Introduction to algebraic operads

Homework 2

Due 11:59PM February 18. *You can either give me your copy by February 16 course or send a scanned version in a* unique and well readable *PDF file to thibaut.mazuir@hu-berlin.de* .

Difficult questions are indicated by a (★) *or a* (★★).

Problem 1: Inhomogeneous Koszul duality (12 pt)

We define a quadratic linear data (V, R) *to be a vector space* V *together with a linear subspace* $R \subset V \oplus V^{\otimes 2}$. *We denote* $qR \subset V^{\otimes 2}$ *the image of* R *under the projection* $q : V \oplus V^{\otimes 2} \to V^{\otimes 2}$. *The data* (V, qR) *is then in particular quadratic.*

1 pt **1.** Prove that under the assumption

$$(\mathrm{ql}_1) \qquad R \cap V = \{0\}$$

there exists a linear map $\phi : qR \to V$ such that $R = \{x - \phi(x),\ x \in qR\}$.

For a quadratic linear data (V, R), *we denote*

$$A = A(V, R) := T(V)/\langle R\rangle \qquad qA := A(V, qR) = T(V)/\langle qR\rangle \ .$$

In particular qA *is the quadratic algebra associated to the quadratic data* (V, qR). *Under the assumption* (ql_1), *we define the map* $\tilde{\phi} : (qA)^{¡} \to sV$ *as*

$$\tilde{\phi} : (qA)^{¡} = C(sV, s^2 qR) \twoheadrightarrow s^2 qR \xrightarrow{s^{-2}} qR \xrightarrow{\phi} V \xrightarrow{s} sV \ .$$

2 pt **2.** Prove that there exists a unique coderivation $d_\phi : (qA)^{¡} \to T^c(sV)$ which extends the linear map $\tilde{\phi}$.

2 pt **3.** Under the assumption (ql_1), prove that if

$$(R \otimes V + V \otimes R) \cap V^{\otimes 2} \subset qR$$

then $\mathrm{Im}(d_\phi) \subset (qA)^{¡} \subset T^c(sV)$.

The linear map d_ϕ *then defines a coderivation on* $(qA)^{¡}$.

2 pt **4.** Under the assumption (ql_1), prove that if

$$(\mathrm{ql}_2) \qquad (R \otimes V + V \otimes R) \cap V^{\otimes 2} \subset R \cap V^{\otimes 2}$$

then the coderivation d_ϕ squares to zero, $d_\phi^2 = 0$.

2 pt **5.** Prove that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ admits a quadratic linear presentation satisfying conditions (ql_1) and (ql_2).

Following question 4, if a quadratic linear data (V, R) *satisfies conditions* (ql_1) *and* (ql_2)*, we can define the* Koszul dual coalgebra *of* $A = A(V, R)$ *as the dg coalgebra*

$$A^{i} := ((qA)^{i}, d_{\phi}) = (C(sV, s^2 qR), d_{\phi}) \ .$$

6. Prove that the Koszul dual coalgebra of the universal enveloping algebra $U(\mathfrak{g})$ is the cofree cocommutative coalgebra $\Lambda^c(s\mathfrak{g}) \subset T^c(s\mathfrak{g})$ endowed with the Chevalley-Eilenberg differential 3 pt

$$d_{\phi}(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n \ .$$

where $x_1 \wedge \cdots \wedge x_n := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) s x_{\sigma(1)} \otimes \cdots \otimes s x_{\sigma(n)}$.

Problem 2: Higher morphisms between A_∞-algebras (26 pt)

For $n \geqslant 0$*, we define* $\mathbf{\Delta}^n$ *to be the graded vector space*

$$\mathbf{\Delta}^n = \bigoplus_{0 \leqslant i_0 < \cdots < i_k \leqslant n} \mathbb{K}[i_0 < \cdots < i_k]$$

generated by the increasing sequences of $[[0, n]]$*. Its grading is defined as*

$$| \ [i_0 < \cdots < i_k] \ | = k \ .$$

We endow this graded vector space with the following degree -1 and degree 0 linear maps:

$$\partial_{\mathbf{\Delta}^n}([i_0 < \cdots < i_k]) := \sum_{j=0}^{k} (-1)^j [i_0 < \cdots < \widehat{i_j} < \cdots < i_k] \ ,$$

$$\Delta_{\mathbf{\Delta}^n}([i_0 < \cdots < i_k]) := \sum_{j=0}^{k} [i_0 < \cdots < i_j] \otimes [i_j < \cdots < i_k] \ .$$

1. Prove that $(\mathbf{\Delta}^n, \partial_{\mathbf{\Delta}^n}, \Delta_{\mathbf{\Delta}^n})$ is a dg coalgebra. 2 pt

We point out that the combinatorics of the dg coalgebra $\mathbf{\Delta}^n$ *can be easily understood on the* standard n-simplex *defined as the convex hull*

$$\Delta^n = \mathrm{Conv}((\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0), 0 \leqslant k \leqslant n) \subset \mathbb{R}^n$$

by labeling the vertex whose first k *coordinates are equal to 1 by* $[k]$ *and by labeling the face whose vertices are* $[i_0], \ldots, [i_k]$ *with* $i_0 < \cdots < i_k$ *by* $[i_0 < \cdots < i_k]$*. We illustrate this in* Figure 1*. We will thereby sometimes denote an increasing sequence of* $[[0, n]]$ *as a face* $I \subset \Delta^n$ *in the rest of this problem. We then have in particular that* $|I| = \dim(I)$.

2. Prove that for every $n \geqslant 0$ there exist morphisms of dg coalgebras 3 pt

$$\delta_i^n : \mathbf{\Delta}^{n-1} \to \mathbf{\Delta}^n \ , 0 \leqslant i \leqslant n \ ,$$

$$\sigma_i^n : \mathbf{\Delta}^{n+1} \to \mathbf{\Delta}^n \ , 0 \leqslant i \leqslant n \ ,$$

that satisfy the following equalities

$$\begin{cases} \delta_j \delta_i = \delta_i \delta_{j-1} & \text{for } i < j \ , \\ \sigma_j \delta_i = \delta_i \sigma_{j-1} & \text{for } i < j \ , \\ \sigma_j \delta_i = \delta_{i-1} \sigma_j & \text{for } i > j+1 \ , \\ \sigma_j \sigma_i = \sigma_i \sigma_{j+1} & \text{for } i \leqslant j \ , \\ \sigma_j \delta_j = \sigma_j \delta_{j+1} = \mathrm{id} \ . \end{cases}$$

where we omit the upper script n in σ_i and δ_i.

We then say that the collection of dg coalgebras $\{\mathbf{\Delta}^n\}_{n\geqslant 0}$ *forms a* cosimplicial dg coalgebra.

1 pt **3.** Let C and C' be two dg coalgebras. Prove that their tensor product $C \otimes C'$ can naturally be endowed with a dg coalgebra structure.

Let $f_0, f_1 : C \longrightarrow C'$ *be two morphisms of dg coalgebras. We define a* homotopy *from* f_0 *to* f_1 *to be a degree +1 linear map* $h : C \longrightarrow C'$ *such that*

$$\begin{aligned} \Delta_{C'} h &= (f_0 \otimes h + h \otimes f_1)\Delta_C \\ [\partial, h] &= f_1 - f_0 \ , \end{aligned}$$

where we recall that for a linear map $f : C \longrightarrow C'$ *of degree* $|f|$*, we denote* $[\partial, f] := \partial_{C'} f - (-1)^{|f|} f \partial_C$.

2 pt **4.** Prove that the data of two morphisms of dg coalgebras $f_0, f_1 : C \longrightarrow C'$ and of a homotopy between them is equivalent to the datum of a morphism of dg coalgebras $\mathbf{\Delta}^1 \otimes C \longrightarrow C'$.

2 pt **5.** Prove that the datum of a morphism of dg coalgebras $\mathbf{\Delta}^n \otimes C \longrightarrow C'$ is equivalent to the data of linear maps $f_{[i_0 < \cdots < i_k]} : C \longrightarrow C'$ of degree k for every $[i_0 < \cdots < i_k] \subset \Delta^n$, that satisfy the following equations:

$$\begin{aligned} [\partial, f_{[i_0 < \cdots < i_k]}] &= \sum_{j=0}^{k} (-1)^j f_{[i_0 < \cdots < \widehat{i_j} < \cdots < i_k]} \ , \\ \Delta_{C'} f_{[i_0 < \cdots < i_k]} &= \sum_{j=0}^{k} (f_{[i_0 < \cdots < i_j]} \otimes f_{[i_j < \cdots < i_k]})\Delta_C \ . \end{aligned}$$

We recall that an A_∞-algebra *structure on a dg vector space* A *can be defined as a codifferential* D_A *on the reduced tensor coalgebra* $\bar{T}(sA)$ *such that the restriction of* D *to the summand* sA *is equal to* ∂_{sA}*. An* A_∞-morphism *between two* A_∞*-algebras* A *and* B *can then be defined as a morphism of dg coalgebras* $(\bar{T}(sA), D_A) \longrightarrow (\bar{T}(sB), D_B)$.

3 pt **6.** Prove that an homotopy H between two A_∞-morphisms $F, G : (\bar{T}(sA), D_A) \longrightarrow (\bar{T}(sB), D_B)$ can be equivalently defined as a collection of linear maps $h_n : A^{\otimes n} \longrightarrow B$ of degree $-n$ for $n \geqslant 1$

such that

$$[\partial, h_n] = g_n - f_n + \sum_{\substack{i_1+i_2+i_3=n\\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1}\otimes m^A_{i_2}\otimes \mathrm{id}^{\otimes i_3})$$
$$+ \sum_{\substack{i_1+\cdots+i_s+l\\ +j_1+\cdots+j_t=n\\ s+1+t\geqslant 2}} \pm m^B_{s+1+t}(f_{i_1}\otimes\cdots\otimes f_{i_s}\otimes h_l\otimes g_{j_1}\otimes\cdots\otimes g_{j_t})\ .$$

where m^A_n and m^B_n are the operations of the A_∞-algebras A and B, and $f_n, g_n : A^{\otimes n} \to B$ are the operations of the A_∞-morphisms F and G. (The explicit signs need not be computed)

Let I be a face of Δ^n. An overlapping s-partition *of I is defined to be a sequence of faces $(I_l)_{1\leqslant \ell\leqslant s}$ of I such that*

(i) the union of this sequence of faces (seen as increasing sequences of $[[0,n]]$) is I, i.e. $\cup_{1\leqslant\ell\leqslant s} I_l = I$;
(ii) for all $1\leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0<1<2]$ is for instance

$$[0<1<2] = [0]\cup[0]\cup[0<1]\cup[1]\cup[1<2]\cup[2]\ .$$

An overlapping 3-partition for $[0<1<2<3<4<5]$ is for instance

$$[0<1<2<3<4<5] = [0<1]\cup[1<2<3]\cup[3<4<5]\ .$$

7. Let A and B be two A_∞-algebras. Prove that a morphism of dg coalgebras $\mathbf{\Delta}^n\otimes \bar{T}(sA)\to \bar{T}(sB)$ is equivalent to a collection of linear maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1-m+|I|$ for $I\subset\Delta^n$ and $m\geqslant 1$, that satisfy 4 pt

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f^{(m)}_{\partial_j I} + (-1)^{|I|}\sum_{\substack{i_1+i_2+i_3=m\\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1}\otimes m^A_{i_2}\otimes\mathrm{id}^{\otimes i_3})$$
$$+ \sum_{\substack{i_1+\cdots+i_s=m\\ I_1\cup\cdots\cup I_s=I\\ s\geqslant 2}} \pm m^B_s(f^{(i_1)}_{I_1}\otimes\cdots\otimes f^{(i_s)}_{I_s})\ ,$$

where the last sum runs over all overlapping s-partitions $I_1\cup\cdots\cup I_s = I$ of I for $s\geqslant 2$.

*A morphism of dg coalgebras $\mathbf{\Delta}^n\otimes\bar{T}(sA)\to\bar{T}(sB)$ will be called a n-*morphism *from A to B.*

8. ($\star$) Compute the explicit signs in the equations of question 7. 3 pt

*We define the (k,n)-*horn Λ^k_n *to be the simplicial subcomplex of the standard n-simplex Δ^n obtained by removing the face $[0<\cdots<n]$ as well as the face $[0<\cdots<\hat{k}<\cdots<n]$ in Δ^n. This is illustrated in* Figure 2. *A* (k,n)-horn of higher morphisms *between two A_∞-algebras A and B is then defined to be a collection of operations $f_I^{(m)} : A^{\otimes m}\longrightarrow B$ of degree $1-m+|I|$ for $I\subset\Lambda^k_n$ and $m\geqslant 1$ that satisfy the equations of question 7.*

9. ($\star\star$) Prove that for all $0<k<n$, every (k,n)-horn of higher morphisms from A to B can 6 pt

be filled to a n-morphism from A to B.

In other words, we have proven that the simplicial set of higher morphisms *from A to B*

$$\mathrm{HOM}_{\infty\text{-}\mathrm{A}_\infty\text{-}\mathtt{alg}}(A,B)_n := \mathrm{Hom}_{\mathtt{dg\ cog}}(\boldsymbol{\Delta}^n \otimes \overline{T}(sA), \overline{T}(sB))$$

is an ∞-category.

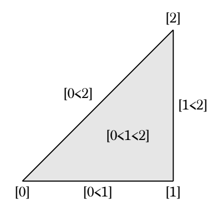

FIGURE 1. The standard 2-simplex Δ^2

The $(1,2)$-horn $\Lambda^1_2 \subset \Delta^2$ The $(0,2)$-horn $\Lambda^0_2 \subset \Delta^2$

FIGURE 2