A few points in topos theory

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Abstract

This paper deals with two problems in topos theory; the construction of finite pseudo-limits and pseudo-colimits in appropriate sub-2-categories of the 2-category of toposes, and the definition and construction of the fundamental groupoid of a topos, in the context of the Galois theory of coverings; we will take results on the fundamental group of étale coverings in [1] as a starting example for the latter. We work in the more general context of bounded toposes over $\mathcal{S}et$ (instead of starting with an effective descent morphism of schemes). Questions regarding the existence of limits and colimits of diagram of toposes arise while studying this problem, but their general relevance makes it worth to study them separately. We expose mainly known constructions, but give some new insight on the assumptions and work out an explicit description of a functor in a coequalizer diagram which was as far as the author is aware unknown, which we believe can be generalised.

This is essentially an overview of study and research conducted at DPMMS, University of Cambridge, Great Britain, between March and August 2006, under the supervision of Martin Hyland.

Contents

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1 Introduction

Toposes were first conceived ([2]) as kinds of "generalised spaces" which could serve as frameworks for cohomology theories; that is, mapping topological or geometrical invariants with an algebraic structure to topological spaces. As such, the first definition was that of a category of sheaves ("good" functions, in that they can be restricted to open sets and glued in a reasonable fashion) on a topological space, in line with the thought that knowledge of the sheaves on a space is as good as knowledge of the space itself. The first generalization is to extend the definition to sheaves on a *site*, *i.e.* a category with a Grothendieck topology (in which the notion of open set is replaced by that of a cover of an object. Throughout this introduction, we will refer to these as *Grothendieck toposes.*

A second, largely independant approach of the Anglo-American school, was to define a topos as a category of "generalised sets"; this yields so-called *elementary toposes* (cartesian closed categories with finite limits and a subobject classifier), which enjoy in all generality many properties of sets, but notably not necessarily the axiom of choice and the law of excluded middle. While Grothendieck toposes have obvious topological and geometrical information, elementary toposes have logical information; one can indeed define, *internally*, a formal language and semantics of a topos, allowing to rigourously make arguments on objects of a topos as if they were sets. Furthermore, it appeared that toposes were the right framework to define some models of formal systems.

It is easy to see that Grothendieck toposes are a particular case of elementary toposes, and that some constructions first made in the case of Grothendieck toposes can be naturally carried in the general case. This is, in a way, the essence of topos theory; we have two visions of a particular class of objects, one which is inherently geometrical and the other inherently logical, and we wish to understand how these visions yield either the same or complementary information. A useful approach is to start with the utmost generality and try to understand how particular geometrical properties of toposes can be expressed elementarily (in formal categorical terms). An example that we will state is the fact that, once we adopt a *relative* view (*i.e.* we work with morphisms of toposes over a base rather than absolute objects), a Grothendieck topos really is an elementary topos with an object of generators.

This paper is an overview of the work done during a 5 month internship in Cambridge University, Great Britain, under the supervision of Martin Hyland. After being introduced to topos theory, it started with the study of constructions of the fundamental groupoid of a topos due to Bunge, Moerdijk, ([8]) and Dubuc ([11]) during which I realised some technical details as well as some fundamental lines of thought behind these constructions were very unclear (among others, the existence of limits and colimits in subcategories of the category of toposes, and the role of points in the construction of a fundamental groupoid); also, the link between generalised Galois theory ([6]) and the theory of the fundamental group is not as obvious as it seems on first glance.

The first section is devoted to giving general definitions and results in topos theory, that can be found in books on the subject ([4], [2]). It only assumes a moderate background in category theory. However, this section is not meant as an introduction to topos theory (which would need a whole book); rather, it is intended to set the vocabulary we will use. Then, we will focus on describing a few constructions of limits and colimits of diagrams of toposes, in particular cases. Next, we will introduce and discuss (omitting some tedious calculations) the construction of the fundamental groupoid of an atomic topos with a point, and sketch the corresponding construction if we do not assume the existence of a point. We will finish by a discussion on the links between Galois theory and these results, especially relating to points, stressing the fact that the results we have do not form yet a general coherent theory, and suggest some lines of thought that would be interesting to consider as a new approach.

Acknowledgements

I would like to thank Martin Hyland for his availability during these 5 months; his insight and willingness to work out even largely technical problems with me greatly helped my understanding of the subject; indeed, in my experience, one of the main difficulty of topos theory as a field is the existence of so many different yet related approaches to problems of which no one fully understand the links between. In a sense, this is not so surprising as it is precisely where the depth of the subject lies.

I would also like to thank Joël Riou and other posters on the newsgroup `ens.forum.sciences.maths.avancees` for a number of very interesting conversations that helped me build an intuition of the subject. Finally, I wish to express my gratitude to the École normale supérieure, Cambridge University and King's College for providing me with stimulating intellectual environments and exceptional material conditions.

2 General knowledge

We first recall a few general definitions and results that we will use throughout the paper. A general reference for these is [4], but [2] is also very useful. We will not enter general considerations on universes (but of course will make sure we do not run into problems!); basically, we can assume we are working in a fixed universe $\mathcal{U}$, and the very few times we will need to enlarge it, we will assume there exist a universe $\mathcal{V}$ large enough for our purposes such that $\mathcal{U} \subset \mathcal{V}$.

Definition 2.1 (Sites). A site $(\mathcal{C}, J)$ is a category $\mathcal{C}$ equipped with a Grothendieck topology J, *i.e.* an application that maps each object C of $\mathcal{C}$ to a set of sieves $J(C)$ of codomain C such that:

- the maximal sieve $t_C = \{f \mid \text{cod } f = C\}$ is in $J(C)$.
- if S is in $J(C)$ and $f : D \to C$ is an arrow in $\mathcal{C}$, then the sieve $f^*(S) = \{h \mid \text{cod } h = D, f \circ h \in S\}$ in in $J(D)$.
- if S is in $J(C)$ and R is a sieve on C such that for any $h : D \to C$ in S, $h^*(R)$ is in $J(D)$, then R is in $J(C)$.

A sieve S in $J(C)$ is said to be a *J-cover* of C; when there is no ambiguity on the topology, we will simply say a *cover*. Also, we will often not explicitely mention the topology, and simply say that $\mathcal{C}$ is a site.

Definition 2.2 (Sheaves on a site). Let $\mathcal{C}$ be a site; a presheaf $F : \mathcal{C}^{\text{op}} \to \mathcal{S}\text{et}$ on $\mathcal{C}$ is a sheaf if it satisifies the following condition: for all objects C of $\mathcal{C}$ and covering sieves S of C, the canonical morphism $\text{Hom}(\mathbf{y}C, F) \to \text{Hom}(S, F)$, where $\mathbf{y}$ is the Yoneda functor and S is naturally identified with a subfunctor of $\mathbf{y}$, is a bijection. If $\mathcal{C}$ has pullbacks, we have an equivalent definition which is closer to the one in the topological case, *viz.*, writing $S = \{f_i : C_i \to C\}_{i \in I}$, F is a sheaf if the diagram

$$F(C) \longrightarrow \prod_{i \in I} F(C_i) \rightrightarrows \prod_{i,j \in I} F(C_i \times_C C_j)$$

is an equaliser.
If $\mathcal{C}$ is a site, we write $\widetilde{\mathcal{C}}$ for the category of sheaves on $\mathcal{C}$.

Definition 2.3 (Toposes, geometric morphisms).

- we will simply call *topos* what some authors call an *elementary topos*, *i.e.*, a cartesian closed category with finite limits and a subobject classifier, that is an object Ω and a map $\text{true} : 1 \to \Omega$ such that for any monomorphism $Y \to X$, there exist a unique morphism $X \to \Omega$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ X & \longrightarrow & \Omega \end{array}$$

  is a pullback. A morphism of toposes (also called a *geometric morphism*) $f : \mathcal{E} \to \mathcal{F}$ is a pair of adjoint functors $f^* \dashv f_*$ ($f^* : \mathcal{F} \to \mathcal{E}$) where f^* is left exact. f^* is called the *inverse image* and f_* the *direct image*. We do not recall the definition of the Mitchell-Bénabou language, nor the Kripke-Joyal semantics associated to it, and (or rather, *because*) we will most of the time write internal arguments in a topos as if it were $\mathcal{S}et$ (taking precautions not to use the axioms of infinity, choice, or excluded middle unless we know they are valid).

- a functor $F : \mathcal{E} \to \mathcal{F}$ between toposes is said to be *logical* if it preserves the topos structure, that is, finite limits and colimits, exponentials and the subobject classifier.

- a morphism of toposes $f : \mathcal{E} \to \mathcal{F}$ is *bounded* if $\mathcal{E}$ has an object of generators (over F), *i.e.*, there exists an object G of $\mathcal{E}$ such that every object of $\mathcal{E}$ is a subquotient of one of the form $f^*I \times G$; *i.e.* for all A in $\mathcal{E}$, such a diagram exists:

$$\begin{array}{ccc} S & \overset{h}{\twoheadrightarrow} & A \\ \downarrow{\scriptstyle m} & & \\ f^*I \times G & & \end{array}$$

  where m is mono and h is epi. Equivalently, this means that $\mathcal{E}$ has a separating family S in $\mathcal{F}$, that is, a family of arrows $\{u\}_{i \in I}$ such as for any $\alpha, \beta : X \rightrightarrows Y$ if $\alpha u_i = \beta u_i$ for all $i \in I$, then $\alpha = \beta$.

  The category of toposes with bounded morphisms is noted $\mathfrak{BTop}$.

Let us note that, in a way, "most" morphisms are bounded; more precisely, we only know of a very few ways to systematically build unbounded morphisms (above $\mathcal{S}et$). Describing them would make us stray too far, but [3] gives some insight on the problem. In practical, this means that working in $\mathfrak{BTop}$ is not a very deep loss of generality, thus we will not always strive to achieve constructions in $\mathfrak{Top}$.

- A functor $u : \mathcal{C} \to \mathcal{D}$ between sites is said to be *continous* if for all $X \in \mathcal{C}$ and for all sheaves F on $\mathcal{D}$, the functor $X \mapsto X \circ u$ is a sheaf on $\mathcal{C}$.

  Given such a functor, if we assume $\mathcal{C}$ and $\mathcal{D}$ are closed under finite limits (which is innocuous given any Grothendieck topos admits such a site), if u also preserves finites limits, it is said to be *a morphism of sites* from $\mathcal{D}$ to $\mathcal{C}$ (note that the morphism of site is in the direction opposite to the direction of the functor).

- A functor $u : \mathcal{C} \to \mathcal{D}$ between sites is said to be *cocontinuous* if for all $C \in \mathcal{C}$ and S a cover of $u(C)$ in $\mathcal{D}$, there exists a cover R of C in $\mathcal{C}$ such that $u(R) \subset S$.

Proposition 2.4. Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism; there exists a *canonical* $\mathcal{F}$-indexed topos $\mathbb{E}$ whose underlying regular topos is $\mathcal{E}$; the index is given by f^*; *i.e.*, for I in $\mathcal{F}$, $\mathbb{E}(I) = \mathcal{E}^{f^*I}$. Therefore, given such a geometric morphism, we can consider $\mathcal{E}$ as an $\mathcal{F}$-topos.

Definition 2.5. If $\gamma : \mathcal{E} \to \mathcal{S}$ is bounded, we will say that $\mathcal{E}$ is a *Grothendieck $\mathcal{S}$-topos*. Given two Grothendieck $\mathcal{S}$-toposes, a morphism between these is a pair of adjoint *$\mathcal{S}$-indexed* functors such that the obvious diagram commutes. Bearing the original definition of a Grothendieck topos as a category of sheaves of sets on a site, this is due to the following result.

Theorem 2.6. Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. f is bounded if and only if $\mathcal{E}$ is equivalent to a category of $\mathcal{F}$-valued sheaves on an internal site in $\mathcal{F}$. That is, there exists an internal category $\mathbb{C}$ in $\mathcal{F}$ and a topology J on $\mathbb{C}$ such that $\mathcal{E}$ is equivalent to the subtopos of $\mathcal{F}^{\mathbb{C}}$ consisting of J-sheaves on $\mathbb{C}$.

Continuing the line of thought of the previous note, restricting our study to $\mathfrak{BTop}$ has the further advantage that we can enjoy both descriptions of a topos; we will see that some constructions are better understood considering sites of definitions than the corresponding toposes.

We will not discuss the construction of sites for toposes in all generality, but let us remark that given a bounded morphism $\gamma : \mathcal{E} \to \mathcal{S}$, there is a *canonical* site in $\mathcal{S}$ for $\mathcal{E}$ given by the category $\mathcal{E}$ equipped with the canonical topology, that is, the finest topology for which representable presheaves are sheaves.

Note 2.7. What was originally called a topos, and studied in [2] is a Grothendieck $\mathcal{S}et$-topos, according to our definition; indeed, any such topos has a unique geometric morphism to $\mathcal{S}et$, which provides the (canonical) indexing making it a $\mathcal{S}et$-topos, and making geometric morphisms as defined previously $\mathcal{S}et$-indexed.

Theorem 2.8 (Morphisms of sites and morphisms of toposes).

- Given two sites $\mathcal{C}$ and $\mathcal{D}$, a morphism of sites $u : \mathcal{D} \to \mathcal{C}$ induces a geometric morphism $f : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$, where $f_* : X \mapsto X \circ u$.

Given a morphism of Grothendieck toposes $f : \mathcal{E} \to \mathcal{F}$, there exists respective sites $\mathcal{C}$ and $\mathcal{D}$ such that f is induced by a morphism of sites from $\mathcal{C}$ to $\mathcal{D}$. In fact, there is an equivalence of categories between morphisms of sites $\mathcal{D} \to \mathcal{E}$ (where $\mathcal{E}$ is equipped with the canonical topology) and geometric morphisms $\mathcal{E} \to \mathcal{F} = \widetilde{D}$.

- Given two sites $\mathcal{C}$ and $\mathcal{D}$, a cocontinous functor $F : \mathcal{C} \to \mathcal{D}$ induces a geometric morphism $f : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$ given by $f^* : X \mapsto \mathbf{a}(F \circ f)$ ($\mathbf{a}$ is the sheafification functor).

We would like to stress that given a morphism of toposes $f : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$, there exists a morphism of sites that yields f, but there is no reason it should be from $\mathcal{C}$ to $\mathcal{D}$. Also, we note that one cannot necessarily find a cocontinous functor between sites that yields f.

When there is no ambiguity, we will indifferently call "topos" an object of $(\mathfrak{B})\mathfrak{Top}$ or $(\mathfrak{B})\mathfrak{Top}/S$.

Definition 2.9 (Points). Let $\gamma : \mathcal{E} \to \mathcal{S}$ be an $\mathcal{S}$-topos; an *$\mathcal{S}$-point* of γ is a section, that is, a morphism $p : \mathcal{S} \to \mathcal{E}$ such that $\gamma p = id$. There is an equivalence of categories between points of a topos $\mathcal{E}$ and its fibre functors, *i.e.* functors $\mathcal{E} \to \mathcal{S}$ (internal to $\mathcal{S}$) that preserve limits and are left exact.

A topos $\mathcal{E}$ is said to have *enough points* if the family of fibre functors corresponding to the points is conservative.

Definition 2.10 (Types of toposes).

- a morphism of toposes $f : \mathcal{E} \to \mathcal{F}$ is said to be an *inclusion* (or an *embedding*) if f_* is full and faithful (or equivalently, the counit of the adjunction $\varepsilon : f^* f_* \to 1$ is an isomorphism). It is said to be a *surjection* if f^* is faithful. It is easy to check that if a geometric morphism is both an inclusion and a surjection, then it is an isomorphism.

- $f : \mathcal{E} \to \mathcal{F}$ is said to be *locally connected* if f^* has a left adjoint $f_!$.

- $f : \mathcal{E} \to \mathcal{F}$ is said to be *atomic*[1] if f^* is logical. Given f is geometric morphism, f^* preserves finite limits and colimits, so we only need to check it preserves exponentials and the subobject classifier. A theorem of Paré asserts that a logical functor has a left adjoint if and only if it has a right adjoint, therefore, atomic morphisms are locally connected.

- $f : \mathcal{E} \to \mathcal{F}$ is said to be *connected* if f^* is full and faithful.

3 On (co)limits of toposes

This section is devoted to the construction of a few cases of limits and colimits in some subcategories of $\mathfrak{Top}$. Formally, we should consider the 2-categorical structure of $\mathfrak{Top}$, and distinguish between 2-limits, pseudo-limits, and lax limits. In pratice, we will focus on pseudo-(co)limits (lax limits do exist in fairly

[1] In the bounded case over $\mathcal{S}et$, this corresponds indeed to having a site comprised of atoms, *i.e.* connected objects with maps between them being epi.

general situations), that is, only require the limit diagram to commute up to isomorphism, and we will loosely call these "(co)limits".

Little seems to be known on the general existence of (co)limits in $\mathfrak{Top}$; neither finite limits or colimits are known to exist, but there is no formal argument or even vague intuition that they do not. The situation is better in $\mathfrak{BTop}/\mathcal{S}$: we will prove that finite limits exist. Finite colimits also exist in $\mathfrak{BTop}/\mathcal{S}$, provided $\mathcal{S}$ has a natural number object. It is not completely clear while this is needed, and the construction itself is not particularly elegant or inspiring, but we will try to explain heuristically why this is the case.

3.1 The construction of finite limits in $\mathfrak{BTop}/\mathcal{S}$

$\mathfrak{BTop}/\mathcal{S}$ obviously has a terminal object, namely $1 : \mathcal{S} \to \mathcal{S}$. We will focus in this subsection on constructing a fibre product. It has been known to exist for at least 30 years, and a purely algebraic construction is given in [3], using comonads and identifying the limit with a category of coalgebras. However, we will rather follow a more geometrical construction using sites due to Gabber and written down by Illusie ([12]).

The main idea in the construction is that one can find sites for the two lax fibre products which differ only in a symmetry in the definition of the topology. We then take the intersection of the two topologies and it appears that it defines a site for the (pseudo-, as always) fibre product. More precisely:

Let $\mathcal{S}$ be a topos (that we will call the *base*), $f : \mathcal{E} \to \mathcal{S}$, $g : \mathcal{F} \to \mathcal{S}$ two bounded (indexed) morphisms. Let $\mathcal{C}$ and $\mathcal{D}$ be sites in $\mathcal{S}$ for $\mathcal{E}$ and $\mathcal{F}$ respectively; we will abuse the notation and write f and g for the corresponding morphisms of sites. Let X be the category whose objects are pairs of morphisms (u, w) where $u : U \to f^*(V)$, $w : W \to g^*(V)$; morphisms in X are given by triples of morphisms that make the obvious diagram commute. For the sake of simplicity, we write such a pair $(U \to V \leftarrow W)$. X is equipped with the topology generated by the following families of covers:

- $\{(U_i \to V \leftarrow W) \to (U \to V \leftarrow W)\}_{i \in I}$ such that $\{U_i \to U\}_{i \in I}$ is a cover.
- $\{(U \to V \leftarrow W_i) \to (U \to V \leftarrow W)\}_{i \in I}$ such that $\{W_i \to W\}_{i \in I}$ is a cover.
- $(U \to V \leftarrow W) \to (U \to V' \leftarrow W')$ where $W' = W \times_{g^*V} V'$.

The morphisms $p_1 : \mathcal{C} \to X$ and $p_2 : \mathcal{D} \to X$ given by $p_1(U) = (U \to 1_S \leftarrow 1_{\mathcal{F}})$, $p_2(W) = (1_{\mathcal{E}} \to 1_S \leftarrow W)$ yield projection morphisms $p_1 : \widetilde{X} \to \mathcal{E}$, $p_2 : \widetilde{X} \to \mathcal{F}$. Now, we claim:

Theorem 3.1. $\widetilde{X}$ is a lax fibre product; we have a 2-cell $\tau : gp_2 \to fp_1$.

Proof. Indeed, given a sheaf F on X and an object V of $\mathcal{S}$, we define

$$(\tau_* F)(V) = F(1_{\mathcal{E}} \to 1_S \leftarrow g^*V) \to F(f^*V \to V \leftarrow g^*V) \leftarrow F(f^*V \to 1_{\mathcal{S}} \leftarrow 1_{\mathcal{F}})$$

where the second map is an isomorphism (see [12]).

It remains to show that the universal property of the lax fibre product is satisfied. Let $\mathcal{T}$ be a topos and assume we have morphisms $a : \mathcal{T} \to \mathcal{E}$, $b : \mathcal{T} \to \mathcal{F}$

and a natural transformation $t : gb \to fa$. We claim there exist a unique triple $(h : \mathcal{T} \to \widetilde{X},\ \alpha : p_1 h \simeq a,\ \beta : p_2 h \simeq b)$ such that $\tau(h) = t$. For uniqueness, the reader is referred to [12]. Let us prove the existence of h. For $Z = (U \to V \leftarrow W)$ we define h^*Z as the fibre product:

$$\begin{array}{ccc} a^*U \times_{(gb)^*V} b^*W & \longrightarrow & b^*W \\ \downarrow & & \downarrow \\ a^*U \longrightarrow (fa)^*V & \xrightarrow{t} & (gb)^*V \end{array}$$

$h^*(\tau)(V)$ is given by $(fa)^*V \leftarrow (fa)^*V \times_{(gb)^*V} (gb)^*V \to (gb)^*V$, therefore is equal to $t((fa)^*V)$. Also, h^* is trivially left exact; thus it remains to show that it extends to the inverse image part of a morphism of toposes; it is sufficient to show that h^* is continuous, which is obvious. $\square$

We will usually write this lax fibre product $\mathcal{E}\overleftarrow{\times}_{\mathcal{S}}\mathcal{F}$.

We define the other lax fibre product, $\mathcal{E}\overrightarrow{\times}_{\mathcal{S}}\mathcal{F}$ in the exact same way, replacing covers of the third type in the site of definition by covers $(U \to V \leftarrow W) \to (U' \to V' \leftarrow W)$ where $U' = U \times_{g^*V} V'$.

Now, let Z be the site defined by the same category as the sites of $\mathcal{E}\overrightarrow{\times}_{\mathcal{S}}\mathcal{F}$ and $\mathcal{E}\overleftarrow{\times}_{\mathcal{S}}\mathcal{F}$, and with the topology defined as the supremum of the two corresponding topologies. The construction of τ described above clearly yields an isomorphism, therefore the topos $\widetilde{Z}$ is the fibre product $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$.

Note 3.2. If X and Y are topological spaces over S, we see that the fibre product of $\widetilde{X}$ and $\widetilde{Y}$ is equivalent to the topos over the fibre product of X and Y, since open sets $U \times_V W$ form a basis of the topology of $U \times_S W$. However, this is not the case in all generality; if X and Y are schemes over S, because the Zariski topology of $X \times_S Y$ is *not* the product of the Zariski topologies of X and Y, the natural map $\widetilde{X \times_S Y} \to \widetilde{X} \times_{\widetilde{S}} \widetilde{Y}$ is not an equivalence. Similarly, the corresponding map between the étale toposes of X and Y is not an equivalence. This suggests that these toposes might not be ideal topological settings for algebraic geometry.

We shall now state, but not prove (these are usually readily checked) some properties that are preserved by change of base.

Proposition 3.3 (Behaviour regarding change of base).

- Open maps, those whose inverse image functor f^* is sub-cartesian closed (*i.e.* $f^*(B^A) \to (f^*B)^{f^*A}$ is monic) and sub-logical ($f^*\Omega_{\mathcal{E}} \to \Omega_{\mathcal{F}}$), are preserved by change of base.
- Atomic maps are preserved by pullback (since a map $f : \mathcal{E} \to \mathcal{S}$ is atomic if and only if it is open and the projections $f : \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \rightrightarrows \mathcal{E}$ are also open).
- Locally connected maps are preserved by change of base.

Let us note (this will be of slight importance later) that surjections are not necessary preserved by change of base.

We now take a look at the construction of colimits in the category of toposes with bounded morphisms. We will work over a base $\mathcal{S}$.

3.2 The construction of finite colimits in $\mathfrak{BTop}/\mathcal{S}$

Coproducts are easily constructed; the underlying category is the product in $\mathfrak{Cat}/\mathcal{S}$. Thus, this subsection is devoted to constructing coequalisers of pairs of morphisms $\mathcal{F} \rightrightarrows \mathcal{E}$. Unfortunately, the situation is much more complicated than that of limits. Indeed, we do not know whether coequalisers exist either in $\mathfrak{BTop}$, or even in $\mathfrak{BTop}/\mathcal{S}$, if S is arbitrary. We will assume in this subsection that S has a natural number object (as far as our set-theoretic style reasoning goes, this allows us to index families of objects over the natural numbers). Before we delve into the formal construction, we will try to give an intuition of why the situation is not as simple as we might expect.

Intuitively, a coequaliser is some kind of quotient object. If we look at the sites of $\mathcal{F}$ and $\mathcal{E}$ as theories and the corresponding toposes as models of these theories, we are indeed building a model for a quotient; this may be an arbitrarily complicated; *e.g.* while equality on natural numbers is trivial to compute, equality on having the same number of prime factors is not. Therefore, we should not expect even the site of definition of a coequaliser to be simple.

We now state and prove the main theorem of this subsection:

Theorem 3.4. Let $f, g : \mathcal{F} \rightrightarrows \mathcal{E}$ be a diagram in $\mathfrak{BTop}/\mathcal{S}$, where $\mathcal{S}$ is assumed to have a natural number object; then the coequaliser of this diagram exists, and is given by the topos $\mathcal{G}$ whose objects are pairs $\langle X, h\rangle$ where X is an object of $\mathcal{E}$ and $h : f^*X \xrightarrow{\sim} g^*X$ and morphisms are arrows $u : \langle X, h\rangle \to \langle Y, k\rangle$ satisfying the obvious coherence condition. The geometric morphism $q : \mathcal{E} \to \mathcal{F}$ is defined by q^* being the forgetful functor.

Proof. We essentially follow an argument of Moerdijk ([5]), giving the precision that we need the base topos to have a natural number object. Let $\mathcal{C}$ and $\mathcal{D}$ be small sites in $\mathcal{S}$ for $\mathcal{E}$ and $\mathcal{F}$ respectively. Let us assume that $\mathcal{C}$ has finite limits[2], and that f and g are given by morphisms of sites $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, *i.e.* internal functors (in $\mathcal{S}$). We use Giraud's criterion to see that $\mathcal{G}$ is indeed a $\mathcal{S}$-topos. All conditions apart from the existence of an object of generators stem from the fact that $\mathcal{E}$ is an $\mathcal{S}$-topos, given f^* and g^* preserve colimits. Therefore we only need to prove that $\mathcal{G}$ has an object of generators.

For X an object of $\mathcal{E}$, let

$$P(X) = \varinjlim_{\substack{x \in X(C) \\ C \in \mathcal{C}}} \mathrm{Hom}(-, FC)$$

$$Q(X) = \varinjlim_{\substack{x \in X(C) \\ C \in \mathcal{C}}} \mathrm{Hom}(-, GC)$$

f^* and g^* are indeed the associated sheaves of $P(X)$ and $Q(X)$ respectively.

Now we enlarge $\mathcal{C}$ to get an object of generators for $\mathcal{G}$. Let I be the index set of a cover in $\mathcal{D}$, and let $\mathcal{C}_0 = \mathcal{C}$, and for $n \geqslant 1$, let $\mathcal{C}_{n+1}$ be the full subcategory of $\mathcal{E}$ whose objects are coproducts $\coprod_{i \in I} C_i$ where $C_i \in \mathcal{C}_n$. Let $\mathcal{C}_\infty$ be the category whose objects are coproducts $\coprod_{n \in \mathbb{N}} C_n$ where $C_n \in \mathcal{C}_n$ for all n. Finally, for all $n \in [\![0; \infty]\!]$, let $\widehat{\mathcal{C}}_n$ be the full subcategory of all quotients of $\mathcal{C}_n$. Clearly the above families can be indexed over $\mathcal{S}$, therefore $\widehat{\mathcal{C}}_\infty$ is indeed internal to $\mathcal{S}$. We

[2]again, there is no loss of generality, given up to enlargement of universe, we can find such a small site.

claim that the indexed family of objects (C, μ) where C is an object of $\widehat{\mathcal{C}_\infty}$ and $\mu : FC \xrightarrow{\sim} GC$ is a generating family for $\mathcal{G}$.

Assume $\alpha, \beta : \langle X, \theta \rangle \rightrightarrows \langle Y, \xi \rangle$ are arrows in $\mathcal{G}$ such that for all $u : (C, \mu) \to (X, \theta)$ with $C \in \widehat{\mathcal{C}_\infty}$, $\alpha u = \beta u$; we want to show that $\alpha = \beta$. Let $C_0 \in \mathcal{C}$, $x_0 \in X(C_0)$. x_0 determines an element $f^*(x_0) \in f^*(X)(FC_0)$ corresponding to the element of $P(X)(FC_0)$ given by id_{FC_0} at (x_0, C_0) in the colimit diagram. Also, $\theta_{FC_0}(x_0)$ is given by a family of elements $y_j \in Q(X)(D_j)$ for a cover $\{D_j \to FC\}_{j \in J}$ in $\mathcal{D}$. y_j, in turn, is represented by an arrow $g_j : D_j \to GC_j$ at the vertex (x_j, C_j) in the colimit diagram that defines $Q(X)(D_j)$. Let $C_1 = \coprod_{j \in J} C_j$; we have a map $x_1 = \{x_j\} : C_1 \to X$, whose image we note U_1, and $\theta \circ f^*(x_0)$ factors through $g^*(U_1)$, as in the following diagram:

$$\begin{array}{ccccc} & & f^*(C_0) & \xrightarrow{f^*(x_0)} & f^*(X) \\ & & \downarrow & & \downarrow \theta \\ g^*(C_1) & \twoheadrightarrow & g^*(U_1) & \rightarrowtail & g^*(X) \end{array}$$

Now we recursively define a sequence of objects $(C_n)_{n \in \mathbb{N}}$ such that for all n, C_n is an object of $\mathcal{C}_n$, and maps $x_n : C_n \to X$ with image U_n, such that we have a factorization:

$$\begin{array}{ccc} f^*(U_n) & \longrightarrow & f^*(X) \\ \downarrow & & \downarrow \theta \\ g^*(U_{n+1}) & \rightarrowtail & g^*(X) \end{array}$$

Let $U = \cup_n U_n \in \widehat{\mathcal{C}_n}$; θ restricts to a map $f^*(U) \to g^*(U)$, so that (U, θ) is a subobject of (X, θ) in $\mathcal{G}$; since $\alpha_{|U} = \beta_{|U}$, $\alpha(x_0) = \beta(x_0)$; since x_0 was arbitrary, we have $\alpha = \beta$.

We now know that $\mathcal{G}$ is a bounded $\mathcal{S}$-topos. It remains to show that it satisfies the universal property of a coequaliser. But this is trivial; $\mathcal{G}$ and $\mathcal{E}$ are toposes bounded over $\mathcal{S}$ and q^*, defined as the forgetful functor $\mathcal{G} \to \mathcal{E}$, obviously preserves colimits; therefore, by the Indexed Special Adjoint Functor theorem ([4]) it has an indexed right adjoint q_*, which makes the pair a geometric morphism. Now let $\mathcal{T}$ be a bounded $\mathcal{S}$-topos, $\Phi : \mathcal{E} \to \mathcal{E}$ a geometric morphism, and $k : f^* \Phi^* \xrightarrow{\sim} g^* \Phi^*$), so that $(\mathcal{T}, \Phi)$ satisfies the property of a coequaliser. The same argument shows that the functor $\delta^* : \mathcal{T} \to \mathcal{G}$ defined by $\delta^* A = \langle \Phi^* A, k_A \rangle$ is the inverse image part of a geometric morphism $\delta : \mathcal{G} \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{F} & \overset{f}{\underset{g}{\rightrightarrows}} & \mathcal{E} & \longrightarrow & \mathcal{G} \\ & & {\scriptstyle (\Phi,k)} \downarrow & \swarrow {\scriptstyle \delta} & \\ & & \mathcal{T} & & \end{array}$$

□

Let us remark that obtaining an explicit description of the subobject classifier for $\mathcal{G}$ is not easy unless we have more information on the maps.

Note that in the previous proof, we do not give explicit descriptions of q_* and δ_*; we will now explicitely construct the latter, and while we believe the method can be generalised to compute a large class of adjoints[3], we do not have a general theorem yet, thus will construct δ_* "by hand".

Recall that $\delta^* A = \langle \Phi^* A, k_A \rangle$, $k_A : f^*\Phi^* A \xrightarrow{\sim} g^*\Phi^* A$. We want to construct a right adjoint δ_* to δ^*, so that $\mathrm{Hom}(\langle \Phi^* A, k_A\rangle, \langle X, h\rangle) \simeq \mathrm{Hom}(A, \delta_*\langle X, h\rangle)$. A map in the left-side Hom-set is a map $\alpha : \Phi^* A \to X$ such that the following diagram commutes:

$$\begin{CD}
f^*\Phi^*A @>{f^*\alpha}>> f^*X \\
@V{k_A}VV @VV{h}V \\
g^*\Phi^*A @>{g^*\alpha}>> g^*X
\end{CD}$$

We transpose this diagram using the adjunction $\Phi^* \dashv \Phi_*$; first, recall that given two adjunctions $F^* \dashv F_*$, $G^* \dashv G_*$ and a natural transformation $\theta : F^* \to G^*$, we have a canonical natural transformation $\bar{\theta} : G_* \to F_*$ given by the composition $F_*\varepsilon^G \circ F_*\theta G_* \circ \eta^F G_*$, called the *mate* of θ across the adjunction. It is clear that if θ is an isomorphism, then $\bar{\theta}$ also is an isomorphism.

We get the following diagram:

$$\begin{CD}
A @>{\bar{\alpha}}>> \Phi_*X \\
@V{\bar{\alpha}}VV @VV{\Phi_*\eta^f}V \\
\Phi_*X @. \Phi_*f_*f^*X \\
@V{\Phi_*\eta^g}VV @VV{\Phi_*f_*h}V \\
\Phi_*g_*g^*X @>>{\Phi_*\bar{k}_X}> \Phi_*f_*g^*X
\end{CD}$$

which is commutative by naturality (this is tedious but rather straightforward to check).

Now let $\delta_*\langle X, h\rangle$ be the equaliser of the above diagram, *i.e.*:

$$\begin{array}{ccccccc}
 & & & \Phi_*g_*g^*X & & & \\
 & & \overset{\Phi_*\eta^g}{\nearrow} & & \overset{\bar{k}}{\searrow} & & \\
\delta_*\langle X, h\rangle & \longrightarrow & \Phi_*X & & & & \Phi_*f_*g^*X \\
 & & \underset{\Phi_*\eta^f}{\searrow} & & \underset{\Phi_*f_*h}{\nearrow} & & \\
 & & & \Phi_*f_*f^*X & & &
\end{array}$$

then δ_* is the right adjoint to δ^* (naturality is given by the universality of the equaliser).

Comparing this to the description of the direct image of the localisation morphism, we remark that the main idea is to translate the adjunction into a suitable commutative diagram for which we can define the desired adjoint

[3]partly because a similar method is used to compute the direct image of the localisation morphism $\mathcal{E}/X \to \mathcal{E}$.

functor to be a limit. Given the rather "automatic" computation involved, we believe there may be a general adjoint functor theorem hidden behind.

We now turn to the study of the fundamental groupoid of a topos, which will use some of the above constructions.

4 The fundamental groupoid of a topos

In the case of topological spaces, the fundamental group (respectively groupoid), $\pi_1 : \mathbf{Top} \to \mathbf{Grp}$ (respectively $\pi_1 : \mathbf{Top} \to \mathbf{Grpd}$) is the functor that maps a topological space to its group of loops (respectively paths) up to homotopy. We do not recall the theory of the fundamental group here (which is not technically that useful in our context, but will help build our intuition), but we will study how we can generalise the notion to toposes. In this section, we will work most of the time with toposes bounded over $\mathcal{S}et$, that is, we work in the category $\mathfrak{BTop}/\mathcal{S}et$. Some constructions apply in a more general context, but given the rather unclear status of the subject, we would rather try and focus on understanding the general concepts and ideas rather than immediately delve into the technicalities that appear when dealing with an arbitrary base, or even more so an unbounded topos.

We will start, perhaps unusually, from the most general theorem we have, and then give more precise but less general results; we believe that even pedagogically, it is a better way to understand the (unfinished and certainly unpolished!) theory of the fundamental groupoid, because of the relative simplicity of the general theorem, and because the links between it and its particular instances are not that clear.

The most general theorem we have is the following:

Theorem 4.1 (The fundamental theorem of generalised Galois theory). Let $f : \mathcal{E} \to \mathcal{S}$ be an arbitrary (bounded) topos. Then there exist a localic[4] groupoid G such that $\mathcal{E}$ is equivalent to the topos $\mathcal{B}G$ of G-sets.

The reader is referred to [6] for a proof; the idea is to find a topos $\mathcal{X}$ and an open surjection $\mathcal{X} \to \mathcal{E}$; the central result to use is that open surjections are effective descent morphisms. The localic groupoid is then obtained in the descent data.

We may want to define the *fundamental groupoid* of a topos to be the[5] groupoid G which gives the equivalence mentioned above. This is rather in line with the topos theoretic interpretation of Galois theory; if $\widetilde{X}$ is the étale topos of $X = \text{Spec } k$, it has been known since [1] that $\widetilde{X}$ is equivalent to $\mathcal{B}G$ where G is the Galois group of the separable closure of k.

A very important example is that of Galois toposes, *i.e.* connected and locally connected toposes generated by locally constant sheaves. The first result on the fundamental groupoid of toposes was established in [1], in this context. The vocabulary there is slightly different as the concept of a topos had not been introduced yet; the context was that of "Galois categories", which are actually Galois toposes, but were considered merely as particular categories equipped

[4] a locale L is simply a complete Heyting algebra; for intuition we may often reason as if it were a topological space – at least in presence of *points*, *i.e.* morphisms $\mathbf{2} \to L$

[5] actually, a representative of the equivalence class of groupoids satisfying the property; indeed, $\mathcal{B}G \simeq BG'$ does not imply $G \simeq G'$; we will see examples later.

with a functor to $\mathcal{S}et$ satisfying some properties. We will state the above result in a slightly more general case, that of an atomic topos with a point, and we will sketch a proof different from ([1]), so that the link with classical Galois theory is more apparent.

4.1 The fundamental group of an atomic topos with a point

Definition 4.2. We denote by $\int F$ the category of elements of F, and D_F the poset obtained by identifying all arrows in each Hom-set of $\int F$. For each object X of $\mathcal{E}$ we write $\lambda_X : FX \to D_F$ defined by $x \mapsto (X, x)$.

Definition 4.3. Natural relations A *natural relation* between F and G is an $R \hookrightarrow F \times G$, *i.e.* a family of relations $RX \hookrightarrow FX \times GX$ such that if we have $f : X \to Y$ and then $(F(f) \times G(f))RX \subset RY$. If R is a function, clearly it is a natural transformation. Alternatively, one can see a relation as a family of functions $\Phi_X : FX \times GX \to 2$ such that $\Phi_X(x_0, x_1) \leqslant \Phi_Y \circ (F(f)x_0, G(f)x_1)$.

Note 4.4. A morphism of posets $D_{\Delta F} \to 2$ corresponds to a natural relation on F.

Theorem 4.5. Let $\gamma : \mathcal{E} \to \mathcal{S}et$ be a connected atomic topos with a point. Then $\mathcal{E}$ is equivalent to the topos of G-sets, where G is the localic group of automorphisms of any base point of $\mathcal{E}$.

Proof. We will omit some tedious calculations, for which we refer to [10]. Let $p : \mathcal{S}et \to \mathcal{E}$ be the base point of $\mathcal{E}$, and let $\mathcal{C}$ be a site for $\mathcal{E}$ such that $F : \mathcal{C} \to \mathcal{S}et$ yields the point p; since $\mathcal{E}$ is atomic, it can be taken to consist only of connected objects with all arrows being epimorphisms, and F can be taken so that it reflects isomorphisms and preserves strict epimorphisms; since $\mathcal{E}$ is atomic, F can also be taken so that for all X, $FX \neq 0$. Since F yields a point, its category of elements is cofiltered, and a poset.

Let $\mathcal{D}(D_{\Delta F})$ be the free-inf lattice on $D_{\Delta F}$. We write $[\langle a_1 \rangle, \ldots, \langle a_n \rangle]$ for the object of $\mathcal{D}(D_{\Delta F})$ corresponding to the subset $(a_1, \ldots, a_n)$ of $D_{\Delta F}$.

Since $\mathrm{lRel}(X) = \mathcal{L}(X \times X) = \widehat{\mathcal{D}}(X \times X)$, natural relations on F correspond to points of the locale $\widehat{\mathcal{D}}(D_{\Delta F})$. Now we add covers forcing a relation to be a bijection; for the precise definition of these covers, we refer the reader to [10]; for instance, the covers generated by $\varnothing \to [(X, \langle x|z \rangle), (X, \langle y|z \rangle)]$ force the relation to be injective. Indeed, the object $[(X_1, \langle x_1|y_1 \rangle), \ldots, (X_n, \langle x_n|y_n \rangle)]$ corresponds to the set $\{\Phi : F \to F | \Phi_{X_i}(x_i) = y_i\}$.Taking sheaves on the resulting site, we get the localic group of automorphisms of F, which we write $\mathrm{lAut}(F)$.

The composition $FX \times FX \overset{\lambda_X}{\to} \mathcal{D}(D_{\Delta F}) \overset{\mathbf{a}}{\to} \mathrm{lAut}(F)$ defines a morphism of locales $\mathrm{lAut}(FX) \to \mathrm{lAut}(F)$, *i.e.* an *action* of $\mathrm{lAut}(F)$ on FX[6]. We write $\mathrm{lFix}(x)$ for the object in $\widetilde{\mathcal{D}}(D_{\Delta F}) = \mathrm{lAut}(F)$ corresponding to $[(X, \langle x|x \rangle)]$. A *morphism of G-sets* for a localic group G is a function $X \to Y$ such that $\mu^*\langle x|y \rangle \leqslant \mu^*\langle f(x)|f(y) \rangle$.

Lemma 4.6. For any $X \in \mathcal{C}$, $(x_0, x_1) \in FX \times FX$, the empty family does not cover $[(X, \langle x_0|x_1 \rangle)]$.

[6]more generally, an action of an localic group G on a set X is a continous morphism of localic groups $G \to \mathrm{lAut}(X \times X)$, which is determined by the values of its inverse image $\mu^* : X \times X \to G$ on the generators of $\mathrm{lAut}(X)$.

The reader is referred to [10] for a proof.

Lemma 4.7 (Lifting lemma). Let X and Y be objects of $\mathcal{C}$, and $x \in FX$, $y \in FY$. If $\mathrm{lFix}(x) \leqslant \mathrm{lFix}(y)$ then there exists a unique arrow $f : X \to Y$ such that $F(f)x = y$.

Again, we refer the reader to [10] for a proof. The central argument is that the canonical functor $D_{\Delta F} \to \mathrm{lAut}(F)$ is full.

Lemma 4.6 implies that $[(X, \langle x_0 | x_1 \rangle)]$ is not 0 in $\mathrm{lAut}(F)$. This proves, in turn, that the action of $\mathrm{lAut}(F)$ on FX is transitive.

The paragraph before the lemma shows that F lifts into a functor $\mu F : \mathcal{C} \to \mathcal{B}G$, for $G = \mathrm{lAut}(F)$. Now, the lemma says that in fact this functor lands in $t\mathcal{B}G$, the category of *transitive* G-sets. Now F is faithful; indeed, take $f : X \to Y$ in $\mathcal{C}$ such that $F(f)$ is an isomorphism. Let $z, t : Z \rightrightarrows X$ be arrows such that $fs = ft$; then $F(s) = F(t)$. Take z in FZ and let $x = F(s)(z) = F(t)(z)$; s and t define in this way arrows $(Z, z) \to (X.x)$ in $\int F$; since $\int F$ is a poset, $s = t$, so F is faithful. Therefore, μF is faithful.

But μF is also full, thanks to the lifting lemma: let $f : (\mu F)X \to (\mu F)Y$ be a morphism of G-sets, take x_0 in FX and let $y_0 = f(x_0)$. By definition of a morphism of G-sets, $\mathrm{lFix}(x_0) \leqslant \mathrm{lFix}(y_0)$ in $\mathrm{lAut}(F)$. By the lifting lemma, we get the desired result.

There remains to show that for any transitive $\mathrm{lAut}(F)$-set S, there exist an $X \in \mathcal{C}$ and a strict epimorphism $(\mu F)X \to S$ in $t\mathcal{B}G$. This is follows from the fact that $\int F$ is cofiltered (we refer the reader to [10] for details). By the comparison lemma, given that $\mathcal{B}G$ is the topos of sheaves for the canonical topology on $t\mathcal{B}G$, we get that $\mathcal{E} \xrightarrow{\sim} \mathcal{B}G$, where G can be taken to be $\mathrm{lAut}(F) = \mathrm{lAut}(p)^{\mathrm{op}}$. □

It is reasonable to *define* the fundamental group of a connected atomic topos with a point p to be $\mathrm{lAut}(p)^{\mathrm{op}}$. In fact, one can take the groupoid of all points, and the result remains. In both cases, this assignement is obviously functorial[7]. In the non-connected case, the result still holds; we write $\mathcal{E}$ as a sum of connected atomic toposes, and apply the theorem to each connected component, then "glue" the result.

Note that the result is indeed a particular case of 4.1, since a point of a connected and atomic topos is an open surjection.

Note 4.8. The example of classical Galois theory (*i.e.*, of the fundamental group of the étale topos of a field) is an instance of this theorem.

Note 4.9. We shall not discuss topos cohomology here, though that is a very interesting subject which would undoubtedly provide technical help. We would like to mention that it is proven ([7]) that in the case of an atomic topos with a point, the fundamental group represent first order cohomology, *i.e.* classifies torsors. That is, for any abelian group K, $\mathrm{Hom}(G, K) \simeq H^1(\mathcal{E}, K)$.

[7] if we are working with a base point, we need to work in the category of connected atomic pointed toposes, *i.e.* pairs $(\mathcal{E}, p)$ of a topos and a base point.

We will now give an overview of what happens in the unpointed case; we will need to first give a few definitions and results on Galois toposes, then sketch a theorem and discuss why we think this is a rather different result.

4.2 The fundamental groupoid of an unpointed locally connected topos

Let us recall a few definitions and result on Galois toposes; these can be found for instance in [11].

Definition 4.10 (Locally constant objects, Galois toposes).

- An object X of a bounded topos $\gamma : \mathcal{E} \to \mathcal{S}et$ is said to be U-split for a cover U (*i.e.* an epimorphic family $\{U_i \to 1\}_{i \in I}$) if it is constant on each U_i. That is, there exist a family of sets $\{S_i\}_{i \in I}$ and isomorphisms in $\mathcal{E}$, $\{\gamma^* S_i \times U_i \to X \times U_i\}_{i \in I}$.
- An object X in a topos is said to be *locally constant* if there exists a cover $U \to 1$ such that it is U-split.
- An object A in a bounded topos $\gamma : \mathcal{E} \to \mathcal{S}et$ is said to be a *Galois object* if it is connected (*i.e.* it is not a nontrivial coproduct) and is an $\mathrm{Aut}(A)$-torsor, that is, $A \to 1$ is epi and $A \times \gamma^* \mathrm{Aut}(A) \to A \times A$ is an isomorphism.
- A topos is said to be a *Galois topos* if it connected, locally connected, and generated by its Galois objects.

Proposition 4.11. A Galois topos has enough points.

Theorem 4.12. Let $\gamma : \mathcal{E} \to \mathcal{S}et$ be a connected and locally connected topos. There exists a Galois topos, which we will denote $\mathcal{G}(\mathcal{E})$, and a geometric morphism $\mathcal{E} \to \mathcal{G}(\mathcal{E})$ whose inverse image is given by inclusion, such that $\mathcal{G}(\mathcal{E})$ is the topos of sums of locally constant objects in $\mathcal{E}$.

Proof. Let $U \to 1$ be an epimorphism in $\mathcal{E}$ and consider the following pushout diagram:

$$\begin{array}{ccc} \mathcal{E}/U & \xrightarrow{\varphi_U} & \mathcal{E} \\ \downarrow{\scriptstyle \rho_U} & & \downarrow{\scriptstyle \sigma_u} \\ \mathcal{S}et/\gamma_! U & \xrightarrow{f_U} & \mathcal{S}_U \end{array}$$

ρ_U and φ_U are the canonical morphism, defined respectively by $\rho_U^*(S \to \gamma_! U) = \gamma^* S \times_{\gamma^* \gamma_! U} U$ and $\varphi_U^*(X) = (X \times U \to U)$.

Now, this diagram being a pushout of toposes, objects of $\mathcal{S}_U$ are triples $(X, S \to \gamma_! U, \theta)$ where $\theta : X \times U \xrightarrow{\sim} \gamma^* S \times_{\gamma^* \gamma_! U} U$ is an isomorphism over U, and a morphism in $\mathcal{S}_U$ is a pair of morphisms $X \to X'$, $S \to S'$, the latter being over U, satisfying the obvious coherence conditions. Then σ_U^* is the projection functor $\mathcal{S}_U \to \mathcal{E}$, which is full and faithful. Therefore $\mathcal{S}_U$ is in fact the topos of U-split objects of $\mathcal{E}$.

Clearly, a morphism $U \to V$ where $U \to 1$ and $V \to 1$ are epimorphisms yields a geometric morphism $\mathcal{S}_V \to \mathcal{S}_U$ whose inverse image is the inclusion functor. We will admit that the poset of coverings $\{U_i \to 1\}$ has a small cofinal system ([9]), which yields a filtered inverse limit of toposes. It is clear that this inverse limit is the topos of sums locally constant objects of $\mathcal{E}$, which is a Galois topos. The geometric morphism $\mathcal{E} \to \mathcal{G}(\mathcal{E})$ is given by the colimit diagram. □

Since $\mathcal{G}(\mathcal{E})$ is a Galois topos, we can apply the result of the previous subsection to it. This yields a group(oid) G, which Bunge and Moerdijk ([8]) suggest as the fundamental groupoid of $\mathcal{E}$, claiming it represent first-order cohomology of $\mathcal{E}$, and citing [7][8] (and not only of $\mathcal{G}(\mathcal{E})$ as we would expect).

Let us try to analyse this result in its general setting. First, let us note that it is not clear how this is a particular instance of Theorem 4.1. While Theorem 4.1 can certainly be applied to $\mathcal{E}$, it will yield a groupoid G' such that $\mathcal{E} \to \mathcal{B}G'$; there is no clear link between G and G'. Much more disturbing, the fundamental group of $\mathcal{E}$, as we just defined following [8] is the same as that of $\mathcal{G}(\mathcal{E})$; we do not understand why the fundamental group should only depend on (sums of) locally constant sheaves, considering for instance the case of a locally connected topos which cannot be described solely by its locally constant sheaves.

Furthermore, we have not unveiled a transitive action of the localic groupoid on any meaningful set; the only case we are provided canonically with such an action is when we work with a base point. However, we must stress that we do not have such an action in Theorem 4.1; it is not clear to the author whether such an action is indeed necessary data of what we would call "a Galois theory"; we certainly do not want to assume the existence of points, so the question boils down to whether we can find a description of the action of the fundamental groupoid on objects that are intrisic to the topos. In the case of a Galois topos, the data is contained in the Galois objects, since they (pro)represent the fibre functors, so we may suggest that the corresponding data in the case of an atomic topos is in the atoms, though we do not have an formal description of the action unless we assume points again.

We finish this analysis by mentioning we can always reduce the situation, if $\mathcal{E} \to \mathcal{S}$ is atomic, to a case where we have a more explicit result. Indeed, we can find a localic cover of $\mathcal{E}$, for instance, to preserve functoriality, the Diaconescu cover $D\mathcal{E}$, defined as the topos of sheaves on the site whose underlying category $\mathbf{String}(\mathcal{C})$ is the poset of words labeled by composable arrows of $\mathcal{C}$, $\mathcal{C}$ being a site for $\mathcal{C}$ (let $s = C_n \to \ldots \to C_0$, $t = C_l \to \ldots \to C_0$, we say that $t \leqslant s$ if t prolongs s to the left). We have a canonical functor $\pi : \mathbf{String}(\mathcal{C}) \to \mathcal{C}$, *viz.* the projection functor. We define a topology on $\mathbf{String}(\mathcal{C})$ saying that a sieve U on s is a cover if and only if for any $t \leqslant s$, the set of arrows $\pi(t' \leqslant t) : \pi(t') \to \pi(t)$ where $t' \in U$ is a cover of $\pi(t)$ in $\mathcal{C}$.

Now we can form the fiber product $\mathcal{E} \times_{\mathcal{S}} D\mathcal{E} \to D\mathcal{E}$ which is bounded and atomic since $D\mathcal{E} \to \mathcal{S}$ is clearly bounded and $\mathcal{E} \to \mathcal{S}$ is atomic. π yields a $D\mathcal{E}$-point of $\mathcal{E} \times_{\mathcal{S}} D\mathcal{E}$, so that we can apply theorem 4.5. We then have the issue of translating results over $D\mathcal{E}$ to results over $\mathcal{S}$. It is yet unclear whether this would yield interesting results; certainly an explicit comparison of the cohomology (at least of first-order) of $\mathcal{E} \times_{\mathcal{S}} D\mathcal{E} \to D\mathcal{E}$ and $\mathcal{E} \to \mathcal{S}$ would

[8] Unfortunately, we have been unable to find any written proof of this non-trivial fact, and while this work was conducted, the author of the cited paper was away from her printed copies.

be helpful in understanding whether we retain enough information through this change of base.

5 Conclusion and future work

We hope to have given a reasonable overview of a few problems in topos theory. The reader should be convinced, by now, that there is much to be done in the subject, and that there are problems of very different natures. As far as limits and colimits are concerned, we think it is important to have a better intuition of what exactly the (co)limits are[9], in term of toposes as well as of sites, because it is fairly unlikely that the problem is conceptual. We hope to be able to find a proper generalisation of the method exposed to compute the direct image of the geometric morphism δ which would allow us to have explicit descriptions of large classes of (co)limits of toposes.

It is quite disappointing that the fibre product of toposes is of little help in algebraic geometry, because it suggests the aforementioned toposes may well not be the proper topological frameworks. We do not know currently whether some other construction might be more appropriate.

The current state of the Galois theory of toposes is rather disturbing, as we hope to have shown without confusing the reader. It seems there is a fundamental misunderstanding of what should be called Galois theory[10] or the fundamental groupoid of a topos; in the case of topological spaces, points play an essential role; however their obvious topos theoretic counterparts should not be assumed to exist, as they do not in even quite simple non-geometrical cases (such as the topos of sheaves on an atomless boolean algebra). It might be useful to try an approach to points of a topos that is closer to that of points of a scheme; that is, work with the "functor of points" $\mathrm{Hom}(-, \mathcal{E})$ of a topos. We do not know of anyone who has tried such an approach, but it seems clear some constructions can be adapted to rely on such "points": a group of automorphisms is trivially definable, for instance, but it remains to see what properties it could have.

We conclude on the fact that what we have introduced is indeed a very small aspect of topos theory, and that many other approaches to the subject are possible and interesting, at all scales; we have for instance ignored all purely logical questions (even in concerning we have introduced), and focused on a rather topological view of the situation. Even retaining a largely topological or geometrical approach, many open problems remain, and there is undoubtedly a wealth of aspects of topos theory which remain to be discovered. Among these, there is ongoing work on the homotopy theory of toposes ([13]) which would certainly be of interest to the author.

[9]perhaps by working out more examples; it turns out that even in simple cases, examples are often quite hard to get a grasp on.

[10]Rather, it seems there is no consensus on what Galois theory is; some believe it refers to points, possibly "phantom" ([11]), others think it is contained in the equivalence with a topos of G-sets.

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