Probabilistic Analysis of Euclidean Capacitated Vehicle Routing

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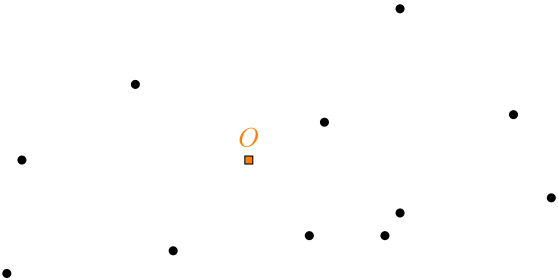
Capacitated Vehicle Routing Problem

Input:

- *a depot* O and n *customers*
- *tour capacity* k



$k = 4$

Capacitated Vehicle Routing Problem

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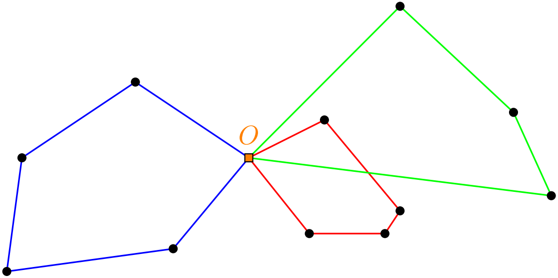
Goal: covering all customers with a *minimum length* collection of tours through depot such that each tour visits *at most* k *customers*



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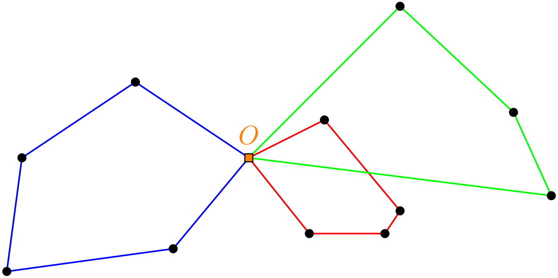

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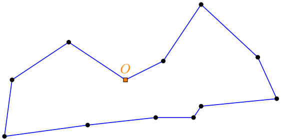
A fundamental problem in operations research, e.g., more than 3700 articles on vehicle routing at DBLP.

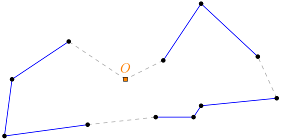
Iterated Tour Partitioning (ITP)

The ITP algorithm [Haimovich and Rinnooy Kan 1985]

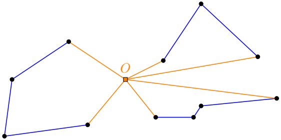
1. compute a traveling salesman tour



2. partition the tour into segments of at most k customers each



3. connect the endpoints of each segment to the depot



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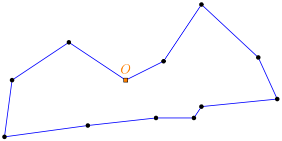

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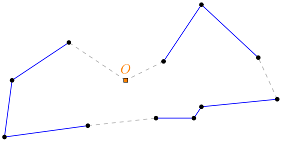

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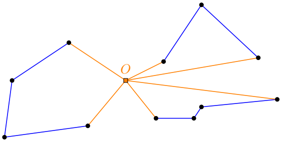

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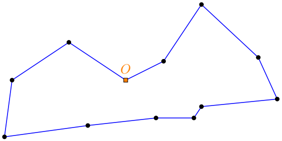

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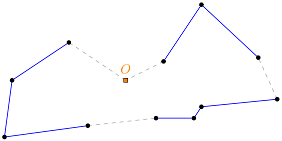

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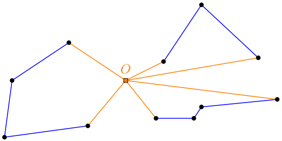
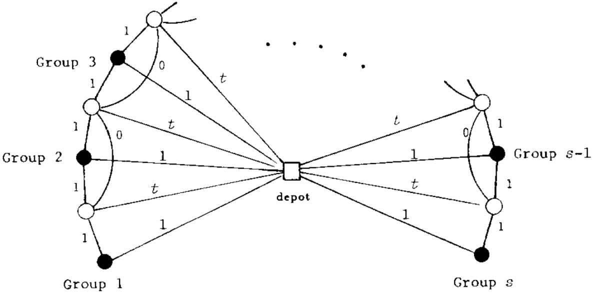


Iterated Tour Partitioning (ITP)

Approximation ratio: upper bound = lower bound = $2 - \frac{1}{k}$
[Altinkemer and Gavish 1990, Li and Simchi-Levi 1990]

Tight instance:



Tight instances seem contrived.

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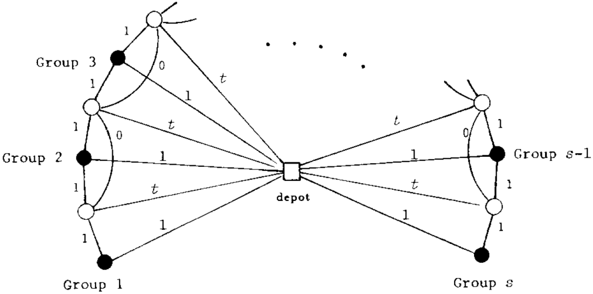

Q: How do we explain the popularity of ITP?

A: Perhaps ITP works particularly well on non-contrived instances.

Iterated Tour Partitioning (ITP)

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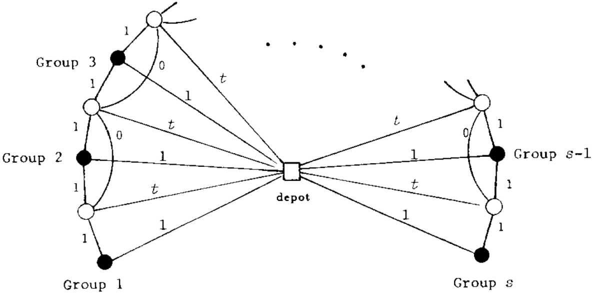
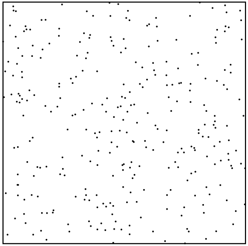


Q: How well does it work on random instances?

ITP in the Random Setting

Random Setting: Customers are *independent, identically distributed (i.i.d.)* random points in the unit square.



Haimovich and Rinnooy Kan 1985

- for $k = o(\sqrt{n})$: $1 + o(1)$
- for $k = \omega(\sqrt{n})$: $1 + o(1)$
- for $k = \Theta(\sqrt{n})$: also effective?

ITP in the Random Setting – New Results

Upper bound on the approximation ratio of ITP in the random setting

- 1.995 [Bompadre, Dror, and Orlin 2007]
- 1.915 [this work]

Lower bound on the approximation ratio of ITP in the random setting

- $1 + c_0$ for some constant $c_0 > 0$ [this work]

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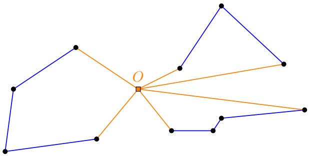
High-Level Analysis

Two factors in the cost of an ITP solution [Haimovich and Rinnooy Kan 1985]:

1. traveling salesman tour cost: TSP
2. radius cost: $\mathrm{radius} = \frac{2}{k} \cdot \sum_x \delta(O, x)$



Properties:

- $\mathrm{ITP} \leq \mathrm{radius} + \mathrm{TSP}$
- $\mathrm{OPT} \geq \mathrm{radius}$
- $\mathrm{OPT} \geq \mathrm{TSP}$

Implication:

- ITP is a 2-approximation.

Properties:

- $\mathrm{ITP} \leq \mathrm{radius} + \mathrm{TSP}$
- $\mathrm{OPT} \geq \mathrm{radius} + 0.085 \cdot \mathrm{TSP}$
- $\mathrm{OPT} \geq \mathrm{TSP}$

Implication:

- ITP is a 1.915-approximation.

Proof for the Lower Bound on OPT (1/4)

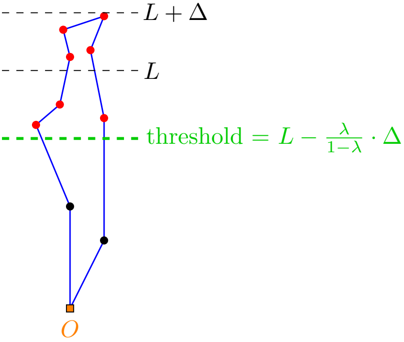
Consider a tour T visiting m points.

$$L := \frac{1}{m} \sum_{x \in T} \delta(O, x) \quad \text{and} \quad \Delta := \max_{x \in T} \delta(O, x) - L.$$

$$\mathrm{cost}(T) \geq 2(L + \Delta)$$

$$W = \left\{ x \in T \;\middle|\; \delta(O, x) \geq \text{threshold} \right\}$$

$$\mathrm{cost}(T) \geq 2 \cdot \text{threshold} + \mathrm{TSP}(W)$$



Facts:

- $|W| \geq \lambda \cdot m$
- $\mathrm{TSP}(W) \geq \sum_{x \in W} \delta(x, W \setminus \{x\})$

Proof for the Lower Bound on OPT (2/4)

Summing over all tours T in an optimal solution, we obtain

$$\mathrm{OPT} \geq \sum_T 2(L_T + \Delta_T)$$

$$\mathrm{OPT} \geq \sum_T 2\left(L_T - \frac{\lambda}{1-\lambda} \cdot \Delta_T\right) + \sum_{x \in U} \delta(x, U \setminus \{x\}), \text{ where } U = \bigcup_T W_T.$$

Linear combination of both inequalities with coefficients λ and $1 - \lambda$:

Structural Theorem

Let V be a set of n points in any metric. For any $\lambda \in (0,1)$, there exists a subset $U \subseteq V$ of cardinality at least $\lambda \cdot n$ such that

$$\mathrm{OPT} \geq \mathrm{radius} + (1-\lambda) \sum_{x \in U} \delta(x, U \setminus \{x\}).$$

Proof for the Lower Bound on OPT (3/4)

Next, we analyze

$$\min_{U} \sum_{x \in U} \delta(x, U \setminus \{x\})$$

in the probabilistic setting for a subset $U \subseteq V$ of cardinality at least $\lambda \cdot n$.

Lemma

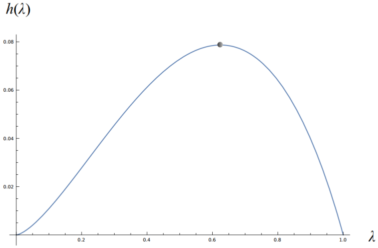
Asymptotically almost surely, $\min_{U} \sum_{x \in U} \delta(x, V \setminus \{x\}) > f(\lambda) \cdot \sqrt{n}$, *where* $f(\lambda) := \frac{1}{2} \operatorname{erf}\left(\sqrt{\ln \frac{1}{1-\lambda}}\right) - (1-\lambda) \cdot \sqrt{\frac{1}{\pi} \cdot \ln \frac{1}{1-\lambda}}$ *in which* $\operatorname{erf}(\cdot)$ *is the Gauss error function.*

Technique: weak law of large numbers on the closest point distance [Penrose and Yukich 2003]

Proof for the Lower Bound on OPT (4/4)

$\mathrm{OPT} \geq \mathrm{radius} + (1-\lambda) \cdot f(\lambda) \cdot \sqrt{n}$, for any λ.

Let $h(\lambda) = (1-\lambda) \cdot f(\lambda)$. The maximum value of $h(\lambda)$ is 0.0786.



- $\mathrm{OPT} \geq \mathrm{radius} + 0.0786 \cdot \sqrt{n}$
- $\mathrm{TSP} < 0.922 \cdot \sqrt{n}$ [Steinerberger 2015]

Conclusion: $\mathrm{OPT} \geq \mathrm{radius} + 0.085 \cdot \mathrm{TSP}.$

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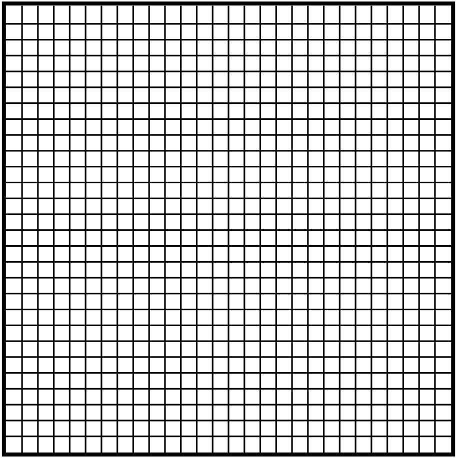
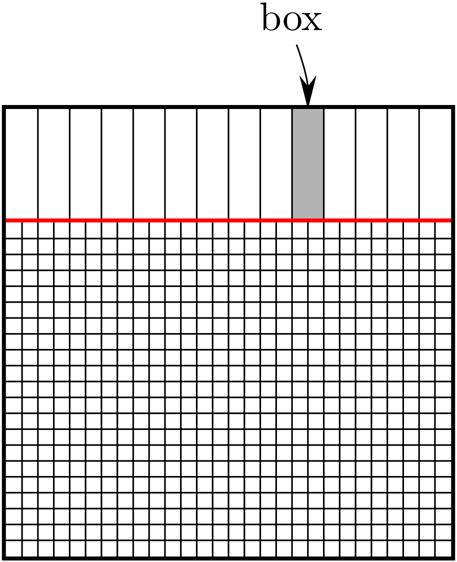
ITP Solution

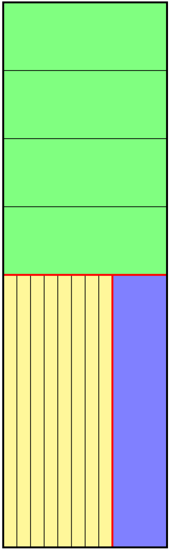
1. Partition the unit square into small squares
2. Make a tour to visit points in each small square



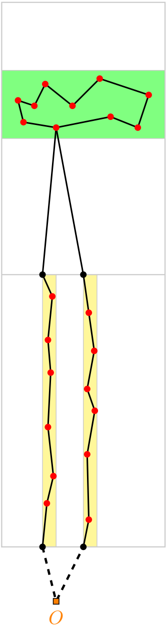
New Solution: Decomposition of the Unit Square



New Solution: Decomposition of a Box



New Solution: A Mixed Tour



- Positive effect on the TSP cost
- Negative effect on the radius cost

Delicate definition of the decomposition

$\Downarrow$

Improvement upon an ITP solution

Solution Costs

Cost of an ITP solution $= \mathrm{radius} + \mathrm{TSP}$

Cost of the new solution $< \mathrm{radius} + \mathrm{TSP} - 0.000068 \cdot \sqrt{n}$

Conclusion: ITP is at best a $(1 + c_0)$-approximation for $c_0 > 0$.

Take-Home Message and Open Problems

We give a partial answer to the question:

How good is the Iterated Tour Partitioning algorithm?

Our partial answer *in the random setting*:

Approximation ratio between $1 + c_0$ *and 1.915.*

Open problems:

1. Reduce the gap between $1 + c_0$ and 1.915 for the performance of ITP in the random setting.
2. Design a PTAS for capacitated vehicle routing in the random setting.