Correlation Clustering and Two-edge-connected Augmentation for Planar Graphs

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My Hobby:
Embedding NP-complete problems in restaurant orders

Chotchkies Restaurant

Appetizers
Mixed Fruit 2.15
French Fries 2.75

Salad 5.35
Hot Wings 3.55

Mozzarella Sticks 4.20
Sampler Plate 5.30

Sandwiches
Barbecue

We'd like exactly $15. 05 worth of appetizers, please.

...Exactly? Uhh...

Here, these papers on the knapsack problem might help you out.

Listen, I have six other tables to get to.

As fast as possible, of course. What something on the traveling salesman?
Approximation Schemes for Planar Graphs

- **Local problems (1977-1983):**
  independent set, vertex cover, dominating set, etc.

- **Connectivity problems (2005-2011):**
  TSP, Steiner, 2-edge-connected subgraph, etc.

- **Other problems (2012-2014):**
  multiway cut, k-center.

- **In our work:**
  correlation clustering, 2-edge-connected augmentation.
Correlation Clustering

**Input:** a graph with edge-weights, where every edge is labelled either $\langle + \rangle$ or $\langle - \rangle$ according to similarity of its endpoints

**Output:** a partition of the vertices that disagrees with the edge labels as little as possible

Motivated by **image segmentation**.

(an example from Berkeley Segmentation Dataset)
Correlation Clustering

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**Our Result**

PTAS for correlation clustering in planar graphs

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**Previous results:**

- **Constant-factor approximation for minor-excluded graphs**
  [Demaine, Emanuel, Fiat, Immorlica, 2006]

- **NP-hardness for planar graphs**
  [Bachrach, Kohli, Kolmogorov, Zadimoghaddam, 2013]

- **APX-hardness for general graphs** [Bansal, Blum, Chawla, 2004]
Two-Edge-Connected Augmentation

Goal: Augment input subset $R$ of edges of a graph so that the result is a collection of 2-edge-connected components.

Our Result
PTAS for two-edge-connected augmentation in planar graphs

Previous results for general graphs:
- APX-hardness [Kortsarz, Krauthgamer, Lee, 2004]
- 2-approximation [Jain, 2001]
Techniques

Reduction from correlation clustering to 2-edge-connected augmentation
- planar duality

PTAS for 2-edge-connected augmentation
- prize-collecting clustering
- brick decomposition
- sphere-cut decomposition
- dynamic programming
Reduction from correlation clustering to 2-edge-connected augmentation:

1. Start from an instance of correlation clustering
2. Construct an instance of 2-edge-connected augmentation:
   - the graph is the dual of the correlation clustering graph
   - the set $R$ contains the duals of the $\langle - \rangle$ edges
3. Solve the 2-edge-connected augmentation instance
4. Switch the labels of the dual edges of the augmentation to obtain a partition
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4. Switch the labels of the dual edges of the augmentation to obtain a partition
Observation: A partition of $V[G]$ has a 1-to-1 correspondence to a collection of 2-edge-connected components of $G^*$. 
Reduction

- Switch a $\langle + \rangle$ edge to a $\langle - \rangle$ edge $\iff$ add an edge in the dual
- Switch a $\langle - \rangle$ edge to a $\langle + \rangle$ edge $\iff$ remove an edge in the dual $\iff$ double that edge in the dual
- Only need to find an augmentation in the dual so that every component is 2-edge-connected.
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Theorem

*There is an approximation-preserving reduction from correlation clustering to 2-edge-connected augmentation.*
Approximation Scheme

Universal paradigm

1. Reduce to bounded treewidth
2. Solve the problem by dynamic programming

Q: How do we reduce to bounded treewidth?

A: For local problems, Baker’s basic framework:
   1. Breadth First Search
   2. Delete edges on one level every $1/\varepsilon$ levels
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Klein’s Dual Framework for Non-Local Problems

1. Compute a brick decomposition such that
   - The total weight of brick boundaries is \( O(OPT) \);
   - **Structure Property:** A near-optimal solution inside each brick is simple.

2. Breadth First Search on the dual graph of the brick graph

3. Delete or contract edges on one level every \( 1/\epsilon \) levels

Ensure that the brick graph has bounded treewidth.
Our Contribution: A New Structural Property on Bricks

Structure Property

For any two vertices $u, v$ on the boundary of a brick, there exists a $u$-to-$v$ Jordan curve inside the brick that intersects the near-optimal solution at only a constant number of points.
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**Structure Property**

For any two vertices $u, v$ on the boundary of a brick, there exists a $u$-to-$v$ Jordan curve inside the brick that intersects the near-optimal solution at only a constant number of points.

**Theorem**

*We have a PTAS for 2-edge-connected augmentation in planar graphs.*
Conclusion

- **Local problems (1977-1983):**
  independent set, vertex cover, dominating set, etc.

- **Connectivity problems (2005-2011):**
  TSP, Steiner, 2-edge-connected subgraph, etc.

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- **Open problem:**
  Steiner version of the 2-edge-connected subgraph problem.
Thank you!