Capacitated Vehicle Routing in Graphic Metrics

Tobias Mömke
University of Augsburg, Germany

Hang Zhou
École Polytechnique, France

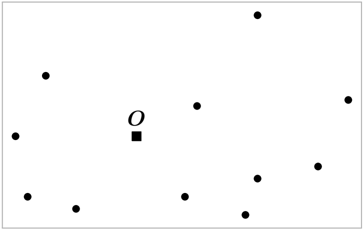
SIAM Symposium on Simplicity in Algorithms (SOSA23)

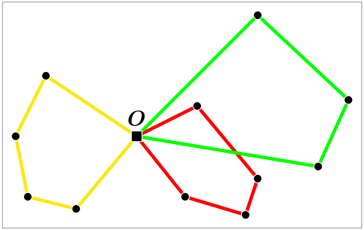
Capacitated vehicle routing problem (CVRP)

Input:

- **depot O**
- **set of n terminals**
- **capacity k**

Minimize total length of tours





Fundamental problem in operations research

Capacitated vehicle routing problem (CVRP)

Unit demands

- **1985** Haimovich and Rinnooy Kan
- **1990** Altinkemer and Gavish
- **1997** Asano, Katoh, Tamaki, and Tokuyama
- **1998** Hamaguchi and Katoh
- **2001** Asano, Katoh, and Kawashima
- **2006** Bompadre, Dror, and Orlin
- **2010** Adamaszek, Czumaj, and Lingas
- **2010** Das and Mathieu
- **2017** Becker, Klein, and Saulpic
- **2018** Becker, Klein, and Saulpic
- **2018** Becker
- **2019** Becker, Klein, and Schild
- **2019** Becker and Paul
- **2020** Cohen-Addad, Filtser, Klein, and Le
- **2021** Blauth, Traub, and Vygen
- **2021** Mathieu and Zhou
- **2022** Jayaprakash and Salavatipour
- **2022** Jayaprakash and Salavatipour
- **2022** Mathieu and Zhou

Unsplittable demands

- **1981** Golden and Wong
- **1987** Altinkemer and Gavish
- **1991** Labbé, Laporte, and Mercure
- **2021** Blauth, Traub, and Vygen
- **2022** Friggstad, Mousavi, Rahgoshay, and Salavatipour
- **2023** Grandoni, Mathieu, and Zhou
- **2023** Mathieu and Zhou

- **general metrics**
- **Euclidean plane**
- **planar graphs**
- **trees**
- **graphs of bounded treewidth**
- **graphs of bounded highway dimension**

Results in general metrics

α : **approximation ratio for TSP**

CVRP in general metrics

- $1 + \left(1 - \frac{1}{k}\right) \cdot \alpha$ **factor [Altinkemer and Gavish, 1990]**
- $1 + \left(1 - \frac{1}{k}\right) \cdot \alpha - \Omega\left(\frac{1}{k^3}\right)$ **factor [Bompadre, Dror, Orlin, 2006]**
- $1 + \alpha - \frac{1}{3000}$ **factor [Blauth, Traub, Vygen, 2022]**

Graphic metrics

Let $G = (V, E)$ be a connected and unweighted graph.

For each $(u, v) \in V^2$, define

$$\delta(u, v) := \text{number of edges on a shortest } u\text{-to-}v \text{ path}.$$

δ is called a graphic metric.

Results in graphic metrics

TSP in graphic metrics

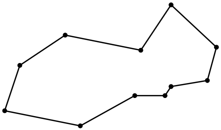
- $1.5 - \epsilon_0$ factor for some $\epsilon_0 > 0$ [Oveis Gharan, Saberi, Singh, 2011]
- 1.461 factor [Mömke and Svensson, 2011]
- $\frac{13}{9}$ factor [Mucha, 2012]
- 1.4 factor [Sebő and Vygen, 2014]

CVRP in graphic metrics

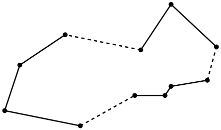
- $2.4 - \frac{1}{3000}$ factor
  [Sebő and Vygen, 2014] + [Blauth, Traub, Vygen, 2022]
- 1.95 factor [our result]

Iterated tour partitioning [Haimovich and Rinnooy Kan 1985]

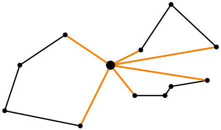
1. **Compute a TSP tour S on all terminals**



2. **Partition S into segments of k terminals**



3. **For each segment, connect its endpoints to the depot**



Fact: **solution** $\leq S + \mathbf{rad}$, where $\mathbf{rad} := \frac{2}{k} \cdot \sum_{v \in V} \delta(O, v)$.

CVRP in graphic metrics – Algorithm

Algorithm for CVRP in graphic metrics

1. $S_1 \leftarrow$ **TSP tour computed by Christofides algorithm**
2. $S_2 \leftarrow$ **TSP tour computed by Sebő-Vygen algorithm**
3. $S \leftarrow$ **cheaper one of** S_1 **and** S_2
4. **Apply iterated tour partitioning on** S

CVRP in graphic metrics – Analysis

Lemma

solution $\leq \mathbf{rad} + \mathbf{0.5} \cdot \boldsymbol{n} + \mathbf{0.95} \cdot \mathbf{opt}.$

Proof. Combining:

- **solution** $\leq \boldsymbol{S} + \mathbf{rad}$
- $\boldsymbol{S} \leq \frac{1}{2}(\boldsymbol{S_1} + \boldsymbol{S_2})$
- $\boldsymbol{S_1} \leq \boldsymbol{n} + \mathbf{0.5} \cdot \mathbf{opt}$ [Christofides]
- $\boldsymbol{S_2} \leq \mathbf{1.4} \cdot \mathbf{opt}$ [Sebő-Vygen] □

Structure Theorem

$\mathbf{opt} \geq \mathbf{rad} + \mathbf{0.5} \cdot \boldsymbol{n}.$

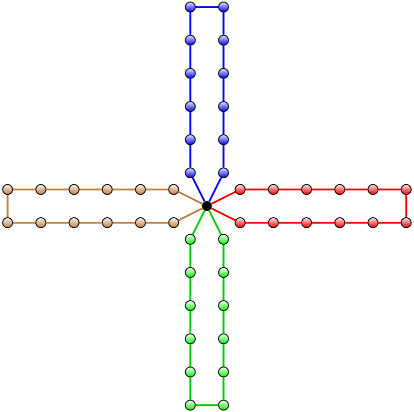
Main result

1.95-approximation for CVRP in graphic metrics.

Tightness of the Structure Theorem

Structure Theorem

$\mathbf{opt} \geq \mathbf{rad} + \mathbf{0.5} \cdot \boldsymbol{n}$.



$\boldsymbol{n} = \mathbf{48}$

$\boldsymbol{k} = \mathbf{12}$

$\mathbf{opt} = \boldsymbol{n} + \frac{\boldsymbol{n}}{\boldsymbol{k}}$

$\mathbf{rad} = \mathbf{0.5} \cdot \boldsymbol{n} + \frac{\boldsymbol{n}}{\boldsymbol{k}}$

Proof of the Structure Theorem

Structure Theorem

$\mathbf{opt} \geq \mathbf{rad} + 0.5 \cdot n.$

↑ **summing over all tours in an optimal solution**

Structure Lemma

Consider any tour T visiting a subset U of m terminals.

Then $\text{cost}\,(T) \geq 2L + 0.5 \cdot m$, where $L := \dfrac{1}{m} \cdot \sum_{v \in U} \delta(O, v)$.

Proof of the Structure Lemma (1/2)

$$H := \max_{v \in U} \{\delta(O, v)\}$$

$a_i :=$ **first terminal on** T **s.t.** $\delta(O, a_i) = i$
$b_i :=$ **last terminal on** T **s.t.** $\delta(O, b_i) = i$
$W := \{a_i\}_i \cup \{b_i\}_i$

Lemma 1

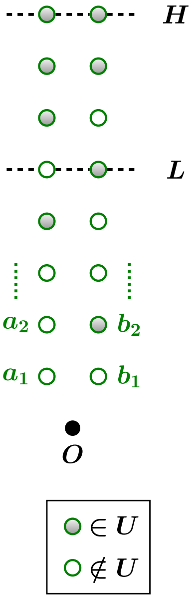
$\text{cost}\,(T) \geq 2H + |U \setminus W|$.

Lemma 2

$|U \cap W| \leq 2\sqrt{m \cdot (H - L)}$.

Combining:

$$\begin{aligned} \text{cost}\,(T) &\geq 2H + m - 2\sqrt{m \cdot (H - L)} \\ &\geq 2L + 0.5 \cdot m \end{aligned}$$



Proof of the Structure Lemma (2/2)

Lemma 2

$|\boldsymbol{U} \cap \boldsymbol{W}| \leq 2\sqrt{\boldsymbol{m} \cdot (\boldsymbol{H} - \boldsymbol{L})}$.

Proof:

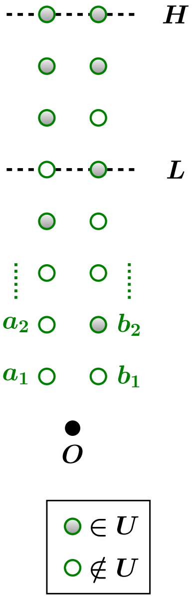
$$\sum_{v \in U \cap W} \delta(O, v) \leq |U \cap W| \cdot H - \frac{|U \cap W|^2}{4}$$

$$\sum_{v \in U \setminus W} \delta(O, v) \leq (m - |U \cap W|) \cdot H$$

Summing:

$$m \cdot L = \sum_{v \in U} \delta(O, v) \leq m \cdot H - \frac{|U \cap W|^2}{4}$$

□



Summary

Structure Theorem

$\mathbf{opt \geq rad + 0.5 \cdot n}$.

Main result

1.95-approximation for CVRP in graphic metrics.

Open question

Study the graphic TSP whose cost depends on n instead of opt.

Thank you!