Euclidean Capacitated Vehicle Routing in the Random Setting: A 1.55-Approximation Algorithm

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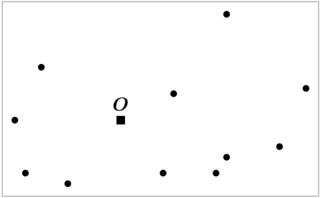
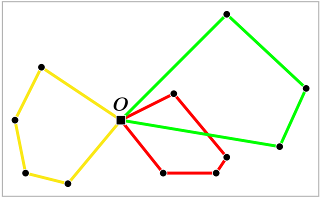
Capacitated Vehicle Routing Problem (CVRP)

Input:

- **depot**
- **set of terminals**
- **capacity k**

Minimize total length of tours





Fundamental problem in operations research

Unit Demand

- 1985 Haimovich and Rinnooy Kan
- 1990 Altinkemer and Gavish
- 1997 Asano, Katoh, Tamaki, and Tokuyama
- 1998 Hamaguchi and Katoh
- 2001 Asano, Katoh, and Kawashima
- 2006 Bompadre, Dror, and Orlin
- 2010 Adamaszek, Czumaj, and Lingas
- 2010 Das and Mathieu
- 2017 Becker, Klein, and Saulpic
- 2018 Becker, Klein, and Saulpic
- 2018 Becker
- 2019 Becker, Klein, and Schild
- 2019 Becker and Paul
- 2020 Cohen-Addad, Filtser, Klein, and Le
- 2022 Blauth, Traub, and Vygen
- 2022 Jayaprakash and Salavatipour
- 2022 Jayaprakash and Salavatipour
- 2022 Mathieu and Z.
- 2023 Mömke and Z.
- 2023 Mathieu and Z.
- 2023 Dufay, Mathieu, and Z.
- 2024 Nie and Z.

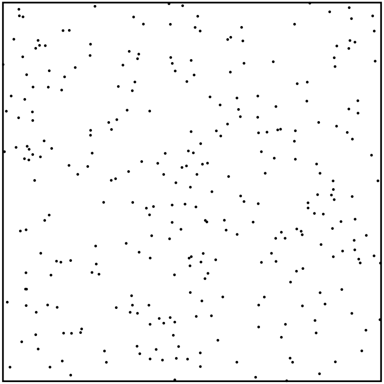
Unsplittable Demand

- 1981 Golden and Wong
- 1987 Altinkemer and Gavish
- 1991 Labbé, Laporte, and Mercure
- 2021 Blauth, Traub, and Vygen
- 2022 Friggstad, Mousavi, Rahgoshay, and Salavatipour
- 2023 Grandoni, Mathieu, and Z.
- 2023 Mathieu and Z.

- general metrics
- Euclidean plane
- trees
- planar graphs
- graphs of bounded treewidth
- graphs of bounded highway dimension
- graphic metrics

Random Setting in the Euclidean Plane

Terminals: **independent, identically distributed** random points in $[0,1]^2$



CVRP in the Random Setting

Previous results

- **2-approximation [Haimovich and Rinnooy Kan 1985]**
- **1.995-approximation [Bompadre, Dror, and Orlin 2007]**
- **1.915-approximation [Mathieu and Z. 2022]**

New results

- **1.55-approximation [this work]**

CVRP in the Random Setting: Our Results

Theorem

Sweep algorithm is at most 1.55-approximation.

Sweep algorithm

1. **Sort** terminals according to the **polar angle** w.r.t. the depot O
2. Partition terminals into groups, each of k/ε **consecutive terminals**
3. For each group, compute a $(1+\varepsilon)$-**approximate solution**

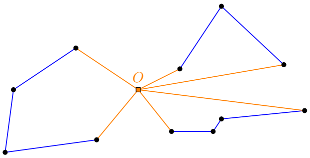
Conjecture

Sweep algorithm is $(1+\varepsilon)$-approximation.

Analysis: Warm Up [Haimovich and Rinnooy Kan 1985]

Two factors in the solution cost:

1. traveling salesman tour cost **TSP**
2. radial cost **rad** $:= \frac{2}{k} \cdot \sum_v \delta(O, v)$



Properties on the optimal cost **OPT**:

- **OPT** $\geq$ **rad**
- **OPT** $\geq$ **TSP**

Implication: **2-approximation algorithm**

Analysis: Our Approach in High Level

Generalization of radial cost and TSP cost

For a real value R:

- **rad**(R) := $\frac{2}{k} \sum_v \min\{\delta(O,v), R\}$
- **TSP**(R) := TSP cost on $\{v : \delta(O,v) \geq R\}$

Structure Theorem

$$\textbf{OPT} \geq \textbf{rad}(R) + \textbf{TSP}(R).$$

Implications:

- $R = 0 \implies$ **OPT** $\geq$ **TSP**
- $R = \infty \implies$ **OPT** $\geq$ **rad**
- R well chosen:

Theorem

Sweep algorithm is at most 1.55-approximation.

Proof of the Structure Theorem

Structure Lemma

Consider a tour S. Let V_S denote the set of points visited by S. Let C be the circle centered at O and with radius R. Let T denote the intersection between S and the exterior of C. Then

$$\text{length}(S) \geq \frac{2}{k} \sum_{v \in V_S} \min\{\delta(O, v), R\} + \text{length}(T).$$

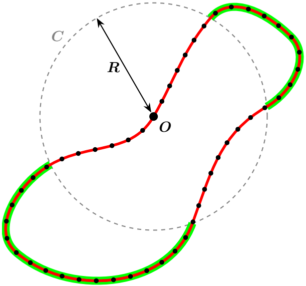
↓ **summing over all tours in an optimal solution**

Structure Theorem

$$\textbf{OPT} \geq \textbf{rad}(\boldsymbol{R}) + \textbf{TSP}(\boldsymbol{R}).$$

Proof of the Structure Lemma (1/2)

Case 1: $T \neq \emptyset$.



S: in red
T: in green

We have

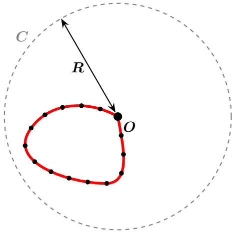
$$\text{length}(S) \geq 2R + \text{length}(T),$$

where

$$2R = \frac{2}{|V_S|} \sum_{v \in V_S} R \geq \frac{2}{k} \sum_{v \in V_S} \min\{\delta(O, v), R\}.$$

Proof of the Structure Lemma (2/2)

Case 2: $T = \emptyset$.



S: in red

We have

$$\text{length}(S) \geq 2 \cdot \max_{v \in V_S} \delta(O, v) \geq \frac{2}{|V_S|} \sum_{v \in V_S} \delta(O, v) \geq \frac{2}{k} \sum_{v \in V_S} \min\{\delta(O, v), R\}.$$

□

Cost of Sweep Algorithm: **Sol**

Sweep algorithm

1. **Sort** terminals according to the **polar angle** w.r.t. the depot O
2. Partition terminals into groups, each of k/ε **consecutive terminals**
3. For each group, compute a $(1+\varepsilon)$-**approximate solution**

Remark

If optimal cost of CVRP is linear with respect to **angular partitioning**, then the sweep algorithm provides a $(1+\varepsilon)$-approximation.

Upper bound [Haimovich and Rinnooy Kan 1985] for CVRP:

$$\textbf{OPT} \leq \textbf{rad} + \textbf{TSP}.$$

TSP is linear with respect to **angular partitioning**, as a corollary of Richard Karp's result, so we have:

Upper Bound on Sol

$$\textbf{Sol} = \sum \textbf{OPT}_i \leq \sum \textbf{rad}_i + \sum \textbf{TSP}_i \approx \textbf{rad} + \textbf{TSP}.$$

Main Theorem: 1.55-Approximation

New Upper Bound on Approximation Ratio

$$\frac{\textsf{Sol}}{\textsf{OPT}} \leq \mathbf{1.55}.$$

Proof:

$$\begin{aligned}\frac{\textsf{Sol}}{\textsf{OPT}} &\leq \min_{R} \frac{\textsf{rad} + \textsf{TSP}}{\textsf{rad}(R) + \textsf{TSP}(R)} \\ &\leq \min_{R} \max\left(\frac{\textsf{rad}}{\textsf{rad}(R)}, \frac{\textsf{TSP}}{\textsf{TSP}(R)}\right) \\ &\leq \mathbf{1.55}.\end{aligned}$$

Converting to Continuous Form

Let v be a uniformly random point in the unit square.

Discrete	Continuous
rad	$\frac{2n}{k}\,\mathbb{E}(\delta(O,v))$
rad($\boldsymbol{R}$)	$\frac{2n}{k}\,\mathbb{E}(\min(\delta(O,v),R))$
TSP	$\beta\sqrt{n}$
TSP($\boldsymbol{R}$)	$\beta\sqrt{n}\,\mathbb{P}(\delta(O,v)\geq R)$

We choose

$$\boldsymbol{R} = \frac{3}{4}\,\mathbb{E}(\delta(O,v)).$$

Two-variable inequalities

Lemma

Let $R = \frac{3}{4}\,\mathbb{E}(\delta(O,v))$, then

$$\mathbb{E}(\delta(O,v)) < 1.55\,\mathbb{E}(\min(\delta(O,v),R)),$$

$$1 < 1.55\,\mathbb{P}(\delta(O,v) \geq R).$$

Linear in the distribution of $v \longrightarrow$ Partition the square and the disk $\{p : \delta(O,p) \leq R\}$ into **disk sectors** and **triangles**.

Closed-Form Formula [Stone 1991]

$$\int_0^a \int_0^{\frac{bx}{a}} \sqrt{x^2+y^2}\,dy\,dx = \frac{a^3}{6}\log\left(\frac{b}{|a|} + \sqrt{1+\frac{b^2}{a^2}}\right) + \frac{ab}{6}\sqrt{a^2+b^2}.$$

Thank you!