Optimization of Bootstrapping in Circuits

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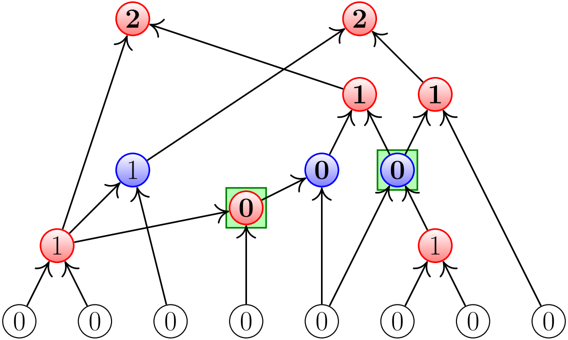
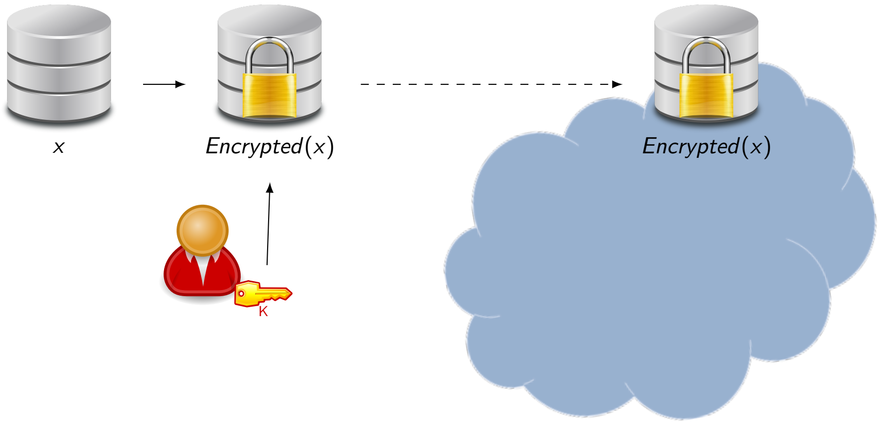


Motivation: Classical Encryption



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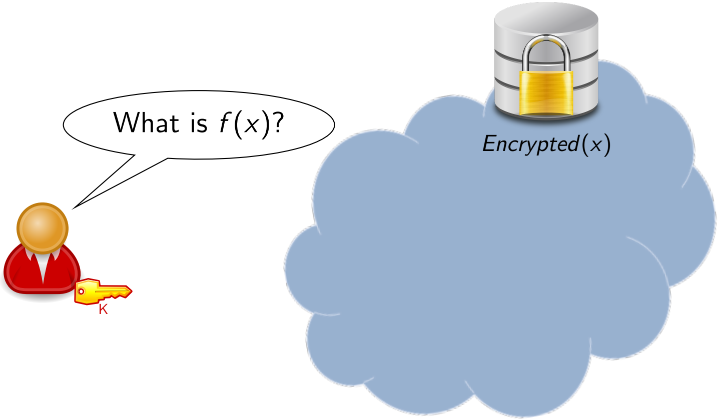

Motivation: Fully Homomorphic Encryption

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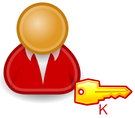
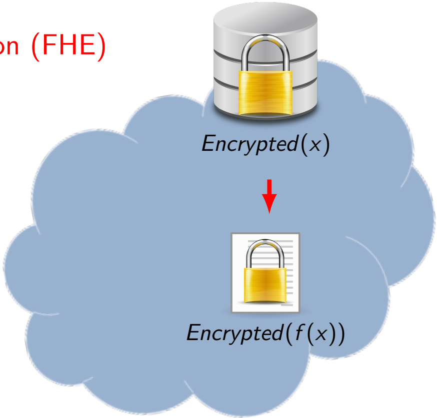
Fully Homomorphic Encryption (FHE)





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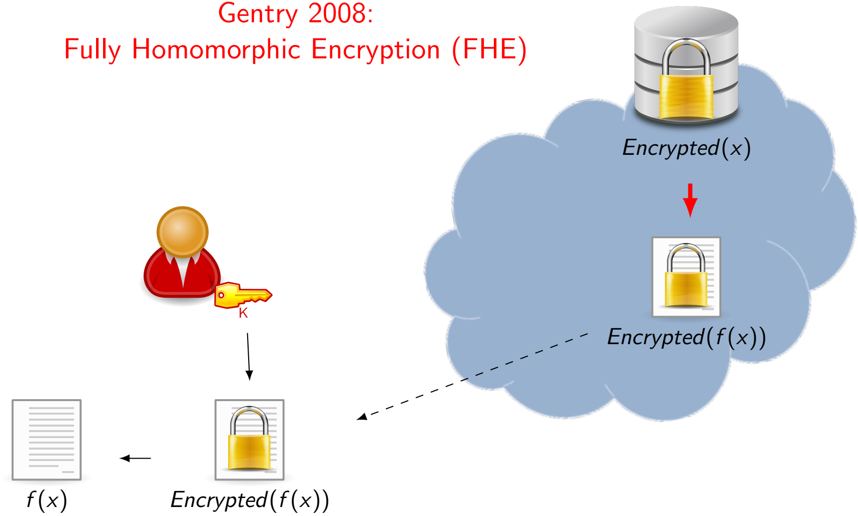
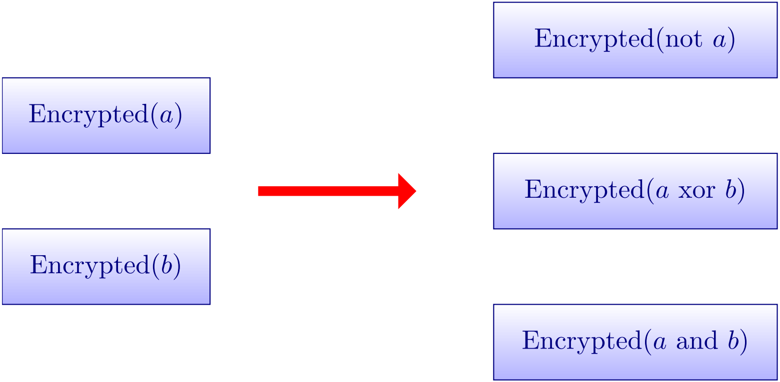
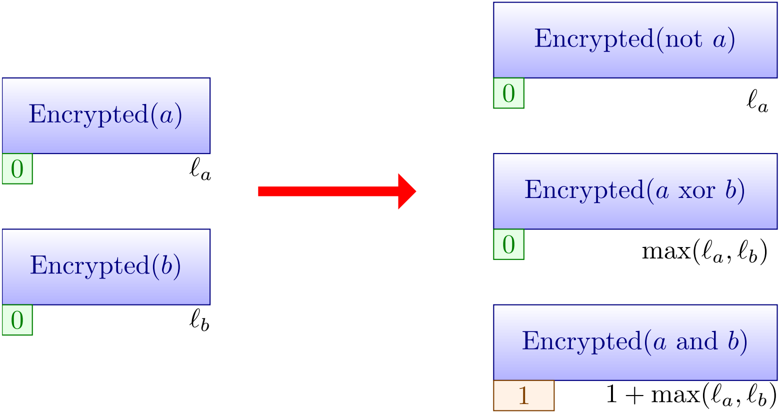


Motivation: Fully Homomorphic Encryption in Details



Motivation: Noise Level



Valid for decryption:

noise level within some parameter L ($L \approx 17$ in practice)

Motivation: Bootstrap Operations

Goal: Minimize the number of bootstrap operations

Bootstrap Problem

Input:

- a directed acyclic graph $G = (V, E)$ with two kinds of vertices:

 $\ell = \max(\cdot, \cdot)$

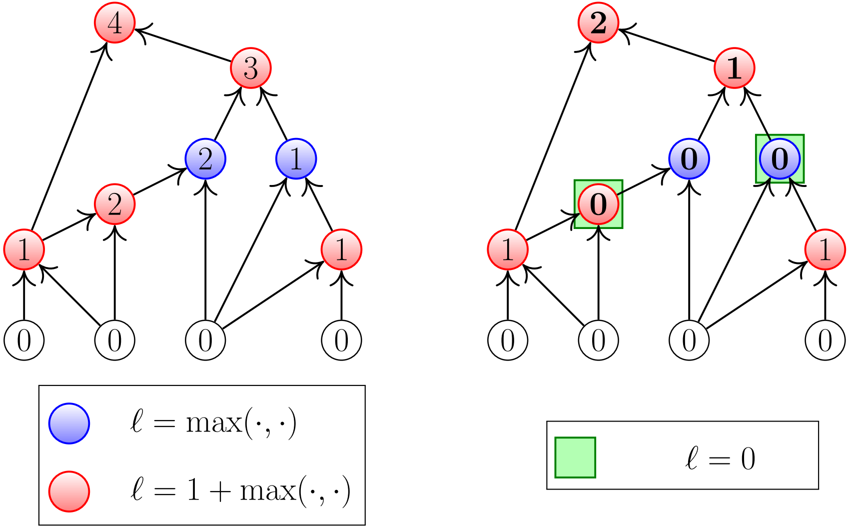
 $\ell = 1 + \max(\cdot, \cdot)$

- an integer parameter L

Output:

- a subset $S \subseteq V$ of minimum cardinality such that bootstrapping S ensures $\ell \leq L$ at every vertex

Example



$L = 2$

Previous Results

- Greedy approaches with approximation ratio $\Omega(|V|)$
  [Gentry Halevi 2011; Gentry Halevi Smart 2012]
- Heuristic method
  [Lepoint Paillier 2013]
- Polynomial time algorithm for $L = 1$ and NP-hardness for $L \geq 2$
  [Paindavoine Vialla 2015]

Our Results

Approximation

Polynomial-time L-approximation algorithm ($L \geq 1$)

Idea: linear program and new rounding scheme

Inapproximability

NP-hard to compute an $(L - \epsilon)$-approximation ($L \geq 2$), assuming the Unique Games Conjecture

Idea: reduction to the DAG vertex deletion problem [Svensson 2013]

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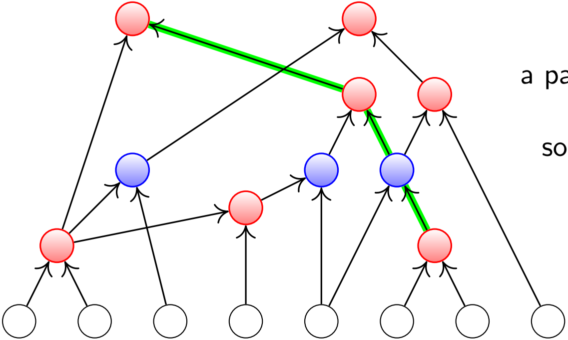
Preliminary Observation



$L = 2$

a path containing 3 red vertices

$\Downarrow$

some vertex is bootstrapped

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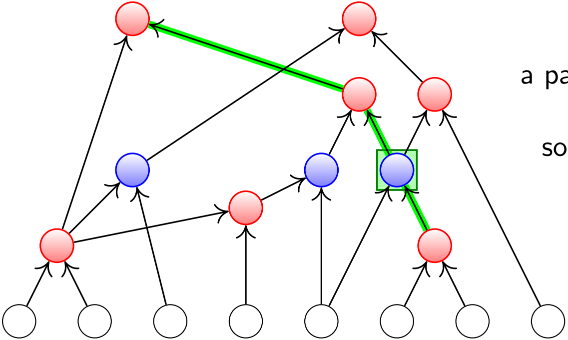

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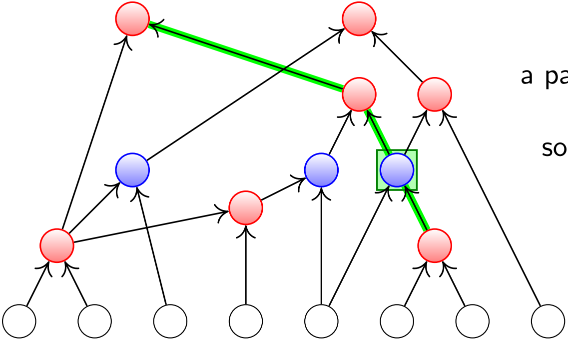

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Interesting path: containing $L + 1$ red vertices

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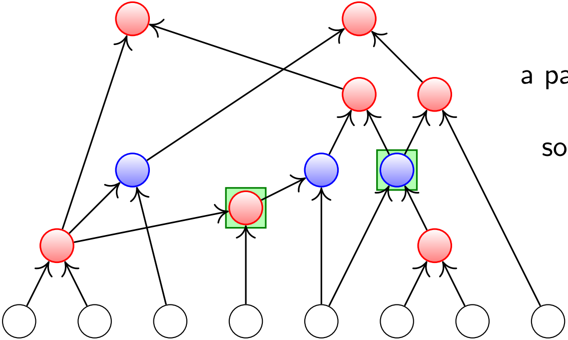

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a path containing 3 red vertices
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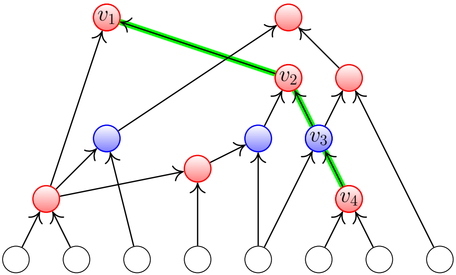
Interesting path: containing $L+1$ red vertices

Observation

bootstrap solution $\iff$ every interesting path has a bootstrapped vertex

Linear Program Relaxation

$$x_v = \begin{cases} 1 & \text{if } v \text{ is bootstrapped} \\ 0 & \text{otherwise} \end{cases}$$

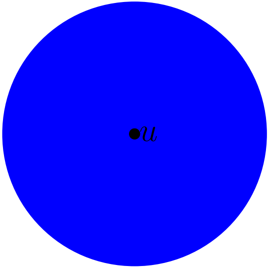


constraint: $x_{v_1} + x_{v_2} + x_{v_3} + x_{v_4} \geq 1$

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & \sum_{v \in p} x_v \geq 1 && \forall \text{ interesting path } p \\ & 0 \leq x_v \leq 1 && \forall\, v \in V \end{aligned}$$

Standard Rounding: Sphere Growing Technique

1. choose a vertex u
2. compute distance from u in metric of $\{x_v\}$
3. bootstrap all vertices at distance ≈ 0.5
4. repeat



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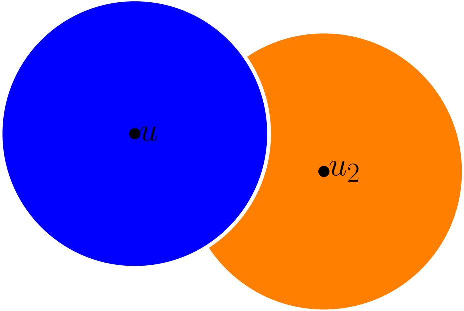

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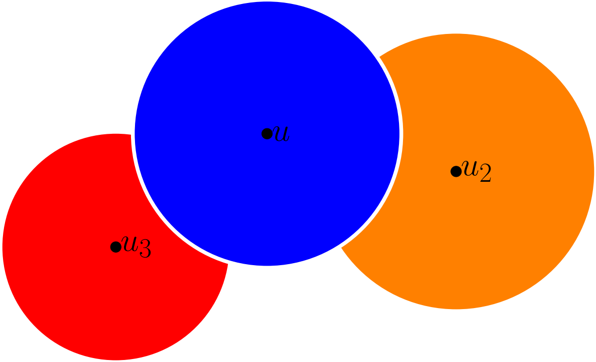

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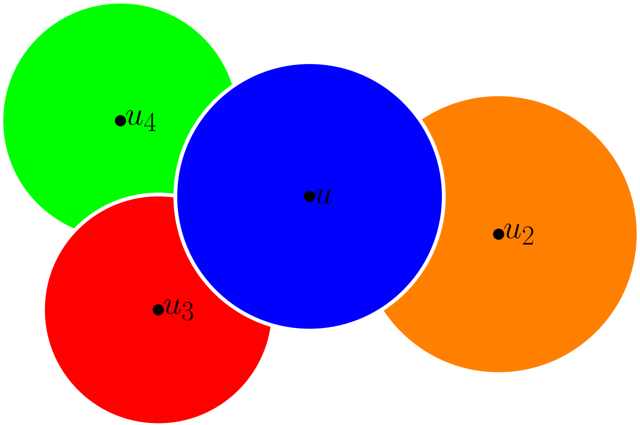
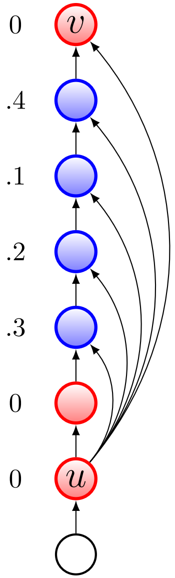


Standard Rounding: Counter Example



$L = 2$

u-to-v distance in the metric is 0.

New Rounding

Definition:

- length of a path: sum of x_v along the path
- For every $i \leq L$, define $f_{v,i}$:

  minimum length of a path that ends at v and contains i red vertices.
- Interval $A_{v,i} := [f_{v,i}\ ,\ f_{v,i} + x_v]$.

Randomized Rounding

1. Pick $t \in [0, 1]$ uniformly at random
2. For every vertex v, bootstrap v if $t \in A_{v,i}$ for some $i \in \{1, \ldots, L\}$.

Correctness

Every interesting path $v_1, \ldots, v_k$ contains a bootstrapped vertex.

Define $A^*_{v_j} := A_{v_j, i_j}$, where $i_j := \#$ red vertices among $v_1, \ldots, v_j$.

Claim

The union of $A^*_{v_j}$ covers the $[0, 1]$-interval.

Correctness

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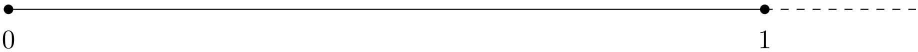
The union of $A^*_{v_j}$ covers the $[0, 1]$-interval.

Proof:

1. $A^*_{v_1}$ starts at 0;
2. every pair of consecutive intervals $A^*_{v_j}$ and $A^*_{v_{j+1}}$ intersect;
3. $A^*_{v_k}$ covers 1.



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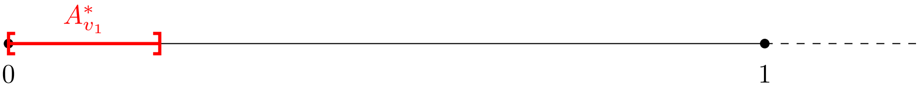

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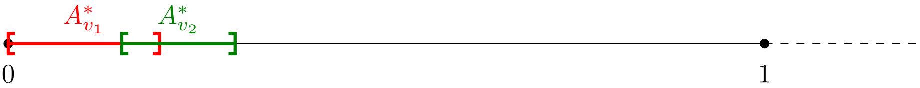

$f_{v_{j+1}, i_{j+1}} \leq f_{v_j, i_j} + x_{v_j}$ by definition of f

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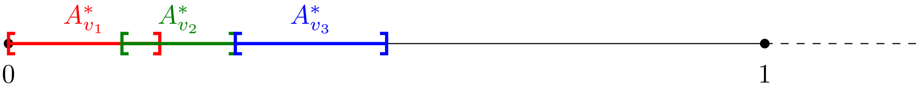

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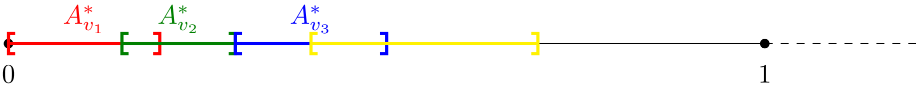

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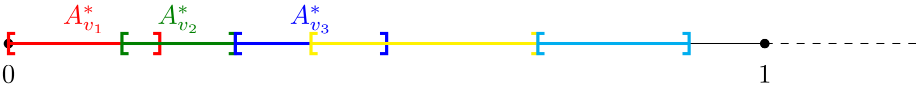

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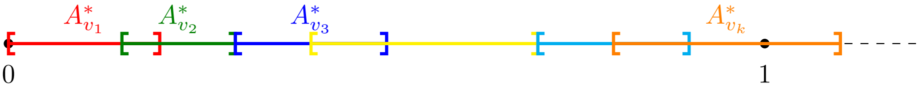

$i_k = L + 1 \quad \Longrightarrow \quad f_{v_k,i_k} \geq 1$ by definition of f and LP constraints

Approximation Ratio

A vertex v is bootstrapped if $t \in A_{v,i}$ for some $i \in \{1, \ldots, L\}$.

$\mathbb{P}[v \text{ is bootstapped}] \leq L \cdot x_v$.

Expected number of bootstrapped vertices:

$$\sum_{v \in V} L \cdot x_v \leq L \cdot \mathrm{OPT}.$$

Derandomization

$\{f_{v,i}\}_{v,i} \cup \{f_{v,i} + x_v\}_{v,i}$ contains $2|V| \cdot L$ values.

$[0, 1]$ interval is decomposed into $O(|V| \cdot L)$ sub-intervals.

Deterministic Rounding

1. For each sub-interval, pick any t and perform the previous rounding;
2. Return the best solution found.

Conclusion

Approximation

Polynomial-time L-approximation algorithm ($L \geq 1$)

Inapproximability

NP-hard to compute an $(L - \epsilon)$-approximation ($L \geq 2$), assuming the Unique Games Conjecture

Thank you!