Sublinear-Time Algorithms for Monomer-Dimer Systems on Bounded Degree Graphs

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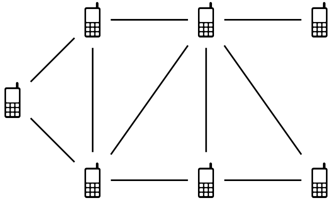
Background

- Sublinear-time algorithms for graph problems
- Optimization vs. Counting and Statistics



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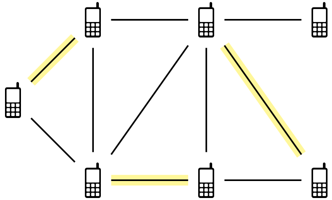

Monomer-Dimer systems

- $G = (V, E)$ undirected graph with $|V| = n$
- Maximum degree Δ
- Monomer $\leftrightarrow$ one vertex, Dimer $\leftrightarrow$ two adjacent vertices
- Monomer-Dimer arrangement $\leftrightarrow$ matching

Definition

- $\mathbb{M}$: set of all matchings of G
- Partition function: $Z(G, \lambda) = \sum_{M \in \mathbb{M}} \lambda^{|M|}$
- Gibbs distribution:

$$\pi_{G,\lambda}(M) = \frac{\lambda^{|M|}}{Z(G, \lambda)}, \quad \forall\, M \in \mathbb{M}$$

- Marginal probability:

$$p_{G,\lambda}(v) := \sum_{M \not\ni v} \pi_{G,\lambda}(M), \quad \forall\, v \in V$$

Previous Results

- Partition function:
  - $\#P$-complete (Valiant 1979, Vadhan 2002)
  - Randomized polynomial-time approximation scheme (Sinclair 1993, Jerrum Sinclair 1997)
  - Deterministic polynomial-time approximation scheme (Bayati *et al.* 2007)
- Matching statistics
  - $\#P$-hard (Sinclair Srivastava 2013)
- Permanent of expander graphs
  - Randomized polynomial-time approximation scheme (Jerrum Sinclair Vigoda 2004)
  - Deterministic polynomial-time approximation algorithm (Gamarnik Katz 2010)

Our Main Results

Local computations for marginal probability

- Approximation algorithm
- Complexity lower bound

Randomized sublinear-time approximation algorithms for

- Partition function
- Matching statistics
- Permanent of expander graphs

Notations

- ϵ-approximation solution : within additive error ϵ
- ϵ-approximation algorithm : outputs an ϵ-approximation solution
  with probability at least $2/3$

- Oracles: $\mathcal{D}(v)$ and $\mathcal{N}(v, i)$
- Our focus: Query complexity

Marginal probability

$$p_{G,\lambda}(v) = \frac{1}{1 + \lambda \sum_{u \in N(G,v)} p_{G \setminus \{v\},\lambda}(u)}$$

What is the probability that ● is matched? $(\lambda = 1)$

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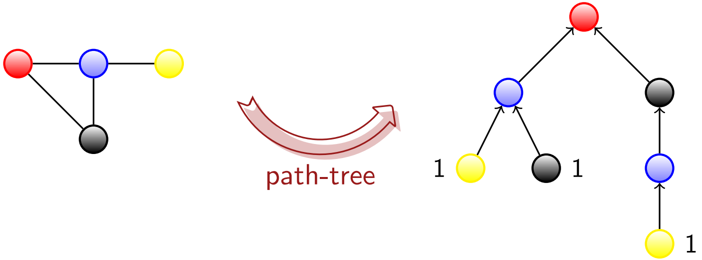

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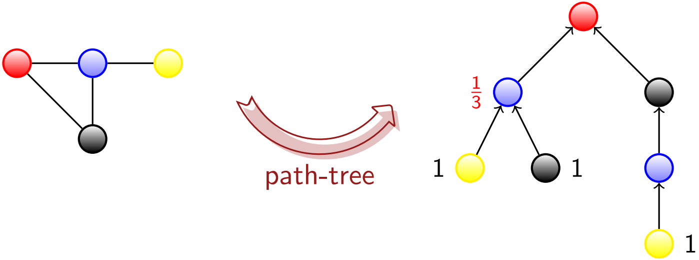

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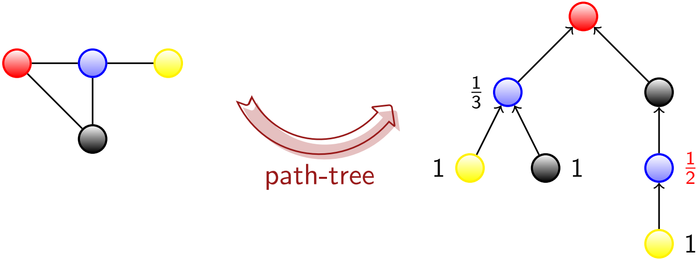

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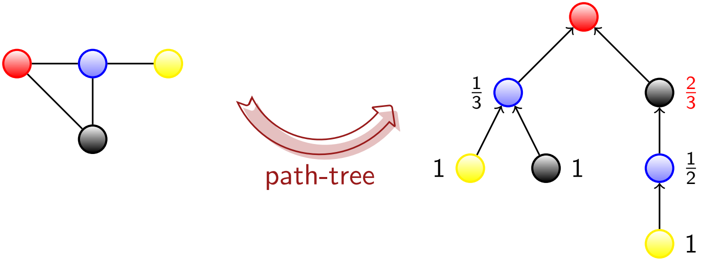

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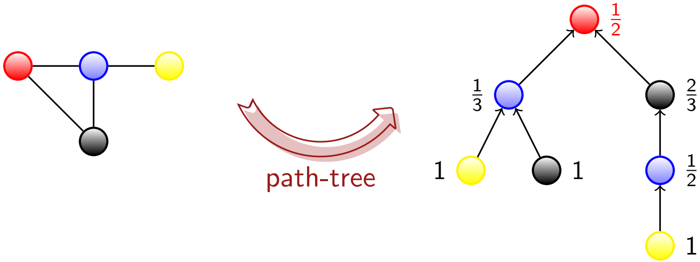

Path-tree

- $T(v)$: the path-tree rooted at v
- $x_u(v) := \dfrac{1}{1 + \lambda \sum_{w \succ u} x_w(v)}$, for every $u \in T(v)$
- $x_v(v) = p_{G,\lambda}(v)$

Truncated path-tree

- $T^h(v)$: $T(v)$ truncated at depth h
- $x_u^h(v)$: the solution of the recursions in $T^h(v)$

Correlation decay property (Bayati *et al.* 2007)

$|\log x_v^h(v) - \log p_{G,\lambda}(v)| \le \epsilon$, for any $h \ge h(\epsilon, \Delta) = \tilde{O}\left(\sqrt{\lambda\Delta}\log(1/\epsilon)\right)$.

Optimality

Proposition

We can compute an ϵ-approximation of $p_{G,\lambda}(v)$ using $O\left(\Delta^{h(\epsilon,\Delta)}\right)$ queries.

Proposition (lower bound)

Any approximation algorithm requires at least the above query complexity.

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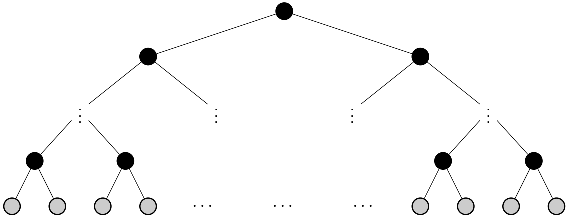
Idea of the lower bound



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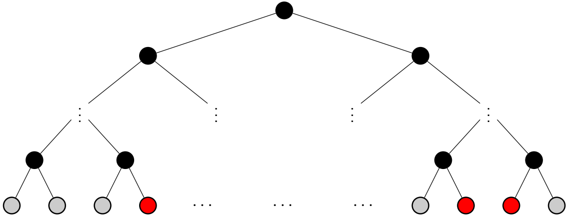

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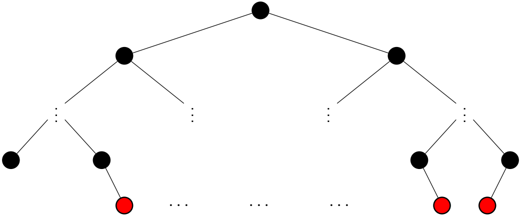

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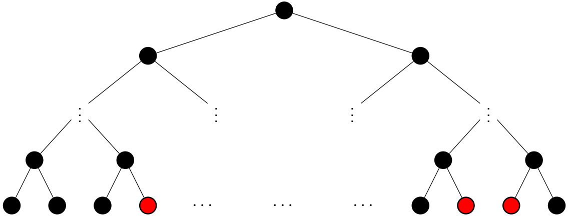

Partition function

$$Z(G,\lambda) = \prod_{1\le k\le n} p^{-1}_{G_k,\lambda}(v_k)$$

Algorithm for $\log Z(G,\lambda)$

- Take $\Theta(1/\epsilon^2)$ samples uniformly at random from $[1,\ldots,n]$;
- Compute an ϵ-approximation of $\log p^{-1}_{G_k,\lambda}(v_k)$ for every sample k;
- Return $n\cdot$(the average of the estimates).

Partition function

Main Theorem

We have an ϵn-approximation algorithm for the logarithm of the partition function with $\tilde{O}\left((1/\epsilon)^{\tilde{O}(\sqrt{\Delta})}\right)$ queries. In addition, any ϵn-approximation algorithm needs $\Omega(1/\epsilon^2)$ queries.

Average matching size

$$
\begin{aligned}
E(G,\lambda) &:= \sum_{M\in\mathbb{M}} |M| \cdot \pi_{G,\lambda}(M) \\
&= n - \frac{1}{2} \sum_{1\le k\le n} p_{G,\lambda}(v_k)
\end{aligned}
$$

Theorem

We have an ϵn-approximation algorithm for the average matching size with $\tilde{O}\left((1/\epsilon)^{\tilde{O}(\sqrt{\Delta})}\right)$ queries. In addition, any ϵn-approximation algorithm needs $\Omega(1/\epsilon^2)$ queries.

Entropy of a matching

$$\begin{aligned} S(G,\lambda) &:= -\sum_{M\in\mathbb{M}} \pi_{G,\lambda}(M)\log \pi_{G,\lambda}(M) \\ &= \log Z(G,\lambda) + \log\lambda \cdot E(G,\lambda) \end{aligned}$$

Corollary

We have an ϵn-approximation algorithm for the entropy of a matching with $\tilde{O}\left((1/\epsilon)^{\tilde{O}(\sqrt{\Delta})}\right)$ queries. In addition, any ϵn-approximation algorithm needs $\Omega(1/\epsilon^2)$ queries.

Application: Permanent of expander graphs

- Bi-partite graph G with vertices $X \cup Y$, $|X| = |Y| = n$
- α-expander graph:
  $|N(S)| \geq (1+\alpha)|S|$, for $S \subset X$ or $S \subset Y$ with $|S| \leq n/2$
- PERM : Permanent of the adjacency matrix of G

Lemma (Gamarnik Katz 2010)

$$1 \leq \frac{Z(G,\lambda)}{\lambda^n \cdot \mathrm{PERM}} \leq e^{O(n\lambda^{-1}\log^{-1}(1+\alpha)\log\Delta)}.$$

Corollary

We have an ϵn-approximation algorithm for $\log \mathrm{PERM}$ with query complexity $\tilde{O}\left((1/\epsilon)^{\tilde{O}(\sqrt{\Delta/(\epsilon\alpha)})}\right)$.

Conclusion

- Marginal probability *(correlation decay property)*
- Sublinear-time approximation for monomer-dimer systems
- Extension to the partition function of independent sets
- Experiments on large real-world graphs

Thank you!