Categorical symplectic geometry seminar

Plan of the talks

Talk 1: Categorical symplectic geometry seminar introduction (17/10). – Thibaut Mazuir

Talk 2: Introduction to symplectic geometry (31/10). – Wennan Zhang

(i) Definition of a symplectic vector space and example of $\mathbb{R}^{2n}$, definition of a symplectic manifold and examples ($\mathbb{R}^{2n}$, cotangent bundle of a smooth manifold with the canonical form, complex projective space with the Fubini-Study form), Darboux theorem

Sections 5.1 to 5.3 in [AD14]*; Example 1.1 in* [AB22]

(ii) Definition of a Lagrangian submanifold and examples in the the three previous cases, Lagrangian neighborhood theorem

Example 1.2 in [AB22]*; Sections 2.3 and 3.4, and Theorem 3.4.13 in* [MS17]

(iii) Definition of a symplectomorphism, definition of Hamiltonian vector fields and Hamiltonian flows

Section 5.4 in [AD14]

(iv) Definition of almost complex structures, definition of tameness and compatibility, example in $\mathbb{R}^{2n}$, contractibility of the space of ω -compatible resp. ω-tame almost complex structures

Section 4.1 in [MS17]

(v) List of open problems in symplectic topology

Sections 13.4 and 13.8 in [Eli19]

Talk 3: Introduction to pseudo-holomorphic curves (07/11). – Kanishka Katipearachchi

This talk consists in presenting a summary of [Oli20]. *We refer to* [MS12] *for the definitions and notions that are not recalled in this paper.*

(i) Definition of a Riemann surface, definition of a pseudo-holomorphic curve with the Cauchy-Riemann equation, energy and area minimization property of a pseudo-holomorphic curve

(ii) Compactness theorem for sequences of pseudo-holomorphic curves in a compact symplectic manifold (Theorem 1 in [Oli20]), bubbling example in details (Section 3.2 in [Oli20]), sketch of the proof of each item of the Theorem, remark about the formation of bubble trees in general

(iii) Definition of a Banach manifold, introduction of the Cauchy-Riemann operator and the Banach manifold viewpoint on pseudo-holomorphic curves, the linearized operator as a Fredholm operator

(iv) Definition of the moduli space of somewhere injective parametrized pseudo-holomorphic curves and finite-dimensional manifold structure theorem with dimension formula for generic almost complex structures

(v) Application: sketch of the proof of Gromov's non-squeezing theorem

Talk 4: Morse homology (14/11). – Nicolas Alexander Weiss

This talk consists in presenting selected pieces of Part I of [AD14].

(i) Definition of a critical point and of the Hessian at a critical point, definition of a Morse function, Morse lemma and definition of the index of a critical point
Examples: local maximum, local minimum and saddle point in dimension 2; the height function on the torus

(ii) Definition of a pseudo-gradient, of the stable and unstable manifolds with dimension formulae and of level sets, explicit representation of these notions in a Morse chart

(iii) Morse trajectories converge to critical points at infinity, the Smale condition, moduli space of Morse trajectories and dimension formula
Example: height function on a tilted torus

(iv) Definition of the moduli spaces of broken trajectories and compactness property, definition of a chain complex and of a chain map, definition of the Morse complex and proof that the Morse differential is indeed a differential

(v) Morse homology is isomorphic to singular homology without proof, and statement of the Morse inequalities

See Section 2 of [Sal90] *for item* (v)

Talk 5: Hamiltonian Floer homology (21/11). – Léo Mousseau

The goal of this talk is to present an outline of the construction of Hamiltonian Floer homology in Part 2 of [AD14]

(i) Definition of a nondegenerate orbit of a time-dependent Hamiltonian, statement of the Arnold conjecture, two assumptions on the symplectic form and the first Chern class

Section 6.1 of [AD14]

(ii) Definition of the space of contractible loops and of the action functional, the critical points of the action functional are the periodic orbits of the Hamiltonian

Section 6.3 of [AD14]

(iii) Gradient of the action functional and the Floer equation, definition of the energy, Floer trajectories always connect periodic orbits, definition of the moduli spaces of Floer trajectories

Sections 6.4 and 6.5 of [AD14]

(iv) Quick outline of the definition of the Maslov index of a nondegenerate orbit in three steps, dimension formula for the moduli spaces of Floer trajectories

Section 7 of [AD14]

(v) Definition of a regular pair (hamiltonian,almost complex structure), definition of the Floer chain complex and proof that $\partial^2 = 0$ using the moduli spaces of broken trajectories

Section 9 of [AD14]

(vi) Idea of the construction of the isomorphism between Morse and Floer homology

Section 3.5 of [Sal99]

(vii) Proof of Arnold conjecture

Talk 6: Lagrangian Floer homology (28/11). – David Suchodoll

The goal of this talk is to give an outline of the construction of Lagrangian Floer homology, the technical assumptions necessary for its definition as well as somme applications and examples, as presented in [Aur14].

(i) Definition of the Novikov ring and field, definition of the underlying vector space and differential of the Floer cochain complex and statement of Theorem 1.5, Remark 1.7 about the choice of coefficients and the exact case

Section 1.2 of [Aur14]

(ii) Definition of the Maslov index as the winding number of the square of the determinant map
Definition of the Maslov index of a pseudo-holomorphic strip, statement of the two assumptions needed to define a $\mathbb{Z}$-grading and quick discussion on the grading by a finite cyclic group

Section 1.3 of [Aur14] *and Section 7 of* [AD14] *for recollections*

(iii) Choice of a Hamiltonian perturbation term and of a 1-parameter family of almost-complex structures, new action functional
Updated definition of the moduli spaces of pseudo-holomorphic strips, discussion on two equivalent viewpoints in Remark 1.10
Existence of generic pairs (H, J)

Section 1.4 of [Aur14] *and Section 7.1 of* [AS19]

(iv) Description of the three types of phenomena occuring in the compactification: strip breaking, disc bubbling and sphere bubbling

Discussion of the assumptions to avoid sphere and disc bubbling
Sketch of Example 1.11 for disc bubbling

Sections 1.4 and 1.5 of [Aur14]

(v) Sketch of the proof of the Theorem 1.5, sketch of the proof of the fact that Lagrangian Floer homology does not depend on the pair (H, J)

Section 1.5 of [Aur14]

(vi) Example 1.12 of [Aur14] comparing Lagrangian Floer theory to Morse theory, list of properties in Section 3.1 of [Smi15], proof of Arnold conjecture for Lagrangian submanifolds

Talk 7: A-infinity categories (05/12). – Maftuna Samatboeva

(i) Differential graded viewpoint on chain complexes, definition of a graded associative algebra and of a dg algebra

Lemma I.2.12 and Subsection II.1.3.1 in [Maz]

(ii) Definition of an A_∞-algebra, proof that it induces a graded associative algebra structure on the level of homology, definition of an A_∞-morphism, proof that it induces a morphism of graded associative algebras

Definitions VI.2.4 and VI.2.6 in [Maz] *(see* (v) *for signs) and illustrate with drawings*

(iii) Statement of the homotopy transfer theorem

Introduction of [Mar06]

(iv) Definition of a category and of a functor, examples

Subsection I.1.1 in [Maz]

(v) Definition of an A_∞-category and of an A_∞-functor, composition of A_∞-functors, discussion on the different notions of identity morphisms

Sections I.1 and I.2 of [Sei08]

(vi) Definition of the A_∞-category of pre-natural transformations, proof that it is indeed an A_∞-category and remarks

Section I.1 of [Sei08]

Talk 8: The operadic principle (12/12). – Guillaume Laplante-Anfossi

(i) Definition of an operad and of an algebra over an operad, examples

(ii) Definition of the associahedra, explicit construction and some combinatorial properties, the compactified moduli spaces of disks with marked boundary points "geometrically realize" the associahedra

(iii) Definition of an operadic bimodule, example of M_∞

(iv) Definition of the multiplihedra, explicit construction and some combinatorial properties, the compactified moduli spaces of quilted disks with marked boundary points "geometrically realize" the multiplihedra

Talk 9: Fukaya categories (19/12). – David Suchodoll

(i) Outline of the construction of the Fukaya category

Section 2 of [Aur14]

(ii) Remarks about cohomological units, weaker technical assumptions as monotonicity, and the notion of curvature when disc bubbling is not precluded

Remark 2.12 of [Aur14] *and Subsection 3.3.4 of* [Smi15]

(iii) If time permits: definition of Morse flow trees, equality between moduli spaces of Morse flow trees and moduli spaces of pseudo-holomorphic discs, equivalence between the topological A_∞-category of Morse functions and the Fukaya category

Introduction and Section 1 of [FO97]

Talk 10: The derived Fukaya category (09/01). – Leonard Leass

(i) Definition of a short exact sequence of chain complexes, statement of the induced long exact sequence

Section I.2.3 in [Maz]

(ii) Definition of the mapping cone of a chain map and relation to the standard mapping cone of topological spaces, definition of an exact triangle of chain complexes and induced long exact sequence

Section 1.5, Definition 10.1.3 and Corollary 10.1.4 in [Wei94]

(iii) Definition of the A_∞-category of twisted complexes associated to an A_∞-category, definition of abstract mapping cones and definition of exact triangles in an A_∞-category

Section 3.2 in [Aur14]

(iv) Definition of a triangulated A_∞-category, A_∞-categories of twisted complexes are triangulated

Subsection 2.1.1 in [Ram]

(v) Definition of the derived category of an A_∞- category, intuition on its structure of triangulated category without details, remark on the split closed derived category, statement of the homological mirror symmetry conjecture

Item 3) in [Aur], *Section 10.2 in* [Wei94], *Section 3.2 in* [Ram]

(vi) Dehn twists induce exact triangles for Fukaya categories

Subsection 3.3.1 in [Aur14]

TALK 11: The work of Biran and Cornea (23/01). – Muhammed E. Gülen

(i) Recollections on Floer homology, technical assumptions of the paper, definitions on Lagrangian cobordisms and statement of Theorem 2.2.1 in [BC13]

Sections 2 and 3 in [BC13]

(ii) Outline of the proof of Theorem 2.2.1

Section 4 in [BC13]

(iii) The trace of a Lagrangian surgery induces a Lagrangian cobordism

Subsection 6.1 in [BC13]

(iv) Statement of Theorem A in [BC14] without proof

(v) Statement of Theorem A in [BC21] and discussion of the technical assumptions and main ideas: marked, unobstructed and immersed Lagrangian, cabling equivalence relation and surgery models assumption

Section 1 in [BC21]

TALK 12: Functoriality between Fukaya categories (30/01). – Thibaut Mazuir

TALK 13: Pseudo-holomorphic foams and Hamiltonian actions in Floer theory (06/02). – Guillem Cazassus

References

[AB22] Mohammed Abouzaid and Nathaniel Bottman. Functoriality in categorical symplectic geometry, 2022. arXiv:2210.11159.

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology. Translated from the French by Reinie Erné.* Universitext. London: Springer; Les Ulis: EDP Sciences, 2014.

[AS19] Alberto Abbondandolo and Felix Schlenk. Floer homologies, with applications. *Jahresber. Dtsch. Math.-Ver.*, 121(3):155–238, 2019.

[Aur] Denis Auroux. Mirror Symmetry: Lecture 18. Lecture notes available at `https://math.berkeley.edu/~auroux/18.969-S09/mirrorsymm-lect18.pdf`.

[Aur14] Denis Auroux. A beginner's introduction to Fukaya categories. In *Contact and symplectic topology. Proceedings of the CAST summer schools and conferences, Nantes, France, June 2011 and Budapest, Hungary, July 2012*, pages 85–136. Cham: Springer; Budapest: János Bolyai Mathematical Society, 2014.

[BC13] Paul Biran and Octav Cornea. Lagrangian cobordism. I. *J. Amer. Math. Soc.*, 26(2):295–340, 2013.

[BC14] Paul Biran and Octav Cornea. Lagrangian cobordism and Fukaya categories. *Geom. Funct. Anal.*, 24(6):1731–1830, 2014.

[BC21] Paul Biran and Octav Cornea. A Lagrangian pictionary. *Kyoto J. Math.*, 61(2):399–493, 2021.

[Cal22] Maxine Elena Calle. An introduction to symplectic geometry for lagrangian floer homology, 2022. Masters dissertation.

[Eli19] Yakov Eliashberg. Rigid and flexible facets of symplectic topology. In *Geometry in history*, pages 493–514. Springer, Cham, 2019.

[FO97] Kenji Fukaya and Yong-Geun Oh. Zero-loop open strings in the cotangent bundle and Morse homotopy. *Asian J. Math.*, 1(1):96–180, 1997.

[Mar06] Martin Markl. Transferring A_∞ (strongly homotopy associative) structures. *Rend. Circ. Mat. Palermo (2) Suppl.*, (79):139–151, 2006.

[Maz] Thibaut Mazuir. Introduction to algebraic operads. Lectures notes available at `http://www.normalesup.org/~mazuir/files/lecturenotes-6.pdf`.

[MS12] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, second edition, 2012.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.

[Oli20] Gonçalo Oliveira. Pseudo-holomorphic curves: a very quick overview. *Complex Manifolds*, 7(1):215–229, 2020.

[Ped17] Andrés Pedroza. A quick view of lagrangian floer homology, 2017. arXiv:1701.02293.

[Ram] Felipe Espreafico G. Ramos. Mirror Symmetry and Fukaya Categories. Draft notes available at `https://mathi.uni-heidelberg.de/~framos/Fukaya.pdf`.

[Sal90] Dietmar Salamon. Morse theory, the Conley index and Floer homology. *Bull. London Math. Soc.*, 22(2):113–140, 1990.

[Sal99] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[Sei08] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Smi15] Ivan Smith. A symplectic prolegomenon. *Bull. Am. Math. Soc., New Ser.*, 52(3):415–464, 2015.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.