\sigma(i+1)} - x_{f(i+1)} \right|^{2-d}. \tag{2.57}$$

By keeping track of the transitions where new vertices are discovered (in a chronological sense) and by noticing that all the others occur with a probability which is not larger than $C_k \max_{i \neq j} |x_i - x_j|^{2-d}$, we have

$$\begin{aligned} \mathbb{P}_0 \left(E = p, \tau_{x_i} < \tau_N \ \forall i = 1 \dots k \right) &\leq (C_k)^{p+1} \left(\max_{i \neq j} |x_i - x_j|^{2-d} \right)^{p-k+1} U(x_1, \dots, x_k) \\ &= (C_k)^{p+1} \left(\max_{i \neq j} |x_i - x_j|^{2-d} \right)^{p-k+1} N^{(2-d)k} U \left(\frac{x_1}{N}, \dots, \frac{x_k}{N} \right). \end{aligned}$$

This proves (2.43).

We notice that (2.42) is immediate from the definition of $(y_1, \dots, y_k) \in (-1,1)^k \mapsto U(y_1, \dots, y_k)$ and we now check that it is integrable. Take (σ, f) admissible. There is only one occurrence of $y_{\sigma(k)}$ in the product, so we can first integrate:

$$\int_{(-1,1)^d} \left| y_{\sigma(k)} - y_{f(k)} \right|^{2-d} dy_{\sigma(k)} \leq \int_{(-2,2)^d + y_{f(k)}} \left| y_{\sigma(k)} - y_{f(k)} \right|^{2-d} dy_{\sigma(k)} = C.$$

We then proceed inductively by integrating next with respect to $y_{\sigma(k-1)}$, and so on. This proves that U is integrable.

We now turn to (2.44). If $x_1 = \lfloor N y_1 \rfloor, \dots, x_k = \lfloor N y_k \rfloor$, for $y_1, \dots, y_k$ non zero and pairwise distinct elements of $(-1,1)^d$, then there exists $\eta \in (0,1)$ such that for all N large enough, $x_i \in V_{(1-\eta)N}, |x_i| \geq \eta N$ and for all $i \neq j, |x_i - x_j| \geq \eta N$. Hence Lemma 2.18 implies

$$\begin{aligned} \mathbb{P}_{x_1} \left(\tau_{x_2} < \tau_N \wedge \min_{j \neq 1} \tau_{x_j} \right) &= \mathbb{P}_{x_1} \left(\tau_{x_2} < \tau_N \right) - \mathbb{P}_{x_1} \left(\exists j \neq 1, \tau_{x_j} < \tau_{x_2} < \tau_N \right) \\ &\geq \mathbb{P}_{x_1} \left(\tau_{x_2} < \tau_N \right) - (k-2) \max_{j \neq 1} \mathbb{P}_{x_1} \left(\tau_{x_j} < \tau_N \right) \mathbb{P}_{x_j} \left(\tau_{x_2} < \tau_N \right) \\ &\geq \mathbb{P}_{x_1} \left(\tau_{x_2} < \tau_N \right) - C_k (\eta N)^{2(2-d)} \end{aligned}$$

which leads to:

$$\lim_{N\to\infty} N^{d-2}\mathbb{P}_{x_1}\left(\tau_{x_2} < \tau_N \wedge \min_{j\notin\{1,2\}} \tau_{x_j}\right) = \lim_{N\to\infty} N^{d-2}\mathbb{P}_{x_1}\left(\tau_{x_2} < \tau_N\right)$$
$$= \frac{a_d}{g}\left(|y_1 - y_2|^{2-d} - q(y_1, y_2)\right).$$

We deduce (2.44) by (2.56). □

We now prove Lemma 2.24.

Proof of Lemma 2.24. Let $x \neq y \in V_N$ and let us denote

$$p_{xy} := \mathbb{P}_x\left(\tau_y < \infty\right) = \mathbb{P}_y\left(\tau_x < \infty\right) \text{ and } \theta_{xy} = \mathbb{E}_x\left[\ell_x^{\tau_y}\right] = \mathbb{E}_y\left[\ell_y^{\tau_x}\right].$$

By decomposing the walk along the different excursions between x and y, by Lemma 2.20 we see that starting from x the joint law of $\left(\ell_x^\infty, \ell_y^\infty\right)$ can be stochastically dominated by:

$$\left(\ell_x^\infty, \ell_y^\infty\right) \preceq \left(\sum_{j=1}^{A} \ell_{x,j}, \sum_{j=1}^{A} \ell_{y,j}\right)$$

where A is a geometric random variable with failure probability

$$\left(p_{xy}\right)^2 = \mathbb{P}_x\left(\exists 0 < s < t, Y_s = y, Y_t = x\right)$$

and $\ell_{x,j}, \ell_{y,j}, j \geq 1$, are i.i.d. exponential variables with mean θ_{xy} independent from A. A is the number of round trips between x and y and $\ell_{x,j}$ is the time spent in x during the j-th round trip. Let us mention that it is not an exact equality in distribution but only a stochastic domination. Indeed, we exactly have: starting from x,

$$\ell_x^\infty \overset{(\mathrm{d})}{=} \sum_{j=1}^{A} \ell_{x,j}, \tag{2.58}$$

but the number of $\ell_{y,j}$'s we have to sum up is A (resp. $A-1$) if the last visited vertex is y (resp. x). However this stochastic domination is sufficient for our purposes.

Let $p \geq 0$. For all $i = 1 \dots p+1$ we stochastically dominate as above $\left(\ell_x^{\infty,i}, \ell_y^{\infty,i}\right)$ by variables with a superscript i and we have

$$\mathbb{P}\left(\sum_{i=1}^{p+1} \ell_x^{\infty,i} \geq 2g\log N + gt, \sum_{i=1}^{p+1} \ell_y^{\infty,i} \geq 2g\log N + gt\right)$$
$$\leq \mathbb{P}\left(\sum_{i=1}^{p+1}\sum_{j=1}^{A^i} \ell_{x,j}^i \geq 2g\log N + gt, \sum_{i=1}^{p+1}\sum_{j=1}^{A^i} \ell_{y,j}^i \geq 2g\log N + gt\right).$$

Conditioned on the value of $\sum_{i=1}^{p+1} A^i$, the variables $\sum_{i=1}^{p+1}\sum_{j=1}^{A^i} \ell_{x,j}^i$ and $\sum_{i=1}^{p+1}\sum_{j=1}^{A^i} \ell_{y,j}^i$ are two independent Gamma variables. We can thus use the claim (2.61) of Lemma 2.25 and

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{p+1} \ell_x^{\infty,i} \geq 2g\log N + gt, \sum_{i=1}^{p+1} \ell_y^{\infty,i} \geq 2g\log N + gt\right)\\
&\leq N^{-4g/\theta_{xy}} e^{-2t} \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{p+1} A_i = n+p+1\right) \sum_{q=0}^{2(n+p)} \frac{1}{q!}\left(4\frac{g}{\theta_{xy}}\log N\right)^q\\
&= N^{-4g/\theta_{xy}} e^{-2t}\left(1-p_{xy}^2\right)^{p+1} \sum_{n=0}^{\infty} p_{xy}^{2n}\binom{n+p}{p} \sum_{q=0}^{2(n+p)} \frac{1}{q!}\left(4\frac{g}{\theta_{xy}}\log N\right)^q\\
&\leq C(p,t) N^{-4g/\theta_{xy}} \sum_{q=0}^{\infty} \frac{1}{q!}\left(4\frac{g}{\theta_{xy}}\log N\right)^q \sum_{n\geq(\lceil q/2\rceil-p)_+} (n+p)\dots(n+1)p_{xy}^{2n}. \qquad (2.59)
\end{aligned}$$

We are going to bound from above the last sum indexed by n. Let us first notice that p_{xy} and θ_{xy} are linked by a simple formula. Indeed, (2.58) implies that $\mathbb{E}_x\left[\ell_x^\infty\right] = \mathbb{E}\left[A\right]\mathbb{E}\left[\ell_{x,1}\right]$, meaning that $g = \theta_{xy}/\left(1-p_{xy}^2\right)$. Then

$$\inf_{x\neq y} g(1-p_{xy})/\theta_{xy} = \inf_{x\neq y} 1/(1+p_{xy}) > 1/2$$

so we can find $\lambda > 1$ such that $\inf_{x\neq y} g(1-\lambda p_{xy})/\theta_{xy} > 1/2$. If the index q in the equation (2.59) is large enough, say $q \geq q_0(p)$, then for all $n \geq \lceil q/2\rceil - p$ we have $2\log(\lambda)n \geq p\log(n+p)$ and we can bound

$$\begin{aligned}
\sum_{n\geq(\lceil q/2\rceil-p)_+} (n+p)\dots(n+1)p_{xy}^{2n} &\leq \sum_{n\geq\lceil q/2\rceil-p} (n+p)^p p_{xy}^{2n}\\
&\leq \sum_{n\geq\lceil q/2\rceil-p} (\lambda p_{xy})^{2n} \leq C(p)\left(\lambda p_{xy}\right)^q.
\end{aligned}$$

If $q < q_0(p)$, we bound the sum indexed by n by some constant depending on p. Overall, coming back to the equation (2.59), we can further bound from above the probability we are interested in by:

$$C'(p,t)N^{-4g/\theta_{xy}}\left((\log N)^{q_0(p)-1} + \sum_{q=q_0(p)}^{\infty}\frac{1}{q!}\left(4\frac{g}{\theta_{xy}}\lambda p_{xy}\log N\right)\right) \leq C''(p,t)N^{-4\frac{g(1-\lambda p_{xy})}{\theta_{xy}}}.$$

We have chosen λ to make sure that the previous exponent is smaller than -2 which is exactly what was required. □

We now state and prove elementary Lemma 2.25 (recall the definition of $f(k\to q)$ in (2.34)).

Lemma 2.25. *1. Poisson distribution: For $\lambda > 0$, consider $P(\lambda)$ a Poisson random variable with parameter λ. Then for all $k \geq 1$,*

$$\mathbb{E}\left[P(\lambda)^k\right] = \sum_{q=1}^{k} f(k \to q)\lambda^q. \tag{2.60}$$

2. Gamma distribution: For $k, p \geq 1$ and $\theta > 0$, consider $\Gamma_1(p,\theta), \dots, \Gamma_k(p,\theta)$ k i.i.d. Gamma random variables with shape parameter p and scale parameter θ, which have the law of the sum of p independent exponential variables with mean θ. Then for all $T > 0$,

$$\mathbb{P}\left(\forall i = 1 \dots k, \Gamma_i(p,\theta) \geq T\right) \leq e^{-k\frac{T}{\theta}} \sum_{q=0}^{k(p-1)} \left(k\frac{T}{\theta}\right)^q / (q!). \tag{2.61}$$

Proof of Lemma 2.25. 1. Poisson distribution: The moment generating function of $P(\lambda)$ is given by: for all $u \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}\left[e^{uP(\lambda)}\right] &= \exp(\lambda(e^u - 1)) = \sum_{q=0}^{\infty} \frac{\lambda^q}{q!}(e^u - 1)^q = \sum_{q=0}^{\infty} \frac{\lambda^q}{q!} \sum_{i=1}^{q} \binom{q}{i} (-1)^{q-i} e^{iu} \\ &= \sum_{q=0}^{\infty} \frac{\lambda^q}{q!} \sum_{i=1}^{q} \binom{q}{i} (-1)^{q-i} \sum_{k=0}^{\infty} i^k \frac{u^k}{k!} = \sum_{k=0}^{\infty} \frac{u^k}{k!} \sum_{q=0}^{k} \lambda^q f(k \to q)\end{aligned}$$

where f is defined in (2.34). This proves (2.60).

2. Gamma distribution: The probability we are interested in is equal to

$$\mathbb{P}\left(\Gamma_1(p,\theta) \geq T\right)^k = e^{-k\frac{T}{\theta}} \left(\sum_{q=0}^{p-1} \left(\frac{T}{\theta}\right)^q / q!\right)^k = e^{-k\frac{T}{\theta}} \sum_{q=0}^{k(p-1)} \left(\frac{T}{\theta}\right)^q \sum_{\substack{0 \leq q_1, \dots, q_k \leq p-1 \\ q_1 + \dots + q_k = q}} \frac{1}{q_1! \dots q_k!}.$$

By looking at the power series of $x \mapsto (e^x)^k$ we find that

$$\sum_{\substack{0 \leq q_1, \dots, q_k \leq p-1 \\ q_1 + \dots + q_k = q}} \frac{1}{q_1! \dots q_k!} \leq \sum_{\substack{q_1, \dots, q_k \geq 0 \\ q_1 + \dots + q_k = q}} \frac{1}{q_1! \dots q_k!} = \frac{k^q}{q!}$$

which concludes the proof of (2.61). □

We finish this paper by stating a lemma of measure theory. We include a proof for completeness and because we have not found any reference for this lemma.

Lemma 2.26. *Let $\phi : [-1,1]^d \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function with compact support. Then there exists a sequence $(\phi_p)_{p \geq 1}$ of functions converging uniformly to ϕ such that for all $p \geq 1$,*

$$\phi_p = \sum_{i=1}^{p} a_i^{(p)} \mathbf{1}_{A_i^{(p)} \times T_i^{(p)}}$$

where $A_i^{(p)} \in \mathcal{B}([-1,1]^d)$ *with the Lebesgue measure of* $\bar{A}_i^{(p)} \backslash (A_i^{(p)})^\circ$ *vanishing,* $T_i^{(p)} \in \mathcal{B}(\mathbb{R})$ *with* $\inf T_i^{(p)} > -\infty$ *and* $a_i^{(p)} \in \mathbb{C}$*.*

Proof. Let $\varepsilon > 0$. As ϕ is $\mathcal{C}^\infty$ with compact support, the Fourier series of ϕ converges uniformly. Without loss of generality, we can assume that the support of ϕ is included in $[-1,1]^{d+1}$. We can thus find $K \geq 1$, $c_{k_x,k_t} \in \mathbb{C}, k_x \in \mathbb{Z}^d, k_t \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$ such that the uniform norm of

$$\phi - \sum_{\substack{k_x \in \mathbb{Z}^d, \|k_x\| \leq K \\ k_t \in \mathbb{Z}, |k_t| \leq K}} c_{k_x,k_t} e^{\pi i k_x \cdot x} e^{\pi i k_t \cdot t} \mathbf{1}_{(t_0,\infty)}$$

is smaller than ε. This procedure separates the variables x and t. Now, writing u_+ and u_- the positive and negative parts of a real number u, we decompose

$$e^{\pi i k_x \cdot x} = (\cos(\pi k_x \cdot x))_+ - (\cos(\pi k_x \cdot x))_- + i\left(\sin(\pi k_x \cdot x)\right)_+ - i\left(\sin(\pi k_x \cdot x)\right)_- .$$

Hence, we conclude this lemma by decomposing these four previous functions into sums of simple functions and we do the same thing for the variable t. We are going to detail this. In particular, we are going to explain how to ensure that the boundary of the Borel sets linked to the simple functions have zero Lebesgue measure. Let $\varphi : \mathbb{R}^d \to [0,\infty)$ be a continuous bounded function. We take $\xi > 0$ such that the Lebesgue measure of $\varphi^{-1}\left(\{k2^{-p} - \xi, k \geq 1, p \geq 1\}\right)$ vanishes. It is possible because the set of non suitable ξ's is at most countable. Now we introduce

$$\psi_p := \sum_{k=0}^{p2^p} k2^{-p} \mathbf{1}_{A_{p,k}} \text{ where } A_{p,k} = \varphi^{-1}\left(\left[k2^{-p} - \xi, (k+1)2^{-p} - \xi\right)\right).$$

Thanks to our choice of ξ, the Lebesgue measure of $\bar{A}_{p,k} \backslash A_{p,k}^\circ$ vanishes. Also, since $\varphi + \xi$ is positive and bounded, $0 \leq (\varphi + \xi) - \psi_p \leq 2^{-p}$ for all p large enough. We have thus uniformly approximated φ by simple functions with Borel sets of the form we desired. This concludes the proof of the lemma. $\square$

Chapter 3

Planar Brownian motion and Gaussian multiplicative chaos

We construct the analogue of Gaussian multiplicative chaos measures for the local times of planar Brownian motion by exponentiating the square root of the local times of small circles. We also consider a flat measure supported on points whose local time is within a constant of the desired thickness level and show a simple relation between the two objects. Our results extend those of [BBK94] and in particular cover the entire L^1-phase or subcritical regime. These results allow us to obtain a nondegenerate limit for the appropriately rescaled size of thick points, thereby considerably refining estimates of [DPRZ01].

3.1 Introduction

3.1.1 Main results

Gaussian multiplicative chaos (GMC) introduced by Kahane [Kah85] consists in defining and studying the properties of random measures formally defined as the exponential of a log-correlated Gaussian field, such as the two-dimensional Gaussian free field (GFF). Since such a field is not defined pointwise but is rather a random generalised function, making sense of such a measure requires some nontrivial work. The theory has expanded significantly in recent years and by now it is relatively well understood, at least in the subcritical case [RV10, DS11, RV11, Sha16, Ber17] and even in the critical case [DRSV14b, DRSV14a, JS17, JSW19, Pow18]. Furthermore, Gaussian multiplicative chaos appears to be a universal feature of log-correlated fields going beyond the Gaussian theory discussed in these papers. Establishing universality for naturally arising models is a very active and important area of research. We mention the work of [SW16] on the Riemann ζ function on the critical line and the work of [FK14, Web15, NSW18, LOS18, BWW18] on large random matrices.

The goal of this paper is to study Gaussian multiplicative chaos for another natural non-Gaussian log-correlated field: (the square root of) the local times of two-dimensional Brownian motion.

Before stating our main results, we start by introducing a few notations. Let $\mathbb{P}_x$ be the law under which $(B_t)_{t\geq 0}$ is a planar Brownian motion starting from $x \in \mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be an open bounded simply connected domain, $x_0 \in D$ a starting point and τ be the first exit time of D:

$$\tau := \inf\{t \geq 0 : B_t \notin D\}.$$

For all $x \in \mathbb{R}^2, t > 0, \varepsilon > 0$, define $L_{x,\varepsilon}(t)$ the local time of $(|B_s - x|, s \geq 0)$ at ε up to time t (here $|\cdot|$ stands for the Euclidean norm):

$$L_{x,\varepsilon}(t) := \lim_{\substack{r\to 0\\ r>0}} \frac{1}{2r} \int_0^t \mathbf{1}_{\{\varepsilon - r \leq |B_s - x| \leq \varepsilon + r\}} ds.$$

One can use classical theory of one-dimensional semimartingales to get existence for a fixed x of $\{L_{x,\varepsilon}(\tau), \varepsilon > 0\}$ as a process. In this article, we need to make sense of $L_{x,\varepsilon}(\tau)$ jointly in x and in ε. It is provided by Proposition 3.5 that we state at the end of this section. If the circle $\partial D(x,\varepsilon)$ is not entirely included in D, we will use the convention $L_{x,\varepsilon}(\tau) = 0$. For all $\gamma \in (0,2)$ we consider the sequence of random measures $\mu_\varepsilon^\gamma(dx)$ on D defined by: for all Borel sets $A \subset D$,

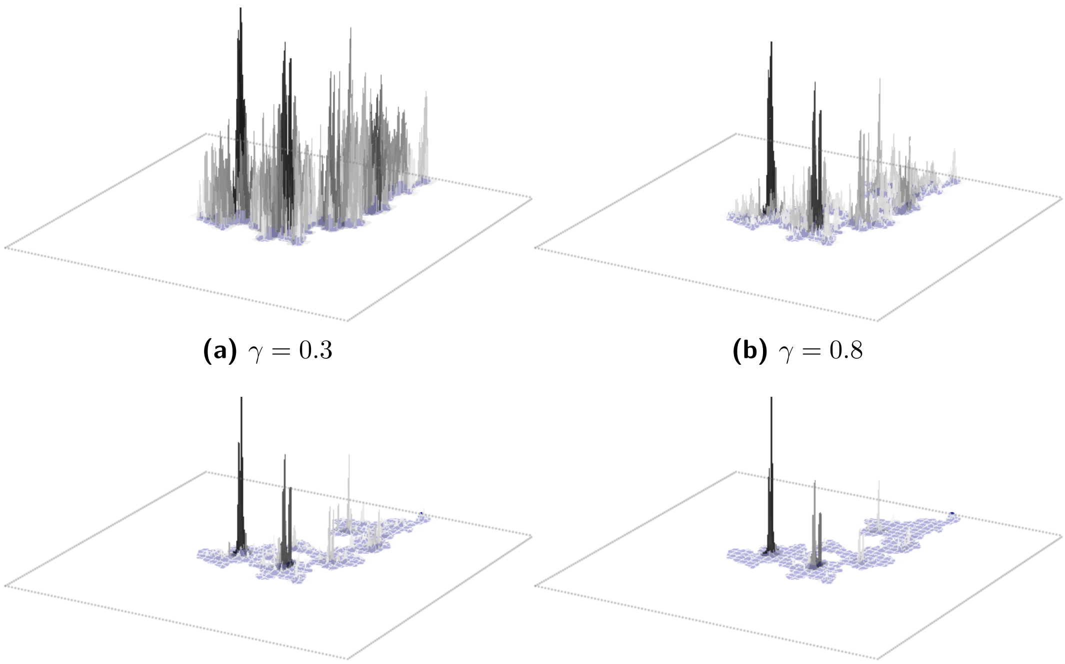
$$\mu_\varepsilon^\gamma(A) := \sqrt{|\log \varepsilon|}\varepsilon^{\gamma^2/2} \int_A e^{\gamma \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{3.1}$$

The presence of the square root in the exponential may appear surprising at first glance, but it is natural nevertheless in view of Dynkin-type isomorphisms (see [Ros14]).

To capture the fractal geometrical properties of a log-correlated field, another natural approach consists in encoding the so-called thick points (points where the field is unusually large) in flat measures supported on those thick points. At criticality, such measures are often called extremal processes. See for instance [BL19], [BL18] in the case of discrete two-dimensional GFF, see also [Abe18] in the case of simple random walk on trees. In our case, we can consider for all $\gamma \in (0,2)$ the sequence of random measures $\nu_\varepsilon^\gamma(dx, dt)$ on $D \times \mathbb{R}$ defined by: for all Borel sets $A \subset D$ and $T \subset \mathbb{R}$,

$$\nu_\varepsilon^\gamma(A \times T) := |\log \varepsilon| \, \varepsilon^{-\gamma^2/2} \int_A \mathbf{1}_{\left\{\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)} - \gamma \log \frac{1}{\varepsilon} \in T\right\}} dx. \tag{3.2}$$

Theorem 3.1. *For all $\gamma \in (0,2)$, the sequences of random measures ν_ε^γ and μ_ε^γ converge as $\varepsilon \to 0$ in probability for the topology of vague convergence on $D \times (\mathbb{R} \cup \{+\infty\})$ and on*



(a) $\gamma = 0.3$ (b) $\gamma = 0.8$

(c) $\gamma = 1.3$ (d) $\gamma = 1.8$

Figure 3.1: Simulation of μ^γ for $\gamma = 0.3$, 0.8, 1.3 and 1.8, for the same underlying sample of Brownian path which is drawn in blue. The domain D is a square and the starting point x_0 is its middle

D respectively towards Borel measures ν^γ and μ^γ.

The measure ν^γ can be decomposed as a product of a measure on D and a measure on $\mathbb{R}$. Moreover, the component on D agrees with μ^γ and the component on $\mathbb{R}$ is exponential:

Theorem 3.2. *For all $\gamma \in (0,2)$, we have $\mathbb{P}_{x_0}$-a.s.,*

$$\nu^\gamma(dx, dt) = (2\pi)^{-1/2} \mu^\gamma(dx) e^{-\gamma t} dt.$$

Moreover, by denoting $R(x,D)$ the conformal radius of D seen from x and $G_D(x_0,x)$ the Green function of D in x_0, x (see (3.8)), we have for all Borel set $A \subset D$,

$$\mathbb{E}_{x_0}\left[\mu^\gamma(A)\right] = \sqrt{2\pi}\gamma \int_A R(x,D)^{\gamma^2/2} G_D(x_0,x) dx \in (0,\infty). \tag{3.3}$$

The decomposition of ν^γ and (3.3) justify that the square root of the local times is the right object to consider. These two properties are very similar to the case of the two-dimensional GFF (see [BL19] and [Ber16], Theorem 2.1 for instance).

Simulations of μ^γ can be seen in Figure 3.1. They have been performed using simple random walk on the square lattice killed when it exits a square composed of 401×401 vertices.

In [BBK94], a slight modification of $\nu_\varepsilon^\gamma(dx,(0,\infty))$ was shown to converge for $\gamma \in (0,1)$ and the authors conjectured that the convergence should hold for the whole range $\gamma \in (0,2)$. One part of Theorem 3.1 settles this question. Let us also mention that the random measure μ^γ has been constructed very recently in [AHS20] through a very different method. In Section 3.1.2 below, we explain carefully the relation between the articles [BBK94], [AHS20] and the current paper.

Remark 3.3. We decided to not include the case $\gamma = 0$ to ease the exposition, but notice that ν_ε^γ is also a sensible measure in this case. By modifying very few arguments in the proofs of Theorems 3.1 and 3.2, one can show that this sequence of random measures converges for the topology of vague convergence on $D \times (0,\infty]$ towards a measure ν^0 which can be decomposed as

$$\nu^0(dx, dt) = \mu^0(dx)\mathbf{1}_{\{t=\infty\}}$$

for some random Borel measure μ^0 on D. With the help of (3.64) in Proposition 3.28 characterising the measure μ^γ, it can be shown that μ^0 is actually $\mathbb{P}_{x_0}$-a.s. absolutely continuous with respect to the occupation measure of Brownian motion, with a deterministic density. This last observation was already made in [AHS20], Section 7.

Define the set of γ-thick points at level ε by

$$\mathcal{T}_\varepsilon^\gamma := \left\{ x \in D : \frac{L_{x,\varepsilon}(\tau)}{\varepsilon(\log \varepsilon)^2} \geq \gamma^2 \right\}. \tag{3.4}$$

This is similar to the notion of thick points in [DPRZ01], except that they look at the occupation measure of small discs rather than small circles. In Chapter 2, the question to show the convergence of the rescaled number of thick points for the simple random walk on the two-dimensional square lattice was raised. As a direct corollary of Theorems 3.1 and 3.2, we answer the analogue of this question in the continuum:

Corollary 3.4. *For all $\gamma \in (0,2)$, we have the following convergence in L^1:*

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \, \varepsilon^{-\gamma^2/2} \mathrm{Leb}\left(\mathcal{T}_\varepsilon^\gamma\right) = \frac{1}{\sqrt{2\pi}\gamma} \mu^\gamma(D)$$

where $\mathrm{Leb}\left(\mathcal{T}_\varepsilon^\gamma\right)$ *denotes the Lebesgue measure of* $\mathcal{T}_\varepsilon^\gamma$*.*

Despite the strong links between the GFF and the local times, this shows a difference in the structure of the thick points of GFF compare to those of planar Brownian motion which cannot be observed through rougher estimates such as the fractal dimension. Indeed, for the GFF, the normalisation is $\sqrt{|\log \varepsilon|}\varepsilon^{-\gamma^2/2}$ instead of $|\log \varepsilon| \, \varepsilon^{-\gamma^2/2}$. See Chapter 2 for more about this.

As announced earlier, in order to define the measures in (3.1) and (3.2), we establish:

Proposition 3.5. *The local time process $L_{x,\varepsilon}(\tau)$, $x \in D$, $0 < \varepsilon < \mathrm{d}(x, \partial D)$, possesses a jointly continuous modification $\tilde{L}_{x,\varepsilon}(\tau)$. In fact, this modification is α-Hölder for all $\alpha < 1/3$.*

The proof of this proposition will be given in Appendix 3.C. In the rest of the paper, when we write $L_{x,\varepsilon}(\tau)$ we actually always mean its continuous modification $\tilde{L}_{x,\varepsilon}(\tau)$.

3.1.2 Relation with other works and further results

The construction of measures supported on the thick points of planar Brownian motion was initiated by the work of Bass, Burdzy and Khoshnevisan [BBK94]. The notion of thick points therein is defined through the number of excursions N_ε^x from x which hit the circle $\partial D(x, \varepsilon)$, before the Brownian motion exits the domain D: more precisely, for $a \in (0, 2)$, they define the set

$$A_a := \left\{ x \in D : \lim_{\varepsilon \to 0} \frac{N_\varepsilon^x}{|\log \varepsilon|} = a \right\}. \tag{3.5}$$

Note that our parametrisation is somewhat different; it is chosen to match the GMC theory. Informally, the relation between the two is given by $a = \gamma^2/2$. Next, we recall that the carrying dimension of a measure β is the infimum of $d > 0$ for which there exists a set A such that $\beta(A^c) = 0$ and the Hausdorff dimension of A is equal to d. They showed:

Theorem B (Theorem 1.1 of [BBK94]). *Assume that the domain D is the unit disc of $\mathbb{R}^2$ and that the starting point x_0 is the origin. For all $a \in (0, 1/2)$, with probability one there exists a random measure β_a, which is carried by A_a and whose carrying dimension is equal to $2 - a$.*

In [BBK94], the measure β_a is constructed as the limit of measures β_a^ε as $\varepsilon \to 0$ which are defined in a very similar manner as our measures $\nu_\varepsilon^\gamma(dx, (0, \infty))$ using local times of circles (see the beginning of Section 3 of [BBK94]). We emphasise here the difference of renormalisation: the local times they consider are half of our local times. We also mention that the range $\{a \in (0, 1/2)\}$ for which they were able to show the convergence of β_a^ε is a strict subset of the so-called L^2-phase of the GMC, which would correspond to $\{a \in (0, 1)\}$ or $\{\gamma \in (0, \sqrt{2})\}$. This is the region where $\beta_a^\varepsilon(D)$ is bounded in L^2, see Theorem 3.2 of [BBK94].

Bass, Burdzy and Khoshnevisan also gave an effective description of their measure β_a in terms of a Poisson point process of excursions. More precisely, they define a probability distribution $\mathbb{Q}_{x,D}^{x_0;a}$ (written $\mathbb{Q}_a^x$ in [BBK94], defined just before Proposition 5.1 of [BBK94]) on continuous trajectories which can be understood heuristically as follows. The trajectory of a process under $\mathbb{Q}_{x,D}^{x_0,a}$ is composed of three independent parts. The first one is a Brownian motion starting from x_0 conditioned to visit x before exiting D and killed at the hitting time of x. The third part is a Brownian motion starting from x and killed when it exits for the first time D. The second part is composed of an infinite number of excursions from x generated by a Poisson point process with the intensity measure being the product of the Lebesgue measure on $[0,a]$ and an excursion law. In Proposition 5.1 of [BBK94], they roughly speaking show that the law of the Brownian motion conditioned on the fact that x has been sampled according to β_a is $\mathbb{Q}_{x,D}^{x_0,a}$. This characterises their measure β_a (Theorem 5.2 of [BBK94]). Once Theorems 3.1 and 3.2 above are established, we can adapt their arguments for the proof of characterisation to conclude the same thing for our measure μ^γ: see Proposition 3.28 for a precise statement. A consequence of Proposition 3.28 is the identification of our measure μ^γ with their measure β_a:

Corollary 3.6. *If the domain D is the unit disc, x_0 the origin, $\gamma \in (0,1)$ and $a = \gamma^2/2$, we have $\mathbb{P}_{x_0}$-a.s. $\mu^\gamma = \sqrt{2\pi}\gamma\beta_a$.*

A consequence of Theorem B is a lower bound on the Hausdorff dimension of the set of thick points A_a: for all $a \in (0,1/2)$, a.s. $\dim(A_a) \geq 2 - a$. The upper bound they obtained ([BBK94], Theorem 1.1 (ii)) is: for all $a > 0$, a.s. $\dim(A_a) \leq \max(0, 2 - a/e)$. They conjectured that the lower bound is sharp and holds for all $a \in (0,2)$. In 2001, Dembo, Peres, Rosen and Zeitouni [DPRZ01] answered positively the analogue of this question for thick points defined through the occupation measure of small discs:

$$\mathcal{T}_a := \left\{ x \in D : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 (\log \varepsilon)^2} \int_0^\tau \mathbf{1}_{\{B_t \in D(x,\varepsilon)\}} dt = a \right\}. \tag{3.6}$$

In particular, their result went beyond the L^2-phase to cover the entire L^1-phase. This allowed them to solve a conjecture by Erdős and Taylor [ET60].

Very recently, Aïdékon, Hu and Shi [AHS20] made a link between the definitions of thick points of [BBK94] and [DPRZ01] (defined in (3.5) and (3.6) respectively) by constructing measures supported on these two sets of thick points. Their approach is superficially very different from ours but we will see that the measure μ^γ we obtained is, perhaps surprisingly, related to theirs in a strong way (Corollary 3.7 below). Their measure is defined through a martingale approach for which the interpretation of the approximation is not immediately transparent (see [AHS20] (4.1), (4.2) and Corollary 3.6).

Let us describe this relation in more details. For technical reasons, in [AHS20], the boundary ∂D of D is assumed to be a finite union of analytic curves. To compare our results with theirs, we will also make this extra assumption in the following and we will call such a domain a nice domain. Consider $z \in \partial D$ a boundary point such that the boundary of D is analytic locally around z; we will call such a point a nice point. They denote by $\mathbb{P}_D^{x_0,z}$ the law of a Brownian motion starting from x_0 and conditioned to exit D through z. They showed:

Theorem C (Theorem 1.1 of [AHS20])**.** *For all $a \in (0,2)$, with $\mathbb{P}_D^{x_0,z}$-probability one there exists a random measure $\mathcal{M}_\infty^a$ which is carried by A_a and by $\mathcal{T}_a$ and whose carrying dimension is equal to $2-a$.*

Their starting point is the interpretation of the measure β_a of [BBK94] described above in terms of Poisson point process of excursions. For $x \in D$, they define a measure $\mathbb{Q}_{x,D}^{x_0,z,a}$ on trajectories similar to $\mathbb{Q}_{x,D}^{x_0,a}$ mentioned above: the only difference is that the last part of the trajectory is a Brownian motion conditioned to exit the domain through z. In a nutshell, they show the absolute continuity of $\mathbb{Q}_{x,D}^{x_0,z,a}$ with respect to $\mathbb{P}_D^{x_0,z}$ (restricted to the event that the Brownian path stays away from x) and define a sequence of measures using the Radon-Nikodym derivative. Their convergence relies on martingales argument rather than on computations on moments. As in [BBK94], they obtain a characterisation of their measure in terms of $\mathbb{Q}_{x,D}^{x_0,z,a}$ ([AHS20], Proposition 5.1) matching with ours (Proposition 3.28). As a consequence, we are able to compare their measure with ours.

Before stating this comparison, let us notice that we can also make sense of our measure μ^γ for the Brownian motion conditioned to exit D through z. Indeed, as noticed in [BBK94], Remark 5.1 (i), our measure μ^γ is measurable with respect to the Brownian path and defined locally. μ^γ is thus well defined for any process which is locally mutually absolutely continuous with respect to the two dimensional Brownian motion killed when it exits for the first time the domain D. The Brownian motion conditioned to exit D through z being such a process, μ^γ makes sense under $\mathbb{P}_D^{x_0,z}$ as a measure on D.

Corollary 3.7. *Let $z \in \partial D$ be a nice point and denote by $H_D(x,z)$ the Poisson kernel of D from x at z, that is the density of the harmonic measure $\mathbb{P}_x(B_\tau \in \cdot)$ with respect to the Lebesgue measure of ∂D at z. For all $\gamma \in (0,2)$, if $a = \gamma^2/2$, we have $\mathbb{P}_D^{x_0,z}$-a.s.,*

$$\mu^\gamma(dx) = \sqrt{2\pi}\gamma \frac{H_D(x_0,z)}{H_D(x,z)} \mathcal{M}_\infty^a(dx).$$

In particular, our measure μ^γ inherits some properties of the measure $\mathcal{M}_\infty^a$ obtained in [AHS20]. Recalling the definitions (3.5) and (3.6) of the two sets of thick points A_a and $\mathcal{T}_a$, we have:

Corollary 3.8. *For all $\gamma \in (0,2)$, the following properties hold:*

(i) Non-degeneracy: with $\mathbb{P}_{x_0}$-probability one, $\mu^\gamma(D) > 0$.

(ii) Thick points: with $\mathbb{P}_{x_0}$-probability one, μ^γ is carried by $A_{\gamma^2/2}$ and by $\mathcal{T}_{\gamma^2/2}$.

(iii) Hausdorff dimension: with $\mathbb{P}_{x_0}$-probability one, the carrying dimension of μ^γ is $2 - \gamma^2/2$.

(iv) Conformal invariance: if $\phi : D \to D'$ is a conformal map between two nice domains, $x_0 \in D$, and if we denote by $\mu^{\gamma,D}$ and $\mu^{\gamma,D'}$ the measures built in Theorem 1 for the domains (D, x_0) and $(D', \phi(x_0))$ respectively, we have

$$\left(\mu^{\gamma,D} \circ \phi^{-1}\right)(dx) \overset{\text{law}}{=} \left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2} \mu^{\gamma,D'}(dx).$$

Let us mention that we present the previous properties *(i)*-*(iii)* as a consequence of Corollary 3.7 to avoid to repeat the arguments, but we could have obtained them without the help of [AHS20]: as in [BBK94], *(i)* and *(ii)* follow from the Poisson point process interpretation of the measure μ^γ (Proposition 3.28) whereas *(iii)* follows from our second moment computations (Proposition 3.20). On the other hand, it is not clear that our approach yields the conformal invariance of the measure without the use of [AHS20].

Finally, while there are strong similarities between μ^γ and the GMC measure associated to a GFF (indeed, our construction is motivated by this analogy), there are also essential differences. In fact, from the point of view of GMC theory, the measure μ^γ is rather unusual in that it is carried by the random fractal set $\{B_t, t \leq \tau\}$ and does not need extra randomness to be constructed, unlike say Liouville Brownian motion or other instances of GMC on random fractals.

3.1.3 Organisation of the paper

We now explain the main ideas of our proofs of Theorems 3.1 and 3.2 and how the paper is organised. The overall strategy of the proof is inspired by [Ber17]. To prove the convergence of the measures ν_ε^γ and μ_ε^γ, it is enough to show that for any suitable $A \subset D$ and $T \subset \mathbb{R}$, the real valued random variables $\nu_\varepsilon^\gamma(A \times T)$ and $\mu_\varepsilon^\gamma(A)$ converge in probability which is the content of Proposition 3.27 (we actually show that they converge in L^1). As in [Ber17], we will consider modified versions $\widetilde{\nu}_\varepsilon^\gamma$ and $\widetilde{\mu}_\varepsilon^\gamma$ of ν_ε^γ and μ_ε^γ by introducing good events (see (3.21) and (3.23)): at a given $x \in D$, the local times are required to be never too thick around x at every scale. We will show that introducing these good events does not change the behaviour of the first moment (Propositions 3.16 and 3.17, Section 3.3) and it makes the sequences $\widetilde{\nu}_\varepsilon^\gamma(A \times T)$ and $\widetilde{\mu}_\varepsilon^\gamma(A)$ bounded in L^2 (Propositions 3.20 and 3.21, Section 3.4). Furthermore, we will see that these two sequences are Cauchy sequences in L^2 (Proposition 3.23, Section 3.5) implying in particular that they converge in L^1. Section 3.6 finishes the proof of Theorems 3.1 and 3.2 and demonstrates the links of our work with the ones of [BBK94] and [AHS20] (Corollaries 3.6, 3.7 and 3.8).

We now explain a few ideas underlying the proof. If the domain D is a disc $D = D(x, \eta)$

centred at x, then it is easy to check (by rotational invariance of Brownian motion and second Ray-Knight isomorphism for local times of one-dimensional Brownian motion) that the local times $L_{x,r}(\tau), r > 0$, have a Markovian structure. More precisely, for all $\eta' \in (0, \eta)$ and all $z \in D(0,\eta) \backslash D(0,\eta')$, under $\mathbb{P}_z$ and conditioned on $L_{x,\eta'}(\tau)$,

$$\left(\frac{L_{x,r}(\tau)}{r}, r = \eta' e^{-s}, s \geq 0 \right) \overset{\text{law}}{=} \left(R_s^2, s \geq 0 \right) \tag{3.7}$$

with $(R_s, s \geq 0)$ being a zero-dimensional Bessel process starting from $\sqrt{L_{x,\eta'}(\tau)/\eta'}$. This is an other clue that exponentiating the square root of the local times should yield an interesting object.

In the case of a general domain D, such an exact description is of course not possible, yet for small enough radii, the behaviour of $L_{x,r}(\tau)$ can be seen to be approximatively given by the one in (3.7). If we assume (3.7) then the construction of μ^γ is similar to the GMC construction for GFF, with the Brownian motions describing circle averages replaced by Bessel processes of suitable dimension. It seems intuitive that the presence of the drift term in a Bessel process should not affect significantly the picture in [Ber17].

To implement our strategy and use (3.7), we need an argument. In the first moment computations (Propositions 3.16 and 3.17), we will need a rough upper bound on the local times; an obvious strategy consists in stopping the Brownian motion when it exits a large disc containing the domain. For the second moment (Proposition 3.23), we will need a much more precise estimate. Let us assume for instance that $D(x, 2) \subset D$. We can decompose the local times $(L_{x,r}(\tau), r < 1)$ according to the different macroscopic excursions from $\partial D(x, 1)$ to $\partial D(x, 2)$ before exiting the domain D. To keep track of the overall number of excursions, we will condition on their initial and final points. Because of this conditioning, the local times of a specific excursion are no longer related to a zero-dimensional Bessel process. But if we now condition further on the fact that the excursion went deep inside $D(x, 1)$, it will have forgotten its initial point and those local times will be again related to a zero-dimensional Bessel process: this is the content of Lemma 3.24 and Appendix 3.A is dedicated to its proof. Let us mention that the spirit of Lemma 3.24 can be tracked back to Lemma 7.4 of [DPRZ01].

As we have just explained, we will use (3.7) to transfer some computations from the local times to the zero-dimensional Bessel process. Throughout the paper, we will thus collect lemmas about this process (Lemmas 3.18, 3.19 and 3.26) that will be proven in Appendix 3.B. Of course, we will not be able to transfer all the computations to the zero-dimensional Bessel process, for instance when we consider two circles which are not concentric. But we will be able to treat the local times as if they were the local times of a continuous time random walk: for a continuous time random walk starting at a given vertex x and killed when it hits for the first time a given set A, the time spent by the walk in x is exactly an exponential variable which is independent of the hitting point of A. We will show that it is also approximatively true for the local times of Brownian motion. This is the content of Section 3.2.

We end this introduction with some notations which will be used throughout the paper.
Notations: If $A, B \subset \mathbb{R}^2$, $x, y \in \mathbb{R}^2$, $\varepsilon > 0$, and $i, j \in \mathbb{Z}$, we will denote by:

- $\tau_A := \inf\{t \geq 0 : B_t \in A\}$ the first hitting time of A. In particular, $\tau = \tau_{\mathbb{R}^2 \setminus D}$;
- $D(x, \varepsilon)$ (resp. $\bar{D}(x, \varepsilon)$, $\partial D(x, \varepsilon)$) the open disc (resp. closed disc, circle) with centre x and radius ε;
- $\mathrm{d}(A, B)$ the Euclidean distance between A and B. If $A = \{x\}$, we will simply write $\mathrm{d}(x, B)$ instead of $\mathrm{d}(\{x\}, B)$;
- $R(x, D)$ the conformal radius of D seen from x;
- $G_D(x, y)$ the Green function in x, y:
$$G_D(x, y) := \pi \int_0^\infty p_s(x, y) ds, \tag{3.8}$$
  where $p_s(x, y)$ is the transition probability of Brownian motion killed at τ. We recall its behaviour close to the diagonal (see Equation (1.2) of [Ber16] for instance):
$$G_D(x, y) = -\log|x - y| + \log R(x, D) + u(x, y) \tag{3.9}$$
  where $u(x, y) \to 0$ as $y \to x$;
- $\mathbb{P}_r$ the law under which $(R_s, s \geq 0)$ is a zero-dimensional Bessel process starting from $r > 0$;
- $[|i, j|]$ the set of integers $\{i, \ldots, j\}$.

Finally, we will write $C, C', \tilde{C}$, etc, positive constants which may vary from one line to another. We will also write $o(1)$ (resp. $O(1)$) real-valued sequences which go to zero as $\varepsilon \to 0$ (resp. which are bounded). If we want to emphasise that such a sequence may depend on a parameter α, we will write $o_\alpha(1)$ (resp. $O_\alpha(1)$).

3.2 Preliminaries

We start off with some preliminary results that will be used throughout the paper.

3.2.1 Green's function

Lemma 3.9. *For all $x \in D$, $r > \varepsilon > 0$ so that $D(x,\varepsilon) \subset D$ and $y \in \partial D(x,\varepsilon)$, we have:*

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau_{\partial D(x,r)})\right] = 2\varepsilon \log \frac{r}{\varepsilon}, \tag{3.10}$$

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right] = 2\varepsilon\left(\log\frac{1}{\varepsilon} + \log R(x,D) + o(1)\right). \tag{3.11}$$

Proof. We start by proving (3.10). By denoting $p_s(y,z)$ the transition probability of Brownian motion killed at $\tau_{\partial D(x,r)}$, we have:

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau_{\partial D(x,r)})\right] = \int_{\partial D(x,\varepsilon)} dz \int_0^\infty ds\ p_s(y,z) = \frac{1}{\pi}\int_{\partial D(x,\varepsilon)} dz\ G_{D(x,r)}(y,z).$$

But the Green function of the disc $D(x,r)$ is equal to (see [Law05], Section 2.4):

$$G_{D(x,r)}(y,z) = \log\frac{\left|1-(\bar{y}-\bar{x})(z-x)/r^2\right|}{|y-z|/r}.$$

Hence

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau_{\partial D(x,r)})\right] = 2\varepsilon\log\frac{r}{\varepsilon} + \frac{1}{\pi}\int_{\partial D(x,\varepsilon)}\log\frac{\varepsilon}{|y-z|}dz + \frac{1}{\pi}\int_{\partial D(x,\varepsilon)}\log\left|1-\frac{(\bar{y}-\bar{x})(z-x)}{r^2}\right|dz.$$

Because the last two integrals vanish, this gives (3.10). The proof of (3.11) is very similar. The only difference is that we consider the Green function of the general domain D. Using the asymptotic (3.9), we conclude in the same way. □

3.2.2 Hitting probabilities

We now turn to the study of hitting probabilities. The following lemma gives estimates on the probability to hit a small circle before exiting the domain D, whereas the next one gives estimates on the probability to hit a small circle before hitting another circle and before exiting the domain D.

Lemma 3.10. *Let $\eta > 0$. For all $\varepsilon > 0$ small enough, for all $x \in D$ such that $\mathrm{d}(x,\partial D) > \eta$ and for all $y \in D \backslash D(x,\varepsilon)$, we have:*

$$\mathbb{P}_y\left(\tau_{\partial D(x,\varepsilon)} < \tau\right) = \left(1 + O_\eta\left(\frac{\varepsilon}{\log\varepsilon}\right)\right) G_D(x,y) \Big/ \log\left(\frac{R(x,D)}{\varepsilon}\right). \tag{3.12}$$

Proof. A similar but weaker statement can be found in [BBK94] (Lemma 2.1) and our proof is really close to theirs. We will take ε smaller than $\eta/2$ to ensure that the circle $\partial D(x,\varepsilon)$ stays far away from ∂D. If the domain D were the unit disc $\mathbb{D}$ and x the origin, then the probability we are interested in is the probability to hit a small circle before hitting the unit circle. The two circles being concentric, we can use the fact that $(\log|B_t|, t \geq 0)$ is a martingale to find that this probability is equal to:

$$\mathbb{P}_y\left(\tau_{\partial D(0,\varepsilon)} < \tau_{\partial\mathbb{D}}\right) = \log|y| / \log\varepsilon. \tag{3.13}$$

In general, we come back to the previous situation by mapping D onto the unit disc $\mathbb{D}$ and x to the origin with a conformal map f_x. By conformal invariance of Brownian motion,

$$\mathbb{P}_y\left(\tau_{\partial D(x,\varepsilon)} < \tau_D\right) = \mathbb{P}_{f_x(y)}\left(\tau_{f_x(\partial D(x,\varepsilon))} < \tau_{\mathbb{D}}\right).$$

As $\partial D(x,\varepsilon)$ is far away from the boundary of D, the contour $f_x(\partial D(x,\varepsilon))$ is included into a narrow annulus

$$D\left(0, |f_x'(x)|\varepsilon + c\varepsilon^2\right) \setminus D\left(0, |f_x'(x)|\varepsilon - c\varepsilon^2\right)$$

for some $c > 0$ depending on η. In particular, using (3.13),

$$\begin{aligned}\mathbb{P}_y\left(\tau_{\partial D(x,\varepsilon)} < \tau_D\right) &\leq \mathbb{P}_{f_x(y)}\left(\tau_{\partial D(0,|f_x'(x)|\varepsilon+c\varepsilon^2)} < \tau_{\mathbb{D}}\right) \\ &= \frac{\log|f_x(y)|}{\log(|f_x'(x)|\varepsilon + c\varepsilon^2)} = \frac{\log|f_x(y)|}{\log(|f_x'(x)|\varepsilon)}\left(1 + O_\eta\left(\frac{\varepsilon}{\log\varepsilon}\right)\right).\end{aligned}$$

The lower bound is obtained is a similar manner which yields the stated claim (3.12) noticing that $R(x,D) = 1/|f_x'(x)|$ and that $G_D(x,y) = -\log|f_x(y)|$ (see [Law05], Section 2.4). □

Remark 3.11. If $x, y \in D$ are at least at a distance η from the boundary of D, the quantities

$$\frac{G_D(x,y)}{-\log|x-y|}, R(x,D) \text{ and } R(y,D)$$

are bounded away from 0 and from infinity uniformly in x, y (depending on η). We thus obtain the simpler estimate:

$$\mathbb{P}_y\left(\tau_{\partial D(x,\varepsilon)} < \tau\right), \mathbb{P}_x\left(\tau_{\partial D(y,\varepsilon)} < \tau\right) = \left(1 + O_\eta\left(\frac{1}{\log\varepsilon}\right)\right)\frac{\log|x-y|}{\log\varepsilon}. \tag{3.14}$$

Depending on the level of accuracy we need, we will use either (3.12) or its rougher version (3.14).

For $x, y \in D$ and $\varepsilon > 0$ define

$$p_{xy}^- := \min_{z \in \partial D(x,\varepsilon)} \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right) \text{ and } p_{xy}^+ := \max_{z \in \partial D(x,\varepsilon)} \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right),$$
$$p_{yx}^- := \min_{z \in \partial D(y,\varepsilon)} \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right) \text{ and } p_{yx}^+ := \max_{z \in \partial D(y,\varepsilon)} \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right).$$

Lemma 3.12. *For all $x, y \in D$, $\varepsilon > 0$ so that $D(x,\varepsilon)$ and $D(y,\varepsilon)$ are disjoint and included in D, for all $z \in D \backslash (D(x,\varepsilon) \cup D(y,\varepsilon))$,*

$$\frac{\mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right) - p_{yx}^+ \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right)}{1 - p_{yx}^+ p_{xy}^-} \leq \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)} \right) \tag{3.15}$$
$$\leq \frac{\mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right) - p_{yx}^- \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right)}{1 - p_{yx}^- p_{xy}^+}.$$

Proof. By Markov property and by definition of p_{yx}^+, we have:

$$\begin{aligned} \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right) &= \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)} \right) + \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau_{\partial D(x,\varepsilon)} < \tau \right) \\ &\leq \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)} \right) + \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \wedge \tau_{\partial D(x,\varepsilon)} \right) p_{yx}^+. \end{aligned}$$

Similarly,

$$\mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right) \geq \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \wedge \tau_{\partial D(x,\varepsilon)} \right) + \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)} \right) p_{xy}^-.$$

Combining those two inequalities yields

$$\mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \right) - p_{yx}^+ \mathbb{P}_z \left(\tau_{\partial D(y,\varepsilon)} < \tau \right) \leq (1 - p_{yx}^+ p_{xy}^-) \mathbb{P}_z \left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)} \right)$$

which is the first inequality stated in (3.15). The other inequality is similar. □

3.2.3 Approximation of local times by exponential variables

In this subsection, we explain how to approximate the local times $L_{x,\varepsilon}(\tau)$ by exponential variables. For $x \in \mathbb{R}^2, \varepsilon > 0, y \in \partial D(x,\varepsilon)$ and any event E, define

$$H_{x,\varepsilon}^y(E) := \frac{1}{2} \lim_{\substack{z \in D(x,\varepsilon) \\ z \to y}} \mathbb{P}_z^*(E)/\mathrm{d}(z, \partial D(x,\varepsilon)) + \frac{1}{2} \lim_{\substack{z \notin \bar{D}(x,\varepsilon) \\ z \to y}} \mathbb{P}_z^*(E)/\mathrm{d}(z, \partial D(x,\varepsilon))$$

where $\mathbb{P}_z^*$ is the probability measure of Brownian motion starting at z and killed when it hits for the first time the circle $\partial D(x,\varepsilon)$. For $A \subset \mathbb{R}^2, x \in \mathbb{R}^2$, we will denote $\omega^A(x, d\xi)$ the harmonic measure of A from x.

Lemma 3.13. *Let $x \in \mathbb{R}^2, \varepsilon > 0$ and $C \subset \mathbb{R}^2$. Assume that $\mathrm{d}(\partial D(x,\varepsilon), C) > 0$ and that there exists $u > 0$ such that for all $y, y' \in \partial D(x,\varepsilon)$ and $E \subset C$,*

$$(1-u)\omega^C(y,E) \leq \omega^C(y',E) \leq (1+u)\omega^C(y,E).$$

Then for all $y \in \partial D(x,\varepsilon)$ and $t > 0$,

$$(1-u)e^{-\max_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau_C<\infty)t} \leq \mathbb{P}_y\left(L_{x,\varepsilon}(\tau_C) > t | B_{\tau_C}\right) \leq (1+u)e^{-\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau_C<\infty)t}.$$

Remark 3.14. The previous lemma states that we can approximate $L_{x,\varepsilon}(\tau_C)$ by an exponential variable which is independent of B_{τ_C}. This is similar to the case of random walks on discrete graphs. If we did not condition on B_{τ_C}, it would not have been necessary to add the multiplicative errors $1-u$ and $1+u$. This statement without conditioning is also a consequence of Lemma 2.2 (i) of [BBK94].

Proof. Since the proof is standard, we will be brief. Take $r > 0$ small enough so that the annulus $\bar{D}(x,\varepsilon+r)\backslash D(x,\varepsilon-r)$ does not intersect C. Consider the different excursions from $\partial D(x,\varepsilon+r)$ to $\partial D(x,\varepsilon-r)$: denote $\sigma_0^{(2)} := 0$ and for all $i \geq 1$,

$$\sigma_i^{(1)} := \inf\left\{t > \sigma_{i-1}^{(2)} : B_t \in \partial D(x,\varepsilon+r)\right\} \text{ and } \sigma_i^{(2)} := \inf\left\{t > \sigma_i^{(1)} : B_t \in \partial D(x,\varepsilon-r)\right\}.$$

The number of excursions $N_r := \max\{i \geq 0 : \sigma_i^{(2)} < \tau_C\}$ before τ_C is related to $L_{x,\varepsilon}(\tau_C)$ by:

$$L_{x,\varepsilon}(\tau_C) = 4\lim_{r\to 0} rN_r \qquad \mathbb{P}_y - \text{a.s.}$$

Hence, for any $f : \mathbb{R}^2 \to [0,\infty)$ continuous bounded function, we have by dominated convergence theorem

$$\mathbb{E}_y\left[\mathbf{1}_{\{L_{x,\varepsilon}(\tau_C)>t\}} f\left(B_{\tau_C}\right)\right] = \lim_{r\to 0}\mathbb{E}_y\left[\mathbf{1}_{\{N_r > \lfloor t/(4r)\rfloor\}} f\left(B_{\tau_C}\right)\right].$$

Because

$$\mathbb{E}_{B_{\sigma^{(2)}_{\lfloor t/(4r)\rfloor}}}\left[f(B_{\tau_C})\right] \leq (1+u+o_{r\to 0}(1))\mathbb{E}_y\left[f(B_{\tau_C})\right] \qquad \mathbb{P}_y - \text{a.s.},$$

and by a repeated application of Markov property, $\mathbb{E}_y\left[\mathbf{1}_{\{N_r>\lfloor t/(4r)\rfloor\}} f\left(B_{\tau_C}\right)\right]$ is at most

$$(1+u+o_{r\to 0}(1))\mathbb{E}_y\left[f(B_{\tau_C})\right] \max_{z\in\partial D(x,\varepsilon+r)} \mathbb{P}_z\left(\sigma_1^{(2)} < \sigma_2^{(1)} < \tau_C\right)^{\left\lfloor \frac{t}{4r}\right\rfloor}. \tag{3.16}$$

If $z \in \partial D(x, \varepsilon + r)$ is at distance r from $z_\varepsilon \in \partial D(x,\varepsilon)$,

$$\begin{aligned}1 - \mathbb{P}_z\left(\sigma_1^{(2)} < \sigma_2^{(1)} < \tau_C\right) &= \mathbb{P}_z\left(\tau_C < \tau_{\partial D(x,\varepsilon-r)}\right) + (1+o_{r\to 0}(1))\mathbb{P}_z\left(\tau_C < \sigma_2^{(1)} \middle| \sigma_1^{(2)} < \tau_C\right) \\ &= 2r(1+o_{r\to0}(1))\left(\lim_{\substack{z' \notin \bar{D}(x,\varepsilon) \\ z' \to z_\varepsilon}} \frac{\mathbb{P}^*_{z'}(\tau_C < \infty)}{\mathrm{d}(z', \partial D(x,\varepsilon))} + \lim_{\substack{z' \in D(x,\varepsilon) \\ z' \to z_\varepsilon}} \frac{\mathbb{P}^*_{z'}(\tau_C < \infty)}{\mathrm{d}(z', \partial D(x,\varepsilon))}\right) \\ &= 4r(1+o_{r\to0}(1)) H^{z_\varepsilon}_{x,\varepsilon}(\tau_C < \infty).\end{aligned}$$

Hence

$$\max_{z \in \partial D(x,\varepsilon+r)} \mathbb{P}_z\left(\sigma_1^{(2)} < \sigma_2^{(1)} < \tau_C\right) = 1 - 4r \min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau_C < \infty) + o_{r\to0}(r).$$

Coming back to (3.16), we have obtained

$$\mathbb{E}_y\left[\mathbf{1}_{\{L_{x,\varepsilon}(\tau_C) > t\}} f\left(B_{\tau_C}\right)\right] \leq (1+u)\mathbb{E}_y\left[f(B_{\tau_C})\right] e^{-\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau_C<\infty) t}$$

which is the required upper bound. The lower bound is obtained in a similar way. □

The next lemma explains how to compute the quantities appearing in the previous lemma. Again, particular cases of this can be found in [BBK94] (Lemmas 2.3, 2.5).

Lemma 3.15. *Let $x \in D, \varepsilon > \delta > 0$ and $A \subset D$ such that $D(x,\varepsilon) \subset D \backslash A$ and denote d the distance between $\partial D(x,\varepsilon)$ and $A \cup \partial D$. Assume $d > 0$. Let B be either A or $A \cup \partial D(x,\delta)$ and denote*

$$u = \begin{cases} \frac{\varepsilon}{\varepsilon + d} \text{ if } B = A, \\ \frac{\varepsilon}{\varepsilon+d} + \frac{\delta}{\varepsilon} \text{ if } B = A \cup \partial D(x,\delta). \end{cases}$$

We have for all $y, y' \in \partial D(x,\varepsilon)$, and $E \subset B \cup \partial D$,

$$\omega^{B\cup\partial D}(y, E) = (1+O(u))\, \omega^{B\cup\partial D}(y', E). \tag{3.17}$$

Moreover, denoting $\tau^B_{\partial D(x,\varepsilon)} := \inf\{t > \tau_B : B_t \in \partial D(x,\varepsilon)\}$ the first hitting time of $\partial D(x,\varepsilon)$ after τ_B, we have for any $z \in \partial D(x,\varepsilon)$,

$$\frac{1}{H^z_{x,\varepsilon}(\tau \wedge \tau_B < \infty)} = (1+O(u)) \max_{y\in\partial D(x,\varepsilon)} \mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right]\left(1 - \int_{\partial D(x,\varepsilon)} \frac{dy}{2\pi\varepsilon} \mathbb{P}_y\left(\tau^B_{\partial D(x,\varepsilon)} < \tau\right)\right). \tag{3.18}$$

Proof. In this proof, we will consider $\eta > 0$ such that $D(x,\varepsilon+\eta) \cap (A \cup \partial D) = \varnothing$.

Let us start by proving (3.17) for $B = A$. Let $y \in \partial D(x,\varepsilon), E \subset A \cup \partial D$. By Markov property applied to the first hitting time of $\partial D(x,\varepsilon+\eta)$, we have

$$\omega^{A\cup\partial D}(y,E) = \int_{\partial D(x,\varepsilon+\eta)} \omega^{\partial D(x,\varepsilon+\eta)}(y,d\xi)\mathbb{P}_\xi\left(B_{\tau_A\wedge\tau} \in E\right).$$

But the measure $\omega^{\partial D(x,\varepsilon+\eta)}(y,d\xi)$ is explicit and its density with respect to the Lebesgue measure on the circle $\partial D(x,\varepsilon+\eta)$ is equal to

$$\frac{1}{2\pi(\varepsilon+\eta)}\frac{(\varepsilon+\eta)^2-|y-x|^2}{|y-\xi|^2} = \frac{1}{2\pi(\varepsilon+\eta)}\left(1+O\left(\frac{\varepsilon}{\varepsilon+\eta}\right)\right).$$

Hence, up to a multiplicative error $1+O(\varepsilon/(\varepsilon+\eta))$, $\omega^{A\cup\partial D}(y,E)$ is independent of $y \in \partial D(x,\varepsilon)$ which is the content of (3.17) for $B = A$. We now prove it for $B = A \cup \partial D(x,\delta)$. The reasoning is going to be similar. Let $y \in \partial D(x,\varepsilon), E \subset B \cup \partial D$. We only need to treat the case of $E \subset \partial D(x,\delta)$ or $E \subset A \cup \partial D$. We will deal with the first one, as the latter is similar. By Markov property applied to $\tau_A \wedge \tau$, we have

$$\begin{aligned}\omega^{B\cup\partial D}(y,E) &= \mathbb{P}_y\left(B_{\tau_{\partial D(x,\delta)}} \in E\right) - \mathbb{P}_y\left(B_{\tau_{\partial D(x,\delta)}} \in E, \tau_{\partial D(x,\delta)} > \tau_A\wedge\tau\right)\\ &= \omega^{\partial D(x,\delta)}(y,E) - \mathbb{E}_y\left[\mathbf{1}_{\left\{\tau_{\partial D(x,\delta)}>\tau_A\wedge\tau\right\}}\omega^{\partial D(x,\delta)}(B_{\tau_A\wedge\tau},E)\right].\end{aligned}$$

Again the measure $\omega^{\partial D(x,\delta)}(y,d\xi)$ is explicit and its density with respect to the Lebesgue measure on the circle $\partial D(x,\delta)$ is equal to

$$\frac{1}{2\pi\delta}\frac{|y-x|^2-\delta^2}{|y-\xi|^2} = \frac{1}{2\pi\delta}\left(1+O\left(\frac{\delta}{\varepsilon}\right)\right).$$

Hence, up to a multiplicative error $1+O(\delta/\varepsilon)$, $\omega^{\partial D(x,\delta)}(y,d\xi)$ is uniform on $\partial D(x,\delta)$ and does not depend on y. As $A \cup \partial D$ is even further from $\partial D(x,\delta)$, the same is true with $\omega^{\partial D(x,\delta)}(z,d\xi)$ for any $z \in A \cup \partial D$. To conclude that $\omega^{B\cup\partial D}(y,E)$ does not depend on y, we observe that

$$\begin{aligned}\mathbb{P}_y\left(\tau_{\partial D(x,\delta)} > \tau_A\wedge\tau\right) &= \mathbb{P}_y\left(\tau_{\partial D(x,\delta)} > \tau_{\partial D(x,\varepsilon+\eta)}\right)\\ &\quad\times\int_{\partial D(x,\varepsilon+\eta)}\omega^{\partial D(x,\varepsilon+\eta)}(y,d\xi)\mathbb{P}_\xi\left(\tau_{\partial D(x,\delta)} > \tau_A\wedge\tau\right).\end{aligned} \tag{3.19}$$

By rotational invariance of Brownian motion, the first term is independent of $y \in \partial D(x,\varepsilon)$. We have already seen that up to a multiplicative error $1+O(\varepsilon/(\varepsilon+\eta))$, $\omega^{\partial D(x,\varepsilon+\eta)}(y,d\xi)$ is uniform on the circle and thus does not depend on y. In the end, it shows that up to a multiplicative error $1+O(\delta/\varepsilon)+O(\varepsilon/(\varepsilon+\eta))$, $\omega^{B\cup\partial D}(y,E)$ is independent of $y \in \partial D(x,\varepsilon)$ which was required by the claim (3.17) in the case $B = A \cup \partial D(x,\delta)$.

We now prove (3.18). We proceed as follows: we bound from above $\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B<\infty)$ and we show that

$$\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B<\infty) \geq \left(1+O\left(\frac{\varepsilon}{\varepsilon+d}\right)\right) \max_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B<\infty) \tag{3.20}$$

which provides a lower bound on $1\big/H^z_{x,\varepsilon}(\tau\wedge\tau_B)$ for any $z\in\partial D(x,\varepsilon)$. The upper bound is obtained in a similar way.

Let us start by proving (3.20). Recall that $\eta>0$ has been chosen such that $D(x,\varepsilon+\eta)\cap(A\cup\partial D)=\varnothing$. Let $z\in D(x,\varepsilon+\eta/2)\backslash D(x,\delta)$. We want to show that the dependence of z on $\mathbb{P}_z\left(\tau\wedge\tau_B<\tau_{\partial D(x,\varepsilon)}\right)$ relies almost exclusively on $|z-x|$. If z is inside $D(x,\varepsilon)$ it is clear: if $B=A$ this probability is equal to zero and if $B=A\cup\partial D(x,\delta)$, it depends only on $|z-x|$ by rotational invariance of Brownian motion. Whereas if z is outside $\bar{D}(x,\varepsilon)$, a similar argument as in (3.19) shows that up to a multiplicative error $1+O\left(|z-x|/(\varepsilon+\eta)\right)$ this probability depends only on $|z-x|$. It concludes the proof of (3.20).

We now bound from below $1\big/\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B<\infty)$. Take a starting point $y\in\partial D(x,\varepsilon)$. We decompose $L_{x,\varepsilon}(\tau)$ according to the different excursions between $\partial D(x,\varepsilon)$ and B. Denote $\sigma_0^{(1)}:=0$ and for all $i\geq 1$,

$$\sigma_i^{(2)}:=\inf\{t\geq\sigma_{i-1}^{(1)}:B_t\in B\} \text{ and } \sigma_i^{(1)}:=\inf\{t\geq\sigma_i^{(2)}:B_t\in\partial D(x,\varepsilon)\}.$$

We also denote $N:=\sup\{i\geq 0:\sigma_i^{(1)}<\tau\}$ the number of excursions from B to $\partial D(x,\varepsilon)$ and $L^i_{x,\varepsilon}$ the local time of $\partial D(x,\varepsilon)$ accumulated during the interval of time $[\sigma_i^{(1)},\sigma_i^{(2)}]$. Using the convention $\sigma_N^{(2)}:=\tau$, we have

$$L_{x,\varepsilon}(\tau)=\sum_{i=0}^N L^i_{x,\varepsilon}.$$

By Lemma 3.13 applied to $C=\partial D\cup B$ and thanks to (3.17),

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right]\leq\sum_{n=0}^\infty \mathbb{P}_y\left(N=n\right)(1+n)(1+O(u))^{1+n}\Big/\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B)\,.$$

As

$$\mathbb{P}_y\left(N\geq n\right)=\left((1+O(u))\int_{\partial D(x,\varepsilon)}\frac{dz}{2\pi\varepsilon}\mathbb{P}_z\left(N\geq 1\right)\right)^n,$$

it leads to

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right]\leq(1+O(u))\left(1-\int_{\partial D(x,\varepsilon)}\frac{dz}{2\pi\varepsilon}\mathbb{P}_z\left(N\geq 1\right)\right)^{-1}\left(\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B)\right)^{-1}$$

which is the required lower bound on $1/\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau\wedge\tau_B)$. $\square$

In the next sections we will consider $\gamma \in (0,2)$, $A \in \mathcal{B}(D)$ and T of the form $T = (b, \infty)$ with $b \in \mathbb{R}$. For $\widetilde{\gamma} > \gamma$, $\varepsilon_0 \in \{e^{-p}, p \geq 1\}$ and $x \in D$, define the good event at x:

$$G_\varepsilon(x, \varepsilon_0) := \left\{ \forall r \in [\varepsilon, \varepsilon_0], \frac{1}{\bar{r}} L_{x,\bar{r}}(\tau) \leq \widetilde{\gamma}^2 \left(\log \bar{r}\right)^2 \right\} \tag{3.21}$$

where for $r > 0$, we denote by $\bar{r} = \inf\left(\{e^{-p}, p \geq 1\} \cap [r, \infty)\right)$. We also define

$$\widetilde{\nu}_\varepsilon^\gamma(dx, dt) = \nu_\varepsilon^\gamma(dx, dt) \mathbf{1}_{G_\varepsilon(x,\varepsilon_0)} \mathbf{1}_{\{|x - x_0| > \varepsilon_0, \mathrm{d}(x, \partial D) > \varepsilon_0\}}. \tag{3.22}$$

To ease computations, we change a bit the definition of good events that we associate to μ_ε^γ:

$$G'_\varepsilon(x, \varepsilon_0, M) := G_\varepsilon(x, \varepsilon_0) \cap \left\{ \left| \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)} - \gamma \log \frac{1}{\varepsilon} \right| \leq M \sqrt{|\log \varepsilon|} \right\}, \tag{3.23}$$

and we define

$$\widetilde{\mu}_\varepsilon^\gamma(dx) = \mu_\varepsilon^\gamma(dx) \mathbf{1}_{G'_\varepsilon(x,\varepsilon_0,M)} \mathbf{1}_{\{|x - x_0| > \varepsilon_0, \mathrm{d}(x, \partial D) > \varepsilon_0\}}. \tag{3.24}$$

This change of good event is purely technical: it will allow us to easily transfer computations linked to $\widetilde{\mu}_\varepsilon^\gamma$ (in Proposition 3.21) to computations linked to $\widetilde{\nu}_\varepsilon^\gamma$ (in Proposition 3.20) rather than repeating arguments which are very similar.

3.3 First moment estimates

In this section, we give estimates on the first moment of $\nu_\varepsilon^\gamma(A \times T)$ and $\mu_\varepsilon^\gamma(A)$ and we show that adding the good events $G_\varepsilon(x, \varepsilon_0)$ and $G'_\varepsilon(x, \varepsilon_0, M)$ does not change the behaviour of the first moment.

Proposition 3.16. *We have the following estimate*

$$\lim_{\varepsilon \to 0} \mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A \times T)\right] = \int_T e^{-\gamma t} \gamma dt \int_A R(x, D)^{\gamma^2/2} G_D(x_0, x) dx. \tag{3.25}$$

Moreover, for all $\varepsilon < \varepsilon_0$,

$$0 \leq \mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A \times T)\right] - \mathbb{E}_{x_0}\left[\widetilde{\nu}_\varepsilon^\gamma(A \times T)\right] \leq p(\varepsilon_0) \tag{3.26}$$

with $p(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$. $p(\varepsilon_0)$ may depend on $\gamma, \widetilde{\gamma}, T$.

Proposition 3.17. *We have the following estimate*

$$\lim_{\varepsilon \to 0} \mathbb{E}_{x_0}\left[\mu_\varepsilon^\gamma(A)\right] = \sqrt{2\pi}\gamma \int_A R(x, D)^{\gamma^2/2} G_D(x_0, x) dx. \tag{3.27}$$

Moreover, for all $\varepsilon < \varepsilon_0$,

$$0 \leq \mathbb{E}_{x_0}\left[\mu_\varepsilon^\gamma(A)\right] - \mathbb{E}_{x_0}\left[\widetilde{\mu}_\varepsilon^\gamma(A)\right] \leq p'(\varepsilon_0, M) \tag{3.28}$$

with $p'(\varepsilon_0, M) \to 0$ *as* $\varepsilon_0 \to 0$ *and* $M \to \infty$. $p'(\varepsilon_0, M)$ *may depend on* $\gamma, \widetilde{\gamma}$.

The estimates (3.25) and (3.27) will be computations on the local times made possible thanks to Section 3.2. To prove (3.26) and (3.28), we will be able to transfer all the computations to the zero-dimensional Bessel process. For this reason, we first start by stating the analogue of (3.26) and (3.28) for this process (recall that we denote $\mathbb{P}_r$ the law under which $(R_t, t \geq 0)$ is a zero-dimensional Bessel process starting from r):

Lemma 3.18. *Let* $\widetilde{\gamma} > \gamma > 0$, $b, \widetilde{b} \in \mathbb{R}$, $r_0, s_0 > 0$ *and define for all* $t > s_0$ *the event*

$$E_t(s_0) := \left\{ \forall s \in \mathbb{N} \cap [s_0, t], R_s \leq \widetilde{\gamma} s + \widetilde{b} \right\}.$$

For all starting point $r \in (0, r_0)$, *for all* $t > s_0$,

$$\mathbb{P}_r\left(E_t(s_0) | R_t \geq \gamma t + b\right) \geq 1 - p(s_0), \tag{3.29}$$

$$\mathbb{E}_r\left[e^{\gamma R_t} \mathbf{1}_{E_t(s_0)}\right] \geq (1 - p(s_0)) \mathbb{E}_r\left[e^{\gamma R_t}\right] \tag{3.30}$$

with $p(s_0) \to 0$ *as* $s_0 \to \infty$. $p(s_0)$ *may depend on* $\gamma, \widetilde{\gamma}, b, \widetilde{b}, r_0$.

In the previous proposition, the starting point r was required to stay bounded away from infinity. To come back to this situation, we will need the following:

Lemma 3.19. *1) Let* $a > 0$. *There exists* $C = C(a) > 0$ *such that for all* $t > 0$, $\lambda \geq at$ *and* $r \in (1, \lambda/2)$,

$$\mathbb{P}_r\left(R_t \geq \lambda\right) \leq C \sqrt{r} e^{\frac{\lambda r}{t}} \frac{1}{\lambda} e^{-\frac{\lambda^2}{2t}}.$$

2) Let $\gamma > 0$. *There exists* $C = C(\gamma) > 0$ *such that for all* $t > 0$, *for all* $r \in (1, \gamma t/2)$,

$$\mathbb{E}_r\left[e^{\gamma R_t}\right] \leq C \sqrt{r} e^{Cr} \frac{1}{\sqrt{t}} e^{\gamma^2 t/2}. \tag{3.31}$$

The first and second points will be used to prove Propositions 3.16 and 3.17 respectively. The two previous lemmas will be proven in Appendix 3.B and we now prove Propositions 3.16 and 3.17.

Proof of Proposition 3.16. We start by proving (3.25). We have

$$\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A \times T)\right] = \int_A \left|\log \varepsilon\right| \varepsilon^{-\gamma^2/2} \mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)} > \gamma \log \frac{1}{\varepsilon} + b \right) dx \tag{3.32}$$

and we are going to estimate the probability

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \gamma\log\frac{1}{\varepsilon} + b\right). \tag{3.33}$$

Assume that $\varepsilon > 0$ is small enough so that $\gamma\left|\log\varepsilon\right| + b > 0$ to ensure that the probability we are interested in is not trivial. Take $a \in (\gamma^2/4, 1)$. If x is at distance at most ε^a from x_0, we bound from above the probability (3.33) by 1 and the contribution to the integral (3.32) of such points is at most $C\varepsilon^{2a-\gamma^2/2}\log 1/\varepsilon$ which goes to zero as $\varepsilon \to 0$.

Let $\eta > 0$. We are now going to deal with points $x \in D$ at distance at least ε^a from x_0 and at distance at least η from the boundary of the domain D. We will then explain how to deal with points close to the boundary. By Markov property, the probability (3.33) is equal to

$$\mathbb{P}_{x_0}\left(\tau_{\partial D(x,\varepsilon)} < \tau\right)\mathbb{E}_{x_0}\left[\mathbb{P}_Y\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \gamma\log\frac{1}{\varepsilon} + b\right)\right]$$

where $Y \in \partial D(x,\varepsilon)$ has the law of $B_{\tau_{\partial D(x,\varepsilon)}}$ starting from x_0 and knowing that $\tau_{\partial D(x,\varepsilon)} < \tau$. Take any $y \in \partial D(x,\varepsilon)$. By Lemma 3.15 we have

$$\min_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau<\infty) = (1+O(\varepsilon/\eta))\max_{z\in\partial D(x,\varepsilon)} H^z_{x,\varepsilon}(\tau<\infty) = (1+O(\varepsilon/\eta))/\mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right].$$

But Lemma 3.9 gives

$$\mathbb{E}_y\left[L_{x,\varepsilon}(\tau)\right] = 2\varepsilon\left(\log 1/\varepsilon + \log R(x,D) + o(1)\right).$$

Hence, with the help of Lemma 3.13, starting from y, $L_{x,\varepsilon}(\tau)$ is stochastically dominated and stochastically dominates exponential variables with mean equal to

$$2\varepsilon\left(\log 1/\varepsilon + \log R(x,D) + o_\eta(1)\right).$$

It implies that

$$\begin{aligned}
&\mathbb{P}_y\left(\sqrt{\tfrac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \gamma\log\tfrac{1}{\varepsilon} + b\right)\\
&= (1+o_\eta(1))\,\mathbb{P}\left(2\left(\log\tfrac{1}{\varepsilon} + \log R(x,D) + o_\eta(1)\right)\mathrm{Exp}(1) > \left\{\gamma\log\tfrac{1}{\varepsilon} + b\right\}^2\right)\\
&= (1+o_\eta(1))\,\varepsilon^{\gamma^2/2}R(x,D)^{\gamma^2/2}e^{-\gamma b}.
\end{aligned}$$

On the other hand, Lemma 3.10 shows that

$$\mathbb{P}_{x_0}\left(\tau_{\partial D(x,\varepsilon)} < \tau\right) = (1+o_\eta(1))G_D(x_0,x)/\left|\log\varepsilon\right|. \tag{3.34}$$

Putting things together leads to

$$\int_A \log\frac{1}{\varepsilon}\varepsilon^{-\gamma^2/2}\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b\right)\mathbf{1}_{\{\mathrm{d}(x,\partial D)>\eta\}}dx$$
$$=(1+o_\eta(1))e^{-\gamma b}\int_A R(x,D)^{\gamma^2/2}G_D(x_0,x)\mathbf{1}_{\{\mathrm{d}(x,\partial D)>\eta\}}dx.$$

To conclude the proof of (3.25), it is enough to show that

$$\limsup_{\varepsilon\to 0}\int_A |\log\varepsilon|\,\varepsilon^{-\gamma^2/2}\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b\right)\mathbf{1}_{\{\mathrm{d}(x,\partial D)\le\eta\}}dx=O(\eta). \tag{3.35}$$

Consider a larger domain $\widetilde{D}$ so that D is compactly included in $\widetilde{D}$. Now, all the points $x\in D$ are far away from the boundary of $\widetilde{D}$ and what we did before shows that

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b\right)\le\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{\widetilde{D}})}>\gamma\log\frac{1}{\varepsilon}+b\right)\le C\varepsilon^{\gamma^2/2}/\left|\log\varepsilon\right|$$

which shows (3.35).

We have finished to prove (3.25) and we now turn to the proof of (3.26). Let $\hat\varepsilon_0>\varepsilon_0$. As we have just seen, the contribution of $\{x\in D:|x-x_0|\le\hat\varepsilon_0 \text{ or } \mathrm{d}(x,\partial D)\le\hat\varepsilon_0\}$ to $\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A\times T)\right]$ is $O(\hat\varepsilon_0)$. Hence $\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A\times T)\right]-\mathbb{E}_{x_0}\left[\tilde\nu_\varepsilon^\gamma(A\times T)\right]$ is equal to

$$O(\hat\varepsilon_0)+\int_A \log\frac{1}{\varepsilon}\varepsilon^{-\gamma^2/2}\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b, G_\varepsilon(x,\varepsilon_0)^c\right)\mathbf{1}_{\{|x-x_0|>\hat\varepsilon_0,\mathrm{d}(x,\partial D)>\hat\varepsilon_0\}}dx.$$

Take $x\in D$ such that $|x-x_0|>\hat\varepsilon_0$ and $\mathrm{d}(x,\partial D)>\hat\varepsilon_0$. Considering a larger domain than D will increase the probability in the above integral. As we want to bound it from above, we can thus assume in the following that $D=D(x,R_0)$ where R_0 is the diameter of our original domain. It is convenient because we can now use (3.7) which relates the local times to a zero-dimensional Bessel process.

We claim that we can take $M>0$ large enough, depending only on $\hat\varepsilon_0$, R_0 and b, such that

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b, L_{x,\hat\varepsilon_0}(\tau)\ge M\right)\le\hat\varepsilon_0\frac{1}{|\log\varepsilon|}\varepsilon^{\gamma^2/2}. \tag{3.36}$$

Indeed, (3.7) and Lemma 3.19 imply that there exists $C=C(\hat\varepsilon_0,b)>0$ such that if ε is small enough and if $\ell\le\frac{\hat\varepsilon_0\gamma^2}{4}\log\left(\frac{\hat\varepsilon_0}{\varepsilon}\right)^2$,

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}>\gamma\log\frac{1}{\varepsilon}+b\,\middle|\,L_{x,\hat\varepsilon_0}(\tau)=\ell\right)\le C\ell^{1/4}e^{C\sqrt{\ell}}\frac{1}{|\log\varepsilon|}\varepsilon^{\gamma^2/2}..$$

As, starting from any point of $\partial D(x,\hat\varepsilon_0)$, $L_{x,\hat\varepsilon_0}(\tau)$ is an exponential variable (with mean depending on $\hat\varepsilon_0$ and R_0),

$$\mathbb{P}_{x_0}\left(L_{x,\hat\varepsilon_0}(\tau) \geq \frac{\hat\varepsilon_0\gamma^2}{4}\log\left(\frac{\hat\varepsilon_0}{\varepsilon}\right)^2\right)$$

goes to zero faster than any polynomial in ε and also

$$\mathbb{E}_{x_0}\left[L_{x,\hat\varepsilon_0}(\tau)^{1/4}e^{C\sqrt{L_{x,\hat\varepsilon_0}(\tau)}}\mathbf{1}_{\left\{L_{x,\hat\varepsilon_0}(\tau)\geq M\right\}}\right]$$

goes to zero as $M\to\infty$. Putting things together then leads to (3.36).

On the other hand, by (3.7) and claim (3.29) of Lemma 3.18 that we use with

$$t\leftarrow \log\frac{\hat\varepsilon_0}{\varepsilon}, s_0\leftarrow \log\frac{\hat\varepsilon_0}{\varepsilon_0}, r_0\leftarrow \frac{M}{\hat\varepsilon_0}, b\leftarrow b+\gamma\log\frac{1}{\hat\varepsilon_0} \text{ and } \widetilde{b}\leftarrow \gamma\log\frac{1}{\hat\varepsilon_0},$$

there exists $p(\varepsilon_0)$ (which may depend on $\gamma,\widetilde\gamma,b,\hat\varepsilon_0,M$) such that $p(\varepsilon_0)\to 0$ as $\varepsilon_0\to 0$ and for all $\varepsilon<\varepsilon_0$,

$$\begin{aligned}
&\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \gamma\log\frac{1}{\varepsilon}+b, G_\varepsilon(x,\varepsilon_0)^c, L_{x,\hat\varepsilon_0}(\tau)\leq M\right)\\
&\leq \mathbb{E}_{x_0}\left[\mathbf{1}_{\left\{L_{x,\hat\varepsilon_0}(\tau)\leq M\right\}}\mathsf{P}_{\sqrt{L_{x,\hat\varepsilon_0}(\tau)/\hat\varepsilon_0}}\left(R_t\geq \gamma t+b+\gamma\log\frac{1}{\hat\varepsilon_0}, E_t(s_0)^c\right)\right]\\
&\leq p(\varepsilon_0)\mathbb{E}_{x_0}\left[\mathbf{1}_{\left\{L_{x,\hat\varepsilon_0}(\tau)\leq M\right\}}\mathsf{P}_{\sqrt{L_{x,\hat\varepsilon_0}(\tau)/\hat\varepsilon_0}}\left(R_t\geq \gamma t+b+\gamma\log\frac{1}{\hat\varepsilon_0}\right)\right]\\
&= p(\varepsilon_0)\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \gamma\log\frac{1}{\varepsilon}+b, L_{x,\hat\varepsilon_0}(\tau)\leq M\right).
\end{aligned}$$

With (3.36) it implies that

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} > \sqrt{\frac{g}{2}}\gamma\log\frac{1}{\varepsilon}+b, G_\varepsilon(x,\varepsilon_0)^c\right)\leq q(\varepsilon_0)\frac{1}{\log\varepsilon}\varepsilon^{\gamma^2/2}$$

for some $q(\varepsilon_0)\to 0$ as $\varepsilon_0\to 0$ which may depend on $\gamma,\widetilde\gamma,b$. It shows that $\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A\times T)\right]-\mathbb{E}_{x_0}\left[\widetilde\nu_\varepsilon^\gamma(A\times T)\right]\leq Cq(\varepsilon_0)$ which finishes the proof of (3.26).

□

We now turn to the proof of Proposition 3.17. As it is similar to what we have just done we will be brief.

Proof of Proposition 3.17. Take $\eta>0$ and $x\in D$ at distance at least η from the boundary. As we saw before, conditioned on $\tau_{\partial D(x,\varepsilon)}<\tau$, $L_{x,\varepsilon}(\tau)$ is approximated by an exponential variable with mean $2\varepsilon(\log 1/\varepsilon + \log R(x,D) + o_\eta(1))$. Hence, denoting $\theta = \log(R(x,D)/\varepsilon) + o(1)$ and with the change of variable $u = \sqrt{t} - \gamma\sqrt{\theta/2}$, we have

$$\begin{aligned}
\mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \middle| \tau_{\partial D(x,\varepsilon)} < \tau \right] &= (1+o_\eta(1)) \int_0^\infty e^{-t} e^{\gamma\sqrt{2\theta t}} dt \\
&= (1+o_\eta(1))\, e^{\gamma^2\theta/2} \int_0^\infty e^{-(\sqrt{t}-\gamma\sqrt{\theta/2})^2} dt \\
&= (1+o_\eta(1))\, \gamma\sqrt{2\theta} e^{\gamma^2\theta/2} \int_{-\gamma\sqrt{\theta/2}}^\infty e^{-u^2} \left(1 + \frac{\sqrt{2}u}{\gamma\sqrt{\theta}}\right) du \\
&= (1+o_\eta(1))\gamma\sqrt{2\pi} R(x,D)^{\gamma^2/2} \sqrt{\log(1/\varepsilon)} \varepsilon^{-\gamma^2/2}.
\end{aligned}$$

In particular, the impact of points x such that $|x - x_0| \leq 1/\log(1/\varepsilon)$ is negligible. For points that are far away from x_0, we can use (3.34) which then shows that

$$\sqrt{\log\left(\frac{1}{\varepsilon}\right)} \varepsilon^{\gamma^2/2} \mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \right] = (1+o_\eta(1))\gamma\sqrt{2\pi} R(x,D)^{\gamma^2/2} G_D(x_0,x).$$

By the same reasoning as in the proof of Proposition 3.16, it concludes the proof of (3.27). We now focus on (3.28). First of all, we notice that requiring $\sqrt{L_{x,\varepsilon}(\tau)/\varepsilon}$ to belong to the interval

$$\left[\gamma \log\frac{1}{\varepsilon} - M\sqrt{\log\frac{1}{\varepsilon}}, \gamma\log\frac{1}{\varepsilon} + M\sqrt{\log\frac{1}{\varepsilon}} \right]$$

has the consequence of restraining the variable t in the above computations to the interval

$$\left[\frac{\gamma^2}{2}\log\frac{1}{\varepsilon} - M\gamma\sqrt{\log\frac{1}{\varepsilon}} + O(1), \frac{\gamma^2}{2}\log\frac{1}{\varepsilon} + M\gamma\sqrt{\log\frac{1}{\varepsilon}} + O(1) \right]$$

which then restrains the variable u to the interval:

$$\left[-\frac{1}{\sqrt{2}}M + o(1), \frac{1}{\sqrt{2}}M + o(1) \right].$$

Therefore, the integral over u is still equal to $(1 + o_{M\to\infty}(1))\sqrt{\pi}$ showing that we can safely forget the event

$$\left\{ \left| \sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)} - \gamma\log\frac{1}{\varepsilon} \right| \leq M\sqrt{\log\frac{1}{\varepsilon}} \right\}$$

in the good event $G'_\varepsilon(x,\varepsilon_0,M)$. To bound from above

$$\mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \right] - \mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \mathbf{1}_{G_\varepsilon(x,\varepsilon_0)} \right],$$

we proceed exactly as in the proof of Proposition 3.16. We notice that this quantity increases with the domain, so we can assume that D is a disc centred at x which allows us to use the link between the local times and the zero-dimensional Bessel process (3.7). We then conclude as in the proof of Proposition 3.16 using claim (3.31) of Lemma 3.19 and claim (3.30) of Lemma 3.18. $\square$

3.4 Uniform integrability

This section is devoted to the following two propositions:

Proposition 3.20. *If $\widetilde{\gamma}$ is close enough to γ, then*

$$\sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\widetilde{\nu}_\varepsilon^\gamma(A\times T)^2\right] < \infty. \tag{3.37}$$

Proposition 3.21. *If $\widetilde{\gamma}$ is close enough to γ, then*

$$\sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\widetilde{\mu}_\varepsilon^\gamma(A)^2\right] < \infty. \tag{3.38}$$

We start by proving Proposition 3.20.

Proof of Proposition 3.20. The proof will be decomposed in three parts. The first part is short and lay the ground work. In particular, it shows that it is enough to control the probability (written in (3.41)) that the local times are large in two circles and small in an other circle. The second part describes the joint law of the local times in those three circles whereas the third part computes the probability (3.41) left in the first part. To shorten the equations, we will denote $L_{x,\varepsilon} := L_{x,\varepsilon}(\tau)$ the local times up to time τ in this proof.

Step 1. Denoting $A_{\varepsilon_0} = \{x \in A : |x-x_0| > \varepsilon_0 \text{ and } \mathrm{d}(x,\partial D) > \varepsilon_0\}$, by definition of $\widetilde{\nu}_\varepsilon^\gamma$ (see (3.22)), $\mathbb{E}_{x_0}\left[\widetilde{\nu}_\varepsilon^\gamma(A\times T)^2\right]$ is equal to

$$\left(\log\frac{1}{\varepsilon}\right)^2 \varepsilon^{-\gamma^2} \int_{A_{\varepsilon_0}\times A_{\varepsilon_0}} dxdy\ \mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}, \sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}} \geq \gamma\log\frac{1}{\varepsilon} + b, G_\varepsilon(x,\varepsilon_0), G_\varepsilon(y,\varepsilon_0)\right).$$

Take $a \in (\gamma^2/4, 1)$. The contribution of points x, y such that $|x-y| \leq \varepsilon^a$ goes to zero as $\varepsilon \to 0$. Indeed, this contribution is not larger than

$$C\left(\log\frac{1}{\varepsilon}\right)^2 \varepsilon^{-\gamma^2}\varepsilon^{2a} \int_{A_{\varepsilon_0}} dx\ \mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}} \geq \gamma\log\frac{1}{\varepsilon} + b\right) = C\log\frac{1}{\varepsilon}\varepsilon^{-\gamma^2/2+2a}\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A_{\varepsilon_0})\right]$$

which goes to zero by the first moment estimate (3.25) of Proposition 3.16. We take now

$x, y \in A_{\varepsilon_0}$ such that $|x-y| > \varepsilon^a$. By symmetry, it is enough to bound from above

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}, \sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}} \geq \gamma \log\frac{1}{\varepsilon} + b, G_\varepsilon(x,\varepsilon_0), G_\varepsilon(y,\varepsilon_0), \tau_{\partial D(x,\varepsilon)} < \tau_{\partial D(y,\varepsilon)}\right). \tag{3.39}$$

Take $M > 0$ large and $R \in (e^{-p}, p \geq 0)$ such that

$$\frac{|x-y|}{eM} \leq R < \frac{|x-y|}{M}. \tag{3.40}$$

We ensure that $R < \varepsilon_0$ by taking M large enough, but M will play another role later. The probability in (3.39) is at most

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}, \sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}} \geq \gamma \log\frac{1}{\varepsilon} + b, \sqrt{\frac{1}{R}L_{x,R}} \leq \tilde{\gamma}\log\frac{1}{R}, \tau_{\partial D(x,\varepsilon)} < \tau_{\partial D(y,\varepsilon)}\right). \tag{3.41}$$

The rest of the proof is dedicated to bound from above this probability. For this purpose, the next paragraph describes the joint law of the local times in those three circles.

Step 2. We are going to decompose those three local times according to the different excursions between $\partial D(x,R)$, $\partial D(x,\varepsilon)$ and $\partial D(y,\varepsilon)$. Denote by $A_{R\to x}$ (resp. $A_{R\to y}$) the number of excursions from $\partial D(x,R)$ to $\partial D(x,\varepsilon)$ (resp. to $\partial D(y,\varepsilon)$) before τ, and denote by

- $L^n_{x,\varepsilon}$ the local time of $\partial D(x,\varepsilon)$ during the n-th excursion from $\partial D(x,\varepsilon)$ to $\partial D(x,R)$,
- $L^n_{y,\varepsilon}$ the local time of $\partial D(y,\varepsilon)$ during the n-th excursion from $\partial D(y,\varepsilon)$ to $\partial D(x,R) \cup \partial D$,
- $L^n_{x,R}$ the local time of $\partial D(x,R)$ during the n-th excursion from $\partial D(x,R)$ to $\partial D(x,\varepsilon) \cup \partial D(y,\varepsilon) \cup \partial D$.

For any $x' \in \partial D(x,\varepsilon)$, we have under $\mathbb{P}_{x'}$

$$L_{x,\varepsilon} = \sum_{n=1}^{1+A_{R\to x}} L^n_{x,\varepsilon}, \; L_{y,\varepsilon} = \sum_{n=1}^{A_{R\to y}} L^n_{y,\varepsilon} \text{ and } L_{x,R} \succ \sum_{n=1}^{A_{R\to x}+A_{R\to y}} L^n_{x,R}. \tag{3.42}$$

The stochastic domination is not exactly an equality because if the last visited circle before τ is $\partial D(x,R)$ (it could be $\partial D(y,\varepsilon)$), the number of excursions from $\partial D(x,R)$ to $\partial D(x,\varepsilon) \cup \partial D(y,\varepsilon)$ before τ is $1 + A_{R\to x} + A_{R\to y}$ rather than $A_{R\to x} + A_{R\to y}$. Lemma 3.13 allows us to approximate (in the precise sense stated therein) the $L^n_{x,\varepsilon}$'s, $L^n_{y,\varepsilon}$'s, $L^n_{x,R}$'s by exponential variables independent of $A_{R\to x}$ and $A_{R\to y}$. We are going to compute the mean of those exponential variables and the transition probabilities between the different three circles.

Let us start with the study of the transition probabilities. We will denote

$$p_{xy} := \frac{\log 1/|x-y|}{\log 1/\varepsilon}. \tag{3.43}$$

Because $|x-y| > \varepsilon^a$, note that p_{xy} is bounded away from 1: $0 < p_{xy} < 1-a$. We first remark that by (3.14) we have

$$\begin{aligned}
\forall z \in \partial D(x,\varepsilon), \mathbb{P}_z\left(\tau_{\partial D(y,\varepsilon)} < \tau\right) &= p_{xy} + O(1/\log\varepsilon),\\
\forall z \in \partial D(y,\varepsilon), \mathbb{P}_z\left(\tau_{\partial D(x,\varepsilon)} < \tau\right) &= p_{xy} + O(1/\log\varepsilon),\\
\forall z \in \partial D(x,R), \mathbb{P}_z\left(\tau_{\partial D(x,\varepsilon)} < \tau\right) = p_{xy} + O(1/\log\varepsilon), &\mathbb{P}_z\left(\tau_{\partial D(y,\varepsilon)} < \tau\right) = p_{xy} + O(1/\log\varepsilon).
\end{aligned}$$

Here and in the following of the proof, the O's may depend on ε_0, M, a. By Lemma 3.12 it thus implies that for all $z \in \partial D(x,R)$,

$$\begin{aligned}
\mathbb{P}_z\left(\tau_{\partial D(x,\varepsilon)} < \tau \wedge \tau_{\partial D(y,\varepsilon)}\right) &= \frac{p_{xy} + O(1/\log\varepsilon) - (p_{xy} + O(1/\log\varepsilon))^2}{1-(p_{xy}+O(1/\log\varepsilon))^2} \qquad (3.44)\\
&= \frac{p_{xy}}{1+p_{xy}} + O\left(\frac{1}{\log\varepsilon}\right),\\
\mathbb{P}_z\left(\tau_{\partial D(y,\varepsilon)} < \tau \wedge \tau_{\partial D(x,\varepsilon)}\right) &= \frac{p_{xy}}{1+p_{xy}} + O\left(\frac{1}{\log\varepsilon}\right). \qquad (3.45)
\end{aligned}$$

Of course, for any $z \in \partial D(x,\varepsilon)$,

$$\mathbb{P}_z\left(\tau_{\partial D(x,R)} < \tau \wedge \tau_{\partial D(y,\varepsilon)}\right) = 1 \tag{3.46}$$

and (3.14) implies that for all $z \in \partial D(y,\varepsilon)$

$$\mathbb{P}_z\left(\tau_{\partial D(x,R)} < \tau \wedge \tau_{\partial D(x,\varepsilon)}\right) = \mathbb{P}_z\left(\tau_{\partial D(x,R)} < \tau\right) = 1 - O\left(\frac{1}{\log|x-y|}\right). \tag{3.47}$$

To summarise, despite the apparent asymmetry between x and y, the circle $\partial D(x,R)$ plays a similar role for $\partial D(x,\varepsilon)$ and $\partial D(y,\varepsilon)$ and the transition probabilities between those three circles are given by (3.44), (3.45), (3.46) and (3.47).

We now move on to the study of the $L^n_{x,\varepsilon}$'s, $L^n_{y,\varepsilon}$'s, $L^n_{x,R}$'s. Starting from any point of $\partial D(x,\varepsilon)$, $L_{x,\varepsilon}(\tau_{\partial D(x,R)})$ is an exponential variable with mean given by (see (3.10) in Lemma 3.9)

$$2\varepsilon\log(R/\varepsilon) = 2(1-p_{xy})\varepsilon\log\frac{1}{\varepsilon}\left(1 + O\left(\frac{1}{\log\varepsilon}\right)\right).$$

Starting from any point of $\partial D(y,\varepsilon)$, Lemma 3.13 allows us to approximate $L_{y,\varepsilon}(\tau \wedge \tau_{\partial D(x,R)})$ by an exponential variable with mean equal to (see Lemma 3.15 applied with $A \leftarrow \partial D(x,R)$

and see (3.11) in Lemma 3.9)

$$\left(1+O\left(\tfrac{\varepsilon}{R}\right)\right)\left(1-p_{xy}+O\left(\tfrac{1}{\log\varepsilon}\right)\right)2\varepsilon\left(\log\tfrac{1}{\varepsilon}+O(1)\right)=2(1-p_{xy})\varepsilon\log\tfrac{1}{\varepsilon}\left(1+O\left(\tfrac{1}{\log\varepsilon}\right)\right).$$

Similarly, starting from any point of $\partial D(x,R)$, we can approximate $L_{x,R}(\tau\wedge\tau_{\partial D(x,\varepsilon)}\wedge\tau_{\partial D(x,\varepsilon)})$ by an exponential variable with mean equal to (we apply Lemma 3.15 with $A\leftarrow\partial D(y,\varepsilon),\varepsilon\leftarrow R,\delta\leftarrow\varepsilon$)

$$\left(1\pm C\tfrac{R}{|x-y|}\right)\left(1-2\tfrac{p_{xy}}{1+p_{xy}}+O\left(\tfrac{1}{\log|x-y|}\right)\right)2R\left(\log\tfrac{1}{R}+O(1)\right)=\left(1\pm\tfrac{C_1}{M}\right)\tfrac{1-p_{xy}}{1+p_{xy}}2R\log\tfrac{1}{R}$$

for some universal constants C,C_1. In the following we will denote $\hat\gamma=\tilde\gamma/\sqrt{1-C_1/M}$. As we can take M as large as we want, we will be able to require $\hat\gamma$ to be as close to γ as we want.

Finally, to use Lemma 3.13 to approximate either $L^n_{x,\varepsilon}$, $L^n_{y,\varepsilon}$ or $L^n_{x,R}$ by exponential variables independent of the exit point, we need to control the error we make in estimating the harmonic measure (what was written u in Lemma 3.13). For this, we use (3.17) of Lemma 3.15 which tells us that the hypothesis of Lemma 3.13 (used in our three above cases) is satisfied with $u=\mathcal{C}/M$ for some $\mathcal{C}>0$.

Step 3. We are now ready to start to compute the probability (3.41). We will denote $\Gamma(n,1),\Gamma(n',1)$ independent Gamma variables with shape parameter n,n' and scale parameter 1. We recall the following elementary fact: for any $n,n'\geq 1$ and $t\geq 0$,

$$\begin{aligned}\mathbb{P}\left(\Gamma(n,1),\Gamma(n',1)\geq t\right)&=e^{-2t}\left(\sum_{i=0}^{n-1}\frac{t^i}{i!}\right)\left(\sum_{j=0}^{n'-1}\frac{t^j}{j!}\right)=e^{-2t}\sum_{k=0}^{n+n'-2}t^k\sum_{\substack{0\leq i\leq n-1\\0\leq j\leq n'-1\\i+j=k}}\frac{1}{i!j!}\\&\leq e^{-2t}\sum_{k=0}^{n+n'-2}t^k\sum_{\substack{i,j\geq 0\\i+j=k}}\frac{1}{i!j!}=e^{-2t}\sum_{k=0}^{n+n'-2}\frac{(2t)^k}{k!}.\end{aligned}$$

By (3.42), we have:

$$\begin{aligned}&\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}},\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}}\geq\gamma\log\frac{1}{\varepsilon}+b,\sqrt{\frac{1}{R}L_{x,R}}\leq\tilde\gamma\log\frac{1}{R},\tau_{\partial D(x,\varepsilon)}<\tau_{\partial D(y,\varepsilon)}\right)\\&\leq\mathbb{P}_{x_0}\left(\tau_{\partial D(x,\varepsilon)}<\tau\wedge\tau_{\partial D(y,\varepsilon)}\right)\sup_{x'\in\partial D(x,\varepsilon)}\sum_{\substack{n_x\geq 0\\n_y\geq 1}}\mathbb{P}_{x'}\left(A_{R\to x}=n_x,A_{R\to y}=n_y\right)\left(1+\tfrac{\mathcal{C}}{M}\right)^{1+2n_x+2n_y}\\&\times\mathbb{P}\Big(\Gamma(n_x+n_y,1)\leq\frac{\hat\gamma^2}{2}\frac{1+p_{xy}}{1-p_{xy}}\log\frac{1}{R},\Gamma(n_x+1,1),\Gamma(n_y,1)\geq\frac{\gamma^2}{2}\frac{\log 1/\varepsilon}{1-p_{xy}}\left(1+O\left(\tfrac{1}{\log\varepsilon}\right)\right)\Big).\end{aligned}$$

The term $(1+\mathcal{C}/M)^{1+2n_x+2n_y}$ in the above inequality comes from the fact that every time we approximate one of $L^n_{x,\varepsilon}$, $L^n_{y,\varepsilon}$, $L^n_{x,R}$ by an exponential variable independent of the last point of the excursion, we have to pay the multiplicative price $(1+\mathcal{C}/M)$. See Lemma 3.13. Now,

$$\begin{aligned}
&\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}, \sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}} \geq \gamma \log\frac{1}{\varepsilon} + b, \sqrt{\frac{1}{R}L_{x,R}} \leq \widetilde{\gamma}\log\frac{1}{R}, \tau_{\partial D(x,\varepsilon)} < \tau_{\partial D(y,\varepsilon)}\right) \\
&\leq \frac{O(1)}{\log\varepsilon} e^{-\frac{\gamma^2}{1-p_{xy}}\log\frac{1}{\varepsilon}} \sum_{n=1}^{\infty} \sup_{x'\in\partial D(x,\varepsilon)} \mathbb{P}_{x'}\left(A_{R\to x} + A_{R\to y} = n\right)\left(1+\tfrac{\mathcal{C}}{M}\right)^{1+2n} \\
&\quad \times \mathbb{P}\left(\Gamma(n,1) \leq \frac{\hat{\gamma}^2}{2}\frac{1+p_{xy}}{1-p_{xy}}\log\frac{1}{R}\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left\{\gamma^2\frac{\log 1/\varepsilon}{1-p_{xy}}\left(1+O\left(\tfrac{1}{\log\varepsilon}\right)\right)\right\}^k \\
&\leq O(1)\frac{p_{xy}}{\log\varepsilon}e^{-\frac{\gamma^2}{1-p_{xy}}\log\frac{1}{\varepsilon}}\sum_{n=1}^{\infty}\left(\tfrac{2p_{xy}}{1+p_{xy}} + O\left(\tfrac{1}{\log\varepsilon}\right)\right)^{n-1}(1+\alpha)^{n-1} \\
&\quad \times \mathbb{P}\left(\Gamma(n,1) \leq \frac{\hat{\gamma}^2}{2}\frac{1+p_{xy}}{1-p_{xy}}\log\frac{1}{|x-y|} + C_2\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left\{\gamma^2\frac{\log 1/\varepsilon}{1-p_{xy}} + C_3\right\}^k.
\end{aligned} \tag{3.48}$$

Here $\alpha > 0$ is of order $1/M$ and can be required to be as small as necessary. We are going to bound from above the last sum indexed by n. We decompose it in three parts that we will denote S_1, S_2 and S_3 respectively: by denoting

$$n_1 := \frac{\hat{\gamma}^2}{2}\frac{1+p_{xy}}{1-p_{xy}}\log\frac{1}{|x-y|} + C_2 \text{ and } n_2 := \gamma^2\frac{\log 1/\varepsilon}{1-p_{xy}} + C_3,$$

S_1 is the sum over $n = 1 \dots n_1$, S_2 corresponds to $n = n_1+1 \dots n_2$ and S_3 is the remaining $n \geq n_2+1$. Let us comment that if $\hat{\gamma}$ is close enough to γ, we have $n_1 < n_2$ because $(1+p_{xy})\hat{\gamma}^2/2 \leq (1+a)\hat{\gamma}^2/2 < \gamma^2$. In the sum S_1, it will be difficult for $L_{x,\varepsilon}(\tau)$ and $L_{y,\varepsilon}(\tau)$ to be large at the same time. In the sum S_2 it will be difficult for all the three events to happen and in the sum S_3, it will be unlikely for $L_{x,R}(\tau)$ to be small.

Later in the proof, we will use the two following elementary inequalities that we record here for ease of reference: for all $n \geq 1$ and $\mu \geq 0$, we have:

$$\text{if } \mu \leq 1, \sum_{k=n}^{\infty}\frac{(\mu n)^k}{k!} \leq (\mu e)^n, \tag{3.49}$$

$$\text{if } \mu \geq 1, \sum_{k=0}^{n-1}\frac{(\mu n)^k}{k!} \leq e(\mu e)^{n-1}. \tag{3.50}$$

• S_1 : We bound from above the probability appearing in the sum by 1 and we exchange the order of the summations: we first sum over $k = 0 \dots n_1 - 1$ and then we sum over

$n \geq k+1$. The sum over n being a geometric sum, it is explicit and it leads to

$$S_1 \leq O(1) \sum_{0 \leq k \leq n_1 - 1} \frac{1}{k!} \left(2\,(1+\alpha)\,\gamma^2 \frac{p_{xy}}{1-p_{xy}^2} \log \frac{1}{\varepsilon} + C_3' \right)^k .$$

We now use (3.50) with

$$\mu = \left(2\,(1+\alpha)\,\gamma^2 \frac{p_{xy}}{1-p_{xy}^2} \log \frac{1}{\varepsilon} + C_3' \right) \Big/ n_1 = 4\,(1+\alpha)\,\frac{\gamma^2}{\hat{\gamma}^2} \frac{1}{(1+p_{xy})^2} \left(1 + O\left(\tfrac{1}{\log|x-y|} \right) \right)$$
$$\geq \left(\frac{2\gamma}{(1+a)\hat{\gamma}} \right)^2 > 1$$

if $\hat{\gamma}$ is close enough to γ. It gives

$$\begin{aligned} S_1 &\leq O(1) \left(4\,(1+\alpha)\,\frac{\gamma^2}{\hat{\gamma}^2} \frac{1}{(1+p_{xy})^2} \left(1 + O\left(\tfrac{1}{\log|x-y|} \right) \right) e \right)^{n_1} \\ &= O(1) \exp\left\{ \frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{1}{|x-y|} \left(1 + 2 \log \frac{2\sqrt{1+\alpha}\gamma}{(1+p_{xy})\hat{\gamma}} \right) \right\} . \end{aligned}$$

- S_2 : For $n \geq n_1 + 1$, we have (see (3.49))

$$\mathbb{P}\left(\Gamma(n,1) \leq \frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{1}{|x-y|} + C_2 \right) \leq \left(\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{O(1)}{|x-y|} \frac{e}{n} \right)^{n-1} e^{-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{O(1)}{|x-y|}} \tag{3.51}$$

and we also have for $n \leq n_2$ (see (3.50))

$$\sum_{k=0}^{n-1} \left\{ \gamma^2 \frac{\log 1/\varepsilon}{1-p_{xy}} + C_3 \right\}^k \leq e \left(\gamma^2 \frac{\log O(1)/\varepsilon}{1-p_{xy}} \frac{e}{n} \right)^{n-1} .$$

Recalling that $p_{xy} = \log|x-y| \,/ \log \varepsilon$, these two inequalities show that S_2 is at most

$$O(1) e^{-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{1}{|x-y|}} \sum_{n=n_1+1}^{n_2} \left((1+\alpha) \tfrac{\gamma^2 \hat{\gamma}^2}{(1-p_{xy})^2} \log\left(\tfrac{O(1)}{|x-y|} \right) \left(p_{xy} \log \tfrac{1}{\varepsilon} + O(1) \right) \right)^{n-1} \left(\tfrac{e}{n} \right)^{2(n-1)}$$
$$\leq O(1) e^{-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \frac{1}{|x-y|}} \sum_{n=1}^{\infty} \left((1+\alpha) \frac{\gamma^2 \hat{\gamma}^2}{(1-p_{xy})^2} \left(\log \frac{O(1)}{|x-y|} \right)^2 \right)^{n-1} \left(\frac{e}{n} \right)^{2(n-1)} .$$

By Stirling's formula, there exists $C > 0$, such that for all $n \geq 2$, $(e/n)^{2(n-1)} \leq C/((n-1)!(n-2)!)$. Also, denoting I_1 the modified Bessel function of the first kind (see (3.74)) and using its asymptotic form (3.76), we notice that

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)!(n-2)!} v^n = 2 v^{5/2} I_1(2\sqrt{v}) \leq C v^{9/4} e^{2\sqrt{v}} .$$

Hence

$$S_2 \leq O(1) \left(\log \frac{1}{|x-y|} \right)^{9/2} \exp \left\{ \left(-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} + 2\sqrt{1+\alpha} \frac{\gamma \hat{\gamma}}{1-p_{xy}} \right) \log \frac{1}{|x-y|} \right\}.$$

- S_3 : We again use (3.51) and we simply bound

$$\sum_{k=0}^{n-1} \frac{1}{k!} \left\{ \gamma^2 \frac{\log 1/\varepsilon}{1-p_{xy}} + C_3 \right\}^k \leq O(1) e^{\gamma^2 \frac{\log 1/\varepsilon}{1-p_{xy}}}$$

to obtain

$$S_3 \leq O(1) \exp \left\{ \gamma^2 \frac{\log 1/\varepsilon}{1-p_{xy}} - \frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} \log \tfrac{1}{|x-y|} \right\} \sum_{n \geq n_2+1} \left((1+\alpha) \hat{\gamma}^2 \frac{p_{xy}}{1-p_{xy}} \log \tfrac{O(1)}{|x-y|} \frac{e}{n} \right)^{n-1}.$$

Again by Stirling's formula, $(e/n)^{n-1} \leq C\sqrt{n}/(n-1)!$ and with an inequality of the kind of (3.49) we have

$$\begin{aligned} S_3 &\leq O(1) \exp \left\{ \gamma^2 \tfrac{\log 1/\varepsilon}{1-p_{xy}} - \frac{\hat{\gamma}^2}{2} \tfrac{1+p_{xy}}{1-p_{xy}} \log \tfrac{1}{|x-y|} \right\} \left((1+\alpha) \tfrac{\hat{\gamma}^2}{\gamma^2} p_{xy} \left(p_{xy} + O\left(\tfrac{1}{\log \varepsilon}\right) \right) e \right)^{\frac{\gamma^2}{1-p_{xy}} \log \frac{1}{\varepsilon}} \\ &= O(1) \exp \left\{ \left(-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} + 2 \frac{\gamma^2}{1-p_{xy}} \right) \log \frac{1}{|x-y|} \right\} \\ &\quad \times \exp \left\{ \frac{\gamma^2}{p_{xy}} \left(2 + \frac{1}{1-p_{xy}} \log \left((1+\alpha) \frac{\hat{\gamma}^2}{\gamma^2} p_{xy} \left(p_{xy} + O\left(\tfrac{1}{\log \varepsilon}\right) \right) \right) \right) \log \frac{1}{|x-y|} \right\}. \end{aligned}$$

But $\sup_{0<p<1-a} 1 + (\log p)/(1-p) < 0$. Hence if $\hat{\gamma}$ is close enough to γ, α close enough to 0 and if ε is small enough

$$2 + \frac{1}{1-p_{xy}} \log \left((1+\alpha) \frac{\hat{\gamma}^2}{\gamma^2} p_{xy} \left(p_{xy} + O\left(\tfrac{1}{\log \varepsilon}\right) \right) \right) < 0$$

which implies that

$$S_3 \leq O(1) \exp \left\{ \left(-\frac{\hat{\gamma}^2}{2} \frac{1+p_{xy}}{1-p_{xy}} + 2 \frac{\gamma^2}{1-p_{xy}} \right) \log \frac{1}{|x-y|} \right\}.$$

Finally, the worst upper bound we have is for S_2 and coming back to (3.48) we have obtained

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}},\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}}\geq\sqrt{\frac{g}{2}}\gamma\log\frac{1}{\varepsilon}+b,\sqrt{\frac{1}{R}L_{x,R}}\leq\sqrt{\frac{g}{2}}\tilde{\gamma}\log\frac{1}{R},\tau_{\partial D(x,\varepsilon)}<\tau_{\partial D(y,\varepsilon)}\right) \tag{3.52}$$

$$\leq O(1)\frac{1}{(\log\varepsilon)^2}\varepsilon^{\gamma^2}\left(\log\frac{1}{|x-y|}\right)^{11/2}\exp\left\{\frac{2\sqrt{1+\alpha}\gamma\hat{\gamma}-\gamma^2-\hat{\gamma}^2(1+p_{xy})/2}{1-p_{xy}}\log\frac{1}{|x-y|}\right\}.$$

We can ensure that the coefficient

$$\frac{2\sqrt{1+\alpha}\gamma\hat{\gamma}-\gamma^2-\hat{\gamma}^2(1+p_{xy})/2}{1-p_{xy}}$$

is as close to $\gamma^2/2$ as we want. In particular, it is smaller than 2 and we have shown that $(\log\varepsilon)^2\varepsilon^{-\gamma^2/2}$ times

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}},\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}}\geq\gamma\log\frac{1}{\varepsilon}+b,\sqrt{\frac{1}{R}L_{x,R}}\leq\tilde{\gamma}\log\frac{1}{R},\tau_{\partial D(x,\varepsilon)}<\tau_{\partial D(y,\varepsilon)}\right)$$

is bounded from above by a quantity independent of ε and integrable. It concludes the proof. $\square$

Remark 3.22. We now do a small remark that will be useful in the proof of Proposition 3.21. If in the inequality (3.48) we had a worse estimate with an extra multiplicative error $(1+O(1/\sqrt{\log(1/\varepsilon)}))^n$ in the sum indexed by n, we could have absorbed this error by increasing slightly the value of α and it would not have changed the final result: we would have still obtained an upper bound which is integrable over x, y.

We now prove Proposition 3.21. We are going to see that this is an easy consequence of the proof of Proposition 3.20 and we will use the notations defined therein.

Proof of Proposition 3.21. By definition of $\tilde{\mu}_\varepsilon^\gamma$ (see (3.24)), $\mathbb{E}_{x_0}\left[\tilde{\mu}_\varepsilon^\gamma(A)^2\right]$ is equal to

$$\log\left(\frac{1}{\varepsilon}\right)\varepsilon^{\gamma^2}\int_{A_{\varepsilon_0}\times A_{\varepsilon_0}}\mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}}\mathbf{1}_{G'_\varepsilon(x,\varepsilon_0,M)}e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}}}\mathbf{1}_{G'_\varepsilon(y,\varepsilon_0,M)}\right]dxdy.$$

As before, if $a\in(\gamma^2/4,1)$, the contribution of points x,y such that $|x-y|\leq\varepsilon^a$ is negligible: it is at most

$$\log\left(\frac{1}{\varepsilon}\right)\varepsilon^{\gamma^2}\varepsilon^{2a}\exp\left(\gamma^2\log\frac{1}{\varepsilon}+\sqrt{\frac{2}{g}}M\gamma\sqrt{\log\frac{1}{\varepsilon}}\right)\int_A\mathbb{E}_{x_0}\left[e^{\gamma\sqrt{\frac{2}{g\varepsilon}L_{x,\varepsilon}}}\right]dx$$
$$=\varepsilon^{-\gamma^2/2+2a-o(1)}\mathbb{E}_{x_0}\left[\nu_\varepsilon^\gamma(A)\right]$$

which converges to zero thanks to the first moment estimate (3.27) of Proposition 3.17. For $x, y \in A_{\varepsilon_0}$ with $|x-y| \geq \varepsilon^a$, we proceed in the exact same way as before. In particular, we have the same description of the joint law of $(L_{x,\varepsilon}, L_{x,R}, L_{y,\varepsilon})$: starting from any point of $\partial D(x,\varepsilon)$ and conditioning on the event that the number of excursions from $\partial D(x,R)$ to $\partial D(x,\varepsilon)$ is n, we can approximate $L_{x,\varepsilon}(\tau)/\varepsilon$ by a Gamma random variable $\Gamma(n+1, 2\theta)$ which is the sum of $n+1$ independent exponential variables with mean 2θ. Here

$$\theta = \log R + O(1) = (1-p_{xy}) \log \frac{1}{\varepsilon} + O(1).$$

The only difference with the case treated in Proposition 3.20 is that we consider

$$\sqrt{\log\left(\frac{1}{\varepsilon}\right)} \varepsilon^{\gamma^2/2} \mathbb{E}\left[e^{\gamma\sqrt{2\theta\Gamma(n+1,1)}} \mathbf{1}_{\left\{ \left| \sqrt{2\theta\Gamma(n+1,1)} - \gamma \log(1/\varepsilon) \right| \leq M\sqrt{\log(1/\varepsilon)} \right\}} \right] \tag{3.53}$$

rather than

$$\log \frac{1}{\varepsilon} \varepsilon^{-\gamma^2/2} \mathbb{P}\left(\Gamma(n+1,1) \geq \frac{\gamma^2}{2\theta} \log \frac{1}{\varepsilon} \right). \tag{3.54}$$

We are actually going to see that the first quantity can be bounded from above by second one, up to an irrelevant factor. This will allow us to conclude the proof thanks to Proposition 3.16. With the change of variable $u = \sqrt{t} - \gamma\sqrt{\theta/2}$, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{\gamma\sqrt{2\theta\Gamma(n+1,1)}} \mathbf{1}_{\left\{ \left| \sqrt{2\theta\Gamma(n+1,1)} - \gamma \log(1/\varepsilon) \right| \leq M\sqrt{\log(1/\varepsilon)} \right\}} \right] \\
&= \int_0^\infty e^{\gamma\sqrt{2\theta t} - t} \frac{t^n}{n!} \mathbf{1}_{\left\{ \left| \sqrt{t} - \gamma \log(1/\varepsilon)/\sqrt{2\theta} \right| \leq M\sqrt{\log(1/\varepsilon)}/\sqrt{2\theta} \right\}} dt \\
&= \frac{(\gamma\sqrt{\theta})^{2n+1}}{n! 2^n} e^{\gamma^2\theta/2} \int_{\mathbb{R}} e^{-u^2/2} \left(1 + \frac{\sqrt{2}u}{\gamma\sqrt{\theta}} \right)^{2n+1} \mathbf{1}_{\left\{ \left| u + \gamma\sqrt{\theta/2} - \gamma\log(1/\varepsilon)/\sqrt{2\theta} \right| \leq M\sqrt{\log(1/\varepsilon)}/\sqrt{2\theta} \right\}} dt.
\end{aligned}$$

In the range of admissible u, we have

$$1 + \frac{\sqrt{2}u}{\gamma\sqrt{\theta}} = \frac{1}{1-p_{xy}} + O\left(\frac{1}{\sqrt{\log(1/\varepsilon)}} \right)$$

and we also have

$$u^2 = \frac{\gamma^2}{2} \frac{p_{xy}}{1-p_{xy}} \log \frac{1}{|x-y|} + O(1) \sqrt{\log \frac{1}{|x-y|}}.$$

Hence

$$\mathbb{E}\left[e^{\gamma\sqrt{2\theta\Gamma(n+1,1)}}\mathbf{1}_{\left\{\left|\sqrt{2\theta\Gamma(n+1,1)}-\gamma\log(1/\varepsilon)\right|\leq M\sqrt{\log(1/\varepsilon)}\right\}}\right]$$
$$=\left(1+O\left(\tfrac{1}{\sqrt{\log(1/\varepsilon)}}\right)\right)^n\tfrac{\sqrt{\theta}}{n!}\left(\tfrac{\gamma^2\theta}{2(1-p_{xy})^2}\right)^n e^{\gamma^2\theta/2}\exp\left(-\tfrac{\gamma^2 p_{xy}}{2(1-p_{xy})}\log\tfrac{1}{|x-y|}+O(1)\sqrt{\log\tfrac{1}{|x-y|}}\right)$$

which then implies that the term in (3.53) is at most

$$\left(1+O\left(\tfrac{1}{\sqrt{\log(1/\varepsilon)}}\right)\right)^n\tfrac{1}{n!}\log\tfrac{1}{\varepsilon}\exp\left(-\tfrac{\gamma^2}{2(1-p_{xy})}\log\tfrac{1}{|x-y|}+O(1)\sqrt{\log\tfrac{1}{|x-y|}}\right)\left(\tfrac{\gamma^2}{2(1-p_{xy})}\log\tfrac{1}{\varepsilon}\right)^n.$$

Recalling that the term in (3.54) is equal to

$$\log\tfrac{1}{\varepsilon}\exp\left(-\tfrac{\gamma^2}{2(1-p_{xy})}\log\tfrac{1}{|x-y|}+O(1)\right)\sum_{k=0}^n\frac{1}{k!}\left(\tfrac{\gamma^2}{2(1-p_{xy})}\log\tfrac{1}{\varepsilon}+O(1)\right)^n,$$

it shows that the term in (3.53) is at most $(1+O(1/\sqrt{|\log\varepsilon|})^n\exp\left(O(1)\sqrt{|\log|x-y||}\right)$ times the term in (3.54). As we mentioned in Remark 3.22, it implies that we obtain the same upper bound as in the proof of Proposition 3.20 with an extra multiplicative error $\exp\left(O(1)\sqrt{\log(1/|x-y|)}\right)$ which is still integrable over x,y. It concludes the proof. $\square$

3.5 Convergence

In this section, we will prove the following proposition:

Proposition 3.23. *If $\tilde{\gamma}$ is close enough to γ, $(\tilde{\nu}_\varepsilon^\gamma(A\times T),\varepsilon>0)$ is a Cauchy sequence in L^2 and moreover,*

$$\lim_{\varepsilon\to0}\mathbb{E}_{x_0}\left[\left(\widetilde{\nu}_\varepsilon^\gamma(A\times(b,\infty))-e^{-\gamma b}\widetilde{\nu}_\varepsilon^\gamma(A\times(0,\infty))\right)^2\right]=0 \tag{3.55}$$

and

$$\limsup_{\varepsilon\to0}\mathbb{E}_{x_0}\left[\left(\frac{1}{\sqrt{2\pi}\gamma}\widetilde{\mu}_\varepsilon^\gamma(A)-\widetilde{\nu}_\varepsilon^\gamma(A\times(0,\infty))\right)^2\right]\leq p(M) \tag{3.56}$$

with $p(M)\to0$ as $M\to\infty$. $p(M)$ may depend on γ.

As mentioned in the introduction, to use the link between the local times and the zero-dimensional Bessel process (3.7), we will use the following lemma proven in Appendix 3.A:

Lemma 3.24. *Let $k,k',n\geq0$ with $k'\geq k+1$ and $n\geq k'-k$. Denote $\eta=e^{-k}$, $\eta'=e^{-k'}$ and for all $i=1\dots k'-k$, $r_i=\eta e^{-i}$. Consider $0<r_n<\cdots<r_{k'-k+1}<r_{k'-k}=\eta'$ and*

for $i = 1 \dots n$, $T_i \in \mathcal{B}([0,\infty))$. *For any* $y \in \partial D(0,\eta/e)$, *we have*

$$1 - p(\eta') \le \frac{\mathbb{P}_y\left(\forall i = 1\dots n, L_{0,r_i}(\tau_{\partial D(0,\eta)}) \in T_i | \tau_{\partial D(0,\eta')} < \tau_{\partial D(0,\eta)}, B_{\tau_{\partial D(0,\eta)}}\right)}{\mathbb{P}_y\left(\forall i = 1\dots n, L_{0,r_i}(\tau_{\partial D(0,\eta)}) \in T_i | \tau_{\partial D(0,\eta')} < \tau_{\partial D(0,\eta)}\right)} \le 1 + p(\eta') \tag{3.57}$$

with $p(\eta') \to 0$ *as* $\eta' \to 0$. $p(\eta')$ *may depend on* η.

Remark 3.25. If we had conditioned on $\tau_{\partial D(0,\eta')} < \tau_{\partial D(0,\eta)}, B_{\tau_{\partial D(0,\eta)}}, L_{0,\eta/e}(\tau_{\partial D(0,\eta)})$ rather than on $\tau_{\partial D(0,\eta')} < \tau_{\partial D(0,\eta)}, B_{\tau_{\partial D(0,\eta)}}$, the same conclusion would have held: up to a multiplicative error $1 + o_{\eta'\to 0}(1)$, we can forget the conditioning on the exit point $B_{\tau_{\partial D(0,\eta)}}$. This is a direct consequence of Lemma 3.24.

We now state the result that we will need on the zero-dimensional Bessel process to prove Proposition 3.23. This lemma will be proven in Appendix 3.B.

Lemma 3.26. *Let* $\tilde{\gamma} > \gamma > 0$, $b, \tilde{b} \in \mathbb{R}$, $s_0 \ge 1$ *an integer and for all* $s \in [|1, s_0|], A_s \in \mathcal{B}(\mathbb{R})$. *Let* $n \ge 1$ *and* $(R_s^{(i)}, s \ge 0), i = 1\dots n$, *independent zero-dimensional Bessel processes. Denote for all* $s \ge 0$,

$$R_s := \sqrt{\sum_{i=1}^n \left(R_s^{(i)}\right)^2}.$$

Then the following two limits exist

$$\begin{aligned} l_1(b) := &\lim_{t\to\infty} t e^{\frac{\gamma^2}{2}t} \\ &\times \mathbb{P}\left(R_t \ge \gamma t + b, \forall s \in [|1,s_0|], R_s \in A_s, \forall s \in [|s_0,t|], R_s \le \tilde{\gamma}s + \tilde{b} \middle| \forall i = 1\dots n, R_{s_0}^{(i)} > 0\right) \end{aligned}$$

and

$$\begin{aligned} l_2(M) := &\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \sqrt{t} e^{-\frac{\gamma^2}{2}t} \\ &\times \mathbb{E}\left[e^{\gamma R_t} \mathbf{1}_{\left\{|R_t - \gamma t| \le M\sqrt{t}\right\}} \mathbf{1}_{\left\{\forall s \in [|1,s_0|], R_s \in A_s, \forall s \in [|s_0,t|], R_s \le \tilde{\gamma}s + \tilde{b}\right\}} \middle| \forall i = 1\dots n, R_{s_0}^{(i)} > 0\right]. \end{aligned}$$

Moreover,

$$l_1(b) e^{b\gamma} = l_1(0) = (1 + p(M)) l_2(M) \tag{3.58}$$

for some universal sequence $p(M)$ *going to 0 as* $M \to \infty$.

We now prove Proposition 3.23.

Proof of Proposition 3.23. For convenience, if $u, v \in \mathbb{R}$, we will write $u = \pm v$ in this proof when we mean $-v \le u \le v$.

We start by proving that $(\tilde{\nu}_\varepsilon^\gamma(A \times T), \varepsilon > 0)$ is a Cauchy sequence in L^2. We want to show that

$$\limsup_{\varepsilon,\delta \to 0} \mathbb{E}_{x_0} \left[\left(\tilde{\nu}_\varepsilon^\gamma(A \times T) - \tilde{\nu}_\delta^\gamma(A \times T) \right)^2 \right] = 0.$$

By expanding the product, we notice that it is enough to show that

$$\limsup_{\varepsilon,\delta \to 0} \mathbb{E}_{x_0} \left[\tilde{\nu}_\varepsilon^\gamma(A \times T) \tilde{\nu}_\varepsilon^\gamma(A \times T) \right] - \mathbb{E}_{x_0} \left[\tilde{\nu}_\varepsilon^\gamma(A \times T) \tilde{\nu}_\delta^\gamma(A \times T) \right] \leq 0.$$

Take $\varepsilon, \delta > 0$. In this proof, we will denote $f_{\varepsilon,\delta}(x,y) := |\log \delta| \, |\log \varepsilon| \, (\delta\varepsilon)^{-\gamma^2/2}$ times

$$\mathbb{P}_{x_0} \left(\sqrt{\frac{L_{x,\varepsilon}(\tau)}{\varepsilon}} \geq \gamma \log \frac{1}{\varepsilon} + b, \sqrt{\frac{L_{y,\delta}(\tau)}{\delta}} \geq \gamma \log \frac{1}{\delta} + b, G_\varepsilon(x, \varepsilon_0), G_\delta(y, \varepsilon_0) \right).$$

Take $\eta \in \{\bar{r}, r < \varepsilon_0\}$ and denote $(A \times A)_\eta$ the subset of $A \times A$ made of "good points":

$$(A \times A)_\eta := \left\{ (x,y) \in A \times A : D(y,\eta) \cap \bigcup_{r \leq \varepsilon_0} \partial D(x, \bar{r}) = \varnothing \right\}. \tag{3.59}$$

If $(x,y) \in (A \times A)_\eta$, the two sequences of circles $(\partial D(x,\bar{r}), r \leq \varepsilon_0)$ and $(\partial D(y,\bar{r}), r \leq \varepsilon_0)$ will not interact between each other inside $D(y,\eta)$. Since the Lebesgue measure of $(A \times A) \backslash (A \times A)_\eta$ goes to 0 when $\eta \to 0$, Proposition 3.20, or more precisely (3.52), implies that

$$\int_{(A \times A) \backslash (A \times A)_\eta} f_{\varepsilon,\varepsilon}(x,y) dx dy \leq \int_{(A \times A) \backslash (A \times A)_\eta} \sup_\varepsilon f_{\varepsilon,\varepsilon}(x,y) dx dy = o_{\eta \to 1}(1).$$

$\mathbb{E}_{x_0} \left[\tilde{\nu}_\varepsilon^\gamma(A \times T) \tilde{\nu}_\varepsilon^\gamma(A \times T) \right] - \mathbb{E}_{x_0} \left[\tilde{\nu}_\varepsilon^\gamma(A \times T) \tilde{\nu}_\delta^\gamma(A \times T) \right]$ is thus at most

$$\leq o_{\eta \to 1}(1) + \int_{(A \times A)_\eta} (f_{\varepsilon,\varepsilon}(x,y) - f_{\varepsilon,\delta}(x,y)) dx dy.$$

Our objective is now to bound from above $f_{\varepsilon,\varepsilon}(x,y) - f_{\varepsilon,\delta}(x,y)$ for $(x,y) \in (A \times A)_\eta$. The two probabilities in $f_{\varepsilon,\varepsilon}(x,y)$ and in $f_{\varepsilon,\delta}(x,y)$ differ only from what is required around y. We are thus going to focus around y. We consider the excursions from $\partial D(y, \eta/e)$ to $\partial D(y,\eta)$: define $\sigma_0^{(2)} := 0$ and for all $i \geq 1$,

$$\sigma_i^{(1)} := \inf \left\{ t > \sigma_{i-1}^{(2)} : B_t \in \partial D(y, \eta/e) \right\} \text{ and } \sigma_i^{(2)} := \inf \left\{ t > \sigma_i^{(1)} : B_t \in \partial D(y, \eta) \right\}.$$

We denote by $N := \max\{i \geq 1 : \sigma_i^{(2)} < \tau\}$ the number of excursions. The local times of circles centred at y inside $D(y, \eta/e)$ accumulated during the i-th excursion, that we will denote by $(L_{y,r}^{(i)}, r \leq \eta/e)$, depend on the starting point $B_{\sigma_i^{(1)}}$ and on the exit point

$B_{\sigma_i^{(2)}}$. But this dependence is weak if the excursion goes deeply inside $D(y, \eta/e)$: this is the content of Lemma 3.24. This is why we consider $\eta' \in (\bar{r}, r < \varepsilon_0)$ much smaller than η and for all $i \geq 1$, we consider the random variable v_i

$$v_i = \begin{cases} 1 \text{ if } B\left[\sigma_i^{(1)}, \sigma_i^{(2)}\right] \cap D(y, \eta'/e) \neq \varnothing, \\ 0 \text{ otherwise.} \end{cases}$$

We claim that there exists N_η independent of x, y, ε such that

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}, \sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}(\tau)} \geq \gamma \log \frac{1}{\varepsilon} + b, N \geq N_\eta\right) \leq \eta \frac{1}{(\log \varepsilon)^2}\varepsilon^{\gamma^2}. \tag{3.60}$$

This is in the same spirit as what we did in Section 3.4. To not interrupt the flow of the proof, we postpone the justification of this claim at the very end of the proof.

It is thus actually enough to bound from above $g_{\varepsilon,\varepsilon}(x,y) - g_{\varepsilon,\delta}(x,y)$ where $g_{\varepsilon,\delta}(x,y)$ is the modification of $f_{\varepsilon,\delta}(x,y)$: $g_{\varepsilon,\delta}(x,y) := \log(1/\delta)\log(1/\varepsilon)(\delta\varepsilon)^{-\gamma^2/2}$ times

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{L_{x,\varepsilon}(\tau)}{\varepsilon}} \geq \gamma \log \frac{1}{\varepsilon} + b, \sqrt{\frac{L_{y,\delta}(\tau)}{\delta}} \geq \gamma \log \frac{1}{\delta} + b, G_\varepsilon(x, \varepsilon_0), G_\delta(y, \varepsilon_0), N < N_\eta\right).$$

We are going to condition on the whole trajectory except the excursions from $\partial D(y, \eta/e)$ to $\partial D(y, \eta)$ which visit $D(y, \eta'/e)$. The only randomness remaining will come from $L_{y,r}(\tau)$ for $r < \eta$. We have:

$$\frac{1}{\log(\delta)\log(\varepsilon)}(\delta\varepsilon)^{\gamma^2/2} g_{\varepsilon,\delta}(x,y) = \mathbb{E}_{x_0}\left[\mathbf{1}_{\left\{\sqrt{\frac{L_{x,\varepsilon}(\tau)}{\varepsilon}} \geq \gamma \log\frac{1}{\varepsilon} + b, G_\varepsilon(x,\varepsilon_0), G_\eta(y,\varepsilon_0), N < N_\eta\right\}}\right.$$
$$\times \mathbb{P}_{x_0}\left(\forall r \in [\eta', \eta/e], \sum_{i=1}^N \mathbf{1}_{\{v_i=1\}} L_{y,\bar{r}}^{(i)} \leq \tilde{\gamma}\bar{r}\left(\log \bar{r}\right)^2 - \sum_{i=1}^N \mathbf{1}_{\{v_i=0\}} L_{y,\bar{r}}^{(i)}\right.$$
$$\left.\left.\sqrt{\tfrac{L_{y,\delta}(\tau)}{\delta}} \geq \gamma \log \tfrac{1}{\delta} + b, G_\delta(y, \eta'), \middle| N, B_{\sigma_i^{(1)}}, B_{\sigma_i^{(2)}}, v_i, \left(\mathbf{1}_{\{v_i=0\}} L_{y,\bar{r}}^{(i)}, r \in [\eta', \eta/e]\right), i = 1 \ldots N\right)\right].$$

We are interested in this last conditional probability. For a given $i \geq 1$, Lemma 3.24 (or more precisely Remark 3.25 following Lemma 3.24) tells us that there exists $p(\eta')$ which may depend on η and which goes to 0 as $\eta' \to 0$, such that for any sequence $(T_r, r < \eta/e)$ of Borel subsets of $\mathbb{R}$,

$$\mathbb{P}\left(L_{y,\delta}^{(i)} \in T_\delta, \forall r \in [\delta, \eta/e), L_{y,\bar{r}}^{(i)} \in T_{\bar{r}} \middle| B_{\sigma_i^{(1)}}, B_{\sigma_i^{(2)}}, v_i = 1, L_{y,\eta/e}^{(i)}\right)$$
$$= (1 \pm p(\eta'))\mathbb{P}\left(L_{y,\delta}^{(i)} \in T_\delta, \forall r \in [\delta, \eta/e), L_{y,\bar{r}}^{(i)} \in T_{\bar{r}} \middle| v_i = 1, L_{y,\eta/e}^{(i)}\right).$$

Now, (3.7) tells us that, conditioned on $v_i = 1$ and $L^{(i)}_{y,\eta/e}$,

$$\left(L^{(i)}_{y,r_s}, r_s = \frac{\eta}{e} e^{-s}, s \geq 0 \right) \overset{\text{law}}{=} \left(\left(R^{(i)}_s \right)^2, s \geq 0 \right)$$

where $R^{(i)}$ is a zero-dimensional Bessel process starting from $\sqrt{eL^{(i)}_{y,\eta/e}/\eta}$ conditioned to be positive at time $s_0 = \log \eta/(e\eta')$. By denoting for all $s \geq 0$,

$$R_s := \sqrt{\sum_{i=1}^N v_i \left(R^{(i)}_s \right)^2},$$

we thus have

$$\begin{aligned}
&(1 \pm p(\eta'))^{-N_\eta} \frac{1}{\log(\delta)\log(\varepsilon)} (\delta\varepsilon)^{\gamma^2/2} g_{\varepsilon,\delta}(x,y) \\
&= \mathbb{E}_{x_0} \Bigg[\mathbf{1}_{\left\{ \sqrt{\frac{L_{x,\varepsilon}(\tau)}{\varepsilon}} \geq \gamma \log \frac{1}{\varepsilon} + b, G_\varepsilon(x,\varepsilon_0), G_\eta(y,\varepsilon_0), N < N_\eta \right\}} \mathbb{P}_{x_0} \Bigg(R_{\log \frac{\eta}{e\delta}} \geq \gamma \log \frac{1}{\delta} + b, \\
&\quad \forall s \in \left[\left| \log \frac{\eta}{\eta'}, \log \frac{\eta}{e\delta} \right|\right], R_s \leq \tilde{\gamma} s + \tilde{\gamma} \log \frac{e}{\eta}, \text{ and } \forall s \in \left[\left| 1, \log \frac{\eta}{\delta} \right|\right], \\
&\quad R_s^2 \leq \tilde{\gamma}^2 \left(s + \log \frac{e}{\eta} \right)^2 - \sum_{i=1}^N \mathbf{1}_{\{v_i = 0\}} L^{(i)}_{y,r_s} \Bigg| N, v_i, \left(\mathbf{1}_{\{v_i=0\}} L^{(i)}_{y,\bar{r}}, r \in [\eta', \eta/e] \right), \forall i = 1 \dots N \Bigg) \Bigg].
\end{aligned}$$

By Lemma 3.26, the conditional probability times $\log(1/\delta)\delta^{-\gamma^2/2}$ converges as $\delta \to 0$. Hence

$$\limsup_{\varepsilon,\delta \to 0} \left\{ (1 + p(\eta'))^{-N_\eta} g_{\varepsilon,\varepsilon}(x,y) - (1 - p(\eta'))^{-N_\eta} g_{\varepsilon,\delta}(x,y) \right\} \leq 0.$$

$g_{\varepsilon,\delta}(x,y)$ and N_η being independent of η' it yields

$$\limsup_{\varepsilon,\delta \to 0} g_{\varepsilon,\varepsilon}(x,y) - g_{\varepsilon,\delta}(x,y) \leq 0.$$

This concludes the proof of the fact that $(\tilde{\nu}^\gamma_\varepsilon(A \times T), \varepsilon > 0)$ is a Cauchy sequence in L^2, assuming the veracity of the claim (3.60). To prove (3.55), we notice that

$$\begin{aligned}
&\mathbb{E}_{x_0} \left[\left(\tilde{\nu}^\gamma_\varepsilon(A \times (b,\infty)) - e^{-\gamma b} \tilde{\nu}^\gamma_\varepsilon(A \times (0,\infty)) \right)^2 \right] \\
&= \left\{ \mathbb{E}_{x_0} \left[\nu_\varepsilon(A \times (b,\infty))^2 \right] - e^{-\gamma b} \mathbb{E}_{x_0} \left[\nu_\varepsilon(A \times (b,\infty)) \tilde{\nu}^\gamma_\varepsilon(A \times (0,\infty)) \right] \right\} \\
&+ e^{-\gamma b} \left\{ e^{-\gamma b} \mathbb{E}_{x_0} \left[\nu_\varepsilon(A \times (0,\infty))^2 \right] - \mathbb{E}_{x_0} \left[\nu_\varepsilon(A \times (b,\infty)) \tilde{\nu}^\gamma_\varepsilon(A \times (0,\infty)) \right] \right\}
\end{aligned}$$

and we want to show that the two terms in brackets go to zero. We proceed in the exact same way as before. We have to control the difference of two probabilities of events which differ only around one point. Around this point, the local times behave as a zero-dimensional squared Bessel process and we use the first equality of claim (3.58) of Lemma 3.26. The proof of (3.56) is similar with the use of the second equality of claim (3.58) of Lemma 3.26 and a claim similar to (3.60) (we omit the details).

We now finish the proof by proving (3.60). As this is a similar reasoning as the ones we saw in Section 3.4, we will be brief. Conditioned on $B_{\sigma_i^{(1)}}$, $B_{\sigma_i^{(2)}}$ and on the fact that the i-th excursion visits $\partial D(y,\varepsilon)$, the local time $L_{y,\varepsilon}^{(i)}$ of $\partial D(y,\varepsilon)$ accumulated during the i-th excursion is approximatively an exponential variable with mean $2\log(O(1)/\varepsilon)$ (see Lemma 3.13 for a precise statement). Moreover, conditioned on the starting and ending points of the excursion, the probability for the excursion to visit $\partial D(y,\varepsilon)$ is at most $O(1)/\log(1/\varepsilon)$. Hence, conditioned on the number of excursions N, $L_{y,\varepsilon}(\tau)$ can be stochastically dominated by a Gamma random variable with scale parameter $1/(2\log(C/\varepsilon))$ and shape parameter having the law of a binomial variable: the sum of N independent Bernouilli random variables with success probability $C/\log(1/\varepsilon)$. By increasing the value of C if necessary, the same is true for $L_{x,\varepsilon}(\tau)$ with N replaced by $N+1$ (we could visit $\partial D(x,\varepsilon)$ before $\partial D(y,\eta/e)$). Hence

$$\begin{aligned}
&\mathbb{P}_{x_0}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)},\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}(\tau)}\geq\gamma\log\frac{1}{\varepsilon}+b, N\geq N_\eta\right)\\
&\leq\sum_{n\geq N_\eta-1}\mathbb{P}_{x_0}\left(N=n-1\right)\left\{\sum_{k=1}^{n}\binom{n}{k}\left(\frac{C}{\log(1/\varepsilon)}\right)^k\varepsilon^{\gamma^2/2}\sum_{l=0}^{k-1}\frac{1}{l!}\left(\frac{\gamma^2}{2}\log\frac{C}{\varepsilon}\right)^l\right\}^2\\
&=\left(\frac{1}{\log\varepsilon}\right)^2\varepsilon^{\gamma^2}\sum_{n\geq N_\eta-1}\mathbb{P}_{x_0}\left(N=n-1\right)\left\{\sum_{k=1}^{n}\binom{n}{k}C'^k\sum_{l=0}^{k-1}\frac{1}{l!}\left(\log\frac{1}{\varepsilon}\right)^{l-k+1}\right\}^2. \qquad (3.61)
\end{aligned}$$

Noticing that the last sum over $l=0\cdots k-1$ is at most (by decomposing it into the sums over $l=0\ldots\lfloor k/2\rfloor-1$ and $l=\lfloor k/2\rfloor\ldots k-1$ for instance)

$$k\left(\log\frac{1}{\varepsilon}\right)^{\lfloor k/2\rfloor-k}+\frac{k}{\lfloor k/2\rfloor!},$$

we see that for any $a>0$, there exists $C=C(a)>0$ such that the sum over $k=1\ldots n$ in brackets is at most $C(a)(1+a)^n$. Moreover, $\mathbb{P}_{x_0}\left(N=n-1\right)\leq p^{n-1}$ for some $p<1$ depending on η. Hence, by considering a small enough so that $(1+a)p<1$, the sum over n in (3.61) is at most $C(p(1+a))^{N_\eta}\leq\eta$ if N_η is large enough. This proves the claim (3.60) and finishes the proof. □

3.6 Vague convergence, identification of the limits and properties of μ^γ

Our proof of Theorems 3.1 and 3.2 relies on the following:

Proposition 3.27. *The sequences $(\nu_\varepsilon^\gamma(A\times T),\varepsilon>0)$ and $(\mu_\varepsilon^\gamma(A),\varepsilon>0)$ converge in L^1. Moreover,*

$$\lim_{\varepsilon\to0} e^{\gamma b}\nu_\varepsilon^\gamma(A\times(b,\infty)) = \lim_{\varepsilon\to0}\nu_\varepsilon^\gamma(A\times(0,\infty)) = \frac{1}{\sqrt{2\pi\gamma}}\lim_{\varepsilon\to0}\mu_\varepsilon^\gamma(A) \qquad \mathbb{P}_{x_0}-\text{a.s.} \tag{3.62}$$

The proof is straightforward from Propositions 3.16, 3.17 and 3.23:

Proof. By (3.26), for any $\varepsilon,\delta>0$ small enough,

$$\mathbb{E}_{x_0}\left[\ \left|\nu_\varepsilon^\gamma(A\times T)-\nu_\delta^\gamma(A\times T)\right|\ \right] \le 2p(\varepsilon_0)+\mathbb{E}_{x_0}\left[\ \left|\widetilde{\nu}_\varepsilon^\gamma(A\times T)-\widetilde{\nu}_\delta^\gamma(A\times T)\right|\ \right].$$

Proposition 3.23 giving

$$\limsup_{\varepsilon,\delta\to0}\mathbb{E}_{x_0}\left[\ \left|\widetilde{\nu}_\varepsilon^\gamma(A\times T)-\widetilde{\nu}_\delta^\gamma(A\times T)\right|\ \right] \le \limsup_{\varepsilon,\delta\to0}\mathbb{E}_{x_0}\left[\left(\widetilde{\nu}_\varepsilon^\gamma(A\times T)-\widetilde{\nu}_\delta^\gamma(A\times T)\right)^2\right]^{1/2}=0,$$

it implies that

$$\limsup_{\varepsilon,\delta\to0}\mathbb{E}_{x_0}\left[\ \left|\nu_\varepsilon^\gamma(A\times T)-\nu_\delta^\gamma(A\times T)\right|\ \right]\le 2p(\varepsilon_0).$$

Since the left hand side term does not depend on ε_0 and since $p(\varepsilon_0)\to0$ as $\varepsilon_0\to0$, it finally implies that

$$\limsup_{\varepsilon,\delta\to0}\mathbb{E}_{x_0}\left[\ \left|\nu_\varepsilon^\gamma(A\times T)-\nu_\delta^\gamma(A\times T)\right|\ \right]\le 0$$

which proves the convergence in L^1 of $(\nu_\varepsilon^\gamma(A\times T),\varepsilon>0)$. Using (3.26) and (3.55), respectively (3.26), (3.28) and (3.56), we can show in the same way that

$$\limsup_{\varepsilon\to0}\mathbb{E}_{x_0}\left[\ \left|e^{\gamma b}\nu_\varepsilon^\gamma(A\times(b,\infty))-\nu_\varepsilon^\gamma(A\times(0,\infty))\right|\ \right]=0,$$

respectively

$$\limsup_{\varepsilon\to0}\mathbb{E}_{x_0}\left[\ \left|\frac{1}{\sqrt{2\pi\gamma}}\mu_\varepsilon^\gamma(A)-\nu_\varepsilon^\gamma(A\times(0,\infty))\right|\ \right]=0.$$

As $(\nu_\varepsilon^\gamma(A\times(0,\infty)),\varepsilon>0)$ converges, this shows the convergence of $(\mu_\varepsilon^\gamma(A),\varepsilon>0)$ and the identification of the limits (3.62). ☐

Proof of Theorems 3.1 and 3.2. By Proposition 3.27, for any $A\in\mathcal{B}(D)$ and T of the form (b,∞) with $b\in\mathbb{R}$, the sequences $(\nu_\varepsilon^\gamma(A\times T),\varepsilon>0)$ and $(\mu_\varepsilon^\gamma(A),\varepsilon>0)$ converge in probability. From this, we obtain the convergence in probability for the vague topology of the random measures $(\nu_\varepsilon^\gamma, \varepsilon > 0)$ and $(\mu_\varepsilon^\gamma, \varepsilon > 0)$ through classical arguments which can be found in [Ber17], Section 6 (the reasoning therein is for the topology of weak convergence but there is no difficulty to adapt it to the topology of vague convergence). This proves Theorem 3.1. We now turn to the proof of Theorem 3.2. We will abusively denote the measure $A \in \mathcal{B}(D) \mapsto \nu^\gamma(A \times (0,\infty))$ by $\nu^\gamma(dx \times (0,\infty))$ and we consider the measure $\bar{\nu}^\gamma$ on $D \times \mathbb{R}$

$$\bar{\nu}^\gamma(dx, dt) := \nu^\gamma(dx \times (0,\infty)) e^{-\gamma t} \gamma dt.$$

The first equality of (3.62) shows that $\mathbb{P}_{x_0}$-a.s. the measures ν^γ and $\bar{\nu}^\gamma$ coincide on the countable π-system of subsets of $D \times \mathbb{R}$ of the form $[x_1, y_1) \times [x_2, y_2) \times (b, \infty)$ with $x_1, x_2, y_1, y_2, b \in \mathbb{Q}$. This π-system generates the Borel σ-field on $D \times \mathbb{R}$ and the measures ν^γ and $\bar{\nu}^\gamma$ are $\mathbb{P}_{x_0}$-a.s. σ-finite. Hence $\nu^\gamma = \bar{\nu}^\gamma$ $\mathbb{P}_{x_0}$-a.s. The same reasoning and the second equality of (3.62) shows that the measures $\nu^\gamma(dx \times (0,\infty))$ and $\mu^\gamma(dx)/(\sqrt{2\pi}\gamma)$ are $\mathbb{P}_{x_0}$-a.s. equal. This finishes to prove Theorem 3.2. □

We now explain how we obtain the links between the work of Bass, Burdzy and Koshnevisan [BBK94] and the one of Aïdékon, Hu and Shi [AHS20] with ours. For this small part, we are going to use their notations that we recall: if $z \in \partial D$ is a nice boundary point, i.e. a point where the boundary ∂D is locally an analytic curve, and $x \in D$,

- $\mathbb{P}_D^{x_0,z}$ denotes the probability measure of Brownian motion starting from x_0 and conditioned to exit D through z (see [AHS20], Notation 2.1 (i)),
- $\mathbb{Q}_{x,D}^{x_0,a}$ is the probability measures of trajectories consisting of, first a Brownian motion starting from x_0 and conditioned to hit x before exiting the domain, second a Poisson point process of excursions from x, and third a Brownian motion starting from x and killed when it exits for the first time the domain (written $\mathbb{Q}_a^x$ in [BBK94], p.606),
- $\mathbb{Q}_{x,D}^{x_0,z,a}$ is similar to $\mathbb{Q}_{x,D}^{x_0,a}$ except that the last part of the trajectory is a Brownian motion conditioned to exit D through z (see [AHS20], Proposition 3.5).

We will also denote $C_*[0,\infty)$ the set of all parametrised continuous planar curves c defined on a finite interval $[0, t_c]$ with $t_c \in (0,\infty)$. $C_*[0,\infty)$ is equipped with the Skorokhod topology. For any event $C \in \mathcal{B}(C_*[0,\infty))$, we have

$$\mathbb{P}_D^{x_0,z}(C) = \lim_{r \to 0} \frac{\mathbb{P}_{x_0}(C, |B_\tau - z| \leq r)}{\mathbb{P}_{x_0}(|B_\tau - z| \leq r)} \text{ and } \mathbb{Q}_{x,D}^{x_0,z,a}(C) = \lim_{r \to 0} \frac{\mathbb{Q}_{x,D}^{x_0,a}(C, |B_\tau - z| \leq r)}{\mathbb{Q}_{x,D}^{x_0,a}(|B_\tau - z| \leq r)}. \tag{3.63}$$

The following proposition characterises the measures μ^γ. Let us emphasise that we only assume that the domain D is bounded and simply connected here.

Proposition 3.28. *For every $\gamma \in (0,2)$ and every non-negative measurable function f on $\mathbb{R}^2 \times C_*[0,\infty)$, we have with $a = \gamma^2/2$,*

$$\mathbb{E}_{x_0}\left[\int_{\mathbb{R}^2} f(x,B)\mu^\gamma(dx)\right] = \sqrt{2\pi}\gamma \int_D \mathbb{E}_{\mathbb{Q}^{x_0,a}_{x,D}}[f(x,B)] R(x,D)^{\gamma^2/2} G_D(x_0,x) dx. \tag{3.64}$$

Proof of Proposition 3.28. Proposition 5.1 of [BBK94] states that if the domain D is the unit disc and if the starting point x_0 is the origin, for any $x \in D$, the distribution of Brownian motion conditioned on

$$\left\{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau) \geq \gamma^2 (\log\varepsilon)^2 - 2\left|\log\varepsilon\right| \log\left|\log\varepsilon\right|\right\}$$

converges to $\mathbb{Q}^{x_0,a}_{x,D}$ as $\varepsilon \to 0$. No restriction on the value of γ is required here and their proof actually works in a general setting of a bounded open simply connected domain and a starting point $x_0 \in D$. Moreover, we notice that if we had conditioned rather on

$$\left\{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau) \geq \gamma^2 (\log\varepsilon)^2\right\}, \tag{3.65}$$

we would have obtained the same result: this can be seen in their equation (5.7) where the term $2\left|\log\varepsilon\right| \log\left|\log\varepsilon\right|$ is killed by bigger order terms. Hence, we also have: the distribution of Brownian motion starting from x_0 and conditioned on (3.65) converges to $\mathbb{Q}^{x_0,a}_{x,D}$ as $\varepsilon \to 0$. We can now conclude as in [BBK94], Theorem 5.2: by standard monotone class argument, it is enough to prove (3.64) for f of the form $f(x,B) = \mathbf{1}_A(x)\mathbf{1}_C(B)$ for some $A \in \mathcal{B}(D)$ and $C \in \mathcal{B}(C_*[0,\infty))$. In that case

$$\begin{aligned}
&\left|\mathbb{E}_{x_0}\left[\int_{\mathbb{R}^2} f(x,B)\nu^\gamma(dx,(0,\infty))\right] - \mathbb{E}_{x_0}\left[\int_{\mathbb{R}^2} f(x,B)\nu^\gamma_\varepsilon(dx,(0,\infty))\right]\right| \\
&\quad \leq \mathbb{E}_{x_0}\left[\mathbf{1}_C(B)\left|\nu^\gamma(A,(0,\infty)) - \nu^\gamma_\varepsilon(A,(0,\infty))\right|\right] \leq \mathbb{E}_{x_0}\left[\left|\nu^\gamma(A,(0,\infty)) - \nu^\gamma_\varepsilon(A,(0,\infty))\right|\right]
\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$ by Proposition 3.27. Hence

$$\begin{aligned}
\mathbb{E}_{x_0}&\left[\int_{\mathbb{R}^2} f(x,B)\nu^\gamma(dx,(0,\infty))\right] = \lim_{\varepsilon\to 0}\mathbb{E}_{x_0}\left[\int_{\mathbb{R}^2} f(x,B)\nu^\gamma_\varepsilon(dx,(0,\infty))\right] \\
&= \lim_{\varepsilon\to 0}\left|\log\varepsilon\right|\varepsilon^{-\gamma^2/2}\int_A \mathbb{P}_{x_0}\left(C \left| \frac{1}{\varepsilon}L_{x,\varepsilon}(\tau) \geq \gamma^2(\log\varepsilon)^2\right.\right)\mathbb{P}_{x_0}\left(\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau) \geq \gamma^2(\log\varepsilon)^2\right)dx \\
&= \int_A \mathbb{Q}^{x_0,a}_{x,D}(C) R(x,D)^{\gamma^2/2} G_D(x_0,x) dx
\end{aligned}$$

by Proposition 3.16, (3.25). Recalling that Theorem 3.2 shows that

$$\mu^\gamma(dx) = \sqrt{2\pi}\gamma\nu^\gamma(dx,(0,\infty)) \qquad \mathbb{P}_{x_0}\text{–a.s.},$$

this finishes to prove (3.64). $\square$

From Proposition 3.28, Corollary 3.6 is immediate:

Proof of Corollary 3.6. When D is the unit disc and $x_0 = 0$, $R(x, D) = 1 - |x|^2$ and $G_D(x_0, x) = -\log |x|$ (see [Law05], Section 2.4). Hence by Proposition 3.28 and by [BBK94], Theorem 5.2, μ^γ and $\sqrt{2\pi}\gamma\beta_a$ both satisfy (3.64). Moreover, these two measures are measurable with respect to the Brownian path. As noticed in [BBK94], Remark 5.2 (i), there is only one measure satisfying these two conditions implying that $\mathbb{P}_{x_0}$-a.s. $\mu^\gamma = \sqrt{2\pi}\gamma\beta_a$. □

The proof of Corollary 3.7 is quite similar:

Proof of Corollary 3.7. For the same reason as before, it is enough to show that for all non-negative measurable function f,

$$\mathbb{E}_{\mathbb{P}_D^{x_0,z}} \left[\int_{\mathbb{R}^2} f(x, B) \mu^\gamma(dx) \right] = \sqrt{2\pi}\gamma \mathbb{E}_{\mathbb{P}_D^{x_0,z}} \left[\int_{\mathbb{R}^2} f(x, B) \frac{H_D(x_0, z)}{H_D(x, z)} \mathcal{M}_\infty^a(dx) \right] \tag{3.66}$$

and we can assume that f is of the form $f(x, B) = \mathbf{1}_A(x)\mathbf{1}_C(B)$ for some $A \in \mathcal{B}(D)$ and $C \in \mathcal{B}(C_*[0,\infty))$. By [AHS20], Proposition 5.1, the right hand side term of (3.66) is equal to

$$\sqrt{2\pi}\gamma \int_D \mathbb{E}_{\mathbb{Q}_{x,D}^{x_0,z,a}} [f(x, B)] R(x, D)^a G_D(x_0, x) dx.$$

On the other hand, by (3.63), (3.64) and by dominated convergence theorem, the left hand side term of (3.66) is equal to

$$\begin{aligned}
&\lim_{r\to 0} \mathbb{E}_{x_0} \left[\int_{\mathbb{R}^2} f(x, B) \mu^\gamma(dx) \mathbf{1}_{\{|B_\tau - z| \le r\}} \right] \Big/ \mathbb{P}_{x_0} \left(|B_\tau - z| \le r \right) \\
&= \lim_{r\to 0} \sqrt{2\pi}\gamma \int_A \frac{\mathbb{Q}_{x,D}^{x_0,a}(B \in A, |B_\tau - z| \le r)}{\mathbb{P}_{x_0}\left(|B_\tau - z| \le r \right)} R(x, D)^{\gamma^2/2} G_D(x_0, x) dx \\
&= \sqrt{2\pi}\gamma \int_A \mathbb{Q}_{x,D}^{x_0,z,a}(B \in A) R(x, D)^{\gamma^2/2} G_D(x_0, x) dx.
\end{aligned}$$

This shows (3.66) and concludes the proof. □

We finish this section by proving Corollary 3.8. We are basically going to collect properties in [AHS20].

Proof of Corollary 3.8. Let $a = \gamma^2/2$. For any nice point $z \in \partial D$, the first three properties are satisfied by $\mathcal{M}_\infty^a$ under $\mathbb{P}_D^{x_0,z}$ (see [AHS20], Proposition 5.4 and Theorem 1.1). By Corollary 3.7 it is thus also the case for μ^γ. To change the probability measure $\mathbb{P}_D^{x_0,z}$ to $\mathbb{P}_{x_0}$, we notice that

$$\mathbb{P}_{x_0}(\cdot) = \int_{\partial D} \mathbb{P}_D^{x_0,z}(\cdot) H_D(x_0, z) dz.$$

So if an event E satisfies $\mathbb{P}_D^{x_0,z}(E) = 0$ for all nice point $z \in \partial D$, $\mathbb{P}_{x_0}(E) = 0$. This concludes the proof of *(i)*-*(iii)*. We now turn to the proof of the claim *(iv)*. It is enough to show that for all non-negative measurable function f,

$$\mathbb{E}_{x_0}\left[\int_{D'} f(x,B)\left(\mu^{\gamma,D}\circ\phi^{-1}\right)(dx)\right] = \mathbb{E}_{\phi(x_0)}\left[\int_{D'} f(x,B)\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\mu^{\gamma,D'}(dx)\right]. \tag{3.67}$$

To help us to do the change of variable $z' = \phi(z)$ in the computations below, we recall that for any $y \in D$ and $z \in \partial D$, $H_{D'}(\phi(y),\phi(z)) = |\phi'(z)|^{-1} H_D(y,z)$ (see [Law05], Section 5.2). By Corollary 3.7, and by the conformal invariance of $\mathcal{M}_\infty^a$ ([AHS20], Proposition 5.3), the left hand side term of (3.67) is then equal to

$$\begin{aligned}
&\int_{\partial D} dz\ H_D(x_0,z)\mathbb{E}_{\mathbb{P}_D^{x_0,z}}\left[\int_{D'} f(x,B)\left(\mu^{\gamma,D}\circ\phi^{-1}\right)(dx)\right]\\
&= \sqrt{2\pi}\gamma\int_{\partial D} dz\ H_D(x_0,z)^2\mathbb{E}_{\mathbb{P}_D^{x_0,z}}\left[\int_{D'} \frac{f(x,B)}{H_D(\phi^{-1}(x),z)}\left(\mathcal{M}_\infty^a\circ\phi^{-1}\right)(dx)\right]\\
&= \sqrt{2\pi}\gamma\int_{\partial D} dz\ H_D(x_0,z)^2\mathbb{E}_{\mathbb{P}_{D'}^{\phi(x_0),\phi(z)}}\left[\int_{D'} \frac{f(x,B)}{H_D(\phi^{-1}(x),z)}\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\mathcal{M}_\infty^a(dx)\right]\\
&= \sqrt{2\pi}\gamma\int_{\partial D} dz\ \left|\phi'(z)\right| H_{D'}(\phi(x_0),\phi(z))\\
&\qquad\times\mathbb{E}_{\mathbb{P}_{D'}^{\phi(x_0),\phi(z)}}\left[\int_{D'} f(x,B)\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\frac{H_{D'}(\phi(x_0),\phi(z))}{H_{D'}(x,\phi(z))}\mathcal{M}_\infty^a(dx)\right]\\
&= \int_{\partial D} dz\ \left|\phi'(z)\right| H_{D'}(\phi(x_0),\phi(z))\mathbb{E}_{\mathbb{P}_{D'}^{\phi(x_0),\phi(z)}}\left[\int_{D'} f(x,B)\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\mu^{\gamma,D'}(dx)\right]\\
&= \int_{\partial D'} dz'\ H_{D'}(\phi(x_0),z')\mathbb{E}_{\mathbb{P}_{D'}^{\phi(x_0),z'}}\left[\int_{D'} f(x,B)\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\mu^{\gamma,D'}(dx)\right]\\
&= \mathbb{E}_{\phi(x_0)}\left[\int_{D'} f(x,B)\left|\phi'(\phi^{-1}(x))\right|^{2+\gamma^2/2}\mu^{\gamma,D'}(dx)\right].
\end{aligned}$$

This shows (3.67). □

Appendix 3.A Proof of Lemma 3.24

We now prove Lemma 3.24.

Proof of Lemma 3.24. To ease notations, we will denote $\tau_\eta := \tau_{\partial D(0,\eta)}, \tau_{\eta'} := \tau_{\partial D(0,\eta')}$ and for all $i = 1\dots n, L_{r_i} := L_{0,r_i}(\tau_\eta)$. Take $C \in \mathcal{B}(\partial D(0,\eta))$. We will denote $\mathrm{Leb}(C)$ for the Lebesgue measure on $\partial D(0,\eta)$ of C. It is enough to show that

$$\begin{aligned}
&\mathbb{P}_y\left(B_{\tau_\eta}\in C, \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i}\in T_i\right) \\
&\qquad = (1+o_{\eta'\to 0}(1))\frac{\mathbb{P}_y\left(B_{\tau_\eta}\in C,\tau_{\eta'}<\tau_\eta\right)}{\mathbb{P}_y\left(\tau_{\eta'}<\tau_\eta\right)}\mathbb{P}_y\left(\tau_{\eta'}<\tau_\eta, \forall i = 1\dots n, L_{r_i}\in T_i\right).
\end{aligned}\tag{3.68}$$

Moreover, establishing (3.68) can be reduced to show that

$$\begin{aligned}\mathbb{P}_y\left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right) \\ = (1 + o_{\eta' \to 0}(1)) \frac{\mathrm{Leb}(C)}{2\pi\eta} \mathbb{P}_y\left(\tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right).\end{aligned} \tag{3.69}$$

Indeed, applying (3.69) to $T_i = [0, \infty)$ for all i gives (which was already contained in Lemma 3.15)

$$\mathbb{P}_y\left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta\right) = (1 + o_{\eta' \to 0}(1)) \mathbb{P}_y\left(\tau_{\eta'} < \tau_\eta\right) \frac{\mathrm{Leb}(C)}{2\pi\eta},$$

which combined with (3.69) leads to (3.68). Finally, after reformulation of (3.69), to finish the proof we only need to prove that

$$\mathbb{P}_y\left(B_{\tau_\eta} \in C | \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right) = (1 + o_{\eta' \to 0}(1)) \frac{\mathrm{Leb}(C)}{2\pi\eta}. \tag{3.70}$$

The skew-product decomposition of Brownian motion (see [Kal02], Corollary 16.7 for instance) tells us that we can write

$$(B_t, t \geq 0) \overset{\text{(d)}}{=} (|B_t| e^{i\theta_t}, t \geq 0) \text{ with } (\theta_t, t \geq 0) = (w_{\sigma_t}, t \geq 0)$$

where $(w_t, t \geq 0)$ is a one-dimensional Brownian motion independent of the radial part $(|B_t|, t \geq 0)$ and $(\sigma_t, t \geq 0)$ is a time-change that is adapted to the filtration generated by $(|B_t|, t \geq 0)$:

$$\sigma_t = \int_0^t \frac{1}{|B_s|^2} ds.$$

In particular, under $\mathbb{P}_y$, we have the following equality in law

$$\left(\tau_\eta, |B_t|, t < \tau_\eta, B_{\tau_\eta}\right) \overset{\text{(d)}}{=} \left(\tau_\eta, |B_t|, t < \tau_\eta, \eta e^{i\theta_0 + i\varsigma\mathcal{N}}\right) \tag{3.71}$$

where θ_0 is the argument of y, $\mathcal{N}$ is a standard normal random variable independent of the radial part $(|B_t|, t \geq 0)$ and

$$\varsigma = \sqrt{\int_0^{\tau_\eta} \frac{1}{|B_s|^2} ds}.$$

We now investigate a bit the distribution of $e^{i\theta_0 + it\mathcal{N}}$ for some $t > 0$. More precisely, we want to give a quantitative description of the fact that if t is large, the previous distribution should approximate the uniform distribution on the unit disc. Using the probability density function of $\mathcal{N}$ and then using Poisson summation formula, we find that the probability density function $f_t(\theta)$ of $e^{i\theta_0+it\mathcal{N}}$ at a given angle θ is given by

$$f_t(\theta) = \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} e^{-(\theta-\theta_0+2\pi n)^2/(2t)} = \frac{1}{2\pi} \sum_{p\in\mathbb{Z}} e^{ip(\theta-\theta_0)} e^{-p^2 t/2}$$
$$= \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} \cos(p(\theta-\theta_0)) e^{-p^2t/2}\right).$$

In particular, we can control the error in the approximation mentioned above by: for all $\theta \in [0, 2\pi]$,

$$\left| f_t(\theta) - \frac{1}{2\pi} \right| \leq \frac{1}{\pi} \sum_{p=1}^{\infty} e^{-p^2t/2} \leq C_1 \max\left(1, \frac{1}{\sqrt{t}}\right) e^{-t/2}$$

for some universal constant $C_1 > 0$.

We now come back to the objective (3.70). Using the identity (3.71) and because the local times L_{r_i} are measurable with respect to the radial part of Brownian motion, we have by triangle inequality

$$\begin{aligned}
&\left| \mathbb{P}_y\left(B_{\tau_\eta} \in C | \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i} \in T_i\right) - \frac{\mathrm{Leb}(C)}{2\pi\eta} \right| \\
&\leq \mathbb{E}_y\left[\int_0^{2\pi} \left| f_\varsigma(\theta) - \frac{1}{2\pi}\right| \mathbf{1}_{\{\eta e^{i\theta}\in C\}} d\theta \right| \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i} \in T_i \Big] \\
&\leq C_1 \frac{\mathrm{Leb}(C)}{\eta} \mathbb{E}_y\left[\max\left(1, \frac{1}{\sqrt{\varsigma}}\right) e^{-\varsigma/2} \right| \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i} \in T_i \Big] \\
&\leq C_1 \frac{\mathrm{Leb}(C)}{\eta} \mathbb{E}_y\left[\max\left(1, \frac{1}{\sqrt{\varsigma'}}\right) e^{-\varsigma'/2} \right| \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i} \in T_i \Big]
\end{aligned}$$

where

$$\varsigma' := \sqrt{\int_{\tau_{r_n}}^{\tau_\eta} \frac{1}{|B_s|^2} ds}.$$

To conclude the proof, we want to show that

$$\mathbb{E}_y\left[\max\left(1, \frac{1}{\sqrt{\varsigma'}}\right) e^{-\varsigma'/2} \right| \tau_{\eta'} < \tau_\eta, \forall i = 1\dots n, L_{r_i} \in T_i \Big] = o_{\eta'\to 0}(1).$$

By conditioning on the trajectory up to $\tau_{\eta'}$, it is enough to show that for any $T_i' \in \mathcal{B}([0,\infty)), i = 1\dots n$, for any $z \in \partial D(0, \eta')$,

$$\mathbb{E}_z\left[\max\left(1, \frac{1}{\sqrt{\varsigma'}}\right) e^{-\varsigma'/2} \right| \forall i = 1\dots n, L_{r_i} \in T_i' \Big] = o_{\eta'\to 0}(1). \tag{3.72}$$

In the following, we fix such T_i' and such a z.

Consider the sequence of stopping times defined by: $\sigma_0^{(2)} := 0$ and for all $i = 1 \dots k' + k$,

$$\sigma_i^{(1)} := \inf \left\{ t > \sigma_{i-1}^{(2)} : |B_t| = \eta' e^{i-1/2} \right\} \text{ and } \sigma_i^{(2)} := \inf \left\{ t > \sigma_i^{(1)} : |B_t| \in \{\eta' e^i, \eta' e^{i-1}\} \right\}.$$

We only keep track of the portions of trajectories during the intervals $\left[\sigma_i^{(1)}, \sigma_i^{(2)}\right]$ by bounding from below ς' by

$$(\varsigma')^2 \geq \sum_{i=1}^{k'-k} \frac{\sigma_i^{(2)} - \sigma_i^{(1)}}{(\eta' e^i)^2} =: L.$$

By Markov property, conditioning on $\{\forall i = 1 \dots n, L_{r_i} \in T_i'\}$ impacts the variables $\sigma_i^{(2)} - \sigma_i^{(1)}$ only through $\left|B_{\sigma_i^{(2)}}\right|$. But one has that there exists $c > 0$ such that for all $i = 1 \dots k' - k$,

$$\mathbb{E}_z \left[\frac{1}{\sigma_i^{(2)} - \sigma_i^{(1)}} \middle| \left|B_{\sigma_i^{(2)}}\right| \right] \leq \frac{c}{(\eta' e^i)^2}.$$

Then for all $S > 0$ we have by Markov's inequality and then by Jensen's inequality applied to $u \mapsto 1/u$:

$$\begin{aligned}
\mathbb{P}_z \left(L < S | \forall i = 1 \dots n, L_{r_i} \in T_i' \right) &\leq S \mathbb{E}_z \left[\frac{1}{L} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i' \right] \\
&\leq \frac{S}{(k'-k)^2} \sum_{i=1}^{k'-k} \mathbb{E}_z \left[\frac{(\eta' e^i)^2}{\sigma_i^{(2)} - \sigma_i^{(1)}} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i' \right] \\
&\leq c \frac{S}{k'-k}.
\end{aligned}$$

In particular, $\mathbb{P}_z \left(L < S | \forall i = 1 \dots n, L_{r_i} \in T_i' \right) \leq o_{\eta' \to 0}(1) S$ and it implies that

$$\begin{aligned}
&\mathbb{E}_z \left[\max \left(1, \frac{1}{\sqrt{\varsigma'}} \right) e^{-\varsigma'/2} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i' \right] \\
&\leq \mathbb{E}_z \left[\max \left(1, \frac{1}{L^{1/4}} \right) e^{-\sqrt{L}/2} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i' \right] \\
&\leq \sum_{p=-\infty}^{\infty} \max \left(1, 2^{(p+1)/4} \right) e^{-2^{-(p+1)/2}/2} \mathbb{P}_z \left(2^{-p-1} \leq L < 2^{-p} | \forall i = 1 \dots n, L_{r_i} \in T_i' \right) \\
&= o_{\eta' \to 0}(1) \sum_{p=-\infty}^{\infty} 2^{-p} \max \left(1, 2^{(p+1)/4} \right) e^{-2^{-(p+1)/2}/2} = o_{\eta' \to 0}(1).
\end{aligned}$$

This shows (3.72) which finishes the proof. □

Appendix 3.B Zero-dimensional Bessel process

This appendix is dedicated to the proofs of the properties we have collected on the zero-dimensional Bessel process throughout the article. Because those properties are fairly classical, we will sometimes be brief. Recall that we denote by $\mathbb{P}_r$ the law under which $(R_s)_{s\geq 0}$ is a zero-dimensional Bessel process starting from r.

In this section, we will denote $q_s(x,y)$ the transition probability of $(R_s)_{s\geq 0}$. It satisfies the following explicit formula (see Proposition 2.5 of [Law18] for instance)

$$q_s(x,y) = \frac{x}{s} e^{-\frac{x^2+y^2}{2s}} I_1\left(\frac{xy}{s}\right) \tag{3.73}$$

where I_1 is a modified Bessel function of the first kind:

$$I_1(u) = \sum_{n\geq 0} \frac{1}{n!(n+1)!} \left(\frac{u}{2}\right)^{2m+1}. \tag{3.74}$$

We also recall (see [Par84]) that for all $v > u > 0$,

$$I_1(v) \leq \frac{v}{u} e^{v-u} I_1(u) \tag{3.75}$$

and that I_1 has the well-known asymptotic form:

$$I_1(u) \underset{u\to\infty}{\sim} \frac{1}{\sqrt{2\pi u}} e^u. \tag{3.76}$$

We start by proving Lemma 3.19.

Proof of Lemma 3.19. Take t, λ and r as in the statement of the lemma. We have

$$\mathbb{P}_r\left(R_t \geq \lambda\right) = \int_\lambda^\infty q_t(r,x)dx.$$

For all $x \geq \lambda$, we have $rx/t \geq a$. Hence, by (3.73) and (3.76), there exists $C = C(a) > 0$ such that for all $x \geq \lambda$,

$$q_t(r,x) \leq C\frac{r}{t} e^{-\frac{r^2+x^2}{2t}} \frac{1}{\sqrt{rx/t}} e^{\frac{rx}{s}} \leq C\sqrt{\frac{r}{\lambda t}} e^{-\frac{(x-r)^2}{2t}}.$$

Using tail estimates of normal random variable, this leads to

$$\mathbb{P}_r\left(R_t \geq \lambda\right) \leq C' \frac{\sqrt{r}}{\lambda} e^{-\frac{(\lambda-r)^2}{2t}} \leq C' \sqrt{r} e^{\frac{\lambda r}{t}} \frac{1}{\lambda} e^{-\frac{\lambda^2}{2t}}.$$

This proves the first claim. The second claim follows from the first one and from

$$\mathbb{E}_r\left[e^{\gamma R_t}\right] \le e^{\frac{\gamma^2 t}{4}} + \int_{e^{\gamma^2 t/4}}^{\infty} \mathbb{P}_r\left(R_t \ge \frac{\log \mu}{\gamma}\right) d\mu.$$

We omit the details. □

We now move on to the proof of Lemma 3.18.

Proof of Lemma 3.18. For ease of notation, we will assume that $\widetilde{b} = 0$ in the proof. We are going to show that there exist $c = c(\gamma, \widetilde{\gamma}) > 0$ and $s_0 = s_0(\gamma, \widetilde{\gamma}, r_0, b) > 0$ such that for all $r \in (0, r_0)$ and $t > s \ge s_0$,

$$\mathbb{P}_r\left(R_t \ge \gamma t + b, R_s \ge \widetilde{\gamma} s\right) \le \frac{1}{c} e^{-cs} \mathbb{P}_r\left(R_t \ge \gamma t + b\right), \tag{3.77}$$

$$\mathbb{E}_r\left[e^{\gamma R_t} \mathbf{1}_{\{R_s \ge \widetilde{\gamma} s\}}\right] \le \frac{1}{c} e^{-cs} \mathbb{E}_r\left[e^{\gamma R_t}\right]. \tag{3.78}$$

Lemma 3.18 is then an easy consequence of these estimates.

Define $\varepsilon = (\widetilde{\gamma} - \gamma)/4 > 0$. Assume that t is large enough so that $\varepsilon t > b$. Take $s < t$ and $\lambda < (\gamma + \varepsilon)t$. We are going to show that

$$\mathbb{P}_r\left(R_t \in [\lambda, (\gamma+\varepsilon)t], R_s \ge \widetilde{\gamma} s\right) \le \frac{1}{c} e^{-cs} \mathbb{P}_r\left(R_t \ge \lambda\right) \tag{3.79}$$

for some $c = c(\gamma, \widetilde{\gamma}) > 0$. We will then see that we can conclude with a proof of (3.77) and (3.78) quite quickly. We have:

$$\mathbb{P}_r\left(R_t \in [\lambda, (\gamma+\varepsilon)t], R_s \ge \widetilde{\gamma} s\right) = \frac{\widetilde{\gamma}}{\gamma} \int_{\gamma s}^{\infty} q_s\left(r, \widetilde{\gamma} x/\gamma\right) \mathbb{P}_{\widetilde{\gamma} x/\gamma}\left(R_{t-s} \in [\lambda, (\gamma+\varepsilon)t]\right) dx.$$

But by (3.75)

$$\begin{aligned}
\mathbb{P}_{\widetilde{\gamma} x/\gamma}\left(R_{t-s} \in [\lambda, (\gamma+\varepsilon)t]\right) &= \int_{\lambda}^{(\gamma+\varepsilon)t} \frac{\widetilde{\gamma} x/\gamma}{t-s} \exp\left(-\frac{(\widetilde{\gamma} x/\gamma)^2 + y^2}{2(t-s)}\right) I_1\left(\frac{\widetilde{\gamma} xy/\gamma}{t-s}\right) dy \\
&\le \left(\frac{\widetilde{\gamma}}{\gamma}\right)^2 \int_{\lambda}^{(\gamma+\varepsilon)t} \frac{x}{t-s} \exp\left(-\frac{(\widetilde{\gamma} x/\gamma)^2 + y^2}{2(t-s)} + \left(\frac{\widetilde{\gamma}}{\gamma} - 1\right)\frac{xy}{t-s}\right) I_1\left(\frac{xy}{t-s}\right) dy
\end{aligned}$$

and

$$q_s(r, \widetilde{\gamma} x/\gamma) \le \frac{\widetilde{\gamma}}{\gamma} \frac{r}{s} \exp\left(-\frac{r^2 + (\widetilde{\gamma} x/\gamma)^2}{2s} + \left(\frac{\widetilde{\gamma}}{\gamma} - 1\right)\frac{rx}{s}\right) I_1\left(\frac{rx}{s}\right). \tag{3.80}$$

After elementary simplifications, we find that $\mathbb{P}_r\left(R_t \in [\lambda, (\gamma+\varepsilon)t], R_s \ge \widetilde{\gamma} s\right)$ is at most

$$\left(\frac{\widetilde{\gamma}}{\gamma}\right)^4 \int_{\gamma s}^{\infty} dx\ q_s(r, x) \int_{\lambda}^{(\gamma+\varepsilon)t} dy\ q_{t-s}(x, y) \exp\left(-\frac{(\frac{\widetilde{\gamma}}{\gamma} - 1)x}{s(t-s)}\left(\frac{\frac{\widetilde{\gamma}}{\gamma} + 1}{2} xt - ys - r(t-s)\right)\right).$$

We have chosen $\varepsilon < (\widetilde{\gamma} - \gamma)/2$ so that for all $x \geq \gamma s$ and $y \in [\lambda, (\gamma+\varepsilon)t]$,

$$\frac{\widetilde{\gamma}/\gamma + 1}{2} xt - ys \geq cts$$

for some $c = c(\gamma, \widetilde{\gamma}) > 0$. Hence if s and t are large enough (depending on $\gamma, \widetilde{\gamma}$ and r_0),

$$\frac{\widetilde{\gamma}/\gamma + 1}{2} xt - ys - r(t-s) \geq c' ts$$

for some $c' = c'(\gamma, \widetilde{\gamma}) > 0$. This implies (3.79).

This finishes almost entirely the proof. Indeed, to prove (3.77) we use (3.78) with $\lambda = \gamma t + b$ and we notice that (3.80) (used with $s = t$ and $\widetilde{\gamma} = \gamma + \varepsilon$) implies that $\mathbb{P}_r\left(R_t \geq (\gamma+\varepsilon)t\right)$ is at most

$$\begin{aligned} &\left(\frac{\gamma+\varepsilon}{\gamma}\right)^2 \int_{\gamma t}^{\infty} q_t(r,x) \exp\left(- \frac{((\gamma+\varepsilon)/\gamma - 1)x}{t} \left(\frac{(\gamma+\varepsilon)/\gamma + 1}{2} x - r \right)\right) \\ &\leq \frac{1}{c} e^{-ct} \mathbb{P}_r\left(R_t \geq \gamma t\right) \end{aligned}$$

for some $c = c(\gamma, \widetilde{\gamma}) > 0$ and if t is large enough. This shows (3.77). For (3.78), we see that (3.79) gives

$$\begin{aligned} \mathbb{E}_r\left[e^{\gamma R_t} \mathbf{1}_{\left\{R_s \geq \widetilde{\gamma} s\right\}} \mathbf{1}_{\{R_t \leq (\gamma+\varepsilon)\}} \right] &= \int_0^\infty \mathbb{P}_r\left(e^{\gamma R_t} \mathbf{1}_{\left\{R_s \geq \widetilde{\gamma} s\right\}} \mathbf{1}_{\{R_t \leq (\gamma+\varepsilon)\}} \geq \lambda \right) d\lambda \\ &= \int_0^\infty \mathbb{P}_r\left(\frac{\log \lambda}{\gamma} \leq R_t \leq (\gamma+\varepsilon)t, R_s \geq \widetilde{\gamma} s \right) d\lambda \\ \leq \frac{1}{c} e^{-cs} \int_0^\infty & \mathbb{P}_r\left(\frac{\log \lambda}{\gamma} \leq R_t \leq (\gamma+\varepsilon)t \right) d\lambda \leq \frac{1}{c} e^{-cs} \mathbb{E}_r\left[e^{\gamma R_t} \right]. \end{aligned}$$

On the other hand, we have by (3.31)

$$\mathbb{E}_r\left[e^{\gamma R_t} \mathbf{1}_{\{R_t \geq (\gamma+\varepsilon)\}} \right] \leq \frac{1}{c} e^{-ct} \mathbb{E}_r\left[e^{\gamma R_t} \right]$$

which concludes the proof of (3.78). This finishes the proof. □

We finish this appendix by proving Lemma 3.26.

Proof of Lemma 3.26. The sum of n independent zero-dimensional squared Bessel processes is still a zero-dimensional squared Bessel process. Hence, by conditioning on

$(R_s^{(i)}, s \leq s_0), i = 1 \dots n$, we have

$$\begin{aligned}
&\mathbb{P}\left(R_t \geq \gamma t + b, \forall s \in [|1, s_0|], R_s \in A_s, \forall s \in [|s_0, t|], R_s \leq \widetilde{\gamma} s + \widetilde{b} \middle| \forall i = 1 \dots n, R_{s_0}^{(i)} > 0\right) \\
&= \mathbb{E}\left[\mathbb{P}_{\sqrt{\sum_{i=1}^n \left(R_{s_0}^{(i)}\right)^2}}\left(R_{t-s_0} \geq \gamma t + b, \forall s \in [|1, t-s_0|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right)\right. \\
&\qquad\qquad\left. \times \mathbf{1}_{\{\forall s \in [|1, s_0|], R_s \in A_s\}} \middle| \forall i = 1 \dots n, R_{s_0}^{(i)} > 0\right].
\end{aligned}$$

Now we focus on the asymptotic of

$$\mathbb{P}_r\left(R_{t-s_0} \geq \gamma t + b, \forall s \in [|1, t-s_0|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right)$$

for a given $r \geq 0$. Take $\varepsilon > 0$. By (3.29) of Lemma 3.18, there exists $s_0' > 0$ such that for all $t \geq s_0' + s_0$,

$$\begin{aligned}
0 \leq \mathbb{P}_r &\left(R_{t-s_0} \geq \gamma t + b, \forall s \in [|1, s_0'|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right) \\
&- \mathbb{P}_r\left(R_{t-s_0} \geq \gamma t + b, \forall s \in [|1, t-s_0|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right) \leq \varepsilon.
\end{aligned}$$

But

$$\begin{aligned}
&\mathbb{P}_r\left(R_{t-s_0} \geq \gamma t + b, \forall s \in [|1, s_0'|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right) \\
&= \mathbb{E}_r\left[\mathbf{1}_{\left\{\forall s \in [|1, s_0'|], R_s \leq \widetilde{\gamma}(s+s_0) + \widetilde{b}\right\}} \mathbb{P}_{R_{s_0'}}\left(R_{t-s_0-s_0'} \geq \gamma t + b\right)\right].
\end{aligned}$$

We could have done the same reasoning with the expectation of $e^{\gamma R_t} \mathbf{1}_{\left\{|R_t - \gamma t| \leq M\sqrt{t}\right\}}$: the only difference is that we would have to replace

$$\mathbb{P}_{R_{s_0'}}\left(R_{t-s_0-s_0'} \geq \gamma t + b\right) \text{ by } \mathbb{E}_{R_{s_0'}}\left[e^{\gamma R_{t-s_0-s_0'}} \mathbf{1}_{\left\{\left|R_{t-s_0-s_0'} - \gamma(t-s_0-s_0')\right| \leq M\sqrt{t-s_0-s_0'}\right\}}\right]$$

(see also claim (3.30) of Lemma 3.18). To conclude the proof, we thus only need to show that for a given $r \geq 0$ and $t_0 \geq 0$,

$$te^{\frac{\gamma^2}{2}t} \mathbb{P}_r\left(R_{t-t_0} \geq \gamma t + b\right) \text{ and } \frac{1}{\gamma\sqrt{2\pi}}\sqrt{t}e^{-\frac{\gamma^2}{2}t} \mathbb{E}_r\left[e^{\gamma R_{t-t_0}} \mathbf{1}_{\left\{\left|R_{t-t_0} - \gamma(t-t_0)\right| \leq M\sqrt{t-t_0}\right\}}\right]$$

converge and that the limits satisfy (3.58). This is a simple computation:

$$\mathbb{P}_r\left(R_t \geq \gamma t + b\right) = \frac{r}{t} e^{-\frac{r^2}{2t}} \int_{\gamma t + b}^{\infty} e^{-\frac{x^2}{2t}} I_1\left(\frac{rx}{t}\right) dx \underset{t \to \infty}{\sim} \frac{r I_1(\gamma r)}{\gamma} e^{-b\gamma} \frac{1}{t} e^{-\frac{\gamma^2}{2}t}$$

implying that

$$te^{\frac{\gamma^2}{2}t}\mathbb{P}_r\left(R_{t-t_0} \geq \gamma t + b\right) \underset{t\to\infty}{\sim} \frac{rI_1(\gamma r)}{\gamma} e^{-b\gamma - \frac{\gamma^2}{2}t_0},$$

whereas

$$\begin{aligned}\mathbb{E}_r\left[e^{\gamma R_t}\mathbf{1}_{\{|R_t-\gamma t|\leq M\sqrt{t}\}}\right] &= \frac{r}{t}e^{-\frac{r^2}{2t}}\int_{\gamma t-M\sqrt{t}}^{\gamma t+M\sqrt{t}} e^{-\frac{x^2}{2t}+\gamma x} I_1\left(\frac{rx}{t}\right)dx \\ &\underset{t\to\infty}{\sim} rI_1(\gamma r)\frac{1}{\sqrt{t}}e^{\frac{\gamma^2}{2}t}\int_{-M}^{M} e^{-y^2/2}dy\end{aligned}$$

implying that

$$\frac{1}{\gamma\sqrt{2\pi}}\sqrt{t}e^{-\frac{\gamma^2}{2}t}\mathbb{E}_r\left[e^{\gamma R_{t-t_0}}\right] \underset{t\to\infty}{\sim} \frac{rI_1(\gamma r)}{\gamma}e^{-\frac{\gamma^2}{2}t_0}(1-p(M))$$

for $p(M) = 1 - \int_{-M}^{M} e^{-y^2/2}dy/\sqrt{2\pi}$. This concludes the proof. $\square$

Appendix 3.C Continuity of the local times. Proof of Proposition 3.5

Consider any norm $\|\cdot\|$ on $\mathbb{R}^2 \times \mathbb{R}$. By Kolmogorov's continuity theorem, to prove Proposition 3.5, it is enough to show:

Lemma 3.29. *For all $p \geq 1$ and $\eta > \eta' > 0$, there exists $C = C(p, \eta, \eta') > 0$ such that for all $x, y \in D$ and $0 < \varepsilon, \delta < \eta'$ such that $D(x, \eta) \cup D(y, \eta) \subset D$,*

$$\mathbb{E}_{x_0}\left[|L_{x,\varepsilon}(\tau) - L_{y,\delta}(\tau)|^p\right] \leq C\left\|(x,\varepsilon)-(y,\delta)\right\|^{p/3}\left|\log\left\|(x,\varepsilon)-(y,\delta)\right\|\right|^p. \tag{3.81}$$

Let us emphasise that the previous lemma considers the local times $L_{x,\varepsilon}(\tau)$ rather than their normalised versions $L_{x,\varepsilon}(\tau)/\varepsilon$. Before proving this lemma, we collect one more time a property on the zero-dimensional Bessel process:

Lemma 3.30. *For all integer $p \geq 1$, there exists $C = C(p) > 0$ such that for all $0 < s < 1$ and for all starting point $r > 0$,*

$$\mathbb{E}_r\left[\left|R_s^2 - r^2\right|^p\right] \leq Cs^{p/2}\max(1, r^{2p}). \tag{3.82}$$

Proof of Lemma 3.30. Take $\lambda > 0$. We are going to bound from above $\mathbb{P}_r\left(|R_s^2 - r^2| > \lambda\right)$.

Denoting $T_\lambda := \inf\{t > 0 : |R_t^2 - r^2| > \lambda\}$, we have:

$$\mathbb{P}_r\left(\left|R_s^2 - r^2\right| > \lambda\right) \leq \mathbb{P}_r\left(\sup_{0\leq t\leq s}\left|R_t^2 - r^2\right| > \lambda\right)$$
$$= \mathbb{P}_r\left(T_\lambda \leq s\right) = \mathbb{P}_r\left(T_\lambda \leq s, \left|R_{T_\lambda}^2 - r^2\right| \geq \lambda\right).$$

And recalling that (see [Law18])

$$d(R_t^2) = 2R_t dW_t$$

where $(W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion, we see that $(R_t^2)_{t\geq 0}$ is a local martingale whose quadratic variation is given by

$$\forall T \geq 0,\ \left\langle R^2\right\rangle_T = 4\int_0^T R_t^2 dt.$$

In particular, $\langle R^2\rangle_{T_\lambda} \leq 4(r^2+\lambda)T_\lambda$ a.s. Also, because $(R_{t\wedge T_\lambda}^2 - r^2, t \geq 0)$ is bounded, for all $u > 0$,

$$\left(e^{u(R_{t\wedge T_\lambda}^2 - r^2) - u^2\left\langle R^2\right\rangle_{t\wedge T_\lambda}/2}, t \geq 0\right)$$

is a martingale uniformly integrable. We thus have by Markov's inequality: for all $u > 0$,

$$\mathbb{P}_r\left(T_\lambda \leq s, R_{T_\lambda}^2 - r^2 \geq \lambda\right) \leq \mathbb{P}_r\left(T_\lambda \leq s, e^{u(R_{T_\lambda}^2 - r^2) - u^2\left\langle R^2\right\rangle_{T_\lambda}/2} \geq e^{u\lambda - u^2\left\langle R^2\right\rangle_{T_\lambda}/2}\right)$$
$$\leq \mathbb{P}_r\left(e^{u(R_{T_\lambda}^2 - r^2) - u^2\left\langle R^2\right\rangle_{T_\lambda}/2} \geq e^{u\lambda - 2u^2(r^2+\lambda)s}\right)$$
$$\leq e^{-u\lambda + 2u^2(r^2+\lambda)s} = \exp\left(-\frac{\lambda^2}{8(r^2+\lambda)s}\right)$$

with the choice of $u = \lambda/(4(r^2+\lambda)s)$. The same reasoning can be applied to the probability $\mathbb{P}_r\left(T_\lambda \leq s, R_{T_\lambda}^2 - r^2 \leq -\lambda\right)$ and we have found

$$\forall\lambda > 0,\ \mathbb{P}_r\left(\left|R_s^2 - r^2\right| > \lambda\sqrt{s}\right) \leq 2\exp\left(-\frac{\lambda^2}{8(r^2+\lambda\sqrt{s})}\right).$$

It then implies that

$$\mathbb{E}_r\left[\left|R_s^2 - r^2\right|^p\right] = s^{p/2}\int_0^\infty \mathbb{P}_r\left(\left|R_s^2 - r^2\right| > \lambda^{1/p}\sqrt{s}\right) d\lambda$$
$$\leq 2s^{p/2}\left(\int_0^1 \exp\left(-\frac{\lambda^{2/p}}{8(r^2+\sqrt{s})}\right) d\lambda + \int_1^\infty \exp\left(-\frac{\lambda^{1/p}}{8(r^2+\sqrt{s})}\right) d\lambda\right)$$
$$\leq Cs^{p/2}\left((r^2+\sqrt{s})^{p/2} + (r^2+\sqrt{s})^p\right)$$

which yields (3.82) recalling that $s \leq 1$. □

We are now ready to prove Lemma 3.29.

Proof of Lemma 3.29. The proof will be decomposed in two steps. The first one will bound from above the left hand side term of (3.81) when $x = y$ whereas the second one will treat the case $\delta = \varepsilon$. In the first part, we will be able to transfer all the computations from the local times to the zero-dimensional Bessel process. For the second part, we will use the result of the first step to compare the local time $L_{x,\varepsilon}(\tau)$ with the occupation measure of a narrow annulus around the circle $\partial D(x, \varepsilon)$. Then an elementary argument of monotonicity will allow us to conclude.

In the entire proof, we will consider $p \geq 1$, $\eta > \eta' > 0$, $x, y \in D$, $0 < \varepsilon, \delta < \eta'$ such that $D(x, \eta') \cup D(y, \eta') \subset D$. Without loss of generality, we will assume that $\varepsilon \geq \delta$. All the constants appearing in the proof may depend on p, η and η'. Before starting off, let us notice that if we fix $K > 0$, the result (3.81) is clear if $|x - y| \vee |\varepsilon - \delta| \geq \varepsilon^{3/2}/K$. Indeed, in that case we have:

$$\begin{aligned}
\mathbb{E}_{x_0}\left[|L_{x,\varepsilon}(\tau) - L_{y,\delta}(\tau)|^p\right] &\leq 2^{p-1}\mathbb{E}_{x_0}\left[L_{x,\varepsilon}(\tau)^p + L_{y,\delta}(\tau)^p\right] \leq C\varepsilon^p \left|\log \varepsilon\right|^p \\
&\leq CK^{p/3}\varepsilon^{p/2} \left|\log \varepsilon\right|^p \left(|x - y| \vee |\varepsilon - \delta|\right)^{p/3} \\
&\leq C' \left\|(x, \varepsilon) - (y, \delta)\right\|^{p/3}.
\end{aligned}$$

In the rest of the proof, we will thus assume that $|x - y| \vee |\varepsilon - \delta| \leq \varepsilon^{3/2}/K$. It will be convenient for us in particular because it forces $\varepsilon - |x - y|^{2/3} - |x - y|$ to be positive (if K is larger than $2^{3/2}$ say).

Step 1. In this step, we assume that $x = y$. To use the links between the local times and the zero-dimensional Bessel process, we consider the different excursions from $\partial D(x, \varepsilon)$ to $\partial D(x, \eta)$: we define $\sigma_0^{(2)} := 0$ and for all $i \geq 1$,

$$\sigma_i^{(1)} := \inf\left\{t > \sigma_{i-1}^{(2)}, B_t \in \partial D(x, \varepsilon)\right\} \text{ and } \sigma_i^{(2)} := \inf\left\{t > \sigma_i^{(1)}, B_t \in \partial D(x, \eta)\right\}.$$

We also denote $N := \max\left\{i \geq 0 : \sigma_i^{(2)} < \tau\right\}$ the number of excursions before exiting the domain D and for all $i \geq 1$, we denote $L_{x,\varepsilon}^i$ and $L_{x,\delta}^i$ the local times of $\partial D(x, \varepsilon)$ and $\partial D(x, \delta)$ accumulated during the i-th excursion. To avoid to condition on N, we do the following rough bound which follows from Jensen's inequality: for $N_0 \geq 1$, $\mathbb{E}_{x_0}\left[|L_{x,\varepsilon}(\tau) - L_{x,\delta}(\tau)|^p\right]$

is equal to

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\left|\sum_{i=1}^{N} L^i_{x,\varepsilon} - L^i_{x,\delta}\right|^p \mathbf{1}_{\{N\leq N_0\}}\right] + \mathbb{E}_{x_0}\left[\left|L_{x,\varepsilon}(\tau) - L_{x,\delta}(\tau)\right|^p \mathbf{1}_{\{N>N_0\}}\right] \\
&\leq (N_0)^{p-1}\sum_{i=1}^{N_0}\mathbb{E}_{x_0}\left[\left|L^i_{x,\varepsilon} - L^i_{x,\delta}\right|^p\right] + \mathbb{E}_{x_0}\left[\left|L_{x,\varepsilon}(\tau) - L_{x,\delta}(\tau)\right|^{2p}\right]^{1/2}\mathbb{P}_{x_0}\left(N>N_0\right)^{1/2} \\
&\leq (N_0)^p \max_{x_0'\in\partial D(x,\varepsilon)}\mathbb{E}_{x_0'}\left[\left|L_{x,\varepsilon}(\tau_{\partial D(x,\eta)}) - L_{x,\delta}(\tau_{\partial D(x,\eta)})\right|^p\right] + C\varepsilon^p\left|\log\varepsilon\right|^p\left(\frac{C'}{\left|\log\varepsilon\right|}\right)^{N_0/2}.
\end{aligned} \tag{3.83}$$

If we choose N_0 to be the first integer larger than

$$2p\log\left(\frac{\varepsilon\left|\log\varepsilon\right|}{\left|\varepsilon-\delta\right|^{1/2}}\right)\Big/\log\left(\frac{\left|\log\varepsilon\right|}{C'}\right),$$

the second term of (3.83) is at most $C\left|\varepsilon-\delta\right|^{p/2}$. Thanks to (3.7) and Lemma 3.30, the first term of (3.83) can be easily controlled: denoting $s=\log(\varepsilon/\delta)$ and $R_0=\sqrt{L_{x,\varepsilon}(\tau_{\partial D(x,\eta)})/\varepsilon}$, for any $x_0'\in\partial D(x,\varepsilon)$, $\mathbb{E}_{x_0'}\left[\left|L_{x,\varepsilon}(\tau_{\partial D(x,\eta)}) - L_{x,\delta}(\tau_{\partial D(x,\eta)})\right|^p\right]$ is at most

$$\begin{aligned}
&2^{p-1}\mathbb{E}_{x_0'}\left[\left|L_{x,\varepsilon}(\tau_{\partial D(x,\eta)}) - \frac{\varepsilon}{\delta}L_{x,\delta}(\tau_{\partial D(x,\eta)})\right|^p\right] + 2^{p-1}\left|\varepsilon-\delta\right|^p\mathbb{E}_{x_0'}\left[\left(\frac{1}{\delta}L_{x,\delta}(\tau_{\partial D(x,\eta)})\right)^p\right] \\
&\leq 2^{p-1}\varepsilon^p\mathbb{E}_{x_0'}\left[\mathsf{E}_{R_0}\left[\left|R_s^2-R_0^2\right|^p\right]\right] + C\left|\varepsilon-\delta\right|^p\left|\log\delta\right|^p \\
&\leq C\varepsilon^p(\log(\varepsilon/\delta))^{p/2}\mathbb{E}_{x_0'}\left[\max\left(1,\left(\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{\partial D(x,\eta)})\right)^p\right)\right] + C\left|\varepsilon-\delta\right|^p\left|\log\delta\right|^p \\
&\leq C\varepsilon^p(\log(\varepsilon/\delta))^{p/2}\left|\log\varepsilon\right|^p + C\left|\varepsilon-\delta\right|^p\left|\log\delta\right|^p.
\end{aligned}$$

Recalling that $|\varepsilon-\delta|\leq\varepsilon^{3/2}/K$, it leads to

$$\mathbb{E}_{x_0'}\left[\left|L_{x,\varepsilon}(\tau_{\partial D(x,\eta)}) - L_{x,\delta}(\tau_{\partial D(x,\eta)})\right|^p\right]\leq C\left|\varepsilon-\delta\right|^{p/2}.$$

Coming back to (3.83), we have just proved that

$$\mathbb{E}_{x_0}\left[\left|L_{x,\varepsilon}(\tau)-L_{x,\delta}(\tau)\right|^p\right]\leq C\left|\log\left|\varepsilon-\delta\right|\right|^p\left|\varepsilon-\delta\right|^{p/2}. \tag{3.84}$$

Step 2. Thanks to the first step we can now assume that $\varepsilon=\delta$. In this step, we will denote for $u\in\mathbb{R}$, $\{u\}_+^p := \max(u,0)^p$. It will be convenient because it is a non-decreasing and convex function. We will also denote $\alpha=\left|x-y\right|^{2/3}$. By taking K large enough and decreasing (resp. increasing) slightly the value of η (resp. η') if necessary, we will be able to use the results of the first part for the circles

$$\{\partial D(x,r), \varepsilon-\alpha-|x-y|<r<\varepsilon+\alpha+|x-y|\} \text{ and } \{\partial D(y,r), \varepsilon-\alpha<r<\varepsilon+\alpha\}.$$

Recall that $\varepsilon-\alpha-|x-y|>0$ thanks to the assumption $|x-y|\leq\varepsilon^{3/2}/K$. We notice that $\mathbb{P}_{x_0}$-a.s.

$$I_x := \int_{\varepsilon-\alpha-|x-y|}^{\varepsilon+\alpha+|x-y|} L_{x,r}(\tau)dr \text{ and } I_y := \int_{\varepsilon-\alpha}^{\varepsilon+\alpha} L_{y,r}(\tau)dr \tag{3.85}$$

are equal to the occupation measures up to time τ of the annuli

$$D(x,\varepsilon+\alpha+|x-y|)\backslash D(x,\varepsilon-\alpha-|x-y|) \text{ and } D(y,\varepsilon+\alpha)\backslash D(y,\varepsilon-\alpha)$$

respectively. As the first annulus contains the second one, $I_x\geq I_y$ $\mathbb{P}_{x_0}$-a.s. We have

$$\begin{aligned}\mathbb{E}_{x_0}\left[\{L_{y,\varepsilon}(\tau)-L_{x,\varepsilon}(\tau)\}_+^p\right] \leq C\mathbb{E}_{x_0}&\left[\left\{\tfrac{1}{2\alpha}I_y-\tfrac{1}{2(\alpha+|x-y|)}I_x\right\}_+^p\right]\\ &+C\mathbb{E}_{x_0}\left[\left\{L_{y,\varepsilon}(\tau)-\tfrac{1}{2\alpha}I_y\right\}_+^p\right]+C\mathbb{E}_{x_0}\left[\left\{\tfrac{1}{2(\alpha+|x-y|)}I_x-L_{x,\varepsilon}(\tau)\right\}_+^p\right].\end{aligned}$$

By our previous observation, the first term on the right hand side is at most

$$C\left(\frac{|x-y|}{\alpha}\right)^p\mathbb{E}_{x_0}\left[\left\{\frac{1}{2(\alpha+|x-y|)}I_x\right\}_+^p\right]\leq C\,|x-y|^{p/3}$$

thanks to our choice of α. The two other terms can be controlled thanks to (3.84): by Jensen's inequality

$$\begin{aligned}\mathbb{E}_{x_0}\left[\left\{L_{y,\varepsilon}(\tau)-\frac{1}{2\alpha}I_y\right\}_+^p\right] &= \mathbb{E}_{x_0}\left[\left\{\frac{1}{2\alpha}\int_{\varepsilon-\alpha}^{\varepsilon+\alpha}\left(L_{y,\varepsilon}(\tau)-L_{y,r}(\tau)\right)dr\right\}_+^p\right]\\ &\leq \frac{1}{2\alpha}\int_{\varepsilon-\alpha}^{\varepsilon+\alpha}\mathbb{E}_{x_0}\left[\{L_{y,\varepsilon}(\tau)-L_{y,r}(\tau)\}_+^p\right]dr\leq C\alpha^{p/2}\,|\log\alpha|^p\end{aligned}$$

and the third term satisfies a similar upper bound. We have thus obtained:

$$\mathbb{E}_{x_0}\left[\{L_{y,\varepsilon}(\tau)-L_{x,\varepsilon}(\tau)\}_+^p\right]\leq C\,|x-y|^{p/3}\,|\log|x-y||^p.$$

By symmetry, the same thing is true for $\mathbb{E}_{x_0}\left[\{L_{x,\varepsilon}(\tau)-L_{y,\varepsilon}(\tau)\}_+^p\right]$ which concludes the proof. □

Chapter 4

Characterisation of planar Brownian multiplicative chaos

We characterise the multiplicative chaos measure $\mathcal{M}$ associated to planar Brownian motion introduced in [BBK94, AHS20, Jeg20a] by showing that it is the only random Borel measure satisfying a list of natural properties. These properties only serve to fix the average value of the measure and to express a spatial Markov property. As a consequence of our characterisation, we establish the scaling limit of the set of thick points of planar simple random walk, stopped at the first exit time of a domain, by showing the weak convergence towards $\mathcal{M}$ of the point measure associated to the thick points. As a corollary, we obtain the convergence of the appropriately normalised number of thick points of random walk to a nondegenerate random variable. The normalising constant is different from that of the Gaussian free field, as conjectured in [Jeg20b]. These results cover the entire subcritical regime.

A key new idea for this characterisation is to introduce measures describing the intersection between different independent Brownian trajectories and how they interact to create thick points.

4.1 Introduction and main results

The study of exceptional points of planar random walk has a long history. In 1960, Erdős and Taylor [ET60] showed that the number of visits of the most visited site of a planar simple random walk after n steps is asymptotically between $(\log n)^2/(4\pi)$ and $(\log n)^2/\pi$ and conjectured that the upper bound is sharp. This conjecture was proven forty years later by Dembo, Peres, Rosen and Zeitouni in the landmark paper [DPRZ01]. These authors also considered the set of thick points of the walk, where the walk has spent a time at least a fraction of $(\log n)^2$, and computed its asymptotic size at the level of exponents. Their proof is based on planar Brownian motion and uses KMT-type approximations to transfer the results to random walk with increments having finite moments of all order.

[Ros05] provided another proof of these results without the use of Brownian motion and [BR07] extended them to planar random walk with increments having finite moment of order $3+\varepsilon$. [Jeg20b] streamlined the arguments by exploiting the links between the local times and the Gaussian free field (GFF) and extended the above results to walks with increments of finite variance and to more general graphs. [AHS20] and [Jeg20a] constructed simultaneously a random measure supported on the set of thick points of Brownian motion extending results of [BBK94]. Finally, [Oka16] studied the most visited points of the inner boundary of the random walk range.

A closely related (but in fact distinct as we will argue below) area of research is the study of planar random walk run until a time close to the cover time. It has become very active since Dembo, Peres, Rosen and Zeitouni [DPRZ04] found the leading order term of the cover time for both planar Brownian motion and random walk settling a conjecture of Aldous [Ald89]. Since then, the understanding of the behaviour of the walk in this regime has considerably improved. We mention a few works. On the torus, the multifractal structure of the set of thin/thick/late points has been studied [DPRZ06, CPV16, Abe15], the subleading order of the cover time has been established [Abe20, BK17] and even the tightness of the cover time associated to Brownian motion on the 2D sphere is known [BRZ19]. For a walk resampled every time it hits the boundary of a planar domain, the scaling limit of the set of thin/thick/late points has been established [AB19]. The picture is even more complete on binary trees where the scaling limit of the cover time [CLS18, DRZ19] as well as the scaling limit of the set of extreme points having maximal local times [Abe18] have been derived.

The current paper is closer to the setup of the first series of articles where the walk is stopped at the first exit time of a planar domain. Its aim is to establish the scaling limit of the thick points of planar simple random walk stopped at the first exit time of a domain by showing that the point measure associated to the thick points converges to a nondegenerate random measure $\mathcal{M}$. This gives much finer information on the set of thick points and, as a corollary, we obtain the convergence of the appropriately normalised number of thick points of random walk to a nondegenerate random variable considerably improving the previously known above-mentioned results. In that sense, it is the final answer to the question raised by Erdős and Taylor.

In this regime a comparison to the GFF is too rough, in contrast with the regime corresponding to times closer to cover time; and indeed, in this latter case the limiting measure is related to the so-called Liouville measure of GFF (see [AB19] and see [RV10, DS11, RV11, Sha16, Ber17] for subcritical Liouville measures and Gaussian multiplicative measures). In our delicate setting of limited time horizon, the limiting measure $\mathcal{M}$, that we can call "Brownian multiplicative chaos" in analogy to Gaussian multiplicative chaos measures, was introduced in [BBK94, AHS20, Jeg20a] and was so far fairly mysterious. On the one hand, it shares a lot of similarities with the Liouville measure such as carrying dimension and conformal invariance. But on the other hand the measure $\mathcal{M}$ is very different in the sense that it is carried and entirely determined by the random fractal composed of a Brownian trace. One of the main result of this paper consists in characterising the law of the measure $\mathcal{M}$. We show that it is the only random Borel measure satisfying a list of natural properties which fix its average value and express a spatial Markov property. This demystifies the measure $\mathcal{M}$ and shows its universal nature.

We start by presenting our results on random walk. We then discuss our characterisation of Brownian multiplicative chaos.

In this paper, we will consider simply connected domains with a boundary composed of a finite number of analytic curves. Such a continuous domain will be called a "nice domain" and a boundary point where the boundary is locally analytic will be called a "nice point".

4.1.1 Scaling limit of thick points of planar random walk

We will extend the definition of the integer part function by setting for $x = (x_1, x_2) \in \mathbb{R}^2$, $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$. For a nice domain U, a reference point $x_0 \in U$ and a large integer N, let U_N and ∂U_N be discrete approximations of U and ∂U defined as follows:

$$U_N := \left\{ \lfloor Nx \rfloor : x \in U, \begin{array}{c} \text{there exists a path in } \mathbb{Z}^2 \text{ from } \lfloor Nx \rfloor \text{ to } \lfloor Nx_0 \rfloor \\ \text{whose distance to the boundary of } NU \text{ is at least } 1 \end{array} \right\}$$

and

$$\partial U_N := \left\{ x \in U_N : \exists y \in \mathbb{Z}^2 \backslash U_N, |x-y| = 1 \right\}.$$

This intricate definition of U_N is just to avoid issues with "thin" boundary pieces. For $z \in \partial U$, we will abusively write $\lfloor Nz \rfloor$ any point of ∂U_N closest to z. Let $(X_t)_{t \geq 0}$ be a continuous time simple random walk on $\mathbb{Z}^2$ with jump rate one (at every vertex, it waits an exponential time with parameter one before jumping) and define its hitting time of ∂U_N and local times:

$$\tau_{\partial U_N} := \inf \{ t \geq 0, X_t \in \partial U_N \} \text{ and for } x \in \mathbb{Z}^2, t > 0, \ell_x^t := \int_0^t \mathbf{1}_{\{X_s = x\}} ds.$$

For $x, z \in \mathbb{C}$, we will denote by $\mathbb{P}_x^{U_N}$ the probability measure associated to the walk $(X_t, t \leq \tau_{\partial U_N})$ starting at $X_0 = \lfloor x \rfloor$ and $\mathbb{P}_{x,z}^{U_N} := \mathbb{P}_x^{U_N} \left(\cdot \middle| X_{\tau_{\partial U_N}} = \lfloor z \rfloor \right)$.

Let $x_0 \in D$ and $z \in \partial D$ be a nice point. Let $a \in (0, 2)$ be a parameter measuring the thickness level,

$$g = \frac{2}{\pi} \quad \text{and} \quad c_0 = \frac{2}{\pi}\Big(\gamma_{\mathrm{EM}} + \frac{1}{2}\log 8\Big) \tag{4.1}$$

be universal constants appearing in the asymptotic behaviour of the discrete Green function (see Lemma 4.11); here γ_{EM} stands for the Euler–Mascheroni constant. We define a random Borel measure $\mu_{x_0;N}^{U,a}$ on $\mathbb{C}$ by setting for all Borel sets $A \subset \mathbb{C}$,

$$\mu_{x_0;N}^{U,a}(A) := \frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\left\{\ell_x^{\tau_{\partial U_N}} \geq g a \log^2 N\right\}} \text{ under } \mathbb{P}_{Nx_0}^{U_N}. \tag{4.2}$$

We also define the conditioned version $\mu_{x_0,z;N}^{U,a}$ of $\mu_{x_0;N}^{U,a}$ by replacing $\mathbb{P}_{Nx_0}^{U_N}$ by $\mathbb{P}_{Nx_0,Nz}^{U_N}$.

One of our main theorems is the following.

Theorem 4.1. *For all $a \in (0,2)$, the sequence $\mu_{x_0;N}^{U,a}, N \geq 1$, (resp. $\mu_{x_0,z;N}^{U,a}, N \geq 1$) converges weakly relatively to the topology of weak convergence (resp. vague convergence) on U. Moreover, the limiting measure has the same distribution as $e^{c_0 a/g}\mathcal{M}_{x_0}^{U,a}$ (resp. $e^{c_0 a/g}\mathcal{M}_{x_0,z}^{U,a}$) built in [BBK94, AHS20, Jeg20a].*

In Section 4.1.2, we recall a precise definition of the above-mentioned Brownian multiplicative chaos measures $\mathcal{M}_{x_0}^{U,a}$ and $\mathcal{M}_{x_0,z}^{U,a}$.

We now emphasise the difficulties inherent to the random walk setting that are not present in the Brownian motion case considered in [BBK94, Jeg20a]. Theorem 4.1 looks very similar to [Jeg20a, Theorem 1.1] (see also [BBK94] for partial results) which studies flat measures $\mathcal{M}_\varepsilon, \varepsilon > 0$, supported on the set of thick points of planar Brownian motion. See Section 4.1.2 for more details about this. But let us emphasise that the approach of [Jeg20a] cannot be adapted to prove Theorem 4.1 and that a new strategy is needed. Indeed, the proof of [Jeg20a, Theorem 1.1] is based on the L^1-convergence of $(\mathcal{M}_\varepsilon(A), \varepsilon > 0)$ for all Borel set $A \subset \mathbb{C}$. This strong form of convergence is crucial to the strategy in [Jeg20a]. Here, it is not even a priori clear how to build the random measures $\mu_{x_0,z;N}^{U,a}, N \geq 1$, on the same probability space so that $(\mu_{x_0,z;N}^{U,a}(A), N \geq 1)$ converges in L^1. For instance, coupling the random walks via the same Brownian motion through KMT-type couplings does not seem to be tractable, or is at least too rough. As mentioned in the introduction, our proof of Theorem 4.1 will rely on a characterisation of the law of Brownian multiplicative chaos, which we describe below.

We first mention however that Abe and Biskup [AB19] have recently established a result with a similar flavour but important differences. Indeed, they consider a random walk on a box with wired boundary conditions (so it is uniformly resampled on the boundary every time it touches the boundary) and run the walk up to a time proportional to the cover time. In this regime, the local times of the walk are very closely related to the Gaussian free field and indeed their limiting measure is the Liouville measure (in contrast to here).

A direct consequence of Theorem 4.1 is the convergence of the appropriately scaled number of random walk's thick points. This answers a question raised in [Jeg20b] and considerably improves the previous known estimates on the fractal dimension [DPRZ01, Ros05, BR07, Jeg20b] of the set of thick points. For $a \in (0, 2)$, we denote

$$\mathcal{T}_N(a) := \left\{ x \in \mathbb{Z}^2, \ell_x^{\tau_{\partial U_N}} \geq \frac{2}{\pi} a \log^2 N \right\}.$$

Recalling the definition (4.1) of g and c_0, we have:

Corollary 4.2. *For all $a \in (0, 2)$, the following convergence holds in distribution: under $\mathbb{P}^{U_N}_{Nx_0}$,*

$$\frac{\log N}{N^{2-a}} \# \mathcal{T}_N(a) \xrightarrow[N \to \infty]{} e^{c_0 a / g} \mathcal{M}^{U,a}_{x_0}(U).$$

Moreover, the limit is nondegenerate, i.e. $\mathcal{M}^{U,a}_{x_0}(U) \in (0, \infty)$ a.s.

As mentioned in [Jeg20b], despite the strong link between the local times and the GFF, this shows a subtle difference in the structure of thick points of random walk compared to those of the GFF which cannot be observed through rougher estimates such as the fractal dimension. Indeed, the analogue of Corollary 4.2 with the local times replaced by half of the GFF squared uses a normalisation factor with $\sqrt{\log N}$ instead of $\log N$. See [BL19].

Remark 4.3. To ease the exposition we decided to focus on the measures $\mu^{U,a}_{x_0;N}$ defined above, but one can consider random measures on $\mathbb{C} \times \mathbb{R}$ defined by: for $A \in \mathcal{B}(\bar{U})$ and $T \in \mathcal{B}(\mathbb{R} \cup \{+\infty\})$,

$$\tilde{\mu}^{U,a}_{x_0;N}(A \times T) := \frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\left\{ \sqrt{\ell_x^{\tau_{\partial U_N}}} - \sqrt{ga} \log N \in T \right\}}.$$

Once the convergence of $\mu^{U,a}_{x_0;N}$ is established, it can be shown that $\tilde{\mu}^{U,a}_{x_0;N}, N \geq 1$, converges, relative to the topology of vague convergence on $\bar{U} \times (\mathbb{R} \cup \{+\infty\})$ to a product measure: $\mathcal{M}^{U,a}_{x_0}$ times an exponential measure. See [Jeg20a] for the case of local times of Brownian motion.

Finally, the convergence of thick points of random walk to Brownian multiplicative chaos opens the door to other scaling limit results. We mention the paper [ABJL21] which builds and studies a multiplicative chaos associated to the so-called Brownian loop soup. When the intensity of the loop soup is critical, [ABJL21] shows that the resulting chaos is closely related to Liouville measure elucidating connections between Brownian multiplicative chaos, Gaussian free field and Liouville measure. This identification of measures heavily relies on the scaling limit results of the current paper. A stronger form of convergence than what is stated in Theorem 4.1 is actually needed in [ABJL21]. This convergence is stated in Theorem 4.23 and is a by-product of our approach to Theorem 4.1. We preferred to defer the exposition of this result to Section 4.5 because it requires the introduction of many more notations.

4.1.2 Brownian multiplicative chaos: background and extension

Background This section recalls the definition of Brownian multiplicative chaos measure $\mathcal{M}_{x_0,z}^{U,a}$ as well as provides the extension of the results of [BBK94, AHS20, Jeg20a] that we need. We follow the construction of [Jeg20a] (see also [BBK94] for partial results and [AHS20] for a different construction). For a nice domain $U \subset \mathbb{C}$ and $x_0 \in U$, let $\mathbb{P}_{x_0}^U$ be the law under which $(B_t, t \leq \tau_{\partial U})$ is a Brownian motion starting at x_0 and stopped at the first exit time of U:

$$\tau_{\partial U} := \inf \left\{ t > 0 : B_t \in \partial U \right\}.$$

For $x_0 \in U$ and a nice point $z \in \partial U$, we will also consider the conditional law $\mathbb{P}_{x_0,z}^U := \mathbb{P}_{x_0}^U\left(\cdot \,|B_{\tau_{\partial U}} = z \right)$ which is rigorously defined for instance in [AHS20, Notation 2.1]. For all $x \in U$ and $\varepsilon > 0$, define the local time $L_{x,\varepsilon}$ of the circle $\partial D(x,\varepsilon)$ up to time $\tau_{\partial U}$:

$$L_{x,\varepsilon} := \lim_{\substack{r \to 0 \\ r > 0}} \frac{1}{2r} \int_0^{\tau_{\partial U}} \mathbf{1}_{\{\varepsilon - r \leq |B_t - x| \leq \varepsilon + r\}} dt$$

with the convention that $L_{x,\varepsilon} = 0$ if the disc $D(x,\varepsilon)$ is not fully included in U. [Jeg20a, Proposition 1.1] shows that these local times are well-defined for all $x \in U$ and $\varepsilon > 0$ simultaneously. For all parameter values $a \in (0,2)$ measuring the thickness level, we can thus define the random measure

$$A \in \mathcal{B}(\mathbb{C}) \mapsto |\log \varepsilon|\, \varepsilon^{-a} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon} \geq 2a|\log \varepsilon|^2 \right\}} dx. \tag{4.3}$$

[Jeg20a] shows that for all $a \in (0,2)$ and under $\mathbb{P}_{x_0,z}^U$, the previous measure converges as $\varepsilon \to 0$ to a nondegenerate random measure $\mathcal{M}_{x_0,z}^{U,a}$, our object of interest. Let us point out that this measure can also be constructed by exponentiating the square root of the local times $L_{x,\varepsilon}$, justifying the name "Brownian multiplicative chaos". This random measure is conformally covariant and, almost surely, it is nondegenerate, supported on the set of thick points of Brownian motion and its carrying dimension equals $2 - a$ (see e.g. [Jeg20a, Corollary 1.4]).

Extension In this paper, a crucial new idea will be to consider the "multipoint" analogue of this measure. We will denote by $\mathcal{S}$ the collection of sets

$$\mathcal{DXZ} = \{(D_i, x_i, z_i), i = 1 \ldots r\} \tag{4.4}$$

where $r \geq 1$, for all $i = 1 \dots r$, D_i is a nice domain, $x_i \in D_i$, $z_i \in \partial D_i$ is a nice boundary point, and the z_i's are pairwise distinct points (i.e. $z_i \neq z_j$ for all $i \neq j$). If $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$, we will (with some abuse of notations) see the set $\mathcal{D}\mathcal{X}\mathcal{Z}$ as a triplet $\mathcal{D}, \mathcal{X}, \mathcal{Z}$ of domains, starting points and exit points. We will for instance write "$D \in \mathcal{D}$" when we mean that we pick a domain that occurs in $\mathcal{D}\mathcal{X}\mathcal{Z}$. Similarly, we will write $\mathcal{D}\mathcal{X}$ when we forget about the exit points.

We now define the multipoint analogue of $\mathcal{M}_{x_0,z}^{U,a}$. Let $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D_i, x_i, z_i), i = 1 \dots r\} \in \mathcal{S}$. For all $i = 1 \dots r$, we consider independent Brownian motions distributed according to $\mathbb{P}_{x_i,z_i}^{D_i}$ and we denote by $L_{x,\varepsilon}^{(i)}$ their associated local times. For all thickness level $a \in (0,2)$ and Borel set $A \subset \mathbb{C}$, we define

$$\mathcal{M}_{\mathcal{X},\mathcal{Z};\varepsilon}^{\mathcal{D},a}(A) := |\log \varepsilon| \, \varepsilon^{-a} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} \sum_{i=1}^r L_{x,\varepsilon}^{(i)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ \forall i = 1 \dots r, L_{x,\varepsilon}^{(i)} > 0 \right\}} dx.$$

We emphasise that, in this definition, the thick points arise from the interaction of the different trajectories. In particular, the single trajectories are not required to be a-thick. In fact, as we will see in Proposition 4.7, a single trajectory will typically be α-thick where α is uniformly distributed in $[0, a]$. Note also that the normalisation is the same as the individual measures (4.3). This indicates that they contribute in the same manner to the occurrence of thick points.

A rather simple modification of [Jeg20a, Theorem 1.1] shows:

Proposition 4.4. *For all $a \in (0,2)$, relative to the topology of weak convergence, the sequence of random measures $\mathcal{M}_{\mathcal{X},\mathcal{Z};\varepsilon}^{\mathcal{D},a}$ converges as $\varepsilon \to 0$ to some random measure $\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$ in probability.*

The proof of this result is contained in Appendix 4.A. Let us comment that $\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$ clearly vanishes almost surely if $\bigcap_{i=1}^r D_i = \varnothing$. Section 4.1.4 investigates some further properties of this multipoint version of Brownian multiplicative chaos. In particular, we explain that we can express $\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$ in terms of the integral of the "intersection" of one-point Brownian multiplicative chaos measures

$$\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}.$$

This "intersection measure" is a natural measure supported on the intersection of the set of thick points associated to each single Brownian motion with suitable thickness level. Further surprising properties of these measures are discussed in Section 4.1.4. See in particular Proposition 4.7.

Finally, we will consider the process of measures $\left(\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S} \right)$. We have already defined the one-dimensional marginals of this process. The definition of the finite-dimensional marginals is done in the following way: if $\mathcal{D}_j\mathcal{X}_j\mathcal{Z}_j \in \mathcal{S}, j = 1 \dots J$, for all (D, x_0, z) appearing in one of the $\mathcal{D}_j\mathcal{X}_j\mathcal{Z}_j$, we always use the same Brownian motion from x_0 to z to define the measures $\mathcal{M}^{\mathcal{D}_j,a}_{\mathcal{X}_j,\mathcal{Z}_j}$. In particular, if $\mathcal{D}\mathcal{X}\mathcal{Z} \cap \mathcal{D}'\mathcal{X}'\mathcal{Z}' = \varnothing$, the measures $\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}$ and $\mathcal{M}^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}$ are independent. This definition is consistent and thus uniquely defines the process $\left(\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$. We mention that we will sometimes write $\mathcal{M}^{a}_{\mathcal{D}\mathcal{X}\mathcal{Z}}$ instead of $\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}$ to clarify the situation.

4.1.3 Characterisation of Brownian multiplicative chaos

We can now state our characterisation of the law of Brownian multiplicative chaos. We start off by introducing some complex analysis notations. Let $a \in (0,2)$ be a thickness level. For any nice domain $D \subset \mathbb{C}$, $x \in D$ and a nice point $z \in \partial D$, we will denote by $\mathrm{CR}(x, D)$ the conformal radius of D seen from x, G^D the Green function of D with zero boundary conditions and $H^D(x,z)dz = \mathbb{P}_x\left(B_{\tau_{\partial D}} \in dz\right)$ the Poisson kernel or harmonic measure of D. See Section 4.1.6 for precise definitions. We set

$$\psi^{D,a}_{x_0,z}(x) := \mathrm{CR}(x,D)^a G^D(x_0,x)\frac{H^D(x,z)}{H^D(x_0,z)}. \tag{4.5}$$

By convention, we will set $\psi^{D,a}_{x_0,z}(x) = 0$ if $x \notin D$. We also introduce, for any $r \geq 1$, a notation for the $(r-1)$-dimensional simplex

$$E(a,r) := \left\{\mathsf{a} = (a_1, \dots, a_r) \in (0,a]^r : a_1 + \dots + a_r = a\right\}. \tag{4.6}$$

The Lebesgue measure on $E(a,r)$ will be denoted by $d\mathsf{a} = da_1 \dots da_{r-1}$.

We are about to consider properties characterising the law of the process $\left(\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}\right)_{\mathcal{D}\mathcal{X}\mathcal{Z}\in\mathcal{S}}$ defined in Section 4.1.2. The most important one will be the spatial Markov property (Property (P_2)). Because it will be notationally heavy, we first present a simple particular case of it which explains the main idea. Let $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$ be of the form $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D, x_0, z)\}$. Let D' be a nice subset of D containing x_0. Then Property (P_2) amounts to:

$$\mathcal{M}^{D,a}_{x_0,z} \quad \text{and} \quad \mathcal{M}^{D',a}_{x_0,Y} + \mathcal{M}^{D,a}_{Y,z} + \mathcal{M}^{a}_{(D',x_0,Y),(D,Y,z)} \tag{4.7}$$

have the same law, where Y has the law of $B_{\tau_{\partial D'}}$ under $\mathbb{P}^D_{x_0,z}$. This comes from the following simple observation. Let $(B_t, t \leq \tau_{\partial D})$ be a Brownian motion in D starting at x_0 and conditioned to exit D through z. We divide $(B_t, t \leq \tau_{\partial D})$ into $(B_t, t \leq \tau_{\partial D'})$ and $(B_t, \tau_{\partial D'} \leq t \leq \tau_{\partial D})$. An a-thick point for the overall trajectory is either entirely generated by one of the two small trajectories and missed by the other one, or comes from the intersection of both.

We now explain our characterisation. Let $\left(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$ be a stochastic process taking values in the set of finite Borel measures. We consider the following properties:

(P_1) (Average value) For all $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D_i, x_i, z_i), i = 1 \dots r\} \in \mathcal{S}$ and for all Borel set $A \subset \mathbb{C}$,

$$\mathbb{E}\left[\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}(A)\right] = \int_A dx \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^{r} \psi_{x_k,z_k}^{D_k,a_k}(x).$$

(P_2) (Markov property) Let $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$, $(D, x_0, z) \in \mathcal{D}\mathcal{X}\mathcal{Z}$ and let D' be a nice subset of D containing x_0. Let Y be distributed according to $B_{\tau_{\partial D'}}$ under $\mathbb{P}_{x_0,z}^{D}$. The joint law of $(\mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z})$ is the same as the joint law given by for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}$,

$$\begin{cases} \mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a} \text{ if } (D, x_0, z) \notin \mathcal{D}'\mathcal{X}'\mathcal{Z}', \\ \mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}} \cup \{(D',x_0,Y)\}} + \mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}} \cup \{(D,Y,z)\}} + \mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}} \cup \{(D',x_0,Y),(D,Y,z)\}} \text{ otherwise,} \end{cases}$$

where in the second line we denote $\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}} = \mathcal{D}'\mathcal{X}'\mathcal{Z}' \setminus \{(D, x_0, z)\}$.

(P_3) (Independence) For all disjoint sets $\mathcal{D}\mathcal{X}\mathcal{Z}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}$, the measures $\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$ and $\mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a}$ are independent.

(P_4) (Non-atomicity) For all $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$, with probability one, simultaneously for all $x \in \mathbb{C}$, $\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}(\{x\}) = 0$.

Theorem 4.5. *Let $a \in (0, 2)$. The process $\left(\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$ from Section 4.1.2 satisfies Properties (P_1)-(P_4). Moreover, if $\left(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$ is another process taking values in the set of finite Borel measures satisfying Properties (P_1)-(P_4), then it has the same law as $\left(\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$.*

Biskup and Louidor [BL19] provide a somewhat similar characterisation of the Liouville measure. The main difference is that Properties (P_2) and (P_3) are replaced by how the spatial Markov property of the Gaussian free field translates to the Liouville measure.

Other characterisations have been formulated before: let D be a fixed nice domain, $x_0 \in D$, $z \in \partial D$ nice and consider the pair given by the measure $\mathcal{M}_{x_0,z}^{D,a}$ together with the Brownian motion $(B_t, t \leq \tau_{\partial D})$ from which it has been built. Then the pair $(\mathcal{M}_{x_0,z}^{D,a}, B)$ is uniquely characterised by

- the measurability of $\mathcal{M}_{x_0,z}^{D,a}$ with respect to the Brownian path B,
- the way the law of the path B is changed given a sample of $\mathcal{M}_{x_0,z}^{D,a}$.

See Theorem 5.2 of [BBK94]. See also Proposition 4.8 for an extension of this characterisation to finitely many trajectories. The advantage of this characterisation is that it considers only one domain, with given starting and ending points and does not need to rely on the multipoint version of Brownian multiplicative chaos. But its drawback is that it refers explicitly to the underlying Brownian motion and it seems to be less applicable in practice. For instance, in the context of our application to random walk, it does not seem easy to apply this characterisation (even measurability is not a priori clear).

Let us also mention that the proof of Theorem 4.5 provides a construction of $\mathcal{M}_{x_0,z}^{D,a}$ through a martingale approximation (see Lemma 4.10). This is very similar to some aspects of the construction of [AHS20] except that they divide the domain into small dyadic squares rather than long narrow rectangles. This might seem to be a cosmetic difference but it is in fact significant since it leads to a decomposition of the Brownian path into excursions from internal to boundary point rather than from boundary to boundary. This is at the heart of what leads to the recursive decomposition of the proof and in turn to the theorem, since the measure $\mathcal{M}_{x_0,z}^{D,a}$ is also itself of this type.

Finally, it is possible that Properties (P_1)-(P_3) are enough to characterise the law, but Property (P_4) is necessary for our current proof; see especially Lemma 4.9. In practice, Property (P_4) is a consequence of uniform-integrability-type estimates that are needed in order to verify Property (P_1).

4.1.4 Further results on multipoint Brownian multiplicative chaos

In this section, we study in greater detail the multipoint version of Brownian multiplicative chaos measures. We start by introducing the "intersection" of Brownian multiplicative chaos measures: a measure whose support is included in the intersection of the support of each intersected measure. Let $\mathcal{DXZ} = \{(D_i, x_i, z_i), i = 1 \ldots r\} \in \mathcal{S}$ and consider independent Brownian motions $B_{x_i,z_i}^{D_i}$ distributed according to $\mathbb{P}_{x_i,z_i}^{D_i}$ for all $i = 1 \ldots r$. Denote by $L_{x,\varepsilon}^{(i)}$ their associated local times. Let $a_i > 0, i = 1 \ldots r$, be thickness levels such that $a := \sum a_i < 2$. We now consider the measure defined by: for all Borel set $A \subset \mathbb{C}$,

$$\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i;\varepsilon}^{D_i,a_i}(A) := |\log \varepsilon|^r \, \varepsilon^{-a} \int_A \prod_{i=1}^r \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(i)} \geq 2a_i |\log \varepsilon|^2 \right\}} dx.$$

Proposition 4.6 below studies the limit of these measures and Proposition 4.7 studies the link between this limiting measure and $\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$ introduced in Section 4.1.2. These results are proven in Appendix 4.A.

Proposition 4.6. *(i) Relatively to the topology of weak convergence, the measure $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i;\varepsilon}^{D_i,a_i}$ converges as $\varepsilon \to 0$ towards a random finite Borel measure $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}$ in probability.*

(ii) Inductive decomposition. If $r \geq 2$, the sequence of random Borel measures

$$A \in \mathcal{B}(\mathbb{C}) \mapsto |\log \varepsilon| \, \varepsilon^{-a_r} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L^{(r)}_{x,\varepsilon} \geq 2a_r |\log \varepsilon|^2 \right\}} \bigcap_{i=1}^{r-1} \mathcal{M}^{D_i,a_i}_{x_i,z_i}(dx) \tag{4.8}$$

converges as $\varepsilon \to 0$ to $\bigcap_{i=1}^r \mathcal{M}^{D_i,a_i}_{x_i,z_i}$ in probability, relative to the topology of weak convergence.

(iii) The measure $\bigcap_{i=1}^r \mathcal{M}^{D_i,a_i}_{x_i,z_i}$ is measurable with respect to $\sigma\left(\mathcal{M}^{D_i,a_i}_{x_i,z_i}, i = 1 \dots r\right)$, the underlying topology being the topology of weak convergence.

(iv) For all $A \in \mathcal{B}(\mathbb{C})$,

$$\mathbb{E}\left[\bigcap_{i=1}^r \mathcal{M}^{D_i,a_i}_{x_i,z_i}(A) \right] = \int_A \prod_{i=1}^r \psi^{D_i,a_i}_{x_i,z_i}(x) dx.$$

(v) With probability one, simultaneously for all Borel set A of Hausdorff dimension strictly smaller than $2 - \sum_{i=1}^r a_i$, $\bigcap_{i=1}^r \mathcal{M}^{D_i,a_i}_{x_i,z_i}(A) = 0$.

(vi) The stochastic process

$$(a_i)_{i=1\dots r} \in \{(\alpha_i)_{i=1\dots r} \in (0,2)^r : \sum \alpha_i < 2\} \mapsto \bigcap_{i=1}^r \mathcal{M}^{D_i,a_i}_{x_i,z_i}$$

taking values in the set of finite Borel measures, equipped with the topology of weak convergence, possesses a measurable modification.

For the following proposition, we consider the measure $\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}$ built from the same Brownian motions as the ones used to defined the previous intersection measures.

Proposition 4.7 (Disintegration). *Let $a \in (0,2)$. If $r \geq 2$, then*

$$\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}} = \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \bigcap_{k=1}^r \mathcal{M}^{D_k,a_k}_{x_k,z_k} \qquad \text{a.s.}$$

Note that the integral of intersection measures above is well-defined thanks to Proposition 4.6, Point *(vi)*.

This result can be compared to the disintegration theorem in measure theory. In words, this proposition shows that the measure $\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}$ "restricted to the event" that, for all $k = 1 \dots r$, the contribution of the k-th trajectory to the overall thickness a is exactly a_k, agrees with the intersection measure $\bigcap_{k=1}^r \mathcal{M}^{D_k,a_k}_{x_k,z_k}$. With the standard disintegration theorem, one is able to make sense of the disintegrated measure for almost every $\mathsf{a} \in E(a,r)$.

Here, the randomness of the measures helps us and we are able to make sense of these measures almost surely, simultaneously for all $\mathsf{a} \in E(a, r)$.

In view of Proposition 4.7, we can rewrite Property (P_2) in the following way. Let $D' \subset D$ be two nice domains, $x_0 \in D'$ and $z \in \partial D$ be a nice point, then

$$\mathcal{M}_{x_0,z}^{D,a} = \mathcal{M}_{x_0,Y}^{D',a} + \mathcal{M}_{Y,z}^{D,a} + \int_0^a \mathcal{M}_{x_0,Y}^{D',a-\alpha} \cap \mathcal{M}_{Y,z}^{D,\alpha} d\alpha$$

with $Y = B_{\tau_{\partial D'}}$. A surprising consequence of Proposition 4.7 is the following.

For all $x \in D'$, if we condition x to be an a-thick point for the overall trajectory $(B_t, t \leq \tau_{\partial D})$ and if we condition the two small trajectories $(B_t, t \leq \tau_{\partial D'})$ and $(B_t, \tau_{\partial D'} \leq t \leq \tau_{\partial D})$ to visit x, then the thickness level of x for one of the two small trajectories will be uniformly distributed in $(0, a)$. To see why this is surprising, consider the following related question. Let h_1 and h_2 be two independent GFFs with zero boundary condition in the domain D. $h := h_1 + h_2$ is now a GFF in D with a variance which has doubled compared to h_1 and h_2. h will therefore have points strictly thicker than any thick point of h_1 and h_2. This situation is very different from the one presented earlier with the local times.

We finish this section by giving an intrinsic characterisation of the intersection measure $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}$. Using Proposition 4.7, this will also provide an intrinsic description of the multipoint measure $\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},\mathsf{a}}$. The characterisation below is a simple extension of the characterisation of the multiplicative chaos associated to one Brownian trajectory, but it is nevertheless an important result since it allows one to quickly identify the measure.

The next result uses the notations introduced above Proposition 4.6. In particular, recall that $B_{x_i,z_i}^{D_i}$ denotes the Brownian motion distributed according to $\mathbb{P}_{x_i,z_i}^{D_i}$ associated to $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}$. For all $i = 1 \dots r$, we view $B_{x_i,z_i}^{D_i}$ as a random element of the set $\mathcal{P}$ of càdlàg paths in $\mathbb{R}^2$ with finite durations. See Section 4.5 for details, in particular concerning the topology associated to $\mathcal{P}$. The following proposition describes the law of the Brownian paths after shifting the probability measure by $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)$ (the so-called rooted measure). As we will see, the resulting trajectories can be written as the concatenations of three independent pieces $B_{x_i,x}^{D_i} \wedge \Xi_x^{D_i,a_i} \wedge B_{x,z_i}^{D_i}$. The first one is a trajectory $B_{x_i,x}^{D_i}$ with law $\mathbb{P}_{x_i,x}^{D_i}$, i.e. a Brownian path conditioned to visit x before exiting D_i. The second part $\Xi_x^{D_i,a_i}$ consists in the concatenation of infinitely many loops rooted at x that are distributed according to a Poisson point process with intensity $a_i \nu_{D_i}(x, x)$. Here $\nu_{D_i}(x, x)$ is a measure on Brownian loops that stay in D_i (see e.g. (2.12) in [AHS20]). Finally, the last part of the trajectory is a Brownian motion $B_{x,z_i}^{D_i}$ distributed according $\mu_{x,z_i}^{D_i}$, that is, a trajectory which starts at x and which is conditioned to exit D_i through z_i.

Proposition 4.8. *Let $F : \mathbb{C} \times \mathcal{P}^r \to \mathbb{R}$ be a bounded measurable function. Then*

$$\mathbb{E}\left[\int_{\mathbb{C}} F(x,(B^{D_i}_{x_i,z_i})_{i=1\dots r}) \bigcap_{i=1}^{r} \mathcal{M}^{D_i,a_i}_{x_i,z_i}(dx)\right] = \int_{\cap_i D_i} \prod_{i=1}^{r} \psi^{D_i,a_i}_{x_i,z_i}(x) \times \mathbb{E}\left[F(z,(B^{D_i}_{x_i,x} \wedge \Xi^{D_i,a_i}_{x} \wedge B^{D_i}_{x,z_i})_{i=1\dots r})\right] dx. \tag{4.9}$$

Moreover, if μ is another random Borel measure which is measurable w.r.t. $B^{D_i}_{x_i,z_i}, i = 1 \dots r$, and which satisfies (4.9) for all bounded measurable function F, then $\mu = \bigcap_{i=1}^{r} \mathcal{M}^{D_i,a_i}_{x_i,z_i}$ almost surely.

As already alluded to, this type of characterisation is of little help when one wants to establish scaling limit results since it relies on the measurability of the underlying Brownian trajectories.

4.1.5 Outline of proofs

We now present the organisation of the paper and explain the main ideas behind the proofs of Theorems 4.1 and 4.5.

Section 4.2 is devoted to the proof of Theorem 4.5. It will start by proving that Brownian multiplicative chaos satisfies Properties (P_1)-(P_4) assuming Propositions 4.4, 4.6 and 4.7 on the multipoint version of Brownian multiplicative chaos. These propositions will be proven in Appendix 4.A. The rest of Section 4.2 will deal with the uniqueness part of Theorem 4.5 and we now sketch its proof. Let $\left(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, \mathcal{DXZ} \in \mathcal{S}\right)$ be a process of Borel measures satisfying Properties (P_1)-(P_4). Let D be a nice domain, $x_0 \in D$ and a nice point $z \in \partial D$. We are going to explain the characterisation of the law of $\mu^{D,a}_{x_0,z}$. The characterisation of the law of more general marginals follows along the same lines. The only extra difficulty lies in the notations. We will start by noticing that Property (P_4) implies that we can find a deterministic direction such that almost surely all the lines parallel to this direction are not seen by the measure $\mu^{D,a}_{x_0,z}$. Without loss of generality, assume that this direction is the vertical one (straightforward adaptations would need to be made in the case of a general direction). We will slice the domain D into many narrow rectangle-type domains $D \cap (q2^{-p},(q+2)2^{-p}) \times \mathbb{R}$, $q \in \mathbb{Z}$. By iterating Property (P_2), we will be able to decompose

$$\mu^{D,a}_{x_0,z} \overset{(\mathrm{d})}{=} \sum_{\mathcal{DXZ} \subset \{(D^p_i, x^p_i, x^p_{i+1}), i \le I_p - 1\}} \mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}.$$

D^p_i will be a narrow rectangle as above centred at x^p_i and $x^p_i, i \geq 1$, will correspond to the successive hitting points of $2^{-p}\mathbb{Z} \times \mathbb{R}$ of a Brownian trajectory. See (4.17) for precise definitions. The idea is then that most of the randomness comes from the points $x^p_i, i \geq 1$, and we do not change the measure so much by replacing each term

$$\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a} \text{ by } \mathbb{E}\left[\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a} \middle| x_i^p, p \geq 1\right].$$

This latter expression is entirely determined by Property (P_1) and does not depend on the process $\left(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$ any more. This conditional expectation encodes a lot of information. For instance, it ensures the measure to be concentrated around the Brownian trajectory. In fact, it provides a martingale approximation of the measure $\mu_{x_0,z}^{D,a}$ as we will see in Lemma 4.10. The proof will then consist in showing that the error in the above approximation tends to zero when $p \to \infty$. The fact that almost surely $\mu_{x_0,z}^{D,a}$ gives zero-mass to any vertical line will be useful for this purpose making sure that we decomposed the initial measure into many small pieces.

We now turn to the random walk part. We will first show the convergence of $\mu_{x_0,z;N}^{U,a}$. The convergence of the unconditioned measures $\mu_{x_0;N}^{U,a}$ will then follow fairly quickly thanks to the weak convergence of the discrete Poisson kernel. To show the convergence of $\mu_{x_0,z;N}^{U,a}$, the overall strategy is simple: we will prove that this sequence is tight and we will then identify the subsequential limits. The tightness is the easy part and relies on a first moment computation. Section 4.3.1 is devoted to it. The identification of the subsequential limits uses Theorem 4.5 and is done in Section 4.3.2. We sketch the main steps of this identification. Let $x_* \in U$ and $z_* \in \partial U$ be a nice point. Let $(N_k, k \geq 1)$ be an increasing sequence of integers so that $(\mu_{x_*,z_*;N_k}^{U,a}, k \geq 1)$ converges. In Lemma 4.15, we will show that we can extract a further subsequence $(N_k', k \geq 1)$ of $(N_k, k \geq 1)$ such that for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}$,

$$\left(\mu_{\mathcal{X},\mathcal{Z};N_k'}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}'\mathcal{X}'\mathcal{Z}'\right)$$

converges. The above measures are the discrete analogue of the multipoint versions of Brownian multiplicative chaos and are defined in (4.23). We denote by $(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S})$ the limiting process of finite Borel measures. Showing that we can extract such a subsequence requires some work since we consider an uncountable number of sequences. Thanks to Theorem 4.5, to conclude the identification of the limiting measure $\mu_{x_*,z_*}^{U,a}$, it is then enough to show that the process $(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S})$ satisfies Properties (P_1)-(P_4). This will roughly follow along the same lines as in the Brownian case. In particular, the uniform integrabilitiy of $\mu_{\mathcal{X},\mathcal{Z};N}^{\mathcal{D},a}(\mathbb{Z}^2)$, $N \geq 1$, which is the content of Proposition 4.17 is key. This comes from a careful truncated second moment estimate which is similar to what was done in [Jeg20a]. The proof of Proposition 4.17 is written in Section 4.4.

4.1.6 Some notations

We finish this introduction with some notations that will be used throughout the paper. Let $D \subset \mathbb{C}$ be a nice domain. For $x \in D$ and a nice point $z \in \partial D$, we will denote by $\mathrm{CR}(x, D)$ the conformal radius of D seen from x, G^D the Green function of D with zero boundary conditions normalised so that $G^D(x,y) \sim -\log|x-y|$ as $|x-y| \to 0$ and $H^D(x,z)dz = \mathbb{P}_x\left(B_{\tau_{\partial D}} \in dz\right)$ the Poisson kernel or harmonic measure of D. These three quantities can be expressed in terms of a conformal map $f_D : D \to \mathbb{D}$ onto the unit disc (see e.g. [Law05, Chapter 2]): for all $x, y \in D$ and for all nice point $z \in \partial D$,

$$\mathrm{CR}(x,D) = \frac{1-|f_D(x)|^2}{|f_D'(x)|}, \tag{4.10}$$

$$G^D(x,y) = \log \frac{\left|1 - f_D(x)\overline{f_D(y)}\right|}{|f_D(y) - f_D(x)|}, \tag{4.11}$$

$$H^D(x,z) = |f_D'(z)| \frac{1-|f_D(x)|^2}{2\pi\,|f_D(x) - f_D(z)|^2}. \tag{4.12}$$

With the notations of Section 4.1.1, we will similarly denote by G^{D_N} and H^{D_N} the discrete Green's function and Poisson kernel defined by: for all $x, y \in \mathbb{Z}^2$,

$$G^{D_N}(x,y) := \mathbb{E}_x\left[\ell_y^{\tau_{\partial D_N}}\right] \text{ and } H^{D_N}(x,y) := \mathbb{P}_x^{D_N}\left(X_{\tau_{\partial D_N}} = y\right). \tag{4.13}$$

In the rest of paper, $a \in (0,2)$ will always denote the thickness level that we look at.

4.2 Characterisation: proof of Theorem 4.5

We start by proving that Brownian multiplicative chaos satisfies Properties (P_1)-(P_4).

Proof of Theorem 4.5, existence. Property (P_1), resp. (P_4), is a direct consequence of Proposition 4.6 *(iv)*, resp. *(v)*, and Proposition 4.7. Property (P_3) follows from the fact that we consider independent Brownian motions.

We now prove Property (P_2). To ease notations, we will only prove this in the simplest case $\mathcal{DXZ} = \{(D, x_0, z)\}$. The general case follows along the same lines. Let D' be a nice subset of D containing x_0. Let B be a Brownian motion under $\mathbb{P}^D_{x_0,z}$, $L_{x,\varepsilon}$ its associated local times and let $L^{(0)}_{x,\varepsilon}$ be the local times of B stopped at the first exit time of D' and

$L_{x,\varepsilon}^{(1)} := L_{x,\varepsilon} - L_{x,\varepsilon}^{(0)}$. We can write

$$\mathcal{M}_{x_0,z;\varepsilon}^{D,a}(dx) = |\log \varepsilon| \, \varepsilon^{-a} \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon} \geq 2a |\log \varepsilon|^2 \right\}} dx = |\log \varepsilon| \, \varepsilon^{-a} \left(\mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} = 0 \right\}} \right. \tag{4.14}$$

$$\left. + \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(0)} = 0 \right\}} + \mathbf{1}_{\left\{ \frac{1}{\varepsilon} \left(L_{x,\varepsilon}^{(0)} + L_{x,\varepsilon}^{(1)} \right) \geq 2a |\log \varepsilon|^2, L_{x,\varepsilon}^{(0)} > 0, L_{x,\varepsilon}^{(1)} > 0 \right\}} \right) dx.$$

If we denote by Y the first hitting point of $\partial D'$ of the Brownian trajectory B, Proposition 4.4 shows that the last term on the right hand side converges in probability towards $\mathcal{M}^a_{(D',x_0,Y),(D,Y,z)}$. We are now going to argue that the first right hand side term converges in probability towards $\mathcal{M}_{x_0,Y}^{D',a}$. Indeed, for all Borel set $A \subset \mathbb{C}$,

$$\begin{aligned} &\mathbb{E} \left[\left| \mathcal{M}_{x_0,Y;\varepsilon}^{D',a}(A) - |\log \varepsilon| \varepsilon^{-a} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} = 0 \right\}} dx \right| \right] \\ &= |\log \varepsilon| \varepsilon^{-a} \int_A \mathbb{P}_{x_0,z}^D \left(\frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0 \right) dx. \end{aligned}$$

We can dominate

$$\sup_\varepsilon |\log \varepsilon| \varepsilon^{-a} \mathbb{P}_{x_0,z}^D \left(\frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0 \right) \leq \sup_\varepsilon |\log \varepsilon| \varepsilon^{-a} \mathbb{P}_{x_0,z}^D \left(\frac{1}{\varepsilon} L_{x,\varepsilon} \geq 2a |\log \varepsilon|^2 \right)$$

which is integrable (see (4.60)). Moreover, for all $x \notin \partial D'$,

$$\begin{aligned} &|\log \varepsilon| \varepsilon^{-a} \mathbb{P}_{x_0,z}^D \left(\frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0 \right) \\ &= \mathbb{E}_{x_0,z}^D \left[|\log \varepsilon| \varepsilon^{-a} \mathbb{P}_{x_0,Y}^{D'} \left(\frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2 \right) \mathbb{P}_{Y,z}^D \left(L_{x,\varepsilon}^{(1)} > 0 \right) \right] \end{aligned}$$

tends to zero as $\varepsilon \to 0$. By dominated convergence theorem, it implies that

$$\mathbb{E} \left[\left| \mathcal{M}_{x_0,Y;\varepsilon}^{D',a}(A) - |\log \varepsilon| \varepsilon^{-a} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} = 0 \right\}} dx \right| \right]$$

tends to zero as $\varepsilon \to 0$. Since $\mathcal{M}_{x_0,Y;\varepsilon}^{D'a}$ converges in probability towards $\mathcal{M}_{x_0,Y}^{D'a}$ (Proposition 4.4), this shows that

$$|\log \varepsilon| \, \varepsilon^{-a} \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(0)} \geq 2a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} = 0 \right\}} dx$$

converges in probability to the same limiting measure. Similarly, the second right hand side term of (4.14) converges in probability towards $\mathcal{M}_{Y,z}^{D,a}$ which overall yields

$$\mathcal{M}_{x_0,z}^{D,a} = \mathcal{M}_{x_0,Y}^{D'a} + \mathcal{M}_{Y,z}^{D,a} + \mathcal{M}_{(D',x_0,Y),(D,Y,z)}^{a}.$$

This is Property (P_2) and it completes the proof. □

The rest of this section is devoted to the uniqueness part of Theorem 4.5.

Proof of Theorem 4.5, uniqueness. Let $\left(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}\right)$ be a process satisfying Properties (P_1)-(P_4). Let D be a nice domain, $x_0 \in D$ and $z \in \partial D$ be a nice point. We are going to identify the law of $\mu_{x_0,z}^{D,a}$. As mentioned in Section 4.1.5, the identification of more general marginals follows along the same lines. The only extra difficulty lies in the notations. We start this proof by noticing that we can find a deterministic angle $\theta \in \mathbb{R}$ such that all the lines with angle θ are not seen by the measure $\mu_{x_0,z}^{D,a}$. Here and in the following, we say that the angle of a line L is θ if we can write $L = x + e^{i\theta}(\{0\} \times \mathbb{R})$ for some $x \in \mathbb{C}$.

Lemma 4.9. *There exists an angle $\theta \in \mathbb{R}$ such that for all $\varepsilon > 0$,*

$$\lim_{p\to\infty} \# \left\{ q \in \mathbb{Z} : \mu_{x_0,z}^{D,a}\left(e^{i\theta}\left(2^{-p}(q+(0,1]) \times \mathbb{R}\right)\right) \geq \varepsilon \right\} = 0 \qquad \text{a.s.}$$

Proof. We proceed by contradiction. Assume that for all $\theta \in \mathbb{R}$, there exists $\varepsilon_\theta > 0$ such that

$$\limsup_{p\to\infty} \# \left\{ q \in \mathbb{Z} : \mu_{x_0,z}^{D,a}\left(e^{i\theta}\left(2^{-p}(q+(0,1]) \times \mathbb{R}\right)\right) \geq \varepsilon_\theta \right\} \geq 1$$

with positive probability p_θ. It implies that for all $\theta \in \mathbb{R}$, the event E_θ that there exists a line L_θ with angle θ such that $\mu_{x_0,z}^{D,a}(L_\theta) \geq \varepsilon_\theta$ has a probability at least p_θ. Moreover, since $[0,\pi)$ is uncountable, there exists $\eta > 0$ such that $\{\theta \in [0,\pi) : p_\theta > \eta, \varepsilon_\theta > \eta\}$ is infinite. Let $\{\theta_k, k \geq 1\}$ be a subset of this set. For all $k \geq 1$, we have by the Paley-Zygmund inequality

$$\begin{aligned}\mathbb{P}\left(\sum_{n\geq 1} \mathbf{1}_{E_{\theta_n}} \geq \frac{\eta}{2} k\right) &\geq \mathbb{P}\left(\sum_{1\leq n\leq k} \mathbf{1}_{E_{\theta_n}} \geq \frac{1}{2}\mathbb{E}\left[\sum_{1\leq n\leq k} \mathbf{1}_{E_{\theta_n}}\right]\right) \\ &\geq \frac{1}{4} \frac{\mathbb{E}\left[\sum_{1\leq n\leq k} \mathbf{1}_{E_{\theta_n}}\right]^2}{\mathbb{E}\left[\left(\sum_{1\leq n\leq k} \mathbf{1}_{E_{\theta_n}}\right)^2\right]} \geq \frac{\eta^2}{4}.\end{aligned}$$

Hence the probability that an infinite number of events $E_{\theta_k}, k \geq 1$, occur is positive. On this event, we have

$$\sum_{k\geq 1} \mu_{x_0,z}^{D,a}(L_{\theta_k}) \geq \eta \sum_{k\geq 1} \mathbf{1}_{E_{\theta_k}} = \infty.$$

But because $\mu_{x_0,z}^{D,a}$ is non-atomic (Property (P_4)), we almost surely have

$$\sum_{k\geq 1} \mu_{x_0,z}^{D,a}(L_{\theta_k}) = \mu_{x_0,z}^{D,a}\left(\bigcup_{k\geq 1} L_{\theta_k}\right) \leq \mu_{x_0,z}^{D,a}(\mathbb{C})$$

which is almost surely finite (Property (P_1) implies that it has a finite first moment). We have obtained an absurdity which concludes the proof. □

This result will be used at the very end of the proof; see (4.22). Roughly speaking, in the course of the proof we will decompose the measure into small pieces and Lemma 4.9 ensures that these pieces are indeed small.

Without loss of generality, we will assume that the specific angle θ provided by Lemma 4.9 is equal to 0. In other words, the measure $\mu_{x_0,z}^{D,a}$ almost surely vanishes on all vertical lines. We will also assume for convenience that $D \subset (0,1) \times \mathbb{R}$.

Let us introduce some notations. We will need to consider small portions of the domain which are well-separated from one another. For this reason, we introduce a Cantor-type set K^∞ which we define now. Let $p_0 \geq 1$ (to be thought of as large) and for all $n \geq 1$, let $\mathfrak{D}_n$ be the set of dyadic points of generation exactly n, i.e.

$$\mathfrak{D}_n = \left\{(2m+1)2^{-n} : m \in \{0, \dots, 2^{n-1} - 1\}\right\}.$$

For instance, $\mathfrak{D}_1 = \{1/2\}$, $\mathfrak{D}_2 = \{1/4, 3/4\}$, $\mathfrak{D}_3 = \{1/8, 3/8, 5/8, 7/8\}$, etc. We now define $K^0 = [0,1]$ and for all $n \geq 1$,

$$K^n := K^{n-1} \setminus \bigcup_{x \in \mathfrak{D}_n} (x - 2^{-(p_0+2n)}, x + 2^{-(p_0+2n)}). \tag{4.15}$$

We then define

$$K^\infty := \bigcap_{n\geq 0} K^n. \tag{4.16}$$

Later in the proof, we will restrict some measures to the set $D \cap K^\infty \times \mathbb{R}$. This will capture almost entirely our measures since the Lebesgue measure of $D \setminus (D \cap K^\infty \times \mathbb{R})$ is at most $C 2^{-p_0}$. Note also that, as $p_0 \to \infty$, K^∞ increases to $[0,1] \setminus \bigcup_{n\geq 1} \mathfrak{D}_n$.

We now start more concretely the proof of Theorem 4.5. Let $p \geq 1$ and $(B_t, t \leq \tau_{\partial D})$ be a Brownian motion distributed according to $\mathbb{P}_{x_0,z}^D$. We are going to keep track of the successive Brownian hitting points of $2^{-p}\mathbb{Z} \times \mathbb{R}$: define $\sigma_0^p := 0$, $x_0^p := x_0$ and $D_0^p := D \cap (2^{-p} \lfloor 2^p x_0 \rfloor + (-2^{-p}, 2^{-p}) \times \mathbb{R})$ and for all $i \geq 1$,

$$\sigma_i^p := \inf\{t > \sigma_{i-1}^p : B_t \notin D_{i-1}^p\}, x_i^p := B_{\sigma_i^p} \text{ and } D_i^p := D \cap \left(x_i^p + \left(-2^{-p}, 2^{-p}\right) \times \mathbb{R}\right). \tag{4.17}$$

Let $I_p := \sup\{i \geq 1 : \sigma_i^p \leq \tau_{\partial D}\}$. Note that $\sigma_{I_p}^p = \tau_{\partial D}$ and $x_{I_p}^p = z$. Let

$$\mathcal{D}^p\mathcal{X}^p\mathcal{Z}^p := \{(D_i^p, x_i^p, x_{i+1}^p), i = 0 \dots I_p - 1\}$$

and let

$$\left(\bar{\mu}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}^p\mathcal{X}^p\mathcal{Z}^p\right)$$

be the process so that conditionally on $x_i^p, i = 1 \dots I_p - 1$, it has the same law as

$$\left(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}^p\mathcal{X}^p\mathcal{Z}^p\right).$$

Note that with the definition (4.17), D_i^p may be formed of several connected components. To be more precise, we define D_i^p as being the connected component that contains x_i^p which is a nice domain belonging to $\mathcal{D}$. x_{i+1}^p being almost surely a nice boundary point of D_i^p and the $x_i^p, i \geq 1$ being almost surely pairwise distinct, the above random measures are well defined. An elementary iteration of Property (P_2) shows that

$$\bar{\mu}_{x_0,z}^{D,a} := \sum_{\mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}^p\mathcal{X}^p\mathcal{Z}^p} \bar{\mu}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}. \tag{4.18}$$

has the same law as $\mu_{x_0,z}^{D,a}$. These definitions are consistent and by Kolmogorov's extension theorem, we can define $x_i^p, i = 0 \dots I_p$, $p \geq 1$, $\bar{\mu}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}$, $\mathcal{D}\mathcal{X}\mathcal{Z} \subset \cup_{p\geq 1}\mathcal{D}^p\mathcal{X}^p\mathcal{Z}^p$ on the same probability space.

In the rest of the proof, we will work on the specific probability space given by Kolmogorov's extension theorem as above. We will drop the bar and simply write

$$\mu_{x_0,z}^{D,a},\ \mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a} \text{ instead of } \bar{\mu}_{x_0,z}^{D,a},\ \bar{\mu}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}.$$

In the following, we will denote by $\mathcal{F}_p$ (resp. $\mathcal{F}_\infty$) the σ-algebra generated by $x_i^p, i = 1 \dots I_p - 1$ (resp. $x_i^p, i = 1 \dots I_p - 1, p \geq 1$) and

$$\mu_p(dx) = \mathbb{E}\left[\mu_{x_0,z}^{D,a}(dx) \middle| \mathcal{F}_p\right]. \tag{4.19}$$

By (4.18) and Property (P_1), $\mu_p(dx)$ does not depend on the process $(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S})$ any more since it is equal to

$$\sum_{r=1}^{I_p} \sum_{\{i_1 \dots i_r\} \subset \{0 \dots I_p - 1\}} \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^{r} \psi_{x_{i_k}^p, x_{i_k+1}^p}^{D_{i_k}^p, a_k}(x) dx. \tag{4.20}$$

The following lemma is a key feature of the proof:

Lemma 4.10. *There exists an a.s. finite random Borel measure μ_∞ such that for all* *bounded measurable function* $f : D \to \mathbb{R}$, $(\langle \mu_p, f \rangle, p \geq 1)$ *is a martingale and converges a.s. to* $\langle \mu_\infty, f \rangle$.

Proof. Let $\mathcal{P}$ be a countable π-system generating the Borel sets of D. For all $A \in \mathcal{P}$, $(\mu_p(A), \mathcal{F}_p)_{p \geq 1}$ is a non-negative martingale thanks to (4.19). Hence, almost surely for all $A \in \mathcal{P}$, $\mu_p(A)$ converges towards some $L(A)$. By standard arguments (see Section 6 of [Ber17] for instance), one can show that it implies that there exists an a.s. finite random Borel measure μ_∞ such that almost surely for all $A \in \mathcal{P}$, $L(A) = \mu_\infty(A)$. It moreover implies that almost surely for all bounded measurable function f, $\langle \mu_p, f \rangle$ converges towards $\langle \mu_\infty, f \rangle$. $\square$

Since μ_∞ is entirely characterised by Properties (P_1)-(P_4), it is enough to show that $\mu_{x_0,z}^{D,a} = \mu_\infty$ a.s. to conclude the proof of Theorem 4.5. Since two finite measures which coincide on a (countable) π-system generating the Borel sets of $\mathbb{C}$ are equal, it is further enough to show that for all Borel set $A \subset \mathbb{C}$, $\mu_{x_0,z}^{D,a}(A) = \mu_\infty(A)$ a.s. We then notice that it is enough to show that for all $t > 0$ and Borel set A,

$$\mathbb{E}\left[e^{-t\mu_{x_0,z}^{D,a}(A)} \middle| \mathcal{F}_\infty \right] = e^{-t\mu_\infty(A)} \qquad \text{a.s.} \tag{4.21}$$

Indeed, it proves that conditionally on $\mathcal{F}_\infty$ the Laplace transform of $\mu_{x_0,z}^{D,a}(A)$ is almost surely equal to the Laplace transform of the constant $\mu_\infty(A)$ on all the positive rational numbers which in turn proves that $\mu_{x_0,z}^{D,a}(A) = \mu_\infty(A)$ a.s. Until the end of the proof we will fix such a Borel set A. We reduce the problem one last time: recall the definition (4.16) of the Cantor-type set K^∞ (which depends on the integer p_0) that we introduced at the beginning of the proof and recall that K^∞ increases with p_0 towards $[0,1] \setminus \bigcup_{n \geq 1} \mathfrak{D}_n$ (see the discussion below (4.16)). By computing the first moment of the variables below, we see that

$$\mu_{x_0,z}^{D,a}\Big(\bigcup_{n \geq 1} \mathfrak{D}_n \times \mathbb{R}\Big) = \mu_\infty\Big(\bigcup_{n \geq 1} \mathfrak{D}_n \times \mathbb{R}\Big) = 0 \qquad \text{a.s.}$$

Therefore, as $p_0 \to \infty$,

$$\mu_{x_0,z}^{D,a}(A \cap K^\infty) \to \mu_{x_0,z}^{D,a}(A) \quad \text{and} \quad \mu_\infty(A \cap K^\infty) \to \mu_\infty(A) \qquad \text{a.s.}$$

In other words, we can safely assume that A is included in K^∞. This assumption will be made for the rest of the proof.

Our objective is to show (4.21). Without loss of generality, we can assume that $t = 1$. One direction is easy: by (4.19), we have

$$\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A) \middle| \mathcal{F}_p \right] = \mu_p(A) \qquad \text{a.s.,}$$

so by Jensen's inequality,

$$\mathbb{E}\left[e^{-\mu_{x_0,z}^{D,a}(A)} \middle| \mathcal{F}_p \right] \geq e^{-\mu_p(A)} \qquad \text{a.s}$$

By Lemma 4.10, $\mu_p(A) \to \mu_\infty(A)$ a.s. So by letting $p \to \infty$ we get

$$\mathbb{E}\left[e^{-\mu_{x_0,z}^{D,a}(A)} \middle| \mathcal{F}_\infty \right] \geq e^{-\mu_\infty(A)} \qquad \text{a.s}$$

For the reverse direction, we use Lemma 3.12 of [BL19] which provides a "reverse Jensen" inequality that we recall.

Lemma D ([BL19], Lemma 3.12)**.** *If $X_1, \ldots, X_n$ are non-negative independent random variables, then for each $\varepsilon > 0$,*

$$\mathbb{E}\left[e^{-\sum_{i=1}^n X_i} \right] \leq \exp\left(-e^{-\varepsilon} \sum_{i=1}^n \mathbb{E}\left[X_i; X_i \leq \varepsilon \right] \right).$$

Let $p \geq 1$ be much larger than p_0 and let $n \geq 1$ be such that $p_0 + 2n = p$ (or such that $p_0 + 2n = p - 1$, depending on the parity). Recall the definition (4.15) of K^n. We will denote by $K^{n,m}, m = 1, \ldots, 2^n$, the connected components of K^n. We notice that conditioned on $\mathcal{F}_p$, the measures $\mu_{x_0,z}^{D,a}(\cdot \cap K^{n,m})$, $m = 1 \ldots 2^n$, are independent. Indeed, looking at (4.18) we see that Property (P_3) implies that conditioned on $\mathcal{F}_p$, $\mu_{x_0,z}^{D,a}(\cdot \cap A_1)$ and $\mu_{x_0,z}^{D,a}(\cdot \cap A_2)$ are independent as soon as the projections of A_1 and A_2 on the real axis are at distance at least 2.2^{-p} from each other. The whole introduction of the set K^∞ is motivated by this fact. Now, because $A \subset K^\infty$ and by Lemma D, we deduce that for each $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{E}\left[e^{-\mu_{x_0,z}^{D,a}(A)} \middle| \mathcal{F}_p \right] \\ &\leq \exp\left(-e^{-\varepsilon} \sum_{m=1}^{2^n} \mathbb{E}\left[\mu_{x_0,z}^{D,a}(A \cap K^{n,m}); \mu_{x_0,z}^{D,a}(A \cap K^{n,m}) \leq \varepsilon \middle| \mathcal{F}_p \right] \right) \qquad \text{a.s.} \end{aligned}$$

To conclude that

$$\mathbb{E}\left[e^{-\mu_{x_0,z}^{D,a}(A)} \middle| \mathcal{F}_\infty \right] \leq e^{-\mu_\infty(A)} \qquad \text{a.s.},$$

it is thus enough to show that a.s.

$$\liminf_{\varepsilon \to 0} \liminf_{p \to \infty} \sum_{m=1}^{2^n} \mathbb{E}\left[\mu_{x_0,z}^{D,a}(A \cap K^{n,m}); \mu_{x_0,z}^{D,a}(A \cap K^{n,m}) \leq \varepsilon \middle| \mathcal{F}_p \right] \geq \mu_\infty(A).$$

We have

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\liminf_{p\to\infty}\sum_{m=1}^{2^n}\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A\cap K^{n,m});\mu_{x_0,z}^{D,a}(A\cap K^{n,m})\leq\varepsilon\middle|\mathcal{F}_p\right]\\&\geq\mu_\infty(A)-\limsup_{\varepsilon\to 0}\limsup_{p\to\infty}\sum_{m=1}^{2^n}\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A\cap K^{n,m});\mu_{x_0,z}^{D,a}(A\cap K^{n,m})>\varepsilon\middle|\mathcal{F}_p\right]\qquad\text{a.s.}\end{aligned}$$

But by Lemma 4.9 and dominated convergence theorem,

$$\begin{aligned}&\mathbb{E}\left[\sum_{m=1}^{2^n}\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A\cap K^{n,m});\mu_{x_0,z}^{D,a}(A\cap K^{n,m})>\varepsilon\middle|\mathcal{F}_p\right]\right]\\&=\sum_{m=1}^{2^n}\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A\cap K^{n,m});\mu_{x_0,z}^{D,a}(A\cap K^{n,m})>\varepsilon\right]\end{aligned}\tag{4.22}$$

tends to zero as $p\to\infty$ (recall that $n\to\infty$ as $p\to\infty$). Hence, by extracting a subsequence if necessary, we have

$$\limsup_{\varepsilon\to 0}\limsup_{p\to\infty}\sum_{m=1}^{2^n}\mathbb{E}\left[\mu_{x_0,z}^{D,a}(A\cap K^{n,m});\mu_{x_0,z}^{D,a}(A\cap K^{n,m})>\varepsilon\middle|\mathcal{F}_p\right]=0\qquad\text{a.s.}$$

which concludes the proof of Theorem 4.5. □

4.3 Application to random walk: proof of Theorem 4.1

We start off by defining the multipoint analogue of $\mu_{x_0,z;N}^{U,a}$. Let $r\geq 1$ and $\mathcal{D}\mathcal{X}\mathcal{Z}=\{(D^i,x_i,z_i),\ i=1\dots r\}\in\mathcal{S}$. Let $X^{(i)}$, $i=1\dots r$, be r independent random walk distributed according to $\mathbb{P}_{Nx_i,Nz_i}^{D_N^i}$ or according to $\mathbb{P}_{Nx_i}^{D_N^i}$ and let $\ell_x^{(i)}$ be their associated local times. We define simultaneously for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}'=\{(D^i,x_i,z_i),i\in I\}\subset\mathcal{D}\mathcal{X}\mathcal{Z}$ the measures given by: for all Borel set A,

$$\mu_{\mathcal{X}',\mathcal{Z}';N}^{\mathcal{D}',a}(A):=\frac{\log N}{N^{2-a}}\sum_{x\in\mathbb{Z}^2}\mathbf{1}_{\{x/N\in A\}}\mathbf{1}_{\left\{\sum_{i\in I}\ell_x^{(i)}\geq ga\log^2 N\right\}}\mathbf{1}_{\left\{\forall i\in I,\ell_x^{(i)}>0\right\}}\tag{4.23}$$

under the probability $\bigotimes_{i=1}^r\mathbb{P}_{Nx_i,Nz_i}^{D_N^i}$. We define similarly the unconditioned measures $\mu_{\mathcal{X}';N}^{\mathcal{D}',a}$, $\mathcal{D}'\mathcal{X}'\subset\mathcal{D}\mathcal{X}$, under $\bigotimes_{i=1}^r\mathbb{P}_{Nx_i}^{D_N^i}$.

4.3.1 Tightness and first moment estimates

In this section we fix a nice domain D. We start by recalling Green's function and Poisson kernel asymptotic behaviours. Recall the notations of Section 4.1.6.

Lemma 4.11 (Green's function)**.** *Let $K \Subset D$. There exist $C, C_K > 0$ such that for all $x, y \in \mathbb{Z}^2$,*

$$G^{D_N}(x,y) \leq g \log \frac{N}{|x-y| \vee 1} + C, \text{ if } x, y \in D_N, \tag{4.24}$$

$$G^{D_N}(x,y) \geq g \log \frac{N}{|x-y| \vee 1} - C_K, \text{ if } \frac{x}{N}, \frac{y}{N} \in K. \tag{4.25}$$

Moreover, for all $x \neq y \in D$, we have

$$\lim_{N \to \infty} G^{D_N}(\lfloor Nx \rfloor, \lfloor Nx \rfloor) - g \log N = g \log \mathrm{CR}(x, D) + c_0, \tag{4.26}$$

$$\lim_{N \to \infty} G^{D_N}(\lfloor Nx \rfloor, \lfloor Ny \rfloor) = g G^D(x,y), \tag{4.27}$$

where c_0 is the universal constant defined in (4.1).

Proof. (4.24) and (4.25) are direct consequences of [Law96] Theorem 1.6.2 and Proposition 1.6.3. (4.26) and (4.27) are contained in Theorem 1.17 of [Bis20]. □

Lemma 4.12 (Poisson kernel)**.** *Let $K \Subset D$ and $\alpha > 0$. For all N large enough, $x, y \in K$ and $z \in \partial D$ a nice point, we have*

$$\left| \frac{H^{D_N}(\lfloor Nx \rfloor, \lfloor Nz \rfloor)}{H^{D_N}(\lfloor Ny \rfloor, \lfloor Nz \rfloor)} - \frac{H^D(x,z)}{H^D(y,z)} \right| \leq \alpha. \tag{4.28}$$

Moreover, for all $x \in D$, the following weak convergence holds:

$$\sum_{z \in \partial D_N} H^{D_N}(\lfloor Nx \rfloor, z) \delta_{z/N}(\cdot) \xrightarrow[N \to \infty]{\text{weakly}} \int_{\partial D} H^D(x,z) \delta_z(\cdot). \tag{4.29}$$

Proof. Statements of the flavour of (4.28) have been extensively studied to show the convergence of loop-erased random walk towards SLE_2. (4.28) is a direct consequence of [YY11, Lemma 1.2] for instance. (4.29) is the content of [Bis20, Lemma 1.23]. □

These two lemmas allow us to derive the first moment estimates that we need. In the following we let $\mathcal{DXZ} = \{(D^i, x_i, z_i), i = 1 \dots r\} \in \mathcal{S}$ and $\mathcal{DX} = \{(D^i, x_i), i = 1 \dots r\}$ and we denote $\ell_x^{(i)}$ the local times associated to the i-th random walk as at the very beginning of Section 4.3. For all nice domain D and $x_0 \in D$, we will also denote for all $x \in \mathbb{C}$,

$$\varphi_{x_0}^{D,a}(x) = G^D(x_0, x) \, \mathrm{CR}(x, D)^a \mathbf{1}_{\{x \in D\}}.$$

Lemma 4.13. *There exists $C > 0$ such that for all $N \geq 1$ and $x \in \mathbb{C}$,*

$$\log(N) N^a \bigotimes_{i=1}^r \mathbb{P}_{Nx_i}^{D_N^i} \left(\sum_{i=1}^r \ell_{\lfloor Nx \rfloor}^{(i)} \geq g a \log^2 N, \forall i = 1 \dots r, \ell_{\lfloor Nx \rfloor}^{(i)} > 0 \right) \tag{4.30}$$
$$\leq C \prod_{i=1}^r \left| \log \left(\frac{|x - x_i|}{C} \vee \frac{1}{N} \right) \right|.$$

Let $K \Subset \cap_{i=0}^{r-1} D^i$. There exists $C > 0$ depending on K, such that for all N large enough and $x \in K$,

$$\log(N) N^a \bigotimes_{i=1}^r \mathbb{P}_{Nx_i, Nz_i}^{D_N^i} \left(\sum_{i=1}^r \ell_{\lfloor Nx \rfloor}^{(i)} \geq g a \log^2 N, \forall i = 1 \dots r, \ell_{\lfloor Nx \rfloor}^{(i)} > 0 \right) \tag{4.31}$$
$$\leq C \prod_{i=1}^r \left| \log \left(\frac{|x - x_i|}{C} \vee \frac{1}{N} \right) \right|.$$

Moreover, for all $x \in \mathbb{C}$,

$$\lim_{N \to \infty} \log(N) N^a \bigotimes_{i=1}^r \mathbb{P}_{Nx_i}^{D_N^i} \left(\sum_{i=1}^r \ell_{\lfloor Nx \rfloor}^{(i)} \geq g a \log^2 N, \forall i = 1 \dots r, \ell_{\lfloor Nx \rfloor}^{(i)} > 0 \right) \tag{4.32}$$
$$= e^{\frac{c_0 a}{g}} \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^r \varphi_{x_k}^{D^k, a_k}(x)$$

and

$$\lim_{N \to \infty} \log(N) N^a \bigotimes_{i=1}^r \mathbb{P}_{Nx_i, Nz_i}^{D_N^i} \left(\sum_{i=1}^r \ell_{\lfloor Nx \rfloor}^{(i)} \geq g a \log^2 N, \forall i = 1 \dots r, \ell_{\lfloor Nx \rfloor}^{(i)} > 0 \right) \tag{4.33}$$
$$= e^{\frac{c_0 a}{g}} \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^r \psi_{x_k, z_k}^{D^k, a_k}(x)$$

where $E(a, r)$ is the $(n-1)$-dimensional simplex defined in (4.6).

Proof of Lemma 4.13. We start by proving (4.31) and (4.33). To ease notations, we will write

$$\mathbb{P} := \bigotimes_{i=1}^r \mathbb{P}_{Nx_i, Nz_i}^{D_N^i}.$$

Let $x \in \mathbb{Z}^2$. We have

$$\mathbb{P} \left(\sum_{i=1}^r \ell_x^{(i)} \geq g a \log^2 N, \forall i = 1 \dots r, \ell_x^{(i)} > 0 \right) \tag{4.34}$$
$$= \prod_{i=1}^r \mathbb{P}_{Nx_i, Nz_i}^{D_N^i} \left(\ell_x^{(i)} > 0 \right) \mathbb{P} \left(\sum_{i=1}^r \ell_x^{(i)} \geq g a \log^2 N \middle| \forall i = 1 \dots r, \ell_x^{(i)} > 0 \right).$$

The Markov property gives that for all $i = 1 \dots r$,

$$\mathbb{P}^{D_N^i}_{Nx_i, Nz_i}\left(\ell_x^{(i)} > 0\right) = \mathbb{P}^{D_N^i}_{Nx_i}\left(\ell_x^{(i)} > 0\right) \frac{\mathbb{P}^{D_N^i}_{x}\left(X^{(i)}_{\tau_{\partial D_N^i}} = Nz_i\right)}{\mathbb{P}^{D_N^i}_{Nx_i}\left(X^{(i)}_{\tau_{\partial D_N^i}} = Nz_i\right)}$$
$$= \frac{G^{D_N^i}(Nx_i, x)}{G^{D_N^i}(x,x)} \frac{H^{D_N^i}(x, Nz_i)}{H^{D_N^i}(Nx_i, Nz_i)}.$$

Moreover, under $\mathbb{P}^{D_N^i}_x$, $\ell_x^{\tau_{\partial D_N^i}}$ is an exponential variable with mean $G^{D_N^i}(x,x)$ which is independent of $X^{(i)}_{\tau_{\partial D_N^i}}$ (see Lemma 4.22). Therefore, conditioning on $X^{(i)}_{\tau_{\partial D_N^i}}$ does not change the law of $\ell_x^{\tau_{\partial D_N^i}}$ and

$$\mathbb{P}\left(\sum_{i=1}^r \ell_x^{(i)} \geq ga \log^2 N \middle| \forall i = 1 \dots r, \ell_x^{(i)} > 0 \right) \tag{4.35}$$
$$= \int_{[0,\infty)^r} dt_1 \dots dt_r e^{-\sum_{i=1}^r t_i} \mathbf{1}_{\left\{ \sum_{i=1}^r G^{D_N^i}(x,x) t_i \geq ga \log^2 N \right\}}.$$

To bound this term from above, we use (4.24) which allows us to bound

$$\mathbf{1}_{\left\{ \sum_{i=1}^r G^{D_N^i}(x,x) t_i \geq ga \log^2 N \right\}} \leq \mathbf{1}_{\left\{ (g \log N + C) \sum_{i=1}^r t_i \geq ga \log^2 N \right\}}$$

which yields

$$\mathbb{P}\left(\sum_{i=1}^r \ell_x^{(i)} \geq ga \log^2 N \middle| \forall i = 1 \dots r, \ell_x^{(i)} > 0 \right) \leq C (\log N)^{r-1} N^{-a}. \tag{4.36}$$

(4.24), (4.25) and (4.28) then concludes the proof of (4.31). To get (4.33), we come back to (4.35) which gives

$$\mathbb{P}\left(\sum_{i=1}^r \ell_x^{(i)} \geq ga \log^2 N \middle| \forall i = 1 \dots r, \ell_x^{(i)} > 0 \right)$$
$$= \mathbb{P}\left(\sum_{i=1}^{r-1} \ell_x^{(i)} \geq ga \log^2 N \middle| \forall i = 1 \dots r-1, \ell_x^{(i)} > 0 \right) + e^{-ga \log^2 N / G^{D_N^r}(x,x)}$$
$$\times \int_{[0,\infty)^{r-1}} dt_1 \dots dt_{r-1} \exp\left(\sum_{i=1}^{r-1} \left(\frac{G^{D_N^i}(x,x)}{G^{D_N^r}(x,x)} - 1 \right) t_i \right) \mathbf{1}_{\left\{ \sum_{i=1}^{r-1} G^{D_N^i}(x,x) t_i \leq ga \log^2 N \right\}}.$$

(4.36) shows that the first right hand side term is at most $C(\log N)^{r-2} N^{-a}$ which is going to be of smaller order than the second term. Using (4.26) and performing the change of variable $s_i = t_i / \log N$ shows that when $x = \lfloor Ny \rfloor$ the second right hand side term is asymptotically equivalent to

$$e^{\frac{c_0 a}{g}} (\log N)^{r-1} N^{-a} \operatorname{CR}(y, D^r)^a \int_{[0,\infty)^{r-1}} ds_1 \dots ds_{r-1} \prod_{i=1}^{r-1} \left(\frac{\operatorname{CR}(y, D^i)}{\operatorname{CR}(y, D^r)} \right)^{s_i} \mathbf{1}_{\left\{\sum_{i=1}^{r-1} s_i \leq a\right\}}.$$

Using (4.28), this shows that

$$\lim_{N\to\infty} \log(N) N^a \mathbb{P}\left(\sum_{i=1}^{r} \ell^{(i)}_{\lfloor Ny \rfloor} \geq ga \log^2 N, \forall i = 1 \dots r, \ell^{(i)}_{\lfloor Ny \rfloor} > 0 \right) = \prod_{i=1}^{r} G^{D^i}(x_i, y) \frac{H^{D^i}(y, z_i)}{H^{D^i}(x_i, z_i)}$$
$$\times e^{\frac{c_0 a}{g}} \int_{[0,a]^{r-1}} ds_1 \dots ds_{r-1} \prod_{i=1}^{r-1} \operatorname{CR}(y, D^i)^{s_i} \operatorname{CR}(y, D^r)^{a - \sum_{i=1}^{r-1} s_i} \mathbf{1}_{\left\{\sum_{i=1}^{r-1} s_i < a\right\}}$$

which proves (4.33).

We omit the proofs of (4.30) and (4.32) which are very similar and even slightly easier since there is no conditioning to deal with. We nevertheless mention that in (4.30), we do not need to restrict ourselves to the bulk of the domains (compared to (4.31)) because the probability increases with the domains. We can thus assume that all the points we consider are deep inside the domains. This finishes the proof. $\square$

We are now ready to prove:

Proposition 4.14 (Tightness). *The sequences*

$$\left((\mu^{\mathcal{D}',a}_{\mathcal{X}';N}, \mathcal{D}'\mathcal{X}' \subset \mathcal{D}\mathcal{X}), N \geq 1 \right) \text{ and } \left((\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}';N}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}), N \geq 1 \right)$$

are tight for the product topology of, respectively, weak and vague convergence on $\bigcap_{D \in \mathcal{D}'} D$*,* $\mathcal{D}' \subset \mathcal{D}$*. Moreover, for any Borel set* $A \subset \mathbb{C}$*,*

$$\lim_{N\to\infty} \mathbb{E}\left[\mu^{\mathcal{D},a}_{\mathcal{X};N}(A) \right] = e^{\frac{c_0 a}{g}} \int_A dx \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^{r} \varphi^{D^k, a_k}_{x_k}(x) \tag{4.37}$$

and if A *is compactly included in* $\cap_{i=1}^r D^i$*,*

$$\lim_{N\to\infty} \mathbb{E}\left[\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z};N}(A) \right] = e^{\frac{c_0 a}{g}} \int_A dx \int_{\mathsf{a} \in E(a,r)} d\mathsf{a} \prod_{k=1}^{r} \psi^{D^k, a_k}_{x_k, z_k}(x) \tag{4.38}$$

where $E(a, r)$ *is the* $(r-1)$*-dimensional simplex defined in* (4.6)*.*

Proof of Proposition 4.14. To prove the desired tightness, it is enough to show that for all $\mathcal{D}'\mathcal{X}' \subset \mathcal{D}\mathcal{X}$ and $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}$ and $K \Subset \bigcap_{D \in \mathcal{D}'} D$, the sequences of real-valued random variables

$$\left(\mu^{\mathcal{D}',a}_{\mathcal{X}';N}(\mathbb{C}), N \geq 1 \right) \text{ and } \left(\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}';N}(K), N \geq 1 \right)$$

are tight. This is a direct consequence of Lemma 4.13: (4.30) and (4.31) show that

$$\mathbb{E}\left[\mu_{\mathcal{X}';N}^{\mathcal{D}',a}(\mathbb{C})\right] \text{ and } \mathbb{E}\left[\mu_{\mathcal{X}',\mathcal{Z}';N}^{\mathcal{D}',a}(K)\right]$$

are uniformly bounded in N. (4.37) and (4.38) follow from dominated convergence theorem and (4.32) and (4.33) respectively. □

4.3.2 Study of the subsequential limits

As described in Section 4.1.5, we start by showing that we can extract a subsequence such that the convergence holds for all domains and starting/stopping points at the same time. The difficulty lies in the fact that we consider uncountably many sequences.

Lemma 4.15. *Let $(N_k, k \geq 1)$ be an increasing sequence of integers. There exists a subsequence $(N_k', k \geq 1)$ of $(N_k, k \geq 1)$ such that for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}$,*

$$(\mu_{\mathcal{X},\mathcal{Z};N_k'}^{\mathcal{D},a}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}'\mathcal{X}'\mathcal{Z}')$$

converges as $k \to \infty$ in distribution, relative to the product topology of vague convergence on $\bigcap_{D\in\mathcal{D}} D$, $\mathcal{D} \subset \mathcal{D}'$.

Before proving this result, we state an elementary lemma for ease of reference:

Lemma 4.16. *Let $(X_k, k \geq 1)$ be a sequence of random variables. Assume that for all $k \geq 1$ and $p \geq 1$, X_k can be written as $X_k = Y_{k,p} + Z_{k,p}$ where $Y_{k,p}$ and $Z_{k,p}$ are two non-negative random variables defined on the same probability space. Assume further that for all $\lambda > 0$,*

$$\lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right] \text{ and } \lim_{p\to\infty}\lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right]$$

exist and that for all $p \geq 1$, $\sup_{k\geq 1} \mathbb{E}\left[Y_{k,p}\right] < \infty$ and $\sup_{k\geq 1} \mathbb{E}\left[Z_{k,p}\right] \to 0$ when $p \to \infty$. Then $(X_k, k \geq 1)$ converges in distribution.

Proof of Lemma 4.16. As $\sup_{k\geq 1} \mathbb{E}\left[X_k\right] < \infty$, $(X_k, k \geq 1)$ is tight. To show that it converges, it is thus enough to show the pointwise convergence of the Laplace transform. Take $\lambda > 0$. Since $Z_{k,p}$ is non-negative,

$$\mathbb{E}\left[e^{-\lambda X_k}\right] \leq \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right]$$

and

$$\limsup_{k\to\infty} \mathbb{E}\left[e^{-\lambda X_k}\right] \leq \lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right] \xrightarrow[p\to\infty]{} \lim_{p\to\infty}\lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right].$$

On the other hand,

$$\mathbb{E}\left[e^{-\lambda X_k}\right] - \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right] = -\mathbb{E}\left[e^{-\lambda Y_{k,p}}\left(1-e^{-\lambda Z_{k,p}}\right)\right] \geq -\lambda \mathbb{E}\left[Z_{k,p}\right]$$

and

$$\liminf_{k\to\infty} \mathbb{E}\left[e^{-\lambda X_k}\right] \geq \lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right] - \lambda \sup_{k\geq 1} \mathbb{E}\left[Z_{k,p}\right] \xrightarrow[p\to\infty]{} \lim_{p\to\infty}\lim_{k\to\infty} \mathbb{E}\left[e^{-\lambda Y_{k,p}}\right].$$

We have shown that $\mathbb{E}\left[e^{-\lambda X_k}\right], k \geq 1$, converges to $\lim_{p\to\infty}\lim_{k\to\infty}\mathbb{E}\left[e^{-\lambda Y_{k,p}}\right]$ which concludes the proof. □

Proof of Lemma 4.15. In this proof, the topologies associated to the unconditioned (resp. conditioned) measures will be the topology of weak convergence (resp. vague convergence) on the underlying domain. We will denote by $\mathfrak{D}$ the collection of simply connected domains that can be written as a finite union of discs with rational centres and radii and

$$\mathcal{S}' := \bigcup_{r\geq 1} \left\{ \{(D_i, x_i), i = 1 \dots r\} : \forall i = 1 \dots r, D_i \in \mathfrak{D}, x_i \in D_i \cap \mathbb{Q}^2 \right\}.$$

Notice that $\mathcal{S}'$ is countable.

Let $\mathcal{D}\mathcal{X} \in \mathcal{S}$. By Proposition 4.14, the sequence $(\mu^{\mathcal{D},a}_{\mathcal{X};N_k}, k \geq 1)$ is tight. Moreover, the associated random walk $\left(N_k^{-1} X^{\mathcal{D}\mathcal{X}}_{N_k^2 t}, t \leq N_k^{-2}\tau^{N_k}_{\mathcal{D}\mathcal{X}}\right), k \geq 1$, is also tight because it converges to Brownian motion. Hence, by Cantor's diagonal argument, we can extract a subsequence of $(N_k, k \geq 1)$ (that we still denote $(N_k, k\geq 1)$ in the following) such that for all $\mathcal{D}'\mathcal{X}' \in \mathcal{S}'$, the joint distribution

$$\left(\mu^{\mathcal{D},a}_{\mathcal{X};N_k}, \left(N_k^{-1} X^{\mathcal{D}\mathcal{X}}_{N_k^2 t}, t \leq N_k^{-2}\tau^{N_k}_{\mathcal{D}\mathcal{X}}\right)\right), \mathcal{D}\mathcal{X} \subset \mathcal{D}'\mathcal{X}', \tag{4.39}$$

converges as $k \to \infty$.

We will conclude the proof with the following two steps.

(i) We will first fix $D_i \in \mathfrak{D}, i = 1 \dots r$ and show that the fact that for all $x_i \in \mathbb{D}_i \cap \mathbb{Q}^2, i = 1 \dots r$, (4.39) converges with $\mathcal{D}'\mathcal{X}' = \{(D_i, x_i)\}$ implies the same statement for all $x_i \in D_i, i = 1 \dots r$.

(ii) We will then fix nice domains D_i and initial points $x_i \in D_i, i = 1 \dots r$, and we will show that the fact that for all $D_i' \in \mathfrak{D}$ containing $x_i, i = 1 \dots r$, (4.39) converges with $\mathcal{D}'\mathcal{X}' = \{(D_i', x_i)\}$ implies that for all pairwise distinct nice points $z_i \in \partial D_i$ and $\mathcal{D}'\mathcal{X}'\mathcal{Z}' = \{(D_i, x_i, z_i)\}$, $\left(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z};N_k}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}'\mathcal{X}'\mathcal{Z}'\right)$ converges as $k \to \infty$.

We will only prove *(ii)* since *(i)* is very similar. See the end of the proof for a few comments about the step *(i)* above. To ease notations, we will moreover only prove *(ii)* for $r = 1$.

The general case $r \geq 1$ follows along the same lines by considering multivariate Laplace transforms.

Let D be a nice domain, $x_0 \in D$ and $z \in \partial D$ be a nice point. Let $(X_t)_{t\geq 0}$ be the associated random walk. We assume that we already know that for all $D' \in \mathfrak{D}$ containing x_0, the joint distribution of

$$\mu_{x_0;N_k}^{D',a}, \left(N_k^{-1} X_{N_k^2 t}, t \leq N_k^{-2} \tau_{D'}^{N_k} \right)$$

converges as $k \to \infty$ and we want to show the convergence of $\mu_{x_0,z;N_k}^{D,a}, k \geq 1$. Let $f \in C_c(D,[0,\infty))$. Our objective is to show that $\left\langle \mu_{x_0,z;N_k}^{D,a}, f \right\rangle, k \geq 1$, converges in law. Let $p \geq 1$ and consider $D^p \in \mathfrak{D}$ such that

$$\{x \in D : \mathrm{dist}(x,\partial D) \geq 2^{-p}\} \subset D^p \subset \{x \in D : \mathrm{dist}(x,\partial D) \geq 2^{-p-1}\}.$$

In the following, we will consider the measure $\mu_{x_0,z;N}^{D^p,a}$ which is defined as $\mu_{x_0;N}^{D^p,a}$ but under the conditional probability $\mathbb{P}_{Nx_0,Nz}^{D_N}$ instead of $\mathbb{P}_{Nx_0}^{D_N^p}$. $(B_t, t \leq \tau_{\partial D^p})$ under $\mathbb{P}_{x_0}^D$ and $(B_t, t \leq \tau_{\partial D^p})$ under $\mathbb{P}_{x_0,z}^D$ are mutually absolutely continuous: if $\mathcal{F}_{\tau_{\partial D^p}}$ denotes the σ-algebra generated by $(B_t, t \leq \tau_{\partial D^p})$, we have (see [AHS20] (2.7) for instance)

$$\frac{d\mathbb{P}_{x_0,z}^D}{d\mathbb{P}_{x_0}^D}\Big|_{\mathcal{F}_{\tau_{\partial D^p}}} = \frac{H_D(B_{\tau_{\partial D^p}}, z)}{H_D(x_0,z)} =: \mathcal{H}.$$

Similarly (direct consequence of Markov property),

$$\frac{d\mathbb{P}_{Nx_0,Nz}^{D_N}}{d\mathbb{P}_{Nx_0}^{D_N}}\Big|_{\mathcal{F}_{\tau_{\partial D_N^p}}} = \frac{H_N(X_{\tau_{D^p}^N}, Nz)}{H_N(Nx_0,Nz)} =: \mathcal{H}_N. \tag{4.40}$$

Hence the convergence of $\left(\left\langle \mu_{x_0;N_k}^{D^p,a}, f \right\rangle, X_{\tau_{D^p}^{N_k}}/N_k \right), k \geq 1$, implies the convergence of $\left\langle \mu_{x_0,z;N_k}^{D^p,a}, f \right\rangle$, $k \geq 1$: by Lemma 4.12, for all $\alpha > 0$ and k large enough,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(- \left\langle \mu_{x_0,z;N_k}^{D^p,a}, f \right\rangle \right) \right] &= \mathbb{E}\left[\mathcal{H}_{N_k} \exp\left(- \left\langle \mu_{x_0;N_k}^{D^p,a}, f \right\rangle \right) \right] \\
&\leq \mathbb{E}\left[\left(\frac{H_D\left(X_{\tau_{D^p}^{N_k}}/N_k, z \right)}{H_D(x_0,z)} + \alpha \right) \exp\left(- \left\langle \mu_{x_0;N_k}^{D^p,a}, f \right\rangle \right) \right] \\
&\xrightarrow[k\to\infty]{} \mathbb{E}\left[(\mathcal{H}+\alpha) \exp\left(- \left\langle \mu_{x_0}^{D^p,a}, f \right\rangle \right) \right]
\end{aligned}$$

and

$$\limsup_{k\to\infty} \mathbb{E}\left[\exp\left(- \left\langle \mu_{x_0,z;N_k}^{D^p,a}, f \right\rangle \right) \right] \leq \mathbb{E}\left[\mathcal{H} \exp\left(- \left\langle \mu_{x_0}^{D^p,a}, f \right\rangle \right) \right].$$

We obtain similarly that the liminf is bounded from below by the above right hand side term implying that $\mathbb{E}\left[\exp\left(-\left\langle \mu^{D^p,a}_{x_0,z;N_k}, f\right\rangle\right)\right]$ converges as $k \to \infty$. Since $D^p, p \geq 1$, is an increasing sequence of domains, for all $k \geq 1$, $\mathbb{E}\left[\exp\left(-\left\langle \mu^{D^p,a}_{x_0,z;N_k}, f\right\rangle\right)\right]$ is non-increasing with p. Hence

$$\lim_{k\to\infty} \mathbb{E}\left[\exp\left(-\left\langle \mu^{D^p,a}_{x_0,z;N_k}, f\right\rangle\right)\right]$$

converges when $p \to \infty$. By Lemma 4.13, we also notice that for all $N \geq 1$ and $p \geq 1$,

$$0 \leq \mathbb{E}\left[\left\langle \mu^{D,a}_{x_0,z;N}, f\right\rangle\right] - \mathbb{E}\left[\left\langle \mu^{D^p,a}_{x_0,z;N}, f\right\rangle\right] \leq o_{p\to\infty}(1).$$

By lemma 4.16, it implies that $\left\langle \mu^{D,a}_{x_0,z;N_k}, f\right\rangle, k \geq 1$, converges in distribution. This concludes the proof of the step *(ii)*.

We finish this proof with a comment about the step *(i)*. The proof is very similar. One would need to first stop the walks at the first hitting times of small discs centred at the starting points x_i. One would need to argue that the main contribution comes from the rest of the trajectories which converge by an h-transform-type of argument as above. We leave the details to the reader. □

As mentioned in Section 4.1.5, to prove that the subsequential limits satisfy Properties (P_1) and (P_4), we need the following result which is proven in Section 4.4:

Proposition 4.17 (Uniform integrability)**.** *For all $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$ and $K \Subset \bigcap_{D\in\mathcal{D}} D$,*

$$\left(\mu^{\mathcal{D},a}_{\mathcal{X};N}(\mathbb{C}), N \geq 1\right) \text{ and } \left(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z};N}(K), N \geq 1\right)$$

are uniformly integrable. Moreover, any subsequential limit $\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}$ of $\left(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z};N}, N \geq 1\right)$ satisfies: almost surely for all Borel set A of Hausdorff dimension less than $2-a$, $\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}(A) = 0$.

Before jumping into the proof of Theorem 4.1, we state the following result which is a quick consequence of (4.29).

Lemma 4.18. *Let $x_0 \in U$ and let $\phi_N : \mathbb{C} \to [0,1]$ be a sequence of functions converging pointwise towards ϕ. Let $\{z_i, i = 1\dots p\} \subset \partial U$ be the points where the boundary ∂U is not analytic. Assume that for all $\alpha > 0$ and for any compact subset K of $\mathbb{C}\backslash\{z_i, i = 1\dots p\}$, there exists $C_{\alpha,K} > 0$ such that for all N large enough and for all $z, z' \in K$,*

$$|\phi_N(z) - \phi_N(z')| \leq C_{\alpha,K}\,|z - z'| + \alpha.$$

Then,

$$\sum_{z\in\partial U_N} H^{U_N}(\lfloor Nx_0\rfloor, z)\phi_N(z/N) \xrightarrow[N\to\infty]{} \int_{\partial U} H^U(x_0, z)\phi(z)dz.$$

dz denotes here the one-dimensional Hausdorff measure on ∂U.

Proof. In this proof, when we say that a set $K \subset \mathbb{C}$ is smooth, we mean that each connected component of the boundary of K is analytic. Let $\alpha, \varepsilon > 0$. Since $0 \leq \phi \leq 1$, there exists a smooth compact subset K of $\mathbb{C} \backslash \{z_i, i = 1 \dots p\}$ such that

$$\int_{\partial U \backslash K} H^U(x,z)\phi(z)dz \leq \alpha.$$

Using the weak convergence (4.29), this upper bound in particular implies

$$\limsup_{N \to \infty} \sum_{z \in \partial U_N} \mathbf{1}_{\{z/N \notin K\}} H^{U_N}(\lfloor Nx \rfloor, z)\phi_N(z/N) \leq \alpha.$$

We now decompose $K = \cup_{i=1}^I K_i$ into smooth compact sets of diameter at most ε and such that for all $i \neq j$, $K_i \cap K_j \cap \partial U$ is composed of at most one point. For all $i = 1 \dots I$, let y_i be any point of K_i. By the weak convergence (4.29), we now have

$$\begin{aligned}
&\limsup_{N \to \infty} \sum_{z \in \partial U_N} \mathbf{1}_{\{z/N \in K\}} H^{U_N}(\lfloor Nx \rfloor, z)\phi_N(z/N) \\
&\leq \alpha + C_{\alpha,K}\ \varepsilon + \limsup_{N \to \infty} \sum_{i=1}^I \phi_N(y_i) \sum_{z \in \partial U_N} \mathbf{1}_{\{z/N \in K_i\}} H^{U_N}(\lfloor Nx \rfloor, z) \\
&\leq \alpha + C_{\alpha,K}\ \varepsilon + \sum_{i=1}^I \phi(y_i) \int_{\partial U \cap K_i} H^U(x,z)dz \\
&\leq 2\alpha + 2C_{\alpha,K}\ \varepsilon + \int_{\partial U \cap K} H^U(x,z)\phi(z)dz.
\end{aligned}$$

We have obtained

$$\limsup_{N \to \infty} \sum_{z \in \partial U_N} H^{U_N}(\lfloor Nx \rfloor, z)\phi_N(z/N) \leq 3\alpha + 2C_{\alpha,K}\ \varepsilon + \int_{\partial U} H^U(x,z)\phi(z)dz.$$

We obtain the desired upper bound by letting $\varepsilon \to 0$ and then $\alpha \to 0$. The lower bound is similar. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $x_0 \in U$. We start by assuming the convergence of $(\mu^{U,a}_{x_0,z;N}, N \geq 1)$ for all nice points $z \in \partial U$ and we are going to explain how we deduce the convergence of $(\mu^{U,a}_{x_0;N}, N \geq 1)$. Let $f \in C(D, [0,\infty))$. It is enough to prove that

$$\mathbb{E}\left[\exp\left(-\left\langle \mu^{U,a}_{x_0;N}, f\right\rangle\right)\right]$$

converges. By Lemma 4.13 (4.30),

$$\lim_{r\to 0}\sup_N \mathbb{E}\left[\mu_{x_0;N}^{U,a}\left(\{x\in U : d(x,\partial U)\le r\}\right)\right]=0.$$

We can thus assume that f has a compact support included in U (see Lemma 4.16). We have

$$\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U,a},f\right\rangle\right)\right]=\sum_{z\in\partial U_N} H^{U_N}(x_0,\lfloor Nz\rfloor)\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0,z/N;N}^{U,a},f\right\rangle\right)\right].$$

To obtain the convergence of the above sum, we are going to show that we can cast our situation into Lemma 4.18. Let $\alpha, r>0$ and define

$$U^r := \{x\in U : \mathrm{dist}(x,\partial U)>r\}.$$

By Lemma 4.13, if r is small enough (possibly depending on U, x_0 and f), we have for all $z\in\partial D$,

$$\begin{aligned}&\left|\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0,z;N}^{U,a},f\right\rangle\right)\right]-\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U^r,a},f\right\rangle\right)\middle| X_{\tau_{\partial U_N}}=\lfloor Nz\rfloor\right]\right|\\&\le \mathbb{E}\left[\left\langle \mu_{x_0,z;N}^{U,a},f\right\rangle\right]-\mathbb{E}\left[\left\langle \mu_{x_0;N}^{U^r,a},f\right\rangle\middle| X_{\tau_{\partial U_N}}=\lfloor Nz\rfloor\right]\le \alpha/3.\end{aligned}$$

We now notice by Lemma 4.12 (4.28) that for all N large enough and $z,z'\in\partial D$,

$$\begin{aligned}&\left|\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U^r,a},f\right\rangle\right)\middle| X_{\tau_{\partial U_N}}=\lfloor Nz\rfloor\right]-\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U^r,a},f\right\rangle\right)\middle| X_{\tau_{\partial U_N}}=\lfloor Nz'\rfloor\right]\right|\\&=\left|\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U^r,a},f\right\rangle\right)\left(\frac{H^{U_N}(X_{\tau_{\partial U_N^r}},\lfloor Nz\rfloor)}{H^{U_N}(x_0,\lfloor Nz\rfloor)}-\frac{H^{U_N}(X_{\tau_{\partial U_N^r}},\lfloor Nz'\rfloor)}{H^{U_N}(x_0,\lfloor Nz'\rfloor)}\right)\right]\right|\\&\le \alpha/3+\sup_{x,y\in U^r}\left|\frac{H^U(x,z)}{H^U(y,z)}-\frac{H^U(x,z')}{H^U(y,z')}\right|.\end{aligned}$$

Using (4.12), we see that for all compact subset K of an analytic portion of ∂U, the above supremum is at most $C_{\alpha,K}|z-z'|$ for all $z,z'\in K$. We have proven that for all N large enough, all such compact subset K and $z,z'\in K$,

$$\left|\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0,z;N}^{U,a},f\right\rangle\right)\right]-\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0,z';N}^{U,a},f\right\rangle\right)\right]\right|\le C_{\alpha,K}|z-z'|+\alpha.$$

We can thus conclude with Lemma 4.18 that

$$\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0;N}^{U,a},f\right\rangle\right)\right]\xrightarrow[N\to\infty]{}\int_{\partial D}H^U(x_0,z)\lim_{N\to\infty}\mathbb{E}\left[\exp\left(-\left\langle \mu_{x_0,z;N}^{U,a},f\right\rangle\right)\right]dz.$$

This finishes the transfer of the convergence of conditioned measures to unconditioned measures.

We now turn to the proof of the convergence of $(\mu^{U,a}_{x_*,z_*;N}, N \geq 1)$ where $x_* \in U$ and $z_* \in \partial U$ is a nice point. Let $(N_k, k \geq 1)$ be an increasing sequence of integers such that $(\mu^{U,a}_{x_*,z_*;N_k}, k \geq 1)$ converges. By Lemma 4.15, by extracting a further subsequence if necessary, we can assume that for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}$,

$$(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z};N_k}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}'\mathcal{X}'\mathcal{Z}')$$

converges as $k \to \infty$ towards some

$$(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, \mathcal{D}\mathcal{X}\mathcal{Z} \subset \mathcal{D}'\mathcal{X}'\mathcal{Z}').$$

By Theorem 4.5, to show that $\mu^{U,a}_{x_*,z_*} \overset{(\mathrm{d})}{=} e^{c_0 a/g} \mathcal{M}^{U,a}_{x_*,z_*}$, it is enough to prove that $(\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S})$ satisfies Properties (P_1)-(P_4).

Property (P_1) is a direct consequence of what we have already done. For instance, for $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D, x_0, z)\} \in \mathcal{S}$, the arguments are as follows. In order to identify the two finite Borel measures

$$\mathbb{E}\left[\mu^{D,a}_{x_0,z}(dx)\right] \quad \text{and} \quad e^{c_0 a/g} \psi^D_{x_0,z}(x) dx,$$

we only need to check that for any continuous bounded nonnegative function $f : \mathbb{C} \to \mathbb{R}$, the integrals of f against these two measures agree. For $r > 0$, let f_r be a continuous function with support compactly included in D which agrees with f on $\{x \in D : d(x, \partial D) \geq r\}$ and such that $0 \leq f_r \leq f$. By Proposition 4.14, for all $r > 0$,

$$\lim_{k \to \infty} \mathbb{E}\left[\left\langle \mu^{D,a}_{x_0,z;N_k}, f_r \right\rangle\right] = e^{c_0 a/g} \int_D f_r(x) \psi^{D,a}_{x_0,z}(x) dx.$$

Since Proposition 4.17 shows that $(\left\langle \mu^{D,a}_{x_0,z;N_k}, f_r \right\rangle, k \geq 1)$ is uniformly integrable, we can interchange the limit and the expectation which gives

$$\mathbb{E}\left[\left\langle \mu^{D,a}_{x_0,z}, f_r \right\rangle\right] = e^{c_0 a/g} \int_D f_r(x) \psi^D_{x_0,z}(x) dx.$$

We then obtain Property (P_1) by letting $r \to 0$ and using monotone convergence theorem.

The proof of Property (P_2) is very similar to the Brownian case. For instance, in the case $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D, x_0, z)\} \in \mathcal{S}$ and D' nice subset of D containing x_0, we can very similarly show that for all continuous function $f : \mathbb{C} \to [0, \infty)$ with compact support included in $D \backslash \partial D'$, and all $y \in \partial D'$, $\left\langle \mu^{D,a}_{x_0,z;N}, f \right\rangle$ under $\mathbb{P}^{D_N}_{\lfloor N x_0 \rfloor, \lfloor N z \rfloor}\left(\cdot \middle| X_{\tau_{\partial D'_N}} = \lfloor N y \rfloor \right)$ has the same law as

$$\left\langle \mu^{D',a}_{x_0,y;N}, f \right\rangle + \left\langle \mu^{D,a}_{y,z;N}, f \right\rangle + \left\langle \mu^{a}_{(D',x_0,y),(D,y,z);N}, f \right\rangle$$

plus smaller order terms which converge to zero in L^1. This shows the conditional version of Property (P_2). To obtain Property (P_2) without having to condition on the hitting point of $\partial D'$, we have to integrate over $y \in \partial D'$. For this, we use the same argument as what we did at the very beginning of the proof to transfer results from the conditioned to the unconditioned measures.

Finally, Property (P_3) follows from the fact that we consider independent random walks and Property (P_4) is a direct consequence of the carrying dimension estimate of Proposition 4.17. This concludes the proof. □

4.4 Uniform integrability: proof of Proposition 4.17

To ease notations, we will prove Proposition 4.17 for $\mathcal{DXZ} = \{(D, x_0, z)\}$. Our approach is very close to the one of [Jeg20a]. We have simplified some minor aspects since we only need to show the uniform integrability of the sequence but not its convergence in L^1. For instance, our definition of "good events" limits the number of certain excursions rather than limiting certain local times.

If $x \in \mathbb{Z}^2$ and $R \geq 1$, we will denote by $C_R(x)$ the contour $\mathbb{Z}^2 \cap \partial(x + [-R, R]^2)$, by $A_N(x \to R)$ the number of excursions from x to $C_R(x)$ before $\tau_{\partial D_N}$ and

$$q_R := \log\left(\frac{N}{R}\right) \Big/ \log N. \tag{4.41}$$

For $b \in (a, 2)$ and $\varepsilon > 0$, we introduce

$$D^\varepsilon := \{x \in D : d_\infty(x, \partial D) > 2\varepsilon \text{ and } |x - x_0| \geq 2\varepsilon\},$$

the good event at x

$$G_N^{b,\varepsilon}(x) := \left\{ \forall R \in (2^p)_{p \geq 1} \cap [N^{1/2-a/4}, \varepsilon N], A_N(x \to R) \leq \frac{b}{2} \frac{1+q_R}{1-q_R} \log \frac{N}{R} \right\}$$

and the modified version of $\mu_{x_0;N}^{D,a}(\mathbb{C})$,

$$\bar{\mu}_{x_0;N}^{D,a}(\mathbb{C}) := \frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in D^\varepsilon\}} \mathbf{1}_{G_N^{b,\varepsilon}(x)} \mathbf{1}_{\left\{\ell_x^{\tau_{\partial D_N}} \geq ga \log^2 N\right\}}.$$

We will see that adding these good events does not change the behaviour of the first moment and makes the second moment finite.

Lemma 4.19. *For all $b > a$,*

$$\lim_{\varepsilon \to 0} \sup_{N \geq 1} \mathbb{E} \left[\left| \mu^{D,a}_{x_0;N}(\mathbb{C}) - \bar{\mu}^{D,a}_{x_0;N}(\mathbb{C}) \right| \right] = 0.$$

Lemma 4.20. *If $b > a$ is close enough to a,*

$$\sup_{N \geq 1} \mathbb{E} \left[\bar{\mu}^{D,a}_{x_0;N}(\mathbb{C})^2 \right] < \infty. \tag{4.42}$$

Moreover, if b is close enough to a, for all $\eta > 0$,

$$\sup_{N \geq 1} \mathbb{E} \left[\int_{\mathbb{C}^2} \frac{1}{|x-y|^{2-2b+a-\eta}} \bar{\mu}^{D,a}_{x_0;N}(dx) \bar{\mu}^{D,a}_{x_0;N}(dy) \right] < \infty. \tag{4.43}$$

We now explain how these two lemmas imply Proposition 4.17.

Proof of Proposition 4.17. Lemma 4.19 and (4.42) imply that $(\mu^{D,a}_{x_0;N}(\mathbb{C}), N \geq 1)$ is uniformly integrable. Moreover, by Frostman's lemma, Lemma 4.19 and the energy estimate (4.43) imply that any subsequential limit $\mu^{D,a}_{x_0}$ of $\mu^{D,a}_{x_0;N}, N \geq 1$, satisfies: almost surely for all Borel set A with Hausdorff dimension smaller than $2-a$, $\mu^{D,a}_{x_0}(A) = 0$.

To finish the proof, we now have to explain how we transfer these results to the conditioned measures $\mu^{D,a}_{x_0,z;N}$, $N \geq 1$. Let $K \Subset D$, $r > 0$ and define $D^r := \{x \in D, d(x, \partial D) > r\}$. We denote by $\mu^{D^r,a}_{x_0,z;N}(K)$ the random variable

$$\frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in K\}} \mathbf{1}_{\left\{ \ell_x^{\tau_{\partial D^r_N}} \geq ga \log^2 N \right\}}$$

under $\mathbb{P}^{D_N}_{x_0,z}$. A similar reasoning as in the proof of Lemma 4.15 shows that

$$0 \leq \mathbb{E} \left[\mu^{D,a}_{x_0,z;N}(K) - \mu^{D^r,a}_{x_0,z;N}(K) \right] \leq p(r)$$

for some $p(r) > 0$ which may depend on a, D, x_0, z and which goes to zero as $r \to 0$. Hence, to show the uniform integrability of $(\mu^{D,a}_{x_0,z;N}(K), N \geq 1)$, it is enough to show that $(\mu^{D^r,a}_{x_0,z;N}(K), N \geq 1)$ is uniformly integrable. Recalling that (see (4.40), (4.28) and (4.12))

$$\left. \frac{d\mathbb{P}^{D_N}_{Nx_0,Nz}}{d\mathbb{P}^{D_N}_{Nx_0}} \right|_{\mathcal{F}_{\tau_{\partial D^r_N}}} = \frac{H_N(X_{\tau_{\partial D^r_N}}, Nz)}{H_N(Nx_0, Nz)} \in [\alpha, 1/\alpha]$$

for some $\alpha = \alpha(r) \in (0,1)$, we then observe that for all $M > 0$,

$$\mathbb{E}\left[\mu_{x_0,z;N}^{D^r,a}(\mathbb{C})\mathbf{1}_{\left\{\mu_{x_0,z;N}^{D^r,a}(\mathbb{C})\geq M\right\}}\right] \leq \frac{1}{\alpha}\mathbb{E}\left[\mu_{x_0;N}^{D^r,a}(\mathbb{C})\mathbf{1}_{\left\{\mu_{x_0;N}^{D^r,a}(\mathbb{C})\geq M\right\}}\right] \leq \frac{1}{\alpha}\mathbb{E}\left[\mu_{x_0;N}^{D,a}(\mathbb{C})\mathbf{1}_{\left\{\mu_{x_0;N}^{D,a}(\mathbb{C})\geq M\right\}}\right].$$

The uniform integrability of $(\mu_{x_0;N}^{D,a}(\mathbb{C}), N \geq 1)$ thus implies the uniform integrability of $(\mu_{x_0,z;N}^{D^r,a}(\mathbb{C}),\ N \geq 1)$.

To obtain the carrying dimension estimate we proceed in a similar manner. If we denote $\bar{\mu}_{x_0,z;N}^{D^r,a}(dx)$ the modified version of $\mu_{x_0,z;N}^{D^r,a}(dx)$ for which we have added the good events $G_N^{b,\varepsilon}(x)$ for the domain D^r, we have as before

$$\sup_{N\geq 1}\mathbb{E}\left[\int_{\mathbb{C}^2}\frac{1}{|x-y|^{2-2b+a-\eta}}\bar{\mu}_{x_0,z;N}^{D^r,a}(dx)\bar{\mu}_{x_0,z;N}^{D^r,a}(dy)\right] \leq \frac{1}{\alpha}\sup_{N\geq 1}\mathbb{E}\left[\int_{\mathbb{C}^2}\frac{1}{|x-y|^{2-2b+a-\eta}}\bar{\mu}_{x_0;N}^{D^r,a}(dx)\bar{\mu}_{x_0;N}^{D^r,a}(dy)\right] < \infty$$

and

$$\limsup_{\varepsilon\to 0}\sup_{N\geq 1}\mathbb{E}\left[\mu_{x_0,z;N}^{D^r,a}(\mathbb{C}) - \bar{\mu}_{x_0,z;N}^{D^r,a}(\mathbb{C})\right] \leq \frac{1}{\alpha}\limsup_{\varepsilon\to 0}\sup_{N\geq 1}\mathbb{E}\left[\mu_{x_0;N}^{D^r,a}(\mathbb{C}) - \bar{\mu}_{x_0;N}^{D^r,a}(\mathbb{C})\right] 0.$$

For the same reasons as before, this shows that any subsequential limit $\mu_{x_0,z}^{D^r,a}$ of $\mu_{x_0,z;N}^{D^r,a}$, $N \geq 1$, satisfies: almost surely for all Borel set A of Hausdorff dimension smaller than $2-a$, $\mu_{x_0,z}^{D^r,a}(A) = 0$. Since this is true for all $r > 0$, it completes the proof the carrying dimension estimate of Proposition 4.17. This concludes the proof. □

The rest of this section is dedicated to the proofs of Lemmas 4.19 and 4.20. We now lay the groundwork. If $A \subset \mathbb{Z}^2$, we will write

$$\tau_A := \inf\{t > 0 : X_t \in A\}$$

and for $x \in \mathbb{Z}^2$, $\tau_x := \tau_{\{x\}}$. Let $N \geq 1$. For $x, y \in D_N$, we will denote

$$p_{xy} := \mathbb{P}_x\left(\tau_y < \tau_{\partial D_N}\right) = G^{D_N}(x,y)/G^{D_N}(y,y). \tag{4.44}$$

If x and y are in the bulk of D_N, Lemma 4.11 implies that

$$p_{xy} = q_{|x-y|}\left(1 + O\left(\frac{1}{\log N}\right)\right). \tag{4.45}$$

We start off with two easy lemmas. The first one is the analogue of [Jeg20a, Lemma 2.3] whereas the second one is well-known and a proof can be found for instance in [Jeg20b, Lemma 4.2.1].

Lemma 4.21. *For all pairwise distinct points x, y, z of D_N,*

$$\mathbb{P}_z\left(\tau_x < \tau_y \wedge \tau_{\partial D_N}\right) = \frac{p_{zx} - p_{zy}p_{yx}}{1 - p_{xy}p_{yx}}.$$

Proof. By Markov property, we have

$$\begin{aligned}\mathbb{P}_z\left(\tau_y < \tau_{\partial D_N}\right) &= \mathbb{P}_z\left(\tau_y < \tau_x \wedge \tau_{\partial D_N}\right) + \mathbb{P}_z\left(\tau_x < \tau_y < \tau_{\partial D_N}\right) \\ &= \mathbb{P}_z\left(\tau_y < \tau_x \wedge \tau_{\partial D_N}\right) + \mathbb{P}_z\left(\tau_x < \tau_y \wedge \tau_{\partial D_N}\right)\mathbb{P}_x\left(\tau_y < \tau_{\partial D_N}\right).\end{aligned}$$

By exchanging the roles of x and y we find that

$$\mathbb{P}_z\left(\tau_x < \tau_{\partial D_N}\right) = \mathbb{P}_z\left(\tau_x < \tau_y \wedge \tau_{\partial D_N}\right) + \mathbb{P}_z\left(\tau_y < \tau_x \wedge \tau_{\partial D_N}\right)\mathbb{P}_y\left(\tau_x < \tau_{\partial D_N}\right).$$

Combining these two equalities yields the stated claim. □

Lemma 4.22. *For all subset $A \subset \mathbb{Z}^2$ and $x \in \mathbb{Z}^2$, starting from x, $\ell_x^{\tau_A}$ is an exponential variable independent of X_{τ_A}.*

We now fix $x, y \in D_N$, $R \in (2^p)_{p \geq 1} \cap [N^{1/2-a/4}, \varepsilon N]$ such that $x/N, y/N \in D^\varepsilon$ and such that $y \notin x + [-R, R]^2$ and we describe the joint law of $(\ell_x^{\tau_{\partial D_N}}, \ell_y^{\tau_{\partial D_N}}, A_N(x \to R), A_N(y \to R))$. For $i \geq 1$, we denote by ℓ_x^i (resp. ℓ_y^i) the local time at x (resp. y) accumulated during the i-th excursion from x to $C_R(x)$ (resp. from y to $C_R(y)$). We have

$$\ell_x^{\tau_{\partial D_N}} = \sum_{i=1}^{A_N(x \to R)} \ell_x^i \text{ and } \ell_y^{\tau_{\partial D_N}} = \sum_{i=1}^{A_N(y \to R)} \ell_y^i. \tag{4.46}$$

By Markov property and by Lemma 4.22, conditioned on $A_N(x \to R)$ and $A_N(y \to R)$, the variables $\ell_x^i, i = 1 \dots A_N(x \to R)$ and $\ell_y^i, i = 1 \dots A_N(y \to R)$ are i.i.d. exponential random variables with mean equal to

$$\mathbb{E}_x\left[\ell_x^{\tau_{C_R(x)}}\right] = \left(1 + O\left(\frac{1}{\log N}\right)\right) g \log R = \left(1 + O\left(\frac{1}{\log N}\right)\right) g(1 - q_R)\log N. \tag{4.47}$$

Moreover, by (4.45), for all $k \geq 1$,

$$\begin{aligned}\mathbb{P}_{Nx_0}\left(A_N(x \to R) \geq k\right) &= \mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} > 0\right)\mathbb{P}_x\left(\ell_x^{\tau_{\partial D_N}} - \ell_x^{\tau_{C_R(x)}} > 0\right)^{k-1} \\ &= \left(1 + O\left(\frac{1}{\log N}\right)\right)^{k-1}\mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} > 0\right) q_R^{k-1}.\end{aligned} \tag{4.48}$$

Similarly, we notice that if $c|x-y| \leq R \leq |x-y|/10$, then Lemma 4.21 and (4.45) show that

$$\begin{aligned} &\mathbb{P}_{Nx_0}\left(A_N(x \to R) + A_N(y \to R) = k\right) \\ &\leq \mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) \left(\frac{2q_R}{1+q_R}\right)^{k-1} \left(1 + O\left(\frac{1}{\log N}\right)\right)^{k-1}. \end{aligned} \tag{4.49}$$

Finally, we state for ease of reference the following two elementary inequalities:

$$\text{if } \mu \leq 1, \sum_{i=n}^{\infty} \frac{(\mu n)^i}{i!} \leq (\mu e)^n, \tag{4.50}$$

$$\text{if } \mu \geq 1, \sum_{i=0}^{n-1} \frac{(\mu n)^i}{i!} \leq e(\mu e)^{n-1}. \tag{4.51}$$

We will moreover denote $\Gamma(k,1)$ a Gamma random variable with shape parameter k and scale parameter 1. This variable has the same law as the sum of k independent exponential variables with parameter 1. Recall that for all $k, k' \geq 1$ and $t > 0$,

$$\mathbb{P}\left(\Gamma(k,1) > t\right) = e^{-t} \sum_{i=0}^{k-1} \frac{t^i}{i!} \tag{4.52}$$

and

$$\begin{aligned} \mathbb{P}\left(\Gamma(k,1) \geq t\right) \mathbb{P}\left(\Gamma(k',1) \geq t\right) &= e^{-2t} \sum_{n=0}^{k+k'-2} t^n \sum_{\substack{0 \leq i \leq k-1 \\ 0 \leq j \leq k'-1 \\ i+j=n}} \frac{1}{i!j!} \\ &\leq e^{-2t} \sum_{n=0}^{k+k'-2} t^n \sum_{\substack{i,j \geq 0 \\ i+j=n}} \frac{1}{i!j!} = e^{-2t} \sum_{n=0}^{k+k'-2} \frac{(2t)^n}{n!} \end{aligned} \tag{4.53}$$

We are now ready to prove Lemmas 4.19 and 4.20.

Proof of Lemma 4.19. Firstly, by Lemma 4.13,

$$\lim_{\varepsilon \to 0} \sup_{N \geq 1} \mathbb{E}_{Nx_0}^{D_N}\left[\frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \notin D^\varepsilon\}} \mathbf{1}_{\left\{\ell_x^{\tau_{\partial D_N}} \geq g a \log^2 N\right\}}\right] = 0.$$

So we only need to show that

$$\lim_{\varepsilon \to 0} \sup_{N \geq 1} \mathbb{E}_{Nx_0}^{D_N}\left[\frac{\log N}{N^{2-a}} \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in D^\varepsilon\}} \mathbf{1}_{G_N^{b,\varepsilon}(x)^c} \mathbf{1}_{\left\{\ell_x^{\tau_{\partial D_N}} \geq g a \log^2 N\right\}}\right] = 0. \tag{4.54}$$

Let $x \in \mathbb{Z}^2$ s.t. $x/N \in D^\varepsilon$. By a union bound,

$$\begin{aligned} &\mathbb{P}^{D_N}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N, G_N^{b,\varepsilon}(x)^c\right) \\ &\leq \sum_{\substack{R\in(2^p)_{p\geq 1} \\ N^{1/2-a/4}\leq R\leq \varepsilon N}} \mathbb{P}^{D_N}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N, A_N(x\to R) > \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}\right). \end{aligned} \tag{4.55}$$

Let $R \in (2^{-p})_{p\geq 1} \cap [N^{1/2-a/4}, \varepsilon N]$. In the discussion following Lemma 4.21 we described the joint law of $(\ell_x^{\tau_{\partial D_N}}, A_N(x\to R))$. Using the notations therein and by (4.52), we have

$$\begin{aligned} &\mathbb{P}^{D_N}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N, A_N(x\to R) > \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}\right) \\ &= O(1)\mathbb{P}^{D_N}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} > 0\right)(1-q_R) \\ &\times \sum_{k>\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \left(1+O\left(\frac{1}{\log N}\right)\right)^{k-1} q_R^{k-1}\mathbb{P}\left(\Gamma(k,1) \geq \frac{a\log N}{1-q_R}\left(1+O\left(\frac{1}{\log N}\right)\right)\right) \\ &= O(1)\frac{G^{D_N}(Nx_0,x)}{G^{D_N}(x,x)}(1-q_R)e^{-a\log N/(1-q_R)} \\ &\times \sum_{k>\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \left(1+O\left(\frac{1}{\log N}\right)\right)^{k-1} q_R^{k-1}\sum_{i=0}^{k-1}\frac{1}{i!}\left(\frac{a\log N}{1-q_R}\right)^i \\ &= O(1)\frac{G^{D_N}(Nx_0,x)}{G^{D_N}(x,x)}e^{-a\log N/(1-q_R)}\left(q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}}\sum_{i=0}^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}-1}\frac{1}{i!}\left(\frac{a\log N}{1-q_R}\right)^i\right. \\ &\qquad\left. + \sum_{i\geq\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}}\frac{1}{i!}\left(q_R\frac{a\log N}{1-q_R}\right)^i\left(1+O\left(\frac{1}{\log N}\right)\right)^i\right) \end{aligned}$$

We are going to bound each individual term of the above expression. Let

$$q_{ab} := \sup\left\{q\in(0,1) : \frac{a}{q} \geq \frac{b(1+q)}{2}\right\} < 1.$$

There exists $\eta = \eta(a,b)$ such that for all $q \in [q_{ab},1], \log q \leq q-1-\eta(q-1)^2$. We deduce that if $q_R \in [q_{ab},1]$,

$$\begin{aligned} N^a q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} &= \exp\left[\left(\frac{a}{q_R}+\frac{b}{2}\frac{1+q_R}{1-q_R}\log q_R\right)\log\frac{N}{R}\right] \\ &\leq \exp\left[\left(\frac{a}{q_R}-\frac{b(1+q_R)}{2}-\frac{\eta b(1-q_R^2)}{2}\right)\log\frac{N}{R}\right] \leq \exp\left[-\eta'\log\frac{N}{R}\right] \end{aligned}$$

for some $\eta' = \eta'(a,b) > 0$. Hence, if $q_R \in [q_{ab}, 1]$, we have

$$e^{-a \log N/(1-p_R)} q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \sum_{i=0}^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}-1} \frac{1}{i!}\left(\frac{a\log N}{1-q_R}\right)^i \leq q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \leq N^{-a}\left(\frac{N}{R}\right)^{-\eta'}.$$

If $q_R < q_{ab}$, we use (4.51) and we get

$$\begin{aligned}
&e^{-a \log N/(1-q_R)} q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \sum_{i=0}^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}-1} \frac{1}{i!}\left(\frac{a\log N}{1-q_R}\right)^i \\
&\leq O(1) e^{-a\log N/(1-q_R)} \left(\frac{2ae}{b(1+q_R)}\right)^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \\
&= O(1) N^{-a} \exp\left[\left(-a + \frac{b}{2}(1+q_R)\left(1 + \log\frac{a}{b} + \log\frac{2}{1+q_R}\right)\right)\frac{1}{1-q_R}\log\frac{N}{R}\right].
\end{aligned}$$

We notice that

$$q \in [0,1] \mapsto -a + \frac{b}{2}(1+q)\left(1+\log\frac{a}{b} + \log\frac{2}{1+q}\right)$$

increases on $[0, 2a/b-1]$, hits 0 at $2a/b-1$ and decreases on $[2a/b-1, 1]$. If b is close enough to a, for all $R \geq N^{1/2-a/4}$,

$$q_R \leq 1/2 + a/4 < 2a/b - 1.$$

We deduce that if $q_R < q_{ab}$,

$$e^{-a \log N/(1-q_R)} q_R^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \sum_{i=0}^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}-1} \frac{1}{i!}\left(\frac{a\log N}{1-q_R}\right)^i \leq N^{-a}\left(\frac{N}{R}\right)^{-\eta'}$$

for some $\eta'' = \eta''(a,b)$. Finally, we use (4.50) to bound

$$\begin{aligned}
&e^{-a\log N/(1-q_R)} \sum_{i \geq \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \frac{1}{i!}\left(q_R\frac{a\log N}{1-q_R}\right)^i \left(1 + O\left(\frac{1}{\log N}\right)\right)^i \\
&\leq e^{-a\log N/(1-q_R)} \left(\frac{2ae}{b(1+q_R)}\right)^{\frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}}
\end{aligned}$$

which is smaller than $N^{-a}(N/R)^{-\eta''}$ according to the previous estimates. Putting things together, we have obtained

$$\mathbb{P}^{D_N}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N, A_N(x\to R) > \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}\right) \leq \frac{G^{D_N}(Nx_0,x)}{G^{D_N}(x,x)}N^{-a}\left(\frac{N}{R}\right)^{-\eta'\wedge\eta''}.$$

Coming back to (4.55), it shows that

$$\mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N, G_N^{b,\varepsilon}(x)^c\right) \leq p(\varepsilon)\frac{G^{D_N}(Nx_0,x)}{G^{D_N}(x,x)}N^{-a}$$

for some $p(\varepsilon) > 0$ depending on a, b, ε going to 0 when $\varepsilon \to 0$. This concludes the proof. □

Proof of Lemma 4.20. We have

$$\mathbb{E}\left[\bar{\mu}^{D,a}_{x_0;N}(\mathbb{C})^2\right] = \frac{\log^2 N}{N^{4-2a}}\sum_{x,y\in\mathbb{Z}^2}\mathbf{1}_{\{x/N,y/N\in D^\varepsilon\}}\mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}},\ell_y^{\tau_{\partial D_N}} \geq ga\log^2 N, G_N^{b,\varepsilon}(x), G_N^{b,\varepsilon}(y)\right). \tag{4.56}$$

The contribution to the above sum of points x, y satisfying $|x-y| \leq N^{1/2-a/4}$ goes to zero. Indeed, thanks to the first moment estimate of Property 4.14, it is at most

$$\frac{\log^2 N}{N^{4-2a}}N^{1-a/2}\sum_{x\in\mathbb{Z}^2}\mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}} \geq ga\log^2 N\right) = \frac{\log N}{N^{1-a/2}}\mathbb{E}\left[\mu^{D_N}_{x_0}(\mathbb{C})\right] \leq C\frac{\log N}{N^{1-a/2}}$$

which goes to zero since $a < 2$. We now take $x, y \in \mathbb{Z}^2$ such that $x/N, y/N \in D^\varepsilon$ and $|x-y| > N^{1/2-a/4}$. The goal is to bound the probability written in (4.56). Take $R \in (2^p)_{p\geq 1} \cap [N^{1/2-a/4}, \varepsilon N]$ so that

$$c\,|x-y| \leq R \leq |x-y|/10$$

with $c > 0$ which may depend on ε and on the domain D. Notice that with this choice of R and because $|x-y| > N^{1/2-a/4}$, the quantity q_R defined in (4.41) stays bounded away from 1. Now, the probability in (4.56) is at most

$$\mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}},\ell_y^{\tau_{\partial D_N}} \geq ga\log^2 N, A_N(x\to R), A_N(y\to R) \leq \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}\right). \tag{4.57}$$

We described the joint law of $(\ell_x^{\tau_{\partial D_N}},\ell_y^{\tau_{\partial D_N}}, A_N(x\to R), A_N(y\to R))$ in the discussion following Lemma 4.21. With the notations therein and with (4.53), the probability (4.57) is equal to

$$\begin{aligned}
&\sum_{\substack{1 \le k_x \le \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R} \\ 1 \le k_y \le \frac{b}{2}\frac{1+q_R}{1-q_R}\log\frac{N}{R}}} \mathbb{P}_{Nx_0}\left(A_N(x \to R) = k_x, A_N(y \to R) = k_y\right) \\
&\qquad \times \mathbb{P}\left(\Gamma(k_x,1), \Gamma(k_y,1) \ge \left(1 + O\left(\frac{1}{\log N}\right)\right) a \log N/(1-q_R)\right) \\
&\le O(1) e^{-2a\log N/(1-q_R)} \sum_{2 \le k \le b\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \mathbb{P}_{Nx_0}\left(A_N(x \to R) + A_N(y \to R) = k\right) \\
&\qquad \times \sum_{i=0}^{k-2} \frac{1}{i!}\left(\frac{2a\log N}{1-q_R}\right)^i
\end{aligned}$$

With (4.49), we get that the probability (4.57) is at most

$$\begin{aligned}
&O(1)e^{-2a\log N/(1-q_R)} \mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) \\
&\times \sum_{2 \le k \le b\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \left(\frac{2q_R}{1+q_R}\right)^{k-1} \sum_{i=0}^{k-2} \frac{1}{i!}\left(\frac{2a\log N}{1-q_R}\right)^i \\
&= O(1)e^{-2a\log N/(1-q_R)} \mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) \frac{q_R}{1-q_R} \sum_{i=0}^{b\frac{1+q_R}{1-q_R}\log\frac{N}{R}-2} \frac{1}{i!}\left(\frac{4aq_R\log N}{1-q_R^2}\right)^i
\end{aligned}$$

by exchanging the two sums. We now use (4.51) with

$$\mu = \frac{4aq_R\log N}{1-q_R^2}\frac{1-q_R}{b(1+q_R)\log(N/R)} = \frac{4a}{b}\frac{1}{(1+q_R)^2}$$

which is bigger than 1 if b is close enough to a (recall that q_R stays bounded away from 1). We obtain that the probability (4.57) is at most

$$\begin{aligned}
&O(1)\mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) q_R e^{-2a\log N/(1-q_R)} \left(e\frac{4a}{b}\frac{1}{(1+q_R)^2}\right)^{b\frac{1+q_R}{1-q_R}\log\frac{N}{R}} \\
&= O(1)\mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) q_R N^{-2a} \\
&\quad \times \exp\left[\Big(-2a + b(1+q_R)\Big(1 + \log\frac{a}{b} + 2\log\frac{2}{1+q_R}\Big)\Big)\frac{1}{1-q_R}\log\frac{N}{R}\right] \\
&\le O(1)\mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) q_R N^{-2a} \\
&\quad \times \exp\left[\Big(-2a + b(1+q_R)\Big(\frac{a}{b} + 2\frac{1-q_R}{1+q_R}\Big)\Big)\frac{1}{1-q_R}\log\frac{N}{R}\right] \\
&= O(1)\mathbb{P}_{Nx_0}\left(\tau_x \wedge \tau_y < \tau_{\partial D_N}\right) q_R N^{-2a}\left(\frac{N}{R}\right)^{2b-a}.
\end{aligned}$$

To wrap things up, we have obtained

$$\frac{\log^2 N}{N^{4-2a}} \sum_{x,y\in\mathbb{Z}^2} \mathbf{1}_{\{x/N,y/N\in D^\varepsilon,|x-y|\geq N^{1/2-a/4}\}} \mathbb{P}_{Nx_0}\left(\ell_x^{\tau_{\partial D_N}}, \ell_y^{\tau_{\partial D_N}} \geq ga\log^2 N, G_N^{b,\varepsilon}(x), G_N^{b,\varepsilon}(y)\right)$$
$$\leq \frac{O(1)}{N^4} \sum_{x,y\in\mathbb{Z}^2} \mathbf{1}_{\{x/N,y/N\in D^\varepsilon,|x-y|\geq N^{1/2-a/4}\}} \log\frac{N}{|x-Nx_0|} \log\frac{N}{|x-y|} \left(\frac{N}{|x-y|}\right)^{2b-a}$$

which is bounded uniformly in N if b is chosen close enough to a so that $2b-a<2$. The energy estimate (4.43) follows as well. This finishes to prove Lemma 4.20. $\square$

4.5 Joint convergence of measures and trajectories

In this section, we state a natural extension of Theorem 4.1 that follows from our approach. Theorem 4.23 below extends Theorem 4.1 in two directions. It considers the joint convergence of the measure together with the associated random walk and it considers finitely many independent random walk trajectories. This generalisation plays a crucial role in the paper [ABJL21] which studies a multiplicative chaos associated to Brownian loop soup.

Let $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D^i, x_i, z_i), i = 1\dots r\} \in \mathcal{S}$ be a collection of domains with starting points and ending points. Let $X^{(i)} = (X_t^{(i)}, 0 \leq t \leq \tau^i), i = 1\dots r$, be r independent random walks distributed according to $\mathbb{P}_{Nx_i,Nz_i}^{D_N^i}$ and, for any $\mathcal{D}'\mathcal{X}'\mathcal{Z}' = \{(D^i, x_i, z_i), i \in I\} \subset \mathcal{D}\mathcal{X}\mathcal{Z}$, recall the definition (4.23) of the measure $\mu_{\mathcal{X}',\mathcal{Z}';N}^{\mathcal{D}',a}$ encoding the set of a-thick points coming from the interaction of the random walks $X^{(i)}$, $i \in I$. We rescale the walk $X^{(i)}$ in time and space and define $X_N^{(i)} = (N^{-1}X_{N^2t}^{(i)}, 0 \leq t \leq \tau^i/N^2)$.

To give a precise meaning of the convergence of the above random walks towards Brownian motion, we need to define a topology on the set $\mathcal{P}$ of càdlàg paths in $\mathbb{R}^2$ with finite durations. If $(\wp_t^1, 0 \leq t \leq T^1)$ and $(\wp_t^2, 0 \leq t \leq T^2)$ are two such paths, we define the distance

$$d(\wp^1, \wp^2) := |\log(T^1/T^2)| + d_{\mathrm{Sk}}((\wp_{tT^1}^1, 0 \leq t \leq 1), (\wp_{tT^2}^2, 0 \leq t \leq 1))$$

where d_{Sk} denotes the Skorokhod distance between càdlàg functions defined on $[0,1]$ with values in $\mathbb{R}^2$ (see e.g. Section 12 of [Bil99]). We equip the set $\mathcal{P}$ with the topology associated to that distance.

Finally, for any Borel set $U \subset \mathbb{R}^2$, we will denote by $\mathfrak{M}(U)$ the set of Borel measures on U equipped with the topology of vague convergence on U.

Theorem 4.23. *For any $\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S}$,*

$$\left(\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}';N}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}, X^{(i)}_N, i = 1 \dots r\right) \in \prod_{\mathcal{D}' \subset \mathcal{D}} \mathfrak{M}\left(\bigcap_{D' \in \mathcal{D}'} D'\right) \times \mathcal{P}^r$$

converges weakly relative to the product topology to

$$\left(e^{c_0 a/g} \mathcal{M}^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}, B^{(i)}, i = 1 \dots r\right),$$

where $B^{(i)}$, $i = 1 \dots r$, are independent Brownian paths distributed according to $\mathbb{P}^{D^i}_{x_i,z_i}$, $i = 1 \dots r$, and $\mathcal{M}^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}$, $\mathcal{D}'\mathcal{X}'\mathcal{Z}'$, are the multipoint Brownian chaos associated to $B^{(i)}$, $i = 1 \dots r$, defined in Section 4.1.2.

We now explain the slight modifications needed in order to prove Theorem 4.23. Firstly, extending the convergence of Theorem 4.23 to the case of finitely many trajectories does not require any modification. Indeed, Proposition 4.6 shows tightness of the sequence $\left(\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}';N}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}\right)$, $N \geq 1$. Let $\left(\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{D}\mathcal{X}\mathcal{Z}\right)$ be any subsequential limit. By Lemma 4.15, we can extract a further subsequence and we obtain an uncountable family $\left(\mu^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}\right)$ of measures. As shown in the proof of Theorem 4.1, this family satisfies Properties (P_1) - (P_4) (up to the multiplicative factor $e^{c_0 a/g}$) which characterise the law of $\left(\mathcal{M}^{\mathcal{D}',a}_{\mathcal{X}',\mathcal{Z}'}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}\right)$ by Theorem 4.5.

It remains to explain how to deal with the joint convergence of the measures together with the underlying random walks. We proceed again by first showing tightness and then identifying the law of the subsequential limits. Tightness is clear since each component converges (we have already seen that the measures converge, and the random walks converge by Donsker invariance principle). The study of the law of the subsequential limit is then very similar to what we have done, as soon as we have an appropriate generalisation of Theorem 4.5 that we explain below in details.

This time, we want to characterise the law of $((\mathcal{M}^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, B^{\mathcal{D}}_{\mathcal{X},\mathcal{Z}}), \mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S})$, where for any $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D^i, x_i, z_i), i = 1 \dots r\} \in \mathcal{S}$, we denoted by $B^{\mathcal{D}}_{\mathcal{X},\mathcal{Z}}$ the collection $(B^{D^i}_{x_i,z_i}, i = 1 \dots r)$ of independent Brownian trajectories associated to the measures. Consider a stochastic process

$$\mathcal{D}\mathcal{X}\mathcal{Z} \in \mathcal{S} \mapsto (\mu^{\mathcal{D},a}_{\mathcal{X},\mathcal{Z}}, B^{\mathcal{D}}_{\mathcal{X},\mathcal{Z}}) \in \mathfrak{M}\left(\bigcap_{D \in \mathcal{D}} D\right) \times \mathcal{P}^{\#\mathcal{D}\mathcal{X}\mathcal{Z}} \tag{4.58}$$

and the following properties:

$(P_1$') (Average value) For all $\mathcal{D}\mathcal{X}\mathcal{Z} = \{(D_i, x_i, z_i), i = 1 \dots r\} \in \mathcal{S}$ and for all Borel set

$A \subset \mathbb{C}$,

$$\mathbb{E}\left[\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}(A)\right] = \int_A dx \int_{\mathsf{a}\in E(a,r)} d\mathsf{a} \prod_{k=1}^r \psi_{x_k,z_k}^{D_k,a_k}(x).$$

(P_2') (Markov property) Let $\mathcal{DXZ} \in \mathcal{S}$, $(D, x_0, z) \in \mathcal{DXZ}$ and let D' be a nice subset of D containing x_0. Let Y be distributed according to $B_{\tau_{\partial D'}}$ under $\mathbb{P}_{x_0,z}^D$. The joint law of $((\mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a}, B_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'}), \mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{DXZ})$ is the same as the joint law given by for all $\mathcal{D}'\mathcal{X}'\mathcal{Z}' \subset \mathcal{DXZ}$,

$$\begin{cases} (\mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a}, B_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'}) \text{ if } (D, x_0, z) \notin \mathcal{D}'\mathcal{X}'\mathcal{Z}', \\ (\mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}}\cup\{(D',x_0,Y)\}} + \mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}}\cup\{(D,Y,z)\}} + \mu^a_{\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}}\cup\{(D',x_0,Y),(D,Y,z)\}}, \tilde{B}_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'}) \text{ otherwise,} \end{cases}$$

where in the second line we denote $\bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}} = \mathcal{D}'\mathcal{X}'\mathcal{Z}' \backslash \{(D, x_0, z)\}$ and $\tilde{B}_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'}$ is the collection of trajectories obtained from $B_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'}$ as follows. For all $(\bar{D}, \bar{x}_0, \bar{z}) \in \bar{\mathcal{D}}\bar{\mathcal{X}}\bar{\mathcal{Z}}$, $B_{\bar{x}_0,\bar{z}}^{\bar{D}}$ is unchanged. $B_{x_0,z}^D$ is replaced by the concatenation of $B_{x_0,Y}^{D'}$ and $B_{Y,z}^D$.

(P_3') (Independence) For all disjoint sets $\mathcal{DXZ}, \mathcal{D}'\mathcal{X}'\mathcal{Z}' \in \mathcal{S}$, the couples $(\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, B_{\mathcal{X},\mathcal{Z}}^{\mathcal{D}})$ and $(\mu_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}',a}, B_{\mathcal{X}',\mathcal{Z}'}^{\mathcal{D}'})$ are independent.

(P_4') (Non-atomicity) For all $\mathcal{DXZ} \in \mathcal{S}$, with probability one, simultaneously for all $x \in \mathbb{C}$, $\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}(\{x\}) = 0$.

(P_5') For all $\{(D, x_0, z)\} \in \mathcal{S}$, $B_{x_0,z}^D \sim \mathbb{P}_{x_0,z}^D$.

Theorem 4.24. *The process $\left((\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, B_{\mathcal{X},\mathcal{Z}}^{\mathcal{D}}), \mathcal{DXZ} \in \mathcal{S}\right)$ from Section 4.1.2 satisfies Properties (P_1')-(P_5'). Moreover, if $\left((\mu_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, B_{\mathcal{X},\mathcal{Z}}^{\mathcal{D}}), \mathcal{DXZ} \in \mathcal{S}\right)$ is another process with target spaces as in* (4.58) *satisfying Properties (P_1')-(P_5'), then it has the same law as $\left((\mathcal{M}_{\mathcal{X},\mathcal{Z}}^{\mathcal{D},a}, B_{\mathcal{X},\mathcal{Z}}^{\mathcal{D}}), \mathcal{DXZ} \in \mathcal{S}\right)$.*

We want to emphasise again that it is crucial that the characterisation does not rely on the measurability of the measures with respect to the Brownian paths.

The proof of Theorem 4.24 is similar to the proof of Theorem 4.5 and we omit it.

Appendix 4.A Multipoint Brownian multiplicative chaos

This section is devoted to the proof of Propositions 4.4, 4.6, 4.7 and 4.8. We start with Proposition 4.6.

Proof of Proposition 4.6. We use the notations of Section 4.1.4 and for $i = 1 \dots r$, we will denote

$$f_i^\varepsilon(x) := |\log \varepsilon| \, \varepsilon^{-a_i} \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(i)} \geq 2a_i |\log \varepsilon|^2 \right\}}.$$

We recall that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[f_i^\varepsilon(x) \right] = \psi_{x_i,z_i}^{D_i,a_i}(x) \tag{4.59}$$

and that we can bound

$$\sup_{\varepsilon > 0} \mathbb{E}\left[f_i^\varepsilon(x) \right] \leq C \psi_{x_i,z_i}^{D_i,a_i}(x) \tag{4.60}$$

for some $C > 0$. See [Jeg20a, Proposition 3.1]. We moreover recall that for all $\eta > 0$, we can decompose

$$f_i^\varepsilon(x) = \rho_i^{\eta,\delta;\varepsilon}(x) + f_i^{\eta,\delta;\varepsilon}(x)$$

where

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{E}\left[\rho_i^{\eta,\delta;\varepsilon}(x) \right] = 0 \tag{4.61}$$

and for all $\delta > 0$, $x \neq y$,

$$\sup_{\varepsilon > 0} \mathbb{E}\left[f_i^{\eta,\delta;\varepsilon}(x) f_i^{\eta,\delta;\varepsilon}(y) \right] \leq C_{\eta,\delta} \, |x-y|^{-a_i-\eta} \tag{4.62}$$

and for all $x \neq y$,

$$\lim_{\varepsilon,\varepsilon' \to 0} \mathbb{E}\left[(f_i^{\eta,\delta;\varepsilon}(x) - f_i^{\eta,\delta;\varepsilon'}(x))(f_i^{\eta,\delta;\varepsilon}(y) - f_i^{\eta,\delta;\varepsilon'}(y)) \right] = 0. \tag{4.63}$$

This follows from the decomposition of the measure using "good" and "bad" events used in [Jeg20a]. Let us detail this decomposition. Let $\delta > 0$ and $b_i > a_i$ be very close to a_i (depending on η). We introduce the good event (see (21) in [Jeg20a])

$$G_\varepsilon(x) := \left\{ \forall r \in [\varepsilon, \delta] \cap \{e^{-n}, n \geq 1\} : \frac{1}{r} L_{x,r}^{(i)} \leq 2b_i |\log r|^2 \right\}$$

and define

$$f_i^{\eta,\delta;\varepsilon}(x) := \mathbf{1}_{G_\varepsilon(x)} f_i^\varepsilon(x) \quad \text{and} \quad \rho_i^{\eta,\delta;\varepsilon}(x) := (1 - \mathbf{1}_{G_\varepsilon(x)}) f_i^\varepsilon(x).$$

Then (4.61) amounts to saying that an a_i-thick point is not b_i-thick (see [Jeg20a, Proposition 3.1]), (4.62) shows that the measure restricted to good events is bounded in L^2 (see (52) of [Jeg20a]) and (4.63) is proved in the course of showing that the measure restricted to good events is Cauchy in L^2 (see [Jeg20a, Proposition 5.1]).

We are now going to prove by induction on $r \geq 1$ the claims *(i)*, *(ii)* and *(iv)* of Proposition 4.6 together with the claim that for all $\alpha < 2 - a$ (recall that $a = a_1 + \dots + a_r$), we can decompose

$$\bigcap_{i=1}^{r} \mathcal{M}_{x_i,z_i}^{D_i,a_i} = \rho_\delta + \mathcal{M}_\delta \tag{4.64}$$

where $\mathbb{E}[\rho_\delta(\mathbb{C})] \to 0$ as $\delta \to 0$ and for all $\delta > 0$

$$\mathbb{E}\left[\int_{\mathbb{C}^2} \frac{1}{|x-y|^\alpha} \mathcal{M}_\delta(dx)\mathcal{M}_\delta(dy)\right] < \infty.$$

This latter claim implies *(v)* by Frostman's lemma. The case $r=1$ follows from [Jeg20a] (in this case, *(ii)* is an empty statement). Let $r \geq 2$ and assume the above results for $r-1$. Let $\alpha, \eta > 0$ be such that $a_r < \alpha < \alpha + \eta < 2 - (a_1 + \cdots + a_{r-1})$. We can decompose

$$\bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i} = \rho_\delta + \mathcal{M}_\delta$$

with $\mathbb{E}[\rho_\delta(\mathbb{C})] \to 0$ as $\delta \to 0$ and for all $\delta > 0$

$$\mathbb{E}\left[\int_{\mathbb{C}^2} \frac{1}{|x-y|^{\alpha+\eta}} \mathcal{M}_\delta(dx)\mathcal{M}_\delta(dy)\right] < \infty. \tag{4.65}$$

(4.62), (4.63) and (4.65) show that for all $A \in \mathcal{B}(\mathbb{C})$, and for all $\delta > 0$,

$$\int_A f_r^{\eta,\delta;\varepsilon}(x)\mathcal{M}_\delta(dx), \varepsilon > 0,$$

is a Cauchy sequence in L^2. This defines a limiting measure $\tilde{\mathcal{M}}_\delta$ which satisfies by Fatou's lemma and (4.62)

$$\mathbb{E}\left[\int_{\mathbb{C}^2} \frac{1}{|x-y|^{\alpha-a_r}} \tilde{\mathcal{M}}_\delta(dx)\tilde{\mathcal{M}}_\delta(dy)\right] < \infty.$$

Moreover,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \mathbb{E}\left[\int_{\mathbb{C}} f_r^\varepsilon(x) \bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx) - \int_{\mathbb{C}} f_r^{\eta,\delta;\varepsilon}(x)\mathcal{M}_\delta(dx)\right] = 0.$$

This shows that for all $A \in \mathcal{B}(\mathbb{C})$, $\int_A f_r^\varepsilon(x) \bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)$ converges in L^1 as $\varepsilon \to 0$ and also shows that the limiting measure can be decomposed as expected in (4.64). This concludes the proof of the convergence of the measure (4.8). This also shows that for all $A \in \mathcal{B}(\mathbb{C})$, we can exchange the expectation and the limit:

$$\mathbb{E}\left[\lim_{\varepsilon\to 0}\int_A f_r^\varepsilon(x) \bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right] = \lim_{\varepsilon\to 0}\int_A \mathbb{E}[f_r^\varepsilon(x)]\, \mathbb{E}\left[\bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right].$$

Now, using *(iv)* for $r-1$, we see that $\mathbb{E}\left[\bigcap_{i=1}^{r-1} \mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right] = \prod_{i=1}^{r-1} \psi_{x_i,z_i}^{D_i,a_i}(x)dx$ and then by dominated convergence theorem and (4.59) and (4.60), we obtain that

$$\mathbb{E}\left[\lim_{\varepsilon\to 0}\int_A f_r^\varepsilon(x)\bigcap_{i=1}^{r-1}\mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right] = \lim_{\varepsilon\to 0}\int_A \mathbb{E}\left[f_r^\varepsilon(x)\right]\prod_{i=1}^{r-1}\psi_{x_i,z_i}^{D_i,a_i}(x)dx$$
$$= \int_A \prod_{i=1}^{r}\psi_{x_i,z_i}^{D_i,a_i}(x)dx.$$

We are now going to show that $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i;\varepsilon}^{D_i,a_i}$ converges to the same limiting measure as (4.8). For this purpose, it is enough to show that for all $A \in \mathcal{B}(\mathbb{C})$,

$$\mathbb{E}\left[\left|\int_A \prod_{i=1}^r f_i^\varepsilon(x)dx - \int_A f_r^\varepsilon(x)\bigcap_{i=1}^{r-1}\mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right|\right]$$

tends to zero as $\varepsilon \to 0$. For each $i = 1\dots r$, consider the decomposition

$$f_i^\varepsilon(x) = \rho_i^{\eta,\delta;\varepsilon}(x) + f_i^{\eta,\delta;\varepsilon}(x)$$

with $\eta > 0$ in (4.62) chosen so that $r\eta + a_1 + \dots a_r < 2$. For $\varepsilon', \delta > 0$, we can bound

$$\mathbb{E}\left[\left|\int_A \prod_{i=1}^r f_i^\varepsilon(x)dx - \int_A f_r^\varepsilon(x)\bigcap_{i=1}^{r-1}\mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right|\right] \tag{4.66}$$
$$\leq \mathbb{E}\left[\left|\int_A f_r^\varepsilon(x)\left(\prod_{i=1}^{r-1} f_i^\varepsilon(x) - \prod_{i=1}^{r-1} f_i^{\varepsilon'}(x)\right)dx\right|\right]$$
$$+\,\mathbb{E}\left[\left|\int_A f_r^\varepsilon(x)\prod_{i=1}^{r-1} f_i^{\varepsilon'}(x)dx - \int_A f_r^\varepsilon(x)\bigcap_{i=1}^{r-1}\mathcal{M}_{x_i,z_i}^{D_i,a_i}(dx)\right|\right].$$

By the case $r-1$, the second right hand side term tends to zero as $\varepsilon' \to 0$. By writing for $i,k = 1\dots r-1$, $\varepsilon_i^k = \varepsilon'$ if $i \leq k-1$ and $\varepsilon_i^k = \varepsilon$ otherwise, we can write

$$\prod_{i=1}^{r-1} f_i^\varepsilon(x) - \prod_{i=1}^{r-1} f_i^{\varepsilon'}(x) = \sum_{k=1}^{r-1}(f_k^\varepsilon(x) - f_k^{\varepsilon'}(x))\prod_{\substack{1\leq i\leq r-1\\ i\neq k}} f_i^{\varepsilon_i^k}(x).$$

By triangle inequality, to bound the first right hand side term of (4.66), it is thus enough to bound

$$\mathbb{E}\left[\left|\int_A (f_1^\varepsilon(x) - f_1^{\varepsilon'}(x))\prod_{i=2}^r f_i^\varepsilon(x)dx\right|\right]$$

and $r-2$ other very similar terms. This is at most

$$\mathbb{E}\left[\left|\int_A (f_1^{\eta,\delta;\varepsilon}(x) - f_1^{\eta,\delta;\varepsilon'}(x)) \prod_{i=2}^r f_i^{\eta,\delta;\varepsilon}(x) dx\right|\right]$$
$$+ \mathbb{E}\left[\left|\int_A (f_1^{\varepsilon}(x) - f_1^{\varepsilon'}(x)) \prod_{i=2}^r f_i^{\varepsilon}(x) dx - \int_A (f_1^{\eta,\delta;\varepsilon}(x) - f_1^{\eta,\delta;\varepsilon'}(x)) \prod_{i=2}^r f_i^{\eta,\delta;\varepsilon}(x) dx\right|\right].$$

By independence and because for all $i = 1 \dots r$, $x \in \mathbb{C}$, $\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{E}\left[\rho_i^{\eta,\delta;\varepsilon}(x)\right] = 0$, dominated convergence theorem (the domination is provided by (4.60)) shows that the $\limsup_{\varepsilon,\varepsilon' \to 0}$ of the second right hand side term goes to zero as $\delta \to 0$. By Cauchy-Schwarz and (4.62), the first term is at most

$$C_{\eta,\delta} \int_{A \times A} \frac{1}{|x-y|^{a_2 + \dots + a_r + (r-1)\eta}} \mathbb{E}\left[(f_1^{\eta,\delta;\varepsilon}(x) - f_1^{\eta,\delta;\varepsilon'}(x))(f_1^{\eta,\delta;\varepsilon}(y) - f_1^{\eta,\delta;\varepsilon'}(y))\right] dx dy$$

which tends to zero as $\varepsilon, \varepsilon' \to 0$ by (4.62), (4.63) and dominated convergence theorem (note that we obtain an integrable domination because $r\eta + a_1 + \dots + a_r < 2$). This concludes the induction proof of *(i)*, *(ii)* and *(iv)*.

Finally, by induction on r, the measurability statement *(iii)* follows from *(ii)*. To conclude the proof, it remains to check *(vi)*. The measurability of the process is clear at the level of approximation, i.e. for all $\varepsilon > 0$,

$$(a_i)_{i=1\dots r} \in \{(\alpha_i)_{i=1\dots r} \in (0,2)^r : \sum \alpha_i < 2\} \mapsto \bigcap_{i=1}^r \mathcal{M}_{x_i,z_i,\varepsilon}^{D_i,a_i}$$

is a measurable process (with appropriate topology). The claim follows from *(i)* since a pointwise limit of measurable functions is measurable. □

Remark 4.25. We now address a remark which will be useful for the proof of Proposition 4.7. The proof of Proposition 4.6 actually shows that for all Borel set $A \subset \mathbb{C}$, $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i;\varepsilon}^{D_i,a_i}(A)$ converges in L^1 as $\varepsilon \to 0$ towards what we denoted $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}(A)$. [Jeg20a] considered not only the spatial configuration of the thick points but also the deviation of the local times by looking at the measure: for all $A \in \mathcal{B}(\mathbb{C})$ and $T \in \mathcal{B}(\mathbb{R} \cup \{+\infty\})$,

$$\bar{\mathcal{M}}_{x_1,z_1;\varepsilon}^{D_1,a_1}(A \times T) := |\log \varepsilon| \, \varepsilon^{-a_1} \int_A \mathbf{1}_{\left\{\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)}} - \sqrt{2a_1}|\log \varepsilon| \in T\right\}} dx.$$

[Jeg20a, Proposition 6.1] shows that for all Borel sets $A \subset \mathbb{C}$ and $T \subset \mathbb{R}$ with $\inf T > -\infty$, $\bar{\mathcal{M}}_{x_1,z_1;\varepsilon}^{D_1,a_1}(A \times T)$ converges in L^1 as $\varepsilon \to 0$ towards

$$\mathcal{M}_{x_1,z_1;\varepsilon}^{D_1,a_1}(A) \int_T \frac{1}{\sqrt{2a_1}} e^{-\sqrt{2a_1} t} dt.$$

Using this result, a straightforward extension of Proposition 4.6 shows similarly that for all Borel sets $A \subset \mathbb{C}$ and $T_i \subset \mathbb{R}$ with $\inf T_i > -\infty$, the following convergence in L^1 holds

$$|\log \varepsilon|^r \, \varepsilon^{-a} \int_A \prod_{i=1}^r \mathbf{1}_{\left\{ \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}^{(i)}} - \sqrt{2a_i} |\log \varepsilon| \in T_i \right\}} dx \xrightarrow[\varepsilon \to 0]{} \bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}(A) \prod_{i=1}^r \int_{T_i} \frac{1}{\sqrt{2a_i}} e^{-\sqrt{2a_i} t_i} dt_i.$$

We are now ready to prove Propositions 4.4 and 4.7.

Proof of Propositions 4.4 and 4.7. To ease the notations, we will restrict ourselves to the case $r = 2$. The general case $r \geq 2$ follows along the same lines. Let $A \subset \mathbb{C}$ be a Borel set. Let $\eta > 0$. We have

$$\begin{aligned}
&|\log \varepsilon| \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} + L_{x,\varepsilon}^{(2)} \geq 2a\varepsilon |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0, L_{x,\varepsilon}^{(2)} > 0 \right\}} \\
&\leq \sum_{\alpha \in \frac{\eta}{|\log \varepsilon|} \mathbb{N}} |\log \varepsilon| \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \in 2\left(\alpha, \alpha + \frac{\eta}{|\log \varepsilon|}\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L_{x,\varepsilon}^{(2)} > 2(a-\alpha)|\log \varepsilon|^2 - g\eta |\log \varepsilon| \right\}} \\
&= \frac{1}{\eta} \int_0^\infty d\alpha \, |\log \varepsilon|^2 \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \in 2\left(\alpha_\varepsilon, \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|}\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L_{x,\varepsilon}^{(2)} > 2(a-\alpha_\varepsilon)|\log \varepsilon|^2 - g\eta |\log \varepsilon| \right\}}
\end{aligned}$$

where $\alpha_\varepsilon = \frac{\eta}{|\log \varepsilon|} \left\lfloor \frac{|\log \varepsilon|}{\eta} \alpha \right\rfloor$. Let K be a large integer. We now have

$$\begin{aligned}
&|\log \varepsilon| \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} + L_{x,\varepsilon}^{(2)} \geq 2a\varepsilon |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0, L_{x,\varepsilon}^{(2)} > 0 \right\}} \\
&\leq \frac{1}{\eta} \int_0^\infty d\alpha \sum_{k=0}^{K-1} \mathbf{1}_{\left\{ \alpha \in \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|} \left[\frac{k}{K}, \frac{k+1}{K}\right) \right\}} \\
&\times |\log \varepsilon|^2 \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \in 2\left(\alpha_\varepsilon, \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|}\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L_{x,\varepsilon}^{(2)} > 2(a-\alpha_\varepsilon)|\log \varepsilon|^2 - 2\eta |\log \varepsilon| \right\}} \\
&\leq \frac{1}{\eta} \int_0^\infty d\alpha \sum_{k=0}^{K-1} \mathbf{1}_{\left\{ \alpha \in \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|} \left[\frac{k}{K}, \frac{k+1}{K}\right) \right\}} |\log \varepsilon|^2 \, \varepsilon^a \\
&\times \int_A dx \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \in 2\left(\alpha - \frac{\eta}{|\log \varepsilon|} \frac{k+1}{K}, \alpha + \frac{\eta}{|\log \varepsilon|}\left(1 - \frac{k}{K}\right)\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L_{x,\varepsilon}^{(2)} > 2(a - \alpha - \frac{\eta}{|\log \varepsilon|} \frac{k}{K}) |\log \varepsilon|^2 - 2\eta |\log \varepsilon| \right\}}.
\end{aligned}$$

For each $\alpha \in (0, a)$ and $k \in \{0, \dots, K-1\}$, $|\log \varepsilon|^2 \, \varepsilon^a$ times

$$\int_A dx \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L_{x,\varepsilon}^{(1)} \in 2\left(\alpha - \frac{\eta}{|\log \varepsilon|} \frac{k+1}{K}, \alpha + \frac{\eta}{|\log \varepsilon|}\left(1 - \frac{k}{K}\right)\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L_{x,\varepsilon}^{(2)} > 2(a - \alpha - \frac{\eta}{|\log \varepsilon|} \frac{k}{K}) |\log \varepsilon|^2 - 2\eta |\log \varepsilon| \right\}}$$

converges in L^1 towards (see Remark 4.25)

$$\begin{aligned}
&\mathcal{M}_{x_0,z_0}^{D_0,\alpha} \cap \mathcal{M}_{x_1,z_1}^{D_1,a-\alpha}(A) e^{\eta(1+k/K)} \int_{-\eta(k+1)/(K\sqrt{2\alpha})}^{\eta(1-k/(K\sqrt{2\alpha}))} e^{-\sqrt{2\alpha} t} dt \\
&= \eta(1 + o(1)) \mathcal{M}_{x_0,z_0}^{D_0,\alpha} \cap \mathcal{M}_{x_1,z_1}^{D_1,a-\alpha}(A)
\end{aligned}$$

where $o(1)$ is independent of α and k and goes to zero as $\eta \to 0$ and then $K \to \infty$. Hence for all $\alpha \in (0, a)$,

$$\frac{1}{\eta} \sum_{k=0}^{K-1} \mathbf{1}_{\left\{\alpha \in \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|}\left[\frac{k}{K}, \frac{k+1}{K}\right)\right\}} |\log \varepsilon|^2 \varepsilon^a$$
$$\times \int_A dx \mathbf{1}_{\left\{\frac{1}{\varepsilon} L^{(1)}_{x,\varepsilon} \in 2\left(\alpha - \frac{\eta}{|\log \varepsilon|}\frac{k+1}{K}, \alpha + \frac{\eta}{|\log \varepsilon|}\left(1 - \frac{k}{K}\right)\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L^{(2)}_{x,\varepsilon} > 2(a - \alpha - \frac{\eta}{|\log \varepsilon|}\frac{k}{K})|\log \varepsilon|^2 - 2\eta |\log \varepsilon|\right\}}$$

converges in L^1 towards

$$(1 + o(1)) \mathcal{M}^{D_0, \alpha}_{x_0, z_0} \cap \mathcal{M}^{D_1, a-\alpha}_{x_1, z_1}(A).$$

If $\alpha > a$, the above term converges in L^1 to zero. We are going to conclude with the following elementary reasoning. If $\alpha \in (0, \infty) \mapsto X^\alpha_\varepsilon, \varepsilon > 0$, are random processes almost surely measurable and defined on the same probability space satisfying: for all $\alpha > 0$, $(X^\alpha_\varepsilon, \varepsilon > 0)$ converges in L^1 and $\int_0^\infty \sup_\varepsilon \mathbb{E}\left[|X^\alpha_\varepsilon|\right] d\alpha < \infty$; then $(\int_0^\infty X^\alpha_\varepsilon d\alpha, \varepsilon > 0)$, converges in L^1. Indeed

$$\limsup_{\varepsilon, \varepsilon' \to 0} \mathbb{E}\left[\left|\int_0^\infty X^\alpha_\varepsilon d\alpha - \int_0^\infty X^\alpha_{\varepsilon'} d\alpha\right|\right] \leq \limsup_{\varepsilon, \varepsilon' \to 0} \int_0^\alpha \mathbb{E}\left[|X^\alpha_\varepsilon - X^\alpha_{\varepsilon'}|\right] d\alpha$$

which vanishes by dominated convergence theorem. We apply this to our specific case for which we have already proven the desired pointwise convergence and for which the domination follows from (4.60). It implies that

$$\int_0^\infty d\alpha \frac{1}{\eta} \sum_{k=0}^{K-1} \mathbf{1}_{\left\{\alpha \in \alpha_\varepsilon + \frac{\eta}{|\log \varepsilon|}\left[\frac{k}{K}, \frac{k+1}{K}\right)\right\}} |\log \varepsilon|^2 \varepsilon^a$$
$$\times \int_A dx \mathbf{1}_{\left\{\frac{1}{\varepsilon} L^{(1)}_{x,\varepsilon} \in 2\left(\alpha - \frac{\eta}{|\log \varepsilon|}\frac{k+1}{K}, \alpha + \frac{\eta}{|\log \varepsilon|}\left(1 - \frac{k}{K}\right)\right] |\log \varepsilon|^2, \frac{1}{\varepsilon} L^{(2)}_{x,\varepsilon} > 2(a - \alpha - \frac{\eta}{|\log \varepsilon|}\frac{k}{K})|\log \varepsilon|^2 - 2\eta |\log \varepsilon|\right\}}$$

converges in L^1 towards

$$(1 + o(1)) \int_0^a \mathcal{M}^{D_0, \alpha}_{x_0, z_0} \cap \mathcal{M}^{D_1, a-\alpha}_{x_1, z_1}(A) d\alpha.$$

By letting $\eta \to 0$ and then $K \to \infty$, we obtain the desired upper bound:

$$|\log \varepsilon| \varepsilon^a \int_A dx \mathbf{1}_{\left\{L^{(1)}_{x,\varepsilon} + L^{(2)}_{x,\varepsilon} \geq 2a\varepsilon |\log \varepsilon|^2, L^{(1)}_{x,\varepsilon} > 0, L^{(2)}_{x,\varepsilon} > 0\right\}}$$

is smaller or equal than a sequence which converges in L^1 towards

$$\int_0^a \mathcal{M}^{D_0, \alpha}_{x_0, z_0} \cap \mathcal{M}^{D_1, a-\alpha}_{x_1, z_1}(A) d\alpha.$$

The lower bound is obtained along the same lines and we have shown that for all Borel set $A \subset \mathbb{C}$, the following convergence in L^1 holds:

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \, \varepsilon^a \int_A dx \mathbf{1}_{\left\{ L_{x,\varepsilon}^{(1)} + L_{x,\varepsilon}^{(2)} \geq 2a\varepsilon |\log \varepsilon|^2, L_{x,\varepsilon}^{(1)} > 0, L_{x,\varepsilon}^{(2)} > 0 \right\}} = \int_0^a \mathcal{M}_{x_0,z_0}^{D_0,\alpha} \cap \mathcal{M}_{x_1,z_1}^{D_1,a-\alpha}(A) d\alpha.$$

This concludes the proof of Propositions 4.4 and 4.7. □

We finish with a proof of Proposition 4.8.

Proof of Proposition 4.8. Recall that $B_{x_i,z_i}^{D_i}$ is a trajectory distributed according to $\mathbb{P}_{x_i,z_i}^{D_i}$ and that $L_{x,\varepsilon}^{(i)}$ denotes the associated local times of circles $\partial D(x,\varepsilon)$. Let $x \in D$. The key point of the proof is that the law of $B_{x_i,z_i}^{D_i}$ conditioned on $\{\frac{1}{\varepsilon} L_{x,\varepsilon}^{(i)} \geq 2a_i |\log \varepsilon|^2\}$ converges weakly to the law of $B_{x_i,x}^{D_i} \wedge \Xi_x^{D_i,a_i} \wedge B_{x,z_i}$ as $\varepsilon \to 0$. This fact was already proven in [BBK94] (see Proposition 5.1 therein). From this and the convergence of $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i;\varepsilon}^{D_i,a_i}$ to $\bigcap_{i=1}^r \mathcal{M}_{x_i,z_i}^{D_i,a_i}$ (Proposition 4.6), one easily obtains Proposition 4.8. See the proof of [Jeg20a, Proposition 6.2] for more details in the case of one single trajectory (no new input is required in the case of several trajectories). □

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