A PTAS for Capacitated Vehicle Routing on Trees

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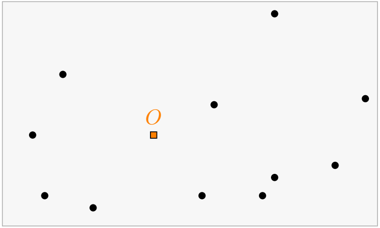
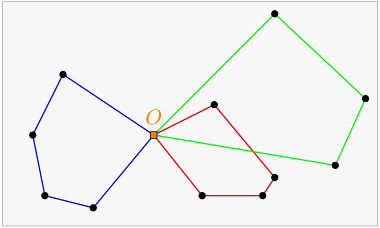
Capacitated vehicle routing

Input:

- **depot O**
- n **terminals**
- **tour capacity k**

Minimize total length of tours





Fundamental problem in operations research

e.g., more than 3900 articles on vehicle routing on DBLP

Capacitated vehicle routing

- **1985** **HR**
- **1990** **AG**
- **1997** **AKTT**
- **2006** **BDO**
- **2010** **ACL**
- **2010** **DM**
- **2017** **BKS**
- **2018** **BKS**
- **2019** **BKS**
- **2020** **CFKL**
- **2021** **BTV**
- **2022** **JS**

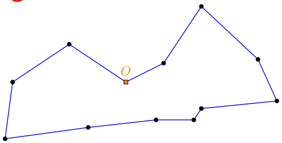
well-studied problem

- **general metric**
- **Euclidean space**
- **planar graph**
- **graph of bounded highway dimension**
- **graph of bounded treewidth**

Note: PTAS only for small capacity k.

Iterated tour partitioning [Haimovich and Rinnooy Kan]

1. Compute a traveling salesman tour



2. Partition the tour into segments of at most k customers each

3. Connect the endpoints of each segment to the depot

Approximation ratio $\approx 1 + \alpha_{\mathrm{TSP}} \geq 2$

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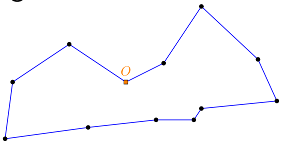

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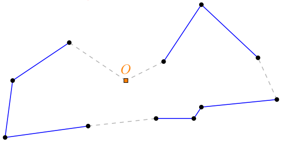

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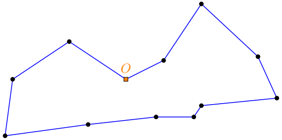

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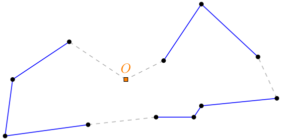

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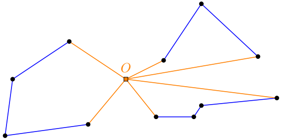

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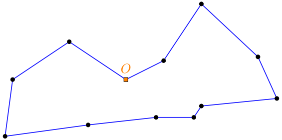

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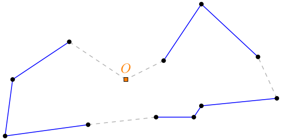

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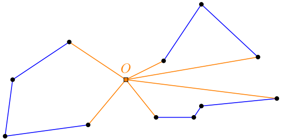


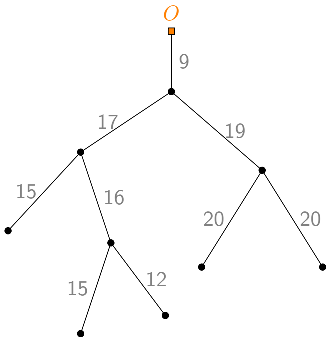
Approximation ratio $\approx 1 + \alpha_{\text{TSP}} \geq 2$

Capacitated vehicle routing on trees

NP-hard [LLM 1991]

Approximation algorithms:

- 1.5-approximation [HK 1998]
- 1.351-approximation [AKK 2001]
- 1.333-approximation [Bec 2018]
- bicriteria PTAS [BP 2019]
- QPTAS [JS 2022]



Our Result

PTAS for Capacitated Vehicle Routing on Trees

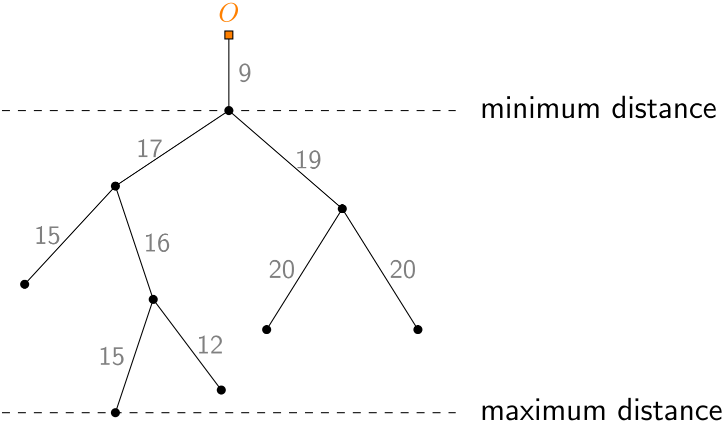
Note: First PTAS for general capacity k.

Capacitated vehicle routing on trees

Jayaprakash and Salavatipour [SODA 2022]:

"*it is not clear if it (the QPTAS) can be turned into a PTAS without significant new ideas.*"

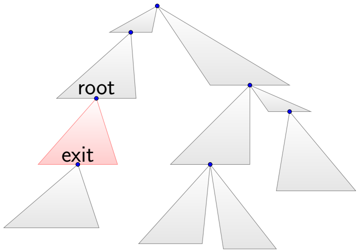
Preprocessing: bounded distance property



Can assume: $\dfrac{\textbf{minimum distance}}{\textbf{maximum distance}} > \epsilon.$

Decomposing the tree into components

Inspiration: clusters decomposition [Becker and Paul]



each **component** has:

- $\approx k/\epsilon$ terminals
- 1 root vertex
- ≤ 1 exit vertex

Structure Theorem

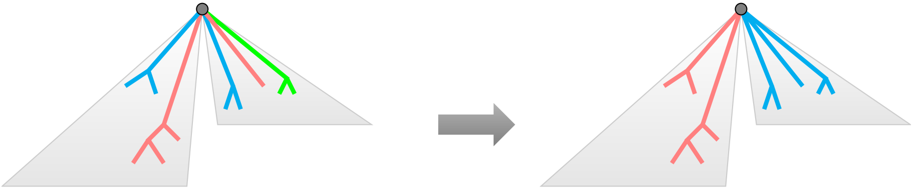
There is a near-optimal solution such that each subtour in a component visits either 0 or at least $\epsilon \cdot k$ terminals.

Proof of the Structure Theorem (1/4)

Definition: a subtour in a component is **small** if it visits more than 0 but less than $\epsilon \cdot k$ terminals.

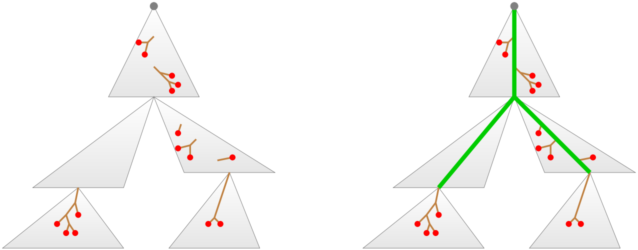
Elimination of small subtours

1. Detach small subtours
2. Combine small subtours within components
3. Reassign combined subtours [Becker and Paul]



Difficulty: **after reassignment some tours may exceed capacity**

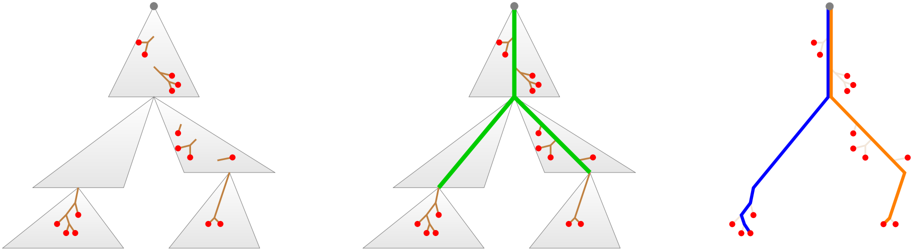
Proof of the Structure Theorem (2/4)



4. Remove some subtours
5. Include spine subtours

brown & **green**: **traveling salesman tour** covering the red terminals

Proof of the Structure Theorem (3/4)

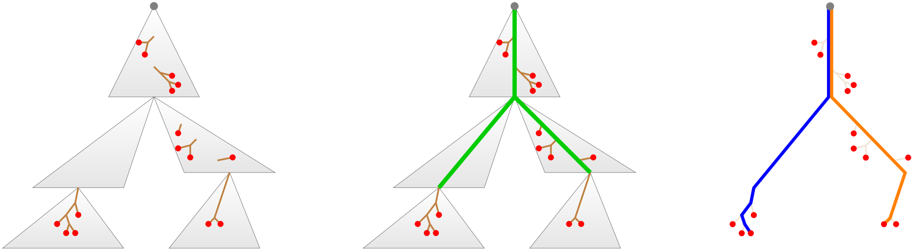


However, the traveling salesman tour may exceed capacity.

Our approach: **iterated tour partitioning**

6. Add connections to the depot (**blue** & **orange**)

Proof of the Structure Theorem (4/4)



Extra cost is negligible:

- **green**: properties of components
- **blue & orange**: bounded distance property

Structure Theorem

There is a near-optimal solution such that each subtour in a component visits either 0 or at least $\epsilon \cdot k$ terminals.

First attempt for the dynamic program

1. Computing solutions inside each component: polynomial time
2. Combining solutions from different components bottom-up:

   exponential time

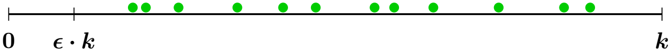
Q: How to improve the running time?

A: Adaptive rounding to reduce the number of subtour demands.

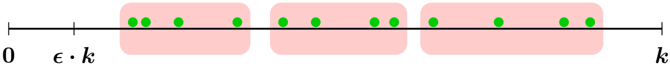
Adaptive rounding [Jayaprakash and Salavatipour]

At each vertex:

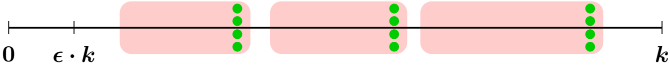
1. **Sort** subtour demands



2. **Make groups** of equal cardinality



3. **Round up** to maximum demand in group



Adaptive rounding analysis

Theorem [Jayaprakash and Salavatipour]

There is a near-optimal solution in which the subtour demands can be rounded up to **polylogarithmic** many values.

$\Longrightarrow$ **QPTAS**

Our Theorem

There is a near-optimal solution in which the subtour demands can be rounded up to **constant** many values.

$\Longrightarrow$ **PTAS**

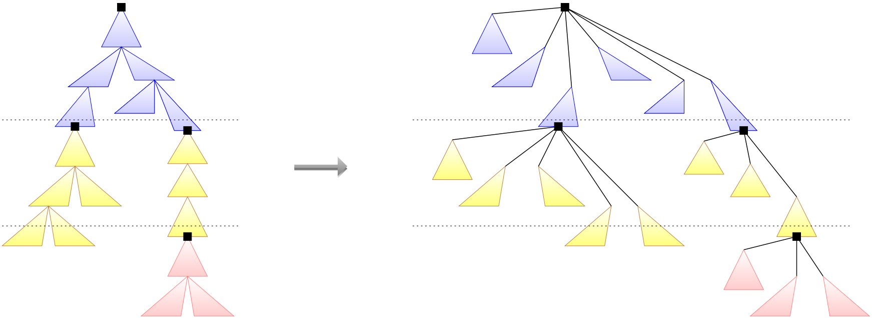
How do we bound the extra cost?

- Structure Theorem
- bounded distance property
- bounded height of components

Bounded height of components

Transform the tree of components so that:

each leaf-root path traverses a bounded number of components



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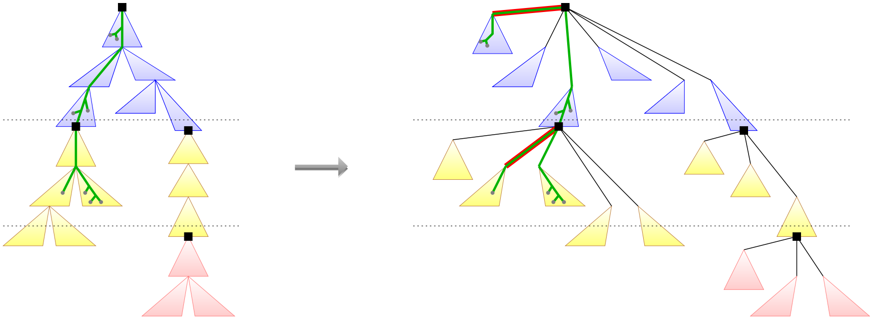
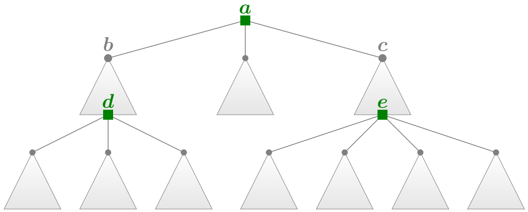


Extra cost of the tour is negligible:

- Structure Theorem
- bounded distance property

Dynamic programming and adaptive rounding



Dynamic program order of computation:

1. each component
2. subtrees rooted at d and e by **adaptive rounding**
3. subtrees rooted at b and c
4. subtree rooted at a by **adaptive rounding**

Polynomial time to obtain a $(1 + \epsilon)$-approximate solution

Summary

1. Preprocessing $\Longrightarrow$ **Bounded distance**
2. Decomposing the tree into components $\Longrightarrow$ **Structure Theorem**
3. Transforming the tree $\Longrightarrow$ **Bounded height of components**
4. Adaptive Rounding $\Longrightarrow$ **Constant many subtour demands**
5. Dynamic programming $\Longrightarrow$ $(1+\epsilon)$**-approximate solution**

Our Result

PTAS for Capacitated Vehicle Routing on Trees

Thank you!