Introduction to algebraic operads

Exercise sheet 2: Standard algebraic structures

Exercise 1 (An equivalence of categories). Prove that the category of nonunital associative algebras is equivalent to the category of augmented associative algebras.

Exercise 2 (Tensor coalgebra). *Let V be a vector space and $\overline{T}^c(V)$ the free reduced tensor coalgebra of V. We define a coderivation of a coalgebra C to be a map $D : C \to C$ such that $(D \otimes \mathrm{id} + \mathrm{id} \otimes D)\Delta_C = \Delta_C D$.*

1. Prove that there is a correspondence

$$\left\{ \begin{array}{c} \text{collections of linear maps} \\ m_n : V^{\otimes n} \to V \ , \ n \geqslant 1 \end{array} \right\} \longleftrightarrow \left\{ \text{ coderivations } D \text{ of } \overline{T}^c(V) \ \right\} .$$

2. Let W be a vector space. Prove that there is a correspondence

$$\left\{ \begin{array}{c} \text{collections of linear maps} \\ f_n : V^{\otimes n} \to W \ , \ n \geqslant 1 \end{array} \right\} \longleftrightarrow \left\{ \text{ morphisms of coalgebras } \overline{T}^c(V) \to \overline{T}^c(W) \ \right\} .$$

Exercise 3 (Minimal models).

1. Prove that the cdg algebra $(\Lambda(v_1, v_2, v_3), \partial)$ defined by $\deg(v_i) = 1$, $\partial(v_1) = v_2 v_3$, $\partial(v_2) = v_3 v_1$ and $\partial(v_3) = v_1 v_2$ is not a Sullivan algebra, and compute a minimal Sullivan model for this cdg algebra.

Recall that the cohomology algebra of spheres is given by $H^*(\mathbb{S}^n, \mathbb{Q}) = \mathbb{Q}[v]/\langle v^2 = 0\rangle$ where $|v| = n$. We will moreover accept that the spheres are formal in the next question.

2. Compute the minimal Sullivan models of the spheres (by separating the even-dimensional and odd-dimensional cases).

Exercise 4 (Hopf algebras). *A bialgebra is said to be* conilpotent *if it is conilpotent as a coassociative coalgebra.*

1. Prove that any conilpotent bialgebra is a Hopf algebra.

2. Prove that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a conilpotent bialgebra.

Given a group G, we define the group algebra $\mathbb{K}[G]$ *to be the vector space $\sum_{g \in G} \mathbb{K}g$ endowed with multiplication defined on basis elements as $g \cdot_{\mathbb{K}[G]} g' = g \cdot_G g'$.*

3. Prove that a group algebra $\mathbb{K}[G]$ is a Hopf algebra that is not conilpotent in general.

Exercise 5 (Hochschild homology and cohomology). *Let A be an associative algebra and M an A-bimodule. The* Hochschild chain complex $C_*(A,M)$ *is defined to be the chain complex whose degree n part is $M \otimes A^{\otimes n}$. Writing an element of $C_n(A,M)$ as*

$$m|a_1|\ldots|a_n \ ,$$

its differential is defined as

$$b_n(m|a_1|\ldots|a_n) = ma_1|a_2|\ldots|a_n + \sum_{i=1}^{n-1}(-1)^i m|\ldots|a_i a_{i+1}|\ldots|a_n + (-1)^n a_n m|a_1|\ldots|a_{n-1} \ .$$

Its homology is called Hochschild homology *and denoted $H_*(A,M)$. When $M = A$ is endowed with its obvious A-bimodule structure, the Hochschild homology $H_*(A,A)$ is denoted $HH_*(A)$, and called the* Hochschild homology of the algebra A.

1. Prove that the maps b_n indeed define a differential on $C_*(A,M)$.

We define the Hochschild cochain complex $C^*(A,M)$ *to be the cochain complex whose degree n part is* $\mathrm{Hom}(A^{\otimes n}, M)$ *and whose differential is defined as*

$$\beta(x)(a_1,\ldots,a_{n+1}) = a_1 x(a_2,\ldots,a_{n+1}) + \sum_{i=1}^{n}(-1)^i x(\ldots,a_i a_{i+1},\ldots) + (-1)^{n+1} x(a_1,\ldots,a_n)a_{n+1}$$

for a linear map $x : A^{\otimes n} \to M$.

2. Check that this formula defines indeed a differential β on $C^*(A,M)$.

We suppose in the rest of the exercise that A is commutative and unital. We moreover define the A-module of Kähler differentials $\Omega^1_{A|\mathbb{K}}$ to be the A-module generated by the symbols da for $a \in A$ and satisfying the relations

$$d(\lambda a + \mu b) = \lambda da + \mu db \qquad\qquad d(ab) = a(db) + b(da)$$

3. Prove that $HH_0(A) = A$ and that $HH_1(A) = \Omega^1_{A|\mathbb{K}}$.

Define a derivation *of A with value in M to be a linear map $D : A \to M$ satisfying the equality*

$$D(ab) = aD(b) + D(a)b \ .$$

4. Prove that the set of derivations $\mathrm{Der}(A,M)$ is an A-module and that there is a correspondence

$$\mathrm{Hom}_{A\text{-}\mathtt{mod}}(\Omega^1_{A|\mathbb{K}}, M) = \mathrm{Der}(A,M) \ .$$

5. Prove that for an element $m \in M$, the map $ad_m : A \to M$ defined as $ad_m(a) := ma - am$ is a derivation. It will be called an *inner derivation*. Prove then that the set of inner derivations $\mathrm{Inn}(A,M)$ is a submodule of $\mathrm{Der}(A,M)$.

6. Prove that $H^1(A,M) = \mathrm{Der}(A,M)/\mathrm{Inn}(A,M)$.

Exercise 6 (Frobenius algebra). *Let V and W be two vector spaces. A* pairing *of V and W is defined to be a linear map $\langle\cdot,\cdot\rangle : V \otimes W \to \mathbb{K}$. It is* non-degenerate in the variable V *if there exists an element $\sum w_i \otimes v_i \in W \otimes V$ such that the map $v \in V \to \sum_i \langle v, w_i\rangle v_i$ is the identity map.* Non-degeneracy in the variable W *is defined similarly. It is* non-degenerate *if it is non-degenerate in both variables.*

1. Prove that the following are equivalent:

(i) The pairing $\langle\cdot,\cdot\rangle$ is non-degenerate.
(ii) W is finite-dimensional and the induced linear map $W \to V^*$ is an isomorphism.
(iii) V is finite-dimensional and the induced linear map $V \to W^*$ is an isomorphism.

Let A be a unital associative algebra. If V is a right A-module and W a left A-module, we say that the pairing is associative *if $\langle va, w\rangle = \langle v, aw\rangle$.*

2. We suppose that A is finite-dimensional as a vector space. Prove that the following are equivalent:

(i) The right A-module A is right A-isomorphic to $A^\vee$.
(ii) The left A-module A is left A-isomorphic to $A^\vee$.
(iii) There exists a linear map $\varepsilon : A \to \mathbb{K}$ whose nullspace does not contain any non-trivial left-ideal.
(iv) There exists an associative non-degenerate pairing $A \otimes A \to \mathbb{K}$.

A is then said to be a *Frobenius algebra*.

3. Let A be a vector space endowed with a multiplication $\mu : A \otimes A \to A$ with unit $\eta : \mathbb{K} \to A$

$$\mu(\eta, \mathrm{id}) = \mathrm{id} = \mu(\mathrm{id}, \eta) ,$$

and with a comultiplication $\Delta : A \to A \to A$ with counit $\varepsilon : A \to \mathbb{K}$

$$(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta ,$$

that satisfy the *Frobenius relation*

$$(\mathrm{id} \otimes \mu)(\Delta \otimes \mathrm{id}) = \Delta\mu = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) .$$

Prove that the multiplication μ is associative and that the unital associative algebra A is a Frobenius algebra.

4. Prove that the conditions of question *3.* are equivalent to the conditions of question *2.*.

Exercise 7 (Gerstenhaber algebra structure on Hochschild cohomology). *Consider the Hochschild cochain complex $C^*(A, A)$. We define the* cup product

$$\cup : C^n(A, A) \otimes C^m(A, A) \longrightarrow C^{n+m}(A, A)$$

for two elements $x : A^{\otimes n} \to A$ and $y : A^{\otimes m} \to A$ as

$$x \cup y(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) := (-1)^{|x||y|} x(a_1, \ldots, a_n) y(a_{n+1}, \ldots, a_{n+m}) .$$

1. Prove that the cup product defines a chain map $C^*(A, A) \otimes C^*(A, A) \to C^*(A, A)$.

We moreover define the Gerstenhaber bracket

$$[\cdot,\cdot] : C^n(A,A) \otimes C^m(A,A) \longrightarrow C^{n+m-1}(A,A)$$

for $x : A^{\otimes n} \to A$ and $y : A^{\otimes m} \to A$ as

$$[x,y] = x \circ y - (-1)^{(|x|+1)(|y|+1)} y \circ x \ ,$$

where $x \circ y = \sum_{i=0}^{n-1} x \circ_i y$ and

$$x \circ_i y(a_1, \dots, a_{m+n-1}) = (-1)^{i(|y|+1)} x(a_1, \dots, a_i, y(a_{i+1}, \dots, a_{i+m}), a_{i+m+1}, \dots, a_{n+m-1}) \ .$$

2. Prove that the Gerstenhaber bracket defines a chain map $C^*(A,A) \otimes C^*(A,A) \to C^{*-1}(A,A)$.

3. Prove that the cup product and the Gerstenhaber bracket define a Gerstenhaber algebra structure on $H^*(A,A)$.

Exercise 8 (Schouten-Nijenhuis bracket). *Let V be a vector space. We define the* exterior algebra ΛV *of V to be the free graded commutative algebra on the graded vector space V seen as being concentrated in degree 1. We denote its product $\wedge$.*

Prove that the exterior algebra $\Lambda\mathfrak{g}$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a Gerstenhaber algebra for the *Schouten-Nijenhuis bracket*

$$[x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m] = \sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} [x_i, y_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge \hat{y_j} \wedge \cdots \wedge y_m \ .$$

Exercise 9 (Chevalley-Eilenberg homology and cohomology). *Let $\mathfrak{g}$ be a Lie algebra. A vector space V endowed with a linear map $[\cdot,\cdot] : V \otimes \mathfrak{g} \to V$ satisfying*

$$[v,[x,y]] = [[v,x],y] - [[v,y],x]$$

is called a right $\mathfrak{g}$-module.

1. Prove that $\mathfrak{g}^{\otimes n}$ and $\Lambda\mathfrak{g}$ are right $\mathfrak{g}$-modules.

Let V be a right $\mathfrak{g}$-module. The Chevalley-Eilenberg chain complex *is defined to be the chain complex $C_*(\mathfrak{g}, V)$ whose degree n part is $V \otimes \Lambda^n\mathfrak{g}$ and whose differential is defined as*

$$\begin{aligned} d(v \otimes g_1 \wedge \cdots \wedge g_n) = &\sum_{i=1}^{n} (-1)^i [v, g_i] \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \\ &+ \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j-1} v \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n \ . \end{aligned}$$

2. Check that the above formula indeed defines a differential on $C_*(\mathfrak{g}, V)$.

3. Prove that $C_*(\mathfrak{g}, V)$ is a right $\mathfrak{g}$-module, that the action of $\mathfrak{g}$ is compatible with the differential and that it is trivial on $H_*(\mathfrak{g}, V)$.

Recall that one can associate an associative algebra $U(\mathfrak{g})$ to the Lie algebra $\mathfrak{g}$, called its universal enveloping algebra. *For M a $U(\mathfrak{g})$-bimodule, we then denote M^{ad} the right $\mathfrak{g}$-module endowed with the action $[m, g] = mg - gm$.*

4. Prove that $H_*(U(\mathfrak{g}), M) = H_*(\mathfrak{g}, M^{ad})$, where $H_*(U(\mathfrak{g}), M)$ is *Hochschild homology* as defined in Exercice 6.

We define the Chevalley-Eilenberg cochain complex *to be the cochain complex $C^*(\mathfrak{g}, V)$ whose degree n part is* $\mathrm{Hom}(\Lambda^n \mathfrak{g}, V)$ *and whose differential is defined as*

$$\begin{aligned}\delta f(g_1, \ldots, g_{n+1}) = &\sum_{i=1}^{n+1} (-1)^i [f(g_1, \ldots, \hat{g}_i, \ldots, g_n), g_i] \\ &+ \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{i+j} f([g_i, g_j], g_1, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_{n+1})\ .\end{aligned}$$

5. Check that the above formula indeed defines a differential on $C^*(\mathfrak{g}, V)$.

We point out that $\Lambda\mathfrak{g} = C_*(\mathfrak{g}, \mathbb{K})$ as graded vector spaces.

6. Prove that the product $\wedge$ is not compatible with the differential ∂_{CE} on $\Lambda\mathfrak{g}$.

Hence $(\Lambda\mathfrak{g}, \partial_{CE})$ is not a cdg algebra.

7. Prove that $(\Lambda\mathfrak{g}, \partial_{CE})$ is a cdg coalgebra for the deconcatenation coproduct.

Exercise 10 (Poisson algebras).

1. Prove that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Poisson algebra.

We define a symplectic manifold *(M, ω) to be the data of a smooth manifold M and of a non-degenerate 2-form ω such that $d\omega = 0$.*

2. Prove that for every smooth function $f \in \mathscr{C}^\infty(M, \mathbb{R})$ there exists a unique vector field $X_f \in \Gamma(TM)$ such that $df(\cdot) = \omega(X_f, \cdot)$.

For f and g two smooth functions, we then define $\{f, g\} := \omega(X_f, X_g)$.

3. Prove that the algebra of smooth functions $\mathscr{C}^\infty(M, \mathbb{R})$ endowed with the bracket $\{\cdot, \cdot\}$ is a Poisson algebra.