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Fix n. Let t be an indeterminate, and consider $B = B(t)$ an algebra structure on $\mathbf{C}(t)^n$ (no assumption such as associativity, Lie, etc). So B can be just viewed as an element of $\mathbf{C}(t)^{\{1,;n\}^3}$. Let F be the (finite) set of poles: then for each $z \in U = \mathbf{C} - F$ one can specialize it as $B(z)$, which defines an algebra structure on $\mathbf{C}^n$.

Let $V_{n,k}^R$ be the space of R-multilinear maps $(R^n)^k \to R^n$, or equivalently the space of R-linear maps $(R^n)^{\otimes k} \to R^n$. It has dimension n^{k+1}. We only consider it for $k = 1, 2$. For a law $b \in V_{n,2}^R$, define the coboundary map $d[B] : V_{n,1}^R \to V_{n,2}^R$ (associated to B) as

$$(d^R[b]f)(x, y) = b(fx, y) + b(x, fy) - fb(x, y).$$

Say that $c \in V_{n,2}^R$ is a 2-coboundary (with respect to b, over R) if c belongs to the image of $d^R[B]$.

Proposition 1. *Either the derivative $B'(t)$ is a 2-coboundary (with respect to $B(t)$, over $\mathbf{C}(t)$), and then the specializations $B(z)$ are pairwise isomorphic for z ranging over a cofinite subset of $\mathbf{C} - F$, or $B'(t)$ is not a 2-coboundary, in which case the isomorphism relation between the $B(z)$, when z ranges over $\mathbf{C} - F$, has finite classes (of bounded cardinal).*

Let q be the rank of the $\mathbf{C}(t)$-linear map $d^{\mathbf{C}(t)}[B(t)] : V_{n,1}^{\mathbf{C}(t)} \to V_{n,2}^{\mathbf{C}(t)}$. For $z \in \mathbf{C} - F$, let $q(z)$ be the rank of the specialized $\mathbf{C}$-linear map $d^{\mathbf{C}}[B(z)] : V_{n,1}^{\mathbf{C}} \to V_{n,2}^{\mathbf{C}}$. Write $U' = \{z \in \mathbf{C} - F : q(z) = q\}$. This is a cofinite subset of $\mathbf{C}$.

Define

$$\Psi : \mathrm{GL}_n(\mathbf{C}) \times U \to V_{n,2}^{\mathbf{C}},\ (M, z) \mapsto M^* B(z)$$

where $M^* b(x, y) = M^{-1} b(M(x), M(y))$.

Lemma 2. *For $z \in U$, the rank $r(M, z)$ of $d\Psi(M, z)$ does not depend on $M \in \mathrm{GL}_n(\mathbf{C})$ so we denote it $r(z)$.*

Preuve. Fix $M_0 \in \mathrm{GL}_n(\mathbf{C})$. Write $s(M, z) = (MM_0, z)$, and $\sigma(D) = M_0^* \sigma$. Hence s is a self-diffeomorphism of $\mathrm{GL}_n(\mathbf{C}) \times U$ and σ is a self-diffeomorphism of $V_{n,2}^{\mathbf{C}}$. Then $\Psi \circ s = \sigma \circ \Psi$. This implies that the rank of the differential of Ψ at (M, z) and at $s(M, z)$ are the same, for all M, z, and for every M_0. Whence the result. $\square$

For $z \in U$, let Ψ_z be the restriction of Ψ to the "hypersurface" $\mathrm{GL}_n(\mathbf{C}) \times \{z\}$. Then the rank of $d\Psi_z$ at (M, z) equals $q(z)$. In particular, $r(z) \in \{q(z), q(z)+1\}$. Hence for $z \in U'$, $r(z) \in \{q, q + 1\}$. So $U_1 = \{z \in U' : r(z) = q + 1\}$ is Zariski-open, hence either empty or cofinite:

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(a) either $r(z) = q$ for all $z \in U'$;

(b) or $r(z) = q + 1$ for all $z \in U_1$, where U_1 is cofinite in U'.

Note that the tangent space of $\mathrm{GL}_n(\mathbf{C})$ at Id can be identified to $V_{n,1}^{\mathbf{C}}$. By direct computation, for $f \in V_{n,1}^{\mathbf{C}}$, we get $d\Psi_z(\mathrm{Id})(f) = B(z)(fx, y)+B(z)(x, fy)-fB(z)(x, y)$.

Lemma 3. *For $z \in U$, we have $r(z) = q(z)$ iff $B'(z)$ is a 2-coboundary (with respect to $B(z)$, over $\mathbf{C}$). (In particular in Case (a) it holds for all $z \in U'$)*

Preuve. We have $r(z) = q(z)$ iff at (Id, z) the image of the differential restricted to $\{\mathrm{Id}\} \times U'$ is included in the image of the differential restricted to $\mathrm{GL}_n(\mathbf{C}) \times \{z\}$, that is, that of Ψ_t. The former image is the line generated by $B'(z)$, while the latter image is the space of 2-coboundaries (by the above computation). □

In particulier, if $B'(z)$ is a 2-coboundary for infinitely many z, then conversely we are in Case (a).

Lemma 4. *In Case (a), all $(\mathbf{C}^n, B(z))$ are isomorphic for $z \in U'$.*

Preuve. By connectedness, it is enough to show that for all z and all z' close enough to z, $(\mathbf{C}^n, B(z'))$ is isomorphic to $(\mathbf{C}^n, B(z))$. The constant rank theorem applies to Ψ: the image of some small ball Y around (M, t) is a q-dimensional submanifold through $\Psi(M, t)$. Let Y' be a ball around (M, t), contained in Y, on which the differential of Ψ_t has rank q. The map Ψ_t sends Y' into the q-dimensional submanifold $\Psi(Y)$, and its differential has rank q. Hence its image is open therein. This precisely says that for every $t \in Y'$, $(\mathbf{C}^n, B(t'))$ is isomorphic to $(\mathbf{C}^n, \Phi(t))$. □

Now, a simple remark:

Lemma 5. *Equivalent statements:*

- *For all $z \in U$ minus a finite subset, $B'(z) \in V_{n,2}^{\mathbf{C}}$ is 2-coboundary*
- *$B'(t) \in V_{n,2}^{\mathbf{C}(t)}$ is a 2-coboundary*

Preuve. Suppose that $B'(t)$ is a 2-coboundary. There exists a linear endomorphism $\delta(t)$ of $\mathbf{C}(t)^n$ such that formally $B'(t)(x, y) = B(t)(\delta(t)x, y)+B(t)(x, \delta(t)y)-\delta(t)B'(t)(x, y)$. Hence this remains true for every specification of t in U minus zeros of denominators of entries of $\delta(t)$.

Conversely suppose that $B'(t)$ is not formally a 2-coboundary, i.e., is not in the image of $d = d(t)$. Let $\hat{d}(t)$ be the block matrix $(d(t)|B'(t))$ (concatenating the matrix $d(t)$ and the column matrix $B'(t)$). Then the rank of the matrice $d(t)$ and $\hat{d}(t)$ are q and $q+1$ respectively. Hence some $(q+1)$-minor of $\hat{d}(t)$ is nonzero. If $P(t)$ is its determinant, let U'' be the set of those $z \in U$ such that $P(s) \neq 0$: this is a cofinite subset of U. Hence for $z \in U''$, $d(z)$ has rank q, and $\hat{d}$ has rank $q + 1$, so that $B'(z)$ is not in the image of $d(z)$. □

It remains to interpret Case (b): we know that in this case, for all $t \in U_1$, $c(t)$ is not a 2-coboundary.

Lemma 6. *In Case (b), the equivalence relation $t \sim t'$ if $(\mathbf{C}^n, B(t))$ and $(\mathbf{C}^n, B(t'))$ are isomorphic has classes of finite bounded cardinal.*

Preuve. Assume otherwise. Then by Lemma 7, there is a cofinite subset U_2 of U_1 such that the $B(t)$ for $t \in U_2$ are pairwise isomorphic.

Fix $t_0 \in U_2$. Define $X = \{(t, M) \in U_1 \times \mathrm{GL}_n(\mathbf{C}) : \ M^*B(t) = B(t_0)$, and $\pi : (t, M) \mapsto t$ be the first projection. So, by assumption, $\pi(X) = U_1$. Let Y be an irreducible component of X such that $\pi|_Y$ is non-constant. Let Z be the set of points of the regular locus of Y at which π has a nonzero differential. So Z is Zariski-dense in Y. Write $U_3 = \pi(Z)$, so U_3 is cofinite in U_1. By the constant rank theorem, there is, for every $t_1 \in U_3$ (which we fix for a moment), a holomorphic map M, defined at a neighborhood V of t_1, valued in Z, such that $\pi(M(t)) = t$ for every $t \in V$. That is, $B(t) = M(t)^*B(t_0)$ for every $t \in V$. In particular $B(t_0) = (M(t_1)^{-1})^*B(t_1)$. So, writing $N(t) = M(t_1)^{-1}M(t)$, we deduce $B(t) = N(t)^*B(t_1)$ for all $t \in V$, with $N(t_1) = \mathrm{id}_{\mathbf{C}^n}$. We differentiate this at $t = t_1$ (both sides being holomorphic in t). The left side yields $B'(t_1)$. The right side precisely yields $d^{\mathbf{C}}[B(t_1)]N'(t_1))$. So $B'(t_1)$ is a 2-coboundary. Since this holds for every $t_1 \in U_3$, we are done. $\square$

I used the standard fact:

Lemma 7. *Let U be a Zariski-open subset of $R \subset U^2$ a constructible equivalence relation (constrictible with respect to the Zariski topology of U^2. Then either R has a cofinite equivalence class, or has finite classes of bounded cardinal.*

Preuve. $R \subset \mathbf{C}^2$ being constructible, it is a finite union of locally closed subset $L_i = U_i \cap F_i$ (U_i Zariski open, F_i Zariski closed, with $U_i \cap F_i$ non-empty). If for some i we have $F_i = \mathbf{C}^2$, L_i is Zariski-open. Let $(x, y) \in L_i$. Then the set of $y' \in \mathbf{C}$ such that xRy' is non-empty, Zariski-open in $\mathbf{C}$, hence is cofinite, hence the class of x is cofinite.

Otherwise, each F_i if finite, or is a Zariski-closed curve, which we can assume to be irreducible, and thus L_i is the complement of a finite subset in F_i. If F_i is a horizontal or vertical line, then there is a cofinite class, which we can discard. Hence, if d_i is the degree of F_i, the intersection of F_i with each horizontal line has cardinal $\leq d_i$. If $d = \sum d_i$ (counting $d_i = 0$ if F_i is finite), we deduce that outside finitely many points (coordinates of finite F_i), the equivalence classes have cardinal $\leq d$. $\square$