A Tight $(1.5 + \epsilon)$-Approximation for Unsplittable Capacitated Vehicle Routing on Trees

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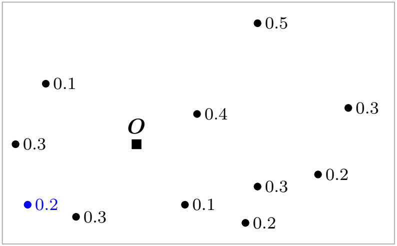
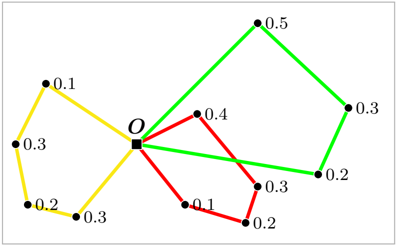
Unsplittable capacitated vehicle routing problem (UCVRP)

Input:

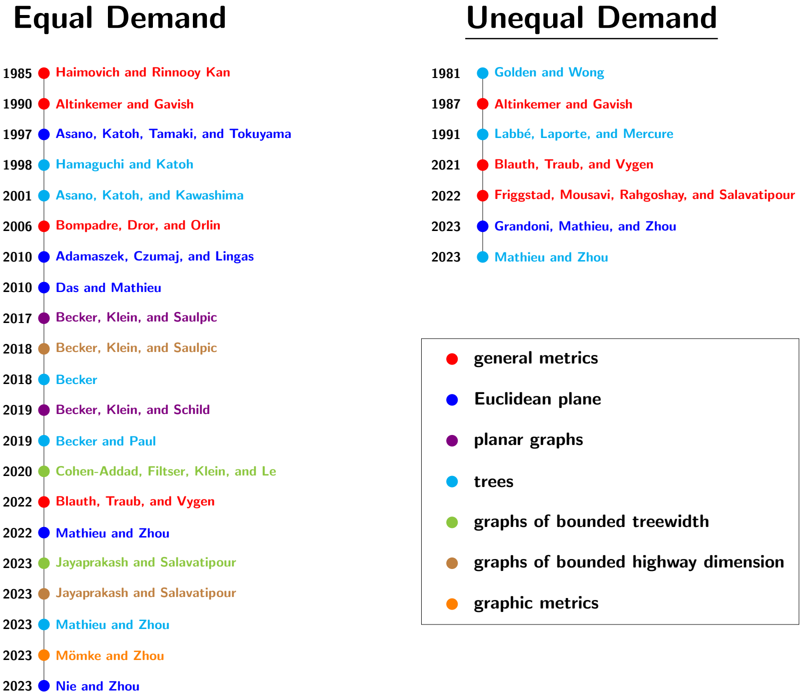
- **depot** O
- n **terminals with unsplittable demands in** $(0, 1]$

Minimize total length of tours s.t. each tour has total demand ≤ 1



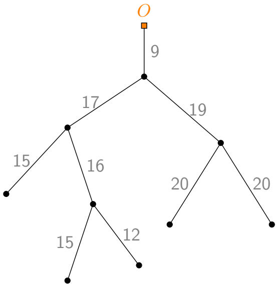


Fundamental problem in operations research


Equal Demand
1985 Haimovich and Rinnooy Kan
1990 Altinkemer and Gavish
1997 Asano, Katoh, Tamaki, and Tokuyama
1998 Hamaguchi and Katoh
2001 Asano, Katoh, and Kawashima
2006 Bompadre, Dror, and Orlin
2010 Adamaszek, Czumaj, and Lingas
2010 Das and Mathieu
2017 Becker, Klein, and Saulpic
2018 Becker, Klein, and Saulpic
2018 Becker
2019 Becker, Klein, and Schild
2019 Becker and Paul
2020 Cohen-Addad, Filtser, Klein, and Le
2022 Blauth, Traub, and Vygen
2022 Mathieu and Zhou
2023 Jayaprakash and Salavatipour
2023 Jayaprakash and Salavatipour
2023 Mathieu and Zhou
2023 Mömke and Zhou
2023 Nie and Zhou
Unequal Demand
1981 Golden and Wong
1987 Altinkemer and Gavish
1991 Labbé, Laporte, and Mercure
2021 Blauth, Traub, and Vygen
2022 Friggstad, Mousavi, Rahgoshay, and Salavatipour
2023 Grandoni, Mathieu, and Zhou
2023 Mathieu and Zhou
general metrics
Euclidean plane
planar graphs
trees
graphs of bounded treewidth
graphs of bounded highway dimension
graphic metrics

UCVRP on trees

- **NP-hard to approximate to better than 1.5**
  [Golden Wang 1981]
- **polynomial-time 2-approximation**
  [Labbé Laporte Mercure 1991]



Our Result

polynomial-time $(1.5 + \epsilon)$-approximation

UCVRP on trees: Related work

Easier versions

unit demand: PTAS [Mathieu Zhou 2022]

splittable demand: PTAS [Mathieu Zhou 2022]

infinite capacity: poly-time [folklore: TSP on trees]

UCVRP on trees is the easiest vehicle routing problem that is APX-hard

Harder versions

general metric: lower bound 1.5 v.s. upper bound $3.194 + \epsilon$
[Friggstad Mousavi Rahgoshay Salavatipour 2022]

Euclidean plane: lower bound 1.5 v.s. upper bound $2 + \epsilon$
[Grandoni Mathieu Zhou 2023]

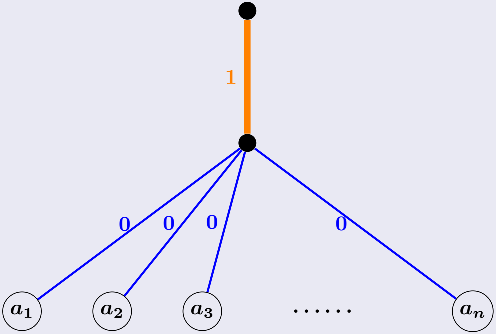
complexity of approximation of all harder vehicle routing problems is open

UCVRP on trees: Hardness of approximation

NP-hard to approximate UCVRP on trees to better than 1.5
[Golden Wang 1981]

Proof

Partition problem: decide whether $\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i$ is possible



capacity:= $\frac{1}{2}\sum_{i=1}^{n} a_i$

UCVRP on trees: Approximation algorithm

Our Result

polynomial-time $(1.5 + \epsilon)$-approximation

Standard general approach

simplifying input: modifying input to have a particular structure

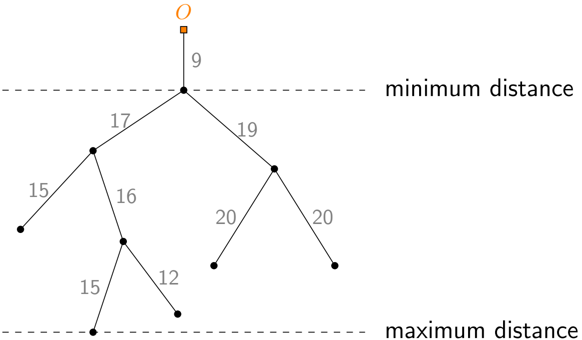
simplifying output: modifying output to have a particular structure

dynamic programming: solving simplified problem optimally

Simplifying input

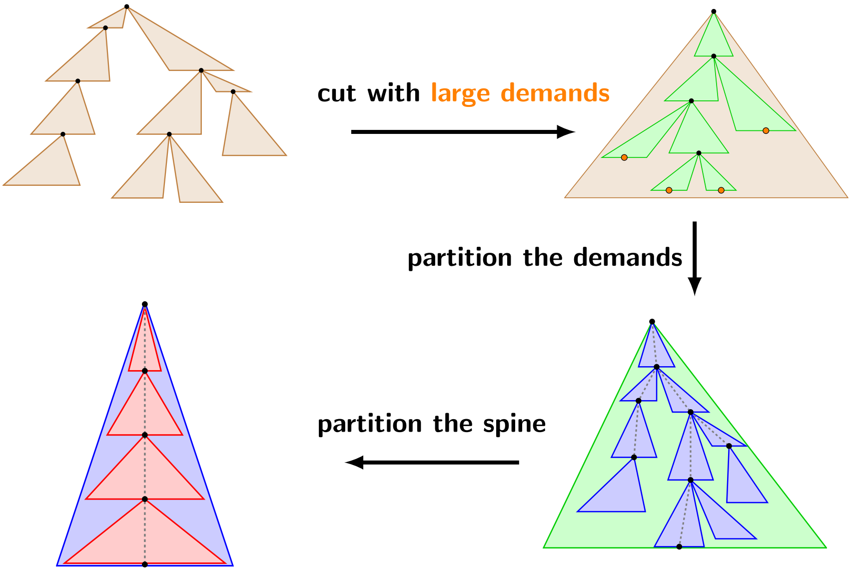
May assume without loss of generality:

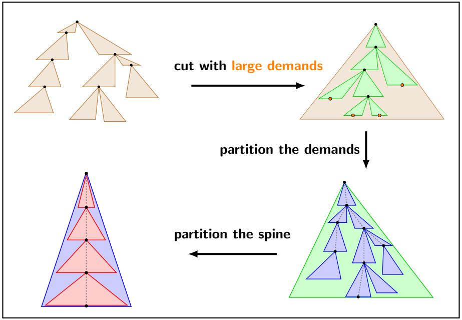
- binary tree
- depot at the root, demands at the leaves
- $\dfrac{\textbf{minimum distance}}{\textbf{maximum distance}} > \epsilon$

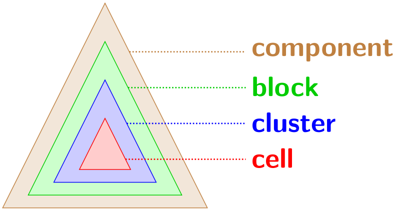


Simplifying output:

Multi-level decomposition


cut with large demands
partition the demands
partition the spine

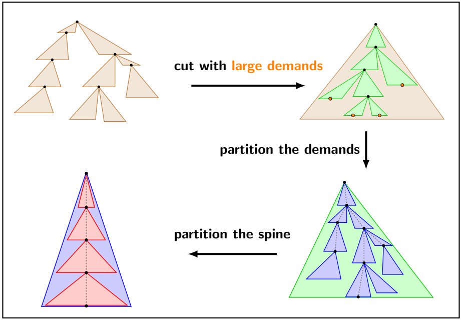


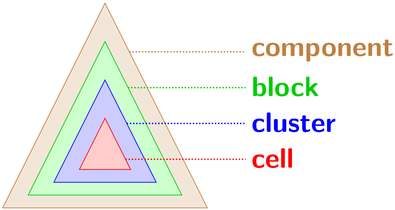


Properties

- **at most $O(1)$ tours** in **component**
- **no large demands** in **block**
- **small total demand** in **cluster**
- **short spine** in **cell**

Combine solutions between components by **adaptive rounding** and **DP**





Structure Theorem in Component

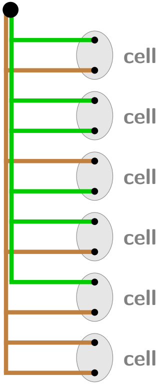
There is a **capacity-preserving** way to modify the solution, **setting aside some terminals**, so that

- in each cell, the terminals are visited by a **single** tour
- the terminals set aside are covered by just **one** additional tour
- the solution cost is increased by at most **50%**

Proof of the Structure Theorem (1/4)

For intuition: special case when all demands are small

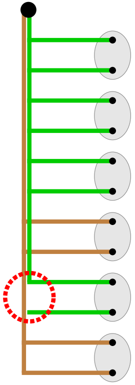
Optimal solution = **green tour** and **brown tour**



assignment lemma:

[Becker Paul 2019]

each cell is **monochromatic**

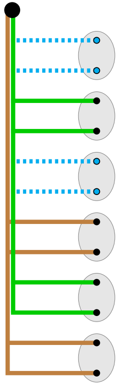


To keep green tour connected, **lengthen** its spine

- **short spine** in cell $\Longrightarrow$ **lengthening is cheap**
- **small total demand** in cluster $\Longrightarrow$ **tour capacity increased slightly**

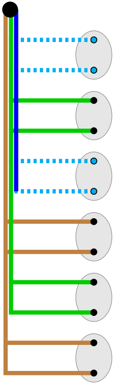
Proof of the Structure Theorem (2/4)

Set aside demands so that each tour is within capacity



Proof of the Structure Theorem (3/4)

Create a new tour for demands set aside by **adding a piece of spine**

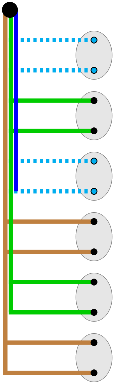


Cost increases by at most 50%:

the **added spine** already used by **two tours**: **green tour** and **brown tour**

Proof of the Structure Theorem (4/4)

Create a new tour for demands set aside by **adding a piece of spine**



Q: Why can the terminals set aside be covered by **one additional tour**?

A: Play with the parameters between

- number of tours in component (large constant), and
- max demand of a cell or cluster (small constant)

UCVRP on trees: Conclusion

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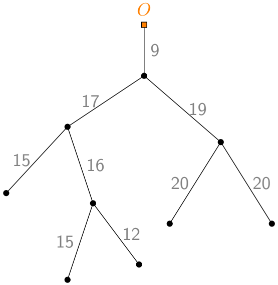

Our Result

polynomial-time $(1.5 + \epsilon)$-approximation

Thank you!