A Simple Algorithm for Graph Reconstruction

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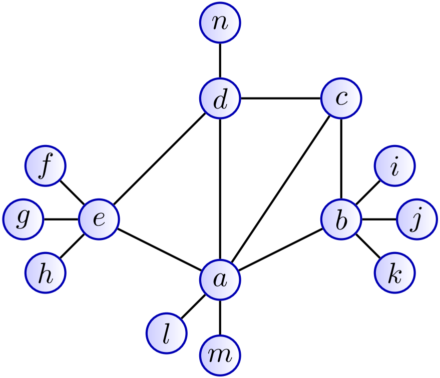
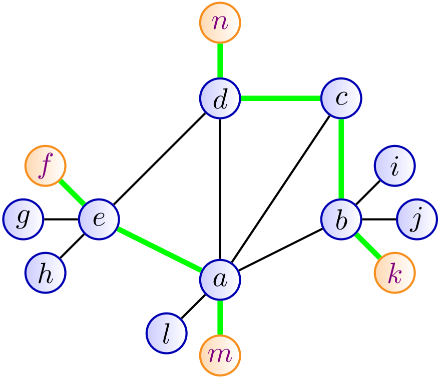
Graph reconstruction problem



- vertex set V is known and edge set E is unknown
- goal: find every edge in E
- assumption: connected and unweighted graph

Graph reconstruction problem



Traceroute

(n,k): $n - \star - \star - \star - k$

(f,m): $f - \star - \star - m$

- distance query: $\{u,v\} \mapsto$ distance $\delta(u,v)$
- efficiency measure: number of queries

Our results

Uniformly random Δ-regular graphs, $\Delta = O(1)$

- **Reconstruction**

query oracle	*complexity*
distance	$\tilde{O}(n)$ [Main Theorem]
all-distances	$O(\log^2 n)$
betweenness	$\tilde{O}(n)$
parallel distance	$\tilde{O}(n)$ queries, $1 + o(1)$ rounds

- **Metric dimension** $\leq \log^2 n$

General graphs of bounded degree

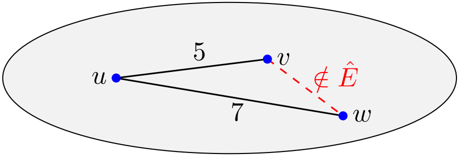
- **Reconstruction**

query oracle	*complexity*
parallel distance	$\tilde{O}(n^{5/3})$ queries, 2 rounds

SIMPLE algorithm

Preliminary observations

- $\delta(v, w) = 1 \implies (v, w)$ is an edge
- $\delta(v, w) > 1 \implies (v, w)$ is not an edge
- $|\delta(u, v) - \delta(u, w)| > 1 \implies (v, w)$ is not an edge



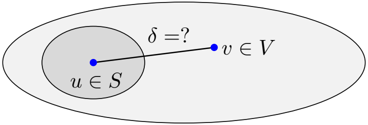
Idea: take multiple vertices u to confirm many non-edges

Potential edges: pairs not confirmed as non-edges by any vertex u

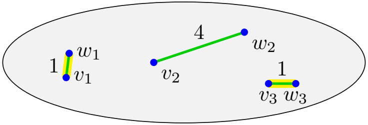
The SIMPLE algorithm

Input parameter $s \in [1, n]$.

1. Take a sample $S \subseteq V$ uniformly at random such that $|S| = s$
2. Query $S \times V$



3. Query every **potential edge** $\{v, w\}$

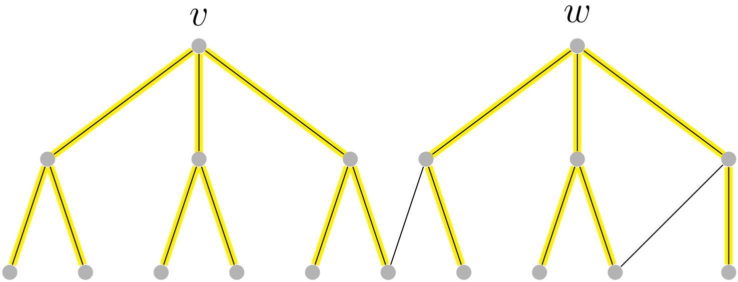


4. Output the potential edges $\{v, w\}$ such that $\delta(v, w) = 1$

Overall query complexity: $\underbrace{s \cdot n}_{2} + \underbrace{\text{number of potential edges}}_{3}$

Let v, w be any non-adjacent pair of vertices.

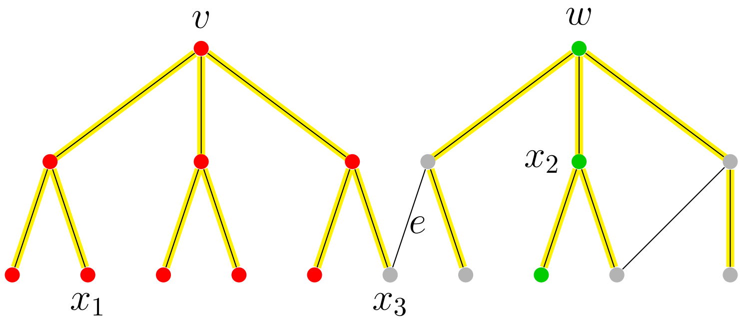
- Breadth First Search (BFS) from v and w simultaneously
- An edge is **exploring**: one endpoint is explored for the first time



Interesting vertices

A vertex x is *v-interesting* (resp. *w-interesting*) if all of the following:

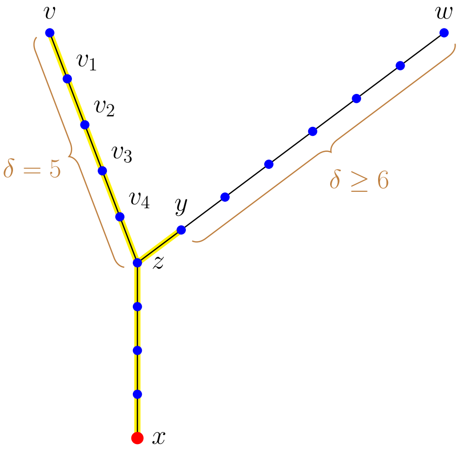
- the shortest path from v (resp. w) to x is unique
- all edges on that path are exploring
- all edges incident to that path are exploring.



Lemma 1

If x is v-interesting or w-interesting, then $|\delta(x,v) - \delta(x,w)| > 1$.

Proof:

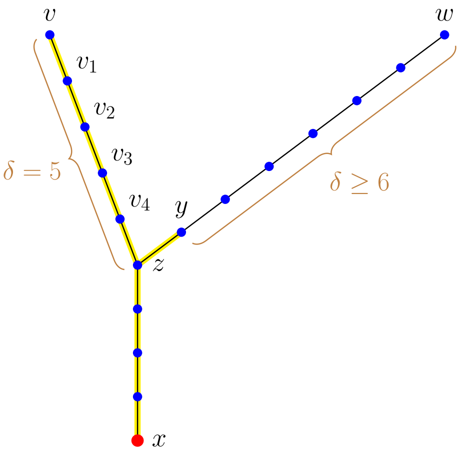


- a shortest x-to-v path
- a shortest x-to-w path
- branching point z
- suffices to show:
  $|\delta(z,v) - \delta(z,w)| > 1$.

Lemma 1

If x is v-interesting or w-interesting, then $|\delta(x,v) - \delta(x,w)| > 1$.

Proof:



Let $\ell(\cdot) = \min\{\delta(\cdot, v), \delta(\cdot, w)\}$

- $\ell(v_1) = 1$
- $\ell(v_2) = 2$
- $\ell(v_3) = 3$
- $\ell(v_4) = 4$
- $\ell(z) = 5$
- $\ell(y) = 6$

- $\delta(y, w) \geq 6$
- $\delta(z, w) \geq 7 = \delta(z, v) + 2$ $\square$

Lemma 1

If x is v-interesting or w-interesting, then $|\delta(x, v) - \delta(x, w)| > 1$.

Lemma 2

With probability $1 - o(n^{-2})$, number of interesting vertices $\geq 3n/\log n$.

Proof of the Main Theorem:

- Let v, w be a non-adjacent pair of vertices.

$s = \log^2 n$ sampled vertices

$\Downarrow$ Lemma 2

$\exists$ an interesting vertex x in the sample, w.h.p.

$\Downarrow$ Lemma 1

(v, w) is not a potential edge, w.h.p.

- Number of potential edges $= |E| + o(1)$
- Query complexity: $s \cdot n +$ (number of potential edges) $= \tilde{O}(n)$. $\square$

Other query models

All-distances query model

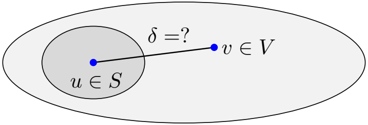
- Querying $S \times V$ takes $|S| = \log^2 n$ all-distances queries
- Query complexity for graph reconstruction: $O(\log^2 n)$

Betweenness query model

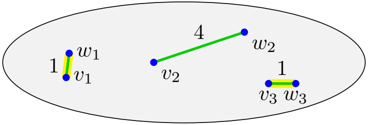
- $\tilde{O}(n)$ betweenness queries simulate an all-distances query *[Abrahamsen, Bodwin, Rotenberg, Stöckel, STACS 2016]*
- Query complexity for graph reconstruction: $\tilde{O}(n)$

Parallel setting

1. Take a sample $S \subseteq V$ uniformly at random such that $|S| = s$
2. Query $S \times V$



3. Query every potential edge $\{v, w\}$



4. Output the potential edges $\{v, w\}$ such that $\delta(v, w) = 1$

Number of rounds

- Two rounds on general graphs
- $1 + o(1)$ rounds on random Δ-regular graphs

Metric dimension

Resolving set $S \subseteq V$ for a graph: for any pair of vertices v and w, there is a vertex $u \in S$ such that $\delta(u,v) \neq \delta(u,w)$.
Metric dimension of a graph: cardinality of a smallest resolving set.

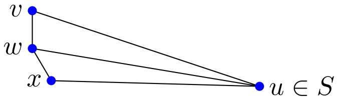
Lemma

A random subset S of $\log^2 n$ vertices is a resolving set for a random Δ-regular graph w.h.p.

Proof:

Case 1: $\delta(v,w) \geq 2$: arguments from graph reconstruction

Case 2: $\delta(v,w) = 1$:



- $\exists$ vertex x adjacent to w but not adjacent to v w.h.p.
- $\delta(v,x) \geq 2 \implies \exists\, u \in S$ with $|\delta(u,v) - (u,x)| \geq 2$
- $|\delta(u,v) - (u,w)| \geq 1$ w.h.p. $\square$

Conclusion: metric dimension of a random Δ-regular graph $\leq \log^2 n$ w.h.p.