Unsplittable Euclidean Capacitated Vehicle Routing

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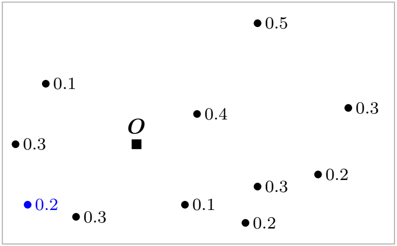
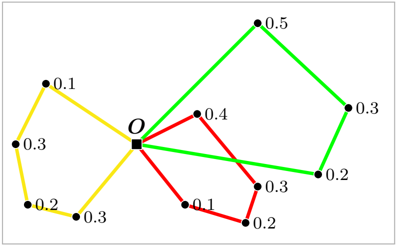
Capacitated vehicle routing problem (CVRP)

Input:

- **depot** O
- n **terminals with unsplittable demands in** $(0, 1]$

Minimize total length of tours s.t. each tour has total demand ≤ 1





Fundamental problem in operations research

Equal demands

- 1985 Haimovich and Rinnooy Kan
- 1990 Altinkemer and Gavish
- 1997 Asano, Katoh, Tamaki, and Tokuyama
- 1998 Hamaguchi and Katoh
- 2001 Asano, Katoh, and Kawashima
- 2006 Bompadre, Dror, and Orlin
- 2010 Adamaszek, Czumaj, and Lingas
- 2010 Das and Mathieu
- 2017 Becker, Klein, and Saulpic
- 2018 Becker, Klein, and Saulpic
- 2018 Becker
- 2019 Becker, Klein, and Schild
- 2019 Becker and Paul
- 2020 Cohen-Addad, Filtser, Klein, and Le
- 2021 Blauth, Traub, and Vygen
- 2021 Mathieu and Zhou
- 2022 Jayaprakash and Salavatipour
- 2022 Jayaprakash and Salavatipour
- 2022 Mathieu and Zhou
- 2023 Mömke and Zhou

Unequal demands

- 1981 Golden and Wong
- 1987 Altinkemer and Gavish
- 1991 Labbé, Laporte, and Mercure
- 2021 Blauth, Traub, and Vygen
- 2022 Friggstad, Mousavi, Rahgoshay, and Salavatipour
- 2023 Mathieu and Zhou

- general metrics
- Euclidean plane
- planar graphs
- trees
- graphs of bounded treewidth
- graphs of bounded highway dimension
- graphic metrics

Previous and new results on the Euclidean CVRP

Arbitrary demands

- **1.5-hard** [folklore]
- **2.694-approximation** [Friggstad et al. 2022]
- **$(2+\epsilon)$-approximation for any $\epsilon > 0$ [our result]**

Big demands only

- **NP-hard** [folklore]
- **2.694-approximation** [Friggstad et al. 2022]
- **$(1+\epsilon)$-approximation for any $\epsilon > 0$ [our result]**

Note: Both our results **match the best known approximation factors** in the corresponding equal demand settings.

Big demands only

Our result

$(1+\epsilon)$-approximation for any $\epsilon > 0$

Polynomial time algorithm under assumptions

Definition

Let $\epsilon > 0$. A demand is **big** if it is at least ϵ, and is **small** otherwise.

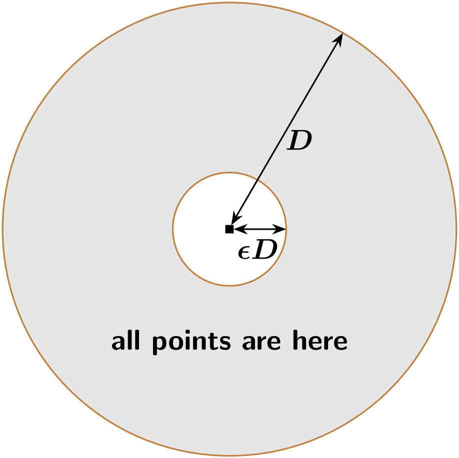
Polynomial time exact algorithm for big demands only, assuming:

1. $O_\epsilon(1)$ **locations**
2. $O_\epsilon(1)$ **distinct demands** at each location

Q: **How to achieve both assumptions?**

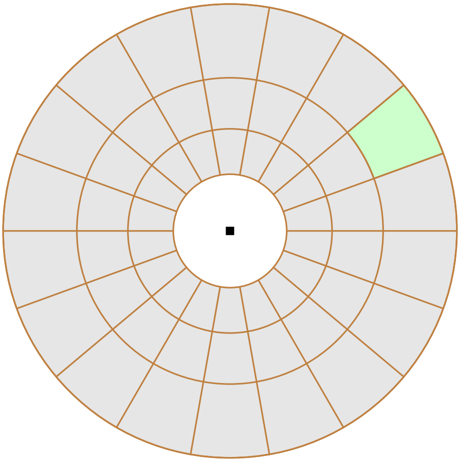
Reducing the number of locations (1/3)

Step 1: Reduce to bounded distance



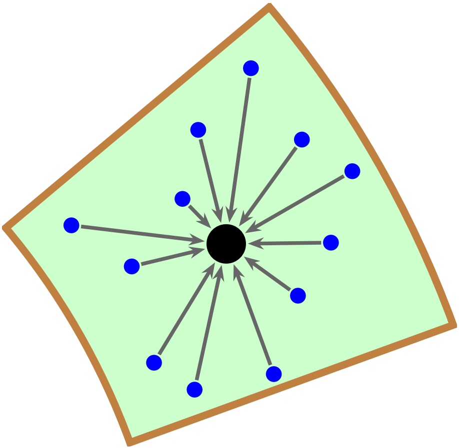
Reducing the number of locations (2/3)

Step 2: Decompose the gray area into $O_\epsilon(1)$ cells



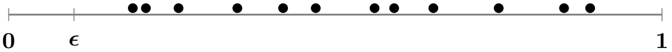
Reducing the number of locations (3/3)

Step 3: Move each terminal to the center of its cell

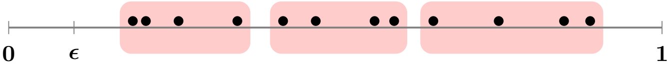


Adaptive rounding

(a) **Sort all demands at a center**



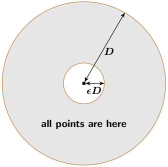
(b) **Make groups of equal cardinality**



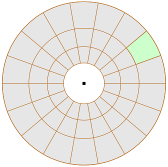
(c) **Round up to maximum demand in group**

Algorithm for big demands only

1. Reduce to **bounded distance**
2. Decompose into **cells**
3. Move each big terminal to the **center** of its cell
4. Apply **adaptive rounding** on the demands at each center
5. Solve the resulting instance in **polynomial time**



1



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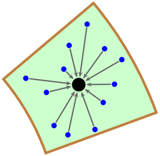

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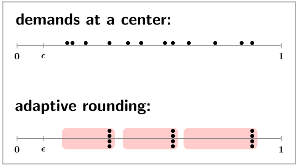


4

Arbitrary demands

Our result

$(2+\epsilon)$**-approximation for any** $\epsilon > 0$

Case 1: Optimal solution has sufficiently many tours (1/5)

Algorithm for big demands only (revisited)

1. Reduce to **bounded distance**
2. Decompose into **cells**
3. Move each big terminal to the **center** of its cell
4. Apply **adaptive rounding** on the demands at each center
5. Solve the resulting instance in **polynomial time**

Q: **How to deal with small demands?**

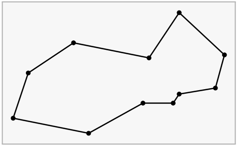
A: **Insert one step between 2 and 3 :**

Cluster the small terminals in each cell into big terminals

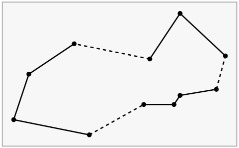
Case 1: Optimal solution has sufficiently many tours (2/5)

Cluster the small terminals in each cell into big terminals:

(a) **Compute a TSP tour on the small terminals in the cell**



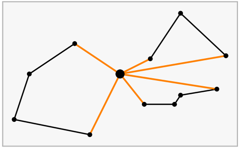
(b) **Partition the tour into segments, each of total demand $\approx \epsilon$**



(c) **For each segment, connect its endpoints to the center of the cell**

$\Downarrow$

a big terminal at the center of the cell

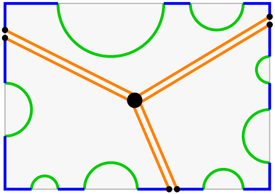


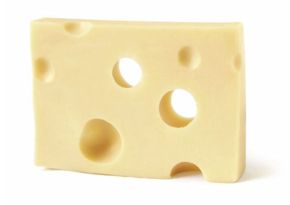
Case 1: Optimal solution has sufficiently many tours (3/5)

$\mathbf{opt}$: cost of an optimal solution to the initial instance

Lemma 1

The cost within cells is at most $(1+2\epsilon)\cdot\mathbf{opt}$.





- **green is the optimal solution**
- **blue $\leq 1.5\cdot$ cell boundaries $\leq \epsilon\cdot\mathbf{opt}$,**
  by Karp and since the optimal solution has many tours
- **orange** $\leq \epsilon\cdot\mathbf{opt}$, since each segment has total demand $\approx \epsilon$

Case 1: Optimal solution has sufficiently many tours (4/5)

$\mathbf{opt}$: cost of an optimal solution to the initial instance
$\mathbf{opt'}$: cost of an optimal solution to the clustered instance

Lemma 2

The cost between cells is at most $(1+\epsilon)\cdot\mathbf{opt'} \leq (1+4\epsilon)\cdot\mathbf{opt}$.

Construction of a near-optimal solution to the clustered instance:

- Compute a set of tours each of **capacity** $1+\epsilon$ [Becker and Paul]
  cost $\leq (1+\epsilon)\cdot\mathbf{opt}$
- Remove one **cluster** (of demand ϵ) from each tour
- Connect the removed clusters by a **TSP tour**
  cost $\leq \epsilon\cdot\mathbf{opt}$, since the optimal solution has many tours
- **Iterated tour partitioning** on the TSP tour
  cost $\leq \epsilon\cdot\mathbf{opt}$, since only ϵ demand is removed from each tour

Case 1: Optimal solution has sufficiently many tours (5/5)

$\mathbf{opt}$: cost of an optimal solution to the initial instance
$\mathbf{opt'}$: cost of an optimal solution to the clustered instance

Lemma 1

The cost within cells is at most $(1 + 2\epsilon) \cdot \mathbf{opt}$.

Lemma 2

The cost between cells is at most $(1 + 4\epsilon) \cdot \mathbf{opt}$.

Conclusion: Overall cost is at most $(2 + 6\epsilon) \cdot \mathbf{opt}$.

Case 2: Optimal solution has a bounded number of tours

Algorithm

1. **Round down** each demand to **an integer multiple of** $\frac{1}{2n}$
2. Compute a $(1+\epsilon)$-approximation to the rounded instance [Arora]
3. **Split each tour into two tours**, each within the capacity

Cost analysis:

- ② : **at most** $(1+\epsilon)\cdot\mathrm{opt}$
- ③ : **doubling the cost**

Conclusion: Overall cost is at most $(2+2\epsilon)\cdot\mathrm{opt}$**.**

Summary on the Euclidean CVRP

Arbitrary demands

- **1.5-hard** [folklore]
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Thank you!