Discriminant of a reflection group and factorisations of a Coxeter element

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Outline

1. Fuss-Catalan numbers of type $W$

2. Factorisations as fibers of a Lyashko-Looijenga covering

3. Maximal and submaximal factorisations of a Coxeter element
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2. Factorisations as fibers of a Lyashko-Looijenga covering
3. Maximal and submaximal factorisations of a Coxeter element
Factorisations in a generated group

Group $G$, generated by $A$
Factorisations in a generated group

Group $G$, generated by $A \rightsquigarrow$ **length function** $\ell_A$ on $G$. 

**Definition** (A-factorisations) $(g_1, \ldots, g_p)$ is an A-factorisation of $g \in G$ if
\[ g_1 \cdots g_p = g; \]
\[ \ell_A(g_1) + \cdots + \ell_A(g_p) = \ell_A(g). \]

**Example:** $(G, A) = (W, S)$ finite Coxeter system.

Strict maximal factorisations of $w_0 \leftrightarrow$ galleries connecting a chamber to its opposite. [Deligne]

**Definition** (Divisibility order $\lesssim_A$) $g \lesssim_A h$ if and only if $g$ is a (left) "A-factor" of $h$.

$[1, h] \lesssim_A = \{ \text{divisors of } h \text{ for } \lesssim_A \} \simeq \{ \text{2-factorisations of } h \}$. 

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\text{Group } G, \text{ generated by } A & \rightsquigarrow \textbf{length function } \ell_A \text{ on } G. \\
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**Definition (Divisibility order $\bowtie_A$)**

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Prototype: noncrossing partitions of an $n$-gon

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A (finite) complex reflection group is a finite subgroup of $\text{GL}(V)$ generated by complex reflections. A complex reflection is an element $s \in \text{GL}(V)$ of finite order, s.t. $\text{Ker}(s - \text{Id}_V)$ is a hyperplane:

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- includes finite (complexified) real reflection groups (aka *finite Coxeter groups*);
Complex reflection groups

\( V \): complex vector space (finite dimension).

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- includes finite (complexified) real reflection groups (aka *finite Coxeter groups*);
- Shephard-Todd’s classification (1954): an infinite series with 3 parameters \( G(de, e, r) \), and 34 exceptional groups.
Noncrossing partitions of type $W$

Now suppose that $W \subseteq \text{GL}(V)$ is a complex reflection group, irreducible and well-generated (i.e. can be generated by $n = \dim V$ reflections).
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- $c$ : a Coxeter element in $W$.  

Definition (Noncrossing partitions of type $W$) $\text{NCP}_W(c) := \{w \in W | w \preceq_R c\}$; the structure does not depend on the choice of the Coxeter element (conjugacy).
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Fuss-Catalan numbers

Kreweras’s formula

- \( W := S_n \);
- \( c : \) an \( n \)-cycle.

The number of \( T \)-factorisations of \( c \) in \( p + 1 \) blocks is the Fuss-Catalan number

\[
\text{Cat}^{(p)}(n) = \prod_{i=2}^{n} \frac{i + pn}{i}.
\]

Proof: [Athanasiadis, Reiner, Bessis...]

Remark: \( \text{Cat}^{(p)}(W) \) counts also the number of maximal faces in the "\( p \)-divisible cluster complex of type \( W \)" (generalization of the simplicial associahedron) [Fomin-Reading].

Related to cluster algebras of finite type if \( W \) is a Weyl group.
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The quotient-space $V/W$

$W \subseteq \text{GL}(V)$ a complex reflection group. $W$ acts on $\mathbb{C}[V]$. 

$\Rightarrow$ isomorphism: $V/W \sim \rightarrow \mathbb{C}^n \bar{v} \mapsto (f_1(v), \ldots, f_n(v))$.
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**Chevalley-Shephard-Todd’s theorem:** there exist invariant polynomials $f_1, \ldots, f_n$, homogeneous and algebraically independent, s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$. 

**Definition**

The degrees $d_1 \leq \cdots \leq d_n = h$ of $f_1, \ldots, f_n$ do not depend on the choice of $f_1, \ldots, f_n$. They are called the invariant degrees of $W$. 

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Discriminant of $W$

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(arrangement of hyperplanes of $W$)
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- discriminant hypersurface (in $V/W \cong \mathbb{C}^n$):

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- discriminant hypersurface (in $V/W \simeq \mathbb{C}^n$): 
  $$\mathcal{H} := \left( \bigcup_{H \in \mathcal{A}} H \right) / W$$
- discriminant $\Delta_W$: equation of the hypersurface $\mathcal{H}$ in $\mathbb{C}[f_1, \ldots, f_n]$. 
  ($\Delta_W = \prod_{H \in \mathcal{A}} \varphi_H^e \in \mathbb{C}[V]^W$)
Example $W = A_3$: discriminant ("swallowtail")

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hypersurface $\mathcal{H}$ (discriminant) $\subseteq W \setminus V \cong \mathbb{C}^3$
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$\bigcup_{H \in A} H \subseteq V$

$\mathcal{H} = \{ \Delta_W = 0 \} \subseteq W \setminus V \cong \mathbb{C}^3$

$\Delta_W(f_1, f_2, f_3) = \text{Disc}(T^4 + f_1 T^2 - f_2 T + f_3 ; T)$
Lyashko-Looijenga map and geometric factorisations

\[ H \subseteq W \setminus V \sim \mathbb{C}^3 \]
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\[ f_n \]

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$$V/W = Y \times \mathbb{C}.$$  

$$\text{LL} : \quad Y \rightarrow E_n := \{\text{multisets of } n \text{ points in } \mathbb{C}\}$$  

$$y \mapsto \{\text{roots, with multiplicities, of } \Delta_W(y, f_n) \text{ in } f_n\}$$
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$$\Delta_W = f_n^n + a_2f_n^{n-2} + a_3f_n^{n-3} + \cdots + a_{n-1}f_n + a_n.$$  

**Definition** (LL as an algebraic (homogeneous) morphism)

$$\text{LL} : \quad \mathbb{C}^{n-1} \quad \rightarrow \quad \mathbb{C}^{n-1}$$  

$$(f_1, \ldots, f_{n-1}) \quad \mapsto \quad (a_2, \ldots, a_n)$$
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$(f_1, \ldots, f_{n-1}) \mapsto (a_2, \ldots, a_n)$

$\text{facto} : Y \rightarrow \text{FACT}(c) := \{\text{strict } R\text{-factorisations of } c\}$
Lyashko-Looijenga map of type $W$

\[ V/W = Y \times \mathbb{C}. \]

\[ LL : \quad Y \rightarrow E_n := \{ \text{multisets of } n \text{ points in } \mathbb{C} \} \]

\[ y \mapsto \{ \text{roots, with multiplicities, of } \Delta_W(y, f_n) \text{ in } f_n \} \]

\[ \Delta_W = f_n^2 + a_2f_n^{n-2} + a_3f_n^{n-3} + \cdots + a_{n-1}f_n + a_n. \]

**Definition** (LL as an algebraic (homogeneous) morphism)

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**facto** : $Y \rightarrow \text{FACT}(c) := \{ \text{strict } R\text{-factorisations of } c \}$

Geometrical compatibilities:

- length of the factors ($\leftrightarrow$ multiplicities in the multiset $LL(y)$);
- conjugacy classes of the factors ($\leftrightarrow$ parabolic strata in $\mathcal{H}$).
Fibers of LL and strict factorisations of $c$

Let $\omega$ be a multiset in $E_n$.

Compatibility $\Rightarrow \forall y \in \text{LL}^{-1}(\omega)$, the distribution of lengths of factors of $\text{facto}(y)$ is the same (composition of $n$).
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**Theorem (Bessis’07)**

The map $\text{facto}$ induces a bijection between the fiber $LL^{-1}(\omega)$ and the set of *strict factorisations* of same “composition” as $\omega$. 
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**Theorem (Bessis’07)**

The map $\text{facto}$ induces a bijection between the fiber $LL^{-1}(\omega)$ and the set of strict factorisations of same “composition” as $\omega$.

Equivalently, the product map:

$$Y \xrightarrow{LL \times \text{facto}} E_n \times \text{FACT}(c)$$

is injective, and its image is the set of “compatible” pairs.
Outline

1. Fuss-Catalan numbers of type $W$
2. Factorisations as fibers of a Lyashko-Looijenga covering
3. Maximal and submaximal factorisations of a Coxeter element
Bifurcation locus \((\mathcal{K})\) of LL

\[
\{x_1, \ldots, x_n\} \in E_n
\]

\[
(w_1, \ldots, w_p) \in \text{FACT}(c)
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An unramified covering

Bifurcation locus:

\[ \mathcal{K} := LL^{-1} (E_n - E_{\text{reg}}) \]
\[ = \{ y \in Y \mid \Delta_W(y, f_n) \text{ has multiple roots w.r.t. } f_n \} \]
\[ = \{ y \in Y \mid D_{LL}(y) = 0 \} \]
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where
\[ D_{LL} := \text{Disc}(\Delta_W(y, f_n) ; f_n). \]
An unramified covering

Bifurcation locus:
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Proposition (Bessis)
- LL : \( Y \rightarrow K \) \( \rightarrow E_n^{\text{reg}} \) is a topological covering, of degree \( n! \ h^n / |W| \).
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- \( LL : Y - \mathcal{K} \to E_n^{\text{reg}} \) is a topological covering, of degree \( n! \ h^n / |W| \);
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Can we compute \(|\text{FACT}_{n-1}(c)|\)?
Irreducible components of $\mathcal{K}$

Want to study the restriction of $\text{LL} : \mathcal{K} \to E_n - E_n^{\text{reg}}$. 

Proposition

There are (canonical) bijections between:

- the set of irreducible components of $\mathcal{K}$ (or irreducible factors of $D_{\text{LL}}$);
- the set of conjugacy classes of elements of $\text{NCP}_W$ of length 2;
- the set of conjugacy classes of parabolic subgroups of $W$ of rank 2.

Explanations

Denote "this" set by $\bar{L}_2$.

Thus: 

$D_{\text{LL}} = \prod_{\Lambda \in \bar{L}_2} D_{\text{r} \Lambda \Lambda}$ (irreducible factors in $C[f_1, \ldots, f_{n-1}]$).
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Submaximal factorisations of type \( \Lambda \)

\[ \text{FACT}^\Lambda_{n-1}(c) := \text{set of factorisations of } c \text{ in } n - 1 \text{ factors, with:} \]

- \( n - 2 \) reflections; and
- \( 1 \) element of length 2 and conjugacy class \( \Lambda \).
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Remark: $\text{FACT}^\Lambda_{n-1}(c) = \text{facto} \left( \{ \text{“generic” points in } \{ D_\Lambda = 0 \} \} \right)$.
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The restriction \( \text{LL}_\Lambda : \mathcal{K}_\Lambda \to E_n - E_n^{\text{reg}} \) corresponds to the extension \( \mathbb{C}[a_2, \ldots, a_n]/(D) \subseteq \mathbb{C}[f_1, \ldots, f_{n-1}]/(D_\Lambda) \).
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**Theorem (R.)**

For any $\Lambda$ in $\tilde{L}_2$,
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For any $\Lambda$ in $\bar{L}_2$,

- $\text{LL}_\Lambda$ is a finite morphism of degree $\frac{(n-2)!}{|W|} \frac{h^{n-1}}{\text{deg } D_\Lambda}$; and
- the number of factorisations of $c$ of type $\Lambda$ is

$$|\text{FACT}^\Lambda_{n-1}(c)| = \frac{(n - 1)!}{|W|} \frac{h^{n-1}}{\text{deg } D_\Lambda}.$$
Submaximal factorisations

Problem: find a general computation of $\sum_{\Lambda \in \bar{\mathcal{L}}_2} \deg D_{\Lambda}$.
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Recall that $D_{LL} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$. 

Proposition (Saito; R.)

Set $J_{LL} := \text{Jac} \left( \left( a_2, \ldots, a_n \right) / (f_1, \ldots, f_{n-1}) \right)$. Then:

$J_{LL} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda - 1}$. 

Virtual reflection groups?
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Virtual reflection groups?

So, $\sum \deg D_{\Lambda} = \deg D_{LL} - \deg J_{LL} = \ldots$
Corollary

Let $W$ be an irreducible, well-generated complex reflection group, of rank $n$. The number of \textbf{strict factorisations of a Coxeter element $c$ in $n - 1$ factors} is:
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Let $W$ be an irreducible, well-generated complex reflection group, of rank $n$. The number of strict factorisations of a Coxeter element $c$ in $n - 1$ factors is:

$$|\text{FACT}_{n-1}(c)| = \frac{(n-1)! h^{n-1}}{|W|} \left( \frac{(n-1)(n-2)}{2} h + \sum_{i=1}^{n-1} d_i \right).$$
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We recover what is predicted by Chapoton’s formula;
Corollary

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- We recover what is predicted by Chapoton’s formula;
- but the proof is more satisfactory and enlightening: we travelled from the numerology of $\text{FACT}_n(c)$ (non-ramified part of LL) to that of $\text{FACT}_{n-1}(c)$, without adding any case-by-case analysis.
Conclusion, questions

- We recover geometrically some combinatorial results known in the real case [Krattenthaler].

Can we go further (compute the $|\text{FACT}_k(c)|$)? Can we interpret Chapoton’s formula as a ramification formula for $LL$?

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Outline

Appendix
- Stratifications
- Comparison reflection groups / LL extensions
Stratification of $V$ with the “flats” (intersection lattice):

$$\mathcal{L} := \{ \bigcap_{H \in A} H \mid B \subseteq A \}.$$
Stratifications

Stratification of $V$ with the “flats” (intersection lattice):
\[ \mathcal{L} := \left\{ \bigcap_{H \in \mathcal{A}} H \mid B \subseteq \mathcal{A} \right\}. \]

**Bijections** [Steinberg]:
stratification $\mathcal{L}$ \iff \{parabolic subgroups of $W$\}

Remark: $H$ is the union of strata of $\bar{\mathcal{L}}$ of codim. 1.

Conjugacy classes of factors of facto \( (y) \) \iff strata containing the intersection points.
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Stratification of $V$ with the “flats” (intersection lattice):

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**Bijectons** [Steinberg]:

stratification $\tilde{\mathcal{L}} = \mathcal{L} / W \iff \text{p.sg.}(W)/\text{conj.}$
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**Bijections** [Steinberg]:
- stratification $\tilde{\mathcal{L}} = \mathcal{L}/W \iff \text{p.sg.}(W)/\text{conj.} \iff \text{NCP}_{W/\text{conj}}.

Remark: $H$ is the union of strata of $\tilde{\mathcal{L}}$ of codim. 1.
Stratification of $\mathcal{V}$ with the “flats” (intersection lattice):

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**Bijections** [Steinberg]:

- **stratification** $\tilde{\mathcal{L}} = \mathcal{L}/\mathcal{W}$
- $\text{codim}(\Lambda) = \text{rank}(\mathcal{W}_\Lambda) = \ell_R(w_\Lambda)$
- $\leftrightarrow \ p.sg.(\mathcal{W})/\text{conj.}$
- $\leftrightarrow \ NCP_{\mathcal{W}}/\text{conj.}$

Remark:

$H$ is the union of strata of $\tilde{\mathcal{L}}$ of codim. 1.

Conjugacy classes of factors of $\mathcal{Y}$ $\leftrightarrow$ strata containing the intersection points.
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**Bijections [Steinberg]:**

- Stratification $\bar{\mathcal{L}} = \mathcal{L} / W$ $\iff$ p.sg.$(W)$/conj. $\iff$ NCP$_W$/conj.
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**Bijections [Steinberg]:**

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Conjugacy classes of factors of $\text{facto}(y) \leftrightarrow$ strata containing the intersection points.
Example of $\mathcal{W} = A_3$: stratification of the discriminant

\[ \bigcup_{H \in \mathcal{A}} H \subseteq V \]

\[ \mathcal{H} = \{ \Delta_{\mathcal{W}} = 0 \} \subseteq \mathcal{W} \setminus V \cong \mathbb{C}^3 \]

\[ \Delta_{\mathcal{W}}(f_1, f_2, f_3) = \text{Disc} \left( T^4 + f_1 T^2 - f_2 T + f_3 ; T \right) \]
Example of $W = A_3$: stratification of the discriminant

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Irreducible components of $\mathcal{K}$, details

$$\bar{\mathcal{L}}_2 := \{ \text{strata of } \bar{\mathcal{L}} \text{ of codim. 2} \}.$$
Irreducible components of $\mathcal{K}$, details

$\mathcal{L}_2 := \{\text{strata of } \mathcal{L} \text{ of codim. } 2\}$.

Steinberg’s theorem $\Rightarrow \mathcal{L}_2$ is in bijection with:

- $\{\text{conjugacy classes of parabolic subgroups of } W \text{ of rank } 2\}$
- $\{\text{conjugacy classes of elements of } N_{CPW} \text{ of length } 2\}$
Irreducible components of $\mathcal{K}$, details

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**Proposition**

The $\phi(\Lambda)$, for $\Lambda \in \bar{L}_2$, are the irreducible components of $\mathcal{K}$ (where $\phi$ is the projection $V/W \rightarrow Y$).
### Reflection group vs. Lyashko-Looijenga extension

<table>
<thead>
<tr>
<th>Reflection group $W$</th>
<th>Extension LL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V \rightarrow V/W$</td>
<td>$Y \rightarrow \mathbb{C}^{n-1}$</td>
</tr>
<tr>
<td>$\mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}[V]^W \subseteq \mathbb{C}[V]$</td>
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</tr>
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<td>degree $</td>
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</tr>
<tr>
<td>$V^{\text{reg}} \rightarrow V^{\text{reg}}/W$</td>
<td>$Y - \mathcal{K} \rightarrow E_n^{\text{reg}}$</td>
</tr>
<tr>
<td>Generic fiber $\simeq W$</td>
<td>$\simeq \text{Red}_R(c)$</td>
</tr>
<tr>
<td>ramified on $\bigcup_{H \in A} H \rightarrow \mathcal{H}$</td>
<td>$\mathcal{K} = \bigcup_{\Lambda \in \bar{L}_2} \varphi(\Lambda) \rightarrow E_n - E_n^{\text{reg}}$</td>
</tr>
<tr>
<td>$\Delta_W = \prod_{H \in A} \alpha_H^{e_H}$</td>
<td>$D_{LL} = \prod_{\Lambda \in \bar{L}<em>2} D</em>{\Lambda}^{r_{\Lambda}}$</td>
</tr>
<tr>
<td>$J_W = \prod \alpha_H^{e_H-1}$</td>
<td>$J_{LL} = \prod D_{\Lambda}^{r_{\Lambda}-1}$</td>
</tr>
<tr>
<td>$e_H =</td>
<td>W_H</td>
</tr>
</tbody>
</table>