Discriminant of a reflection group and factorisations of a Coxeter element

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Outline

1. Fuss-Catalan numbers of type W
2. Factorisations as fibers of a Lyashko-Looijenga covering
3. Maximal and submaximal factorisations of a Coxeter element

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$[1, h]_{\preccurlyeq_A} = \{\text{divisors of } h \text{ for } \preccurlyeq_A\} \simeq \{\text{2-factorisations of } h\}$.

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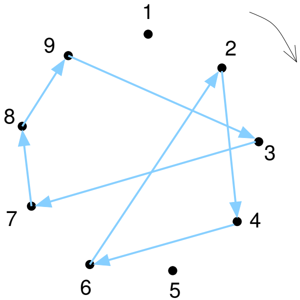

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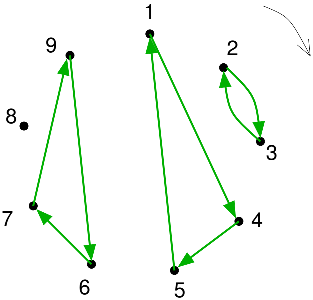

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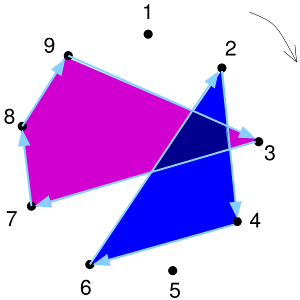

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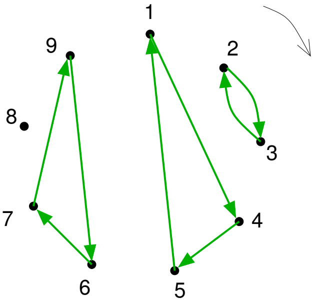

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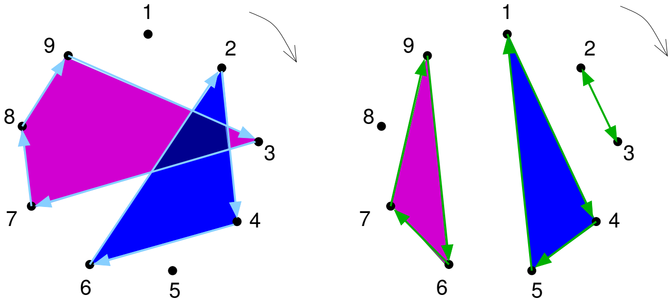

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Complex reflection groups

V : complex vector space (finite dimension).

Definition

A (finite) complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections.
A complex reflection is an element $s \in \mathrm{GL}(V)$ of finite order, s.t. $\mathrm{Ker}(s - \mathrm{Id}_V)$ is a hyperplane:

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- Shephard-Todd's classification (1954): an infinite series with 3 parameters $G(de, e, r)$, and 34 exceptional groups.

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- $\mathrm{NCP}_W(c) \simeq \{\text{2-factorisations of } c\}$;
- the structure does not depend on the choice of the Coxeter element (conjugacy).

Fuss-Catalan numbers

Kreweras's formula

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The number of T-factorisations of c in $p+1$ blocks is the **Fuss-Catalan number**

$$\mathrm{Cat}^{(p)}(n) = \prod_{i=2}^{n} \frac{i+pn}{i} .$$

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Definition

The degrees $d_1 \leq \cdots \leq d_n = h$ of $f_1, \ldots, f_n$ do not depend on the choice of $f_1, \ldots, f_n$. They are called the invariant degrees of W.

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- discriminant Δ_W: equation of the hypersurface $\mathcal{H}$ in $\mathbb{C}[f_1, \ldots, f_n]$. $\quad (\Delta_W = \prod_{H \in \mathcal{A}} \varphi_H^{e_H} \in \mathbb{C}[V]^W)$

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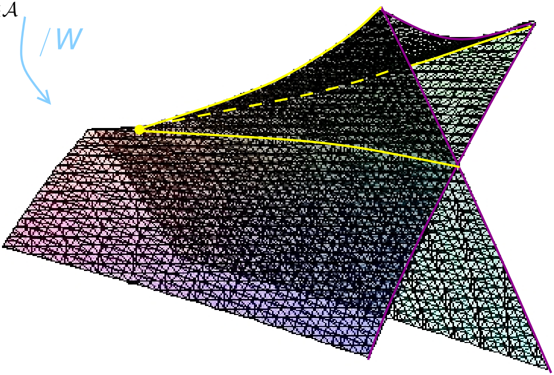

hypersurface $\mathcal{H}$ (discriminant) $\subseteq W \backslash V \simeq \mathbb{C}^3$

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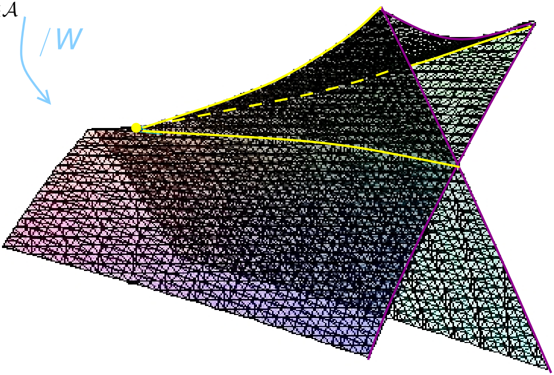
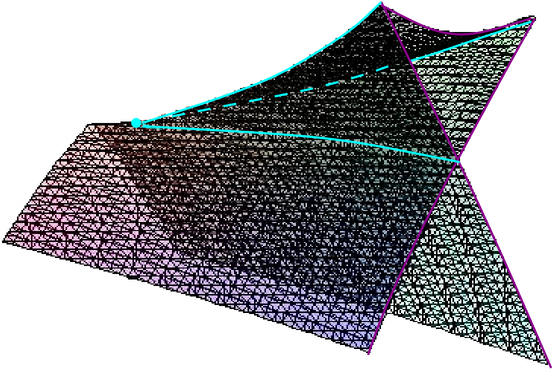


$$\mathcal{H} = \{\Delta_W = 0\} \subseteq W \backslash V \simeq \mathbb{C}^3$$

$$\Delta_W(f_1, f_2, f_3) = \mathrm{Disc}(T^4 + f_1 T^2 - f_2 T + f_3 \; ; \; T)$$

Lyashko-Looijenga map and geometric factorisations



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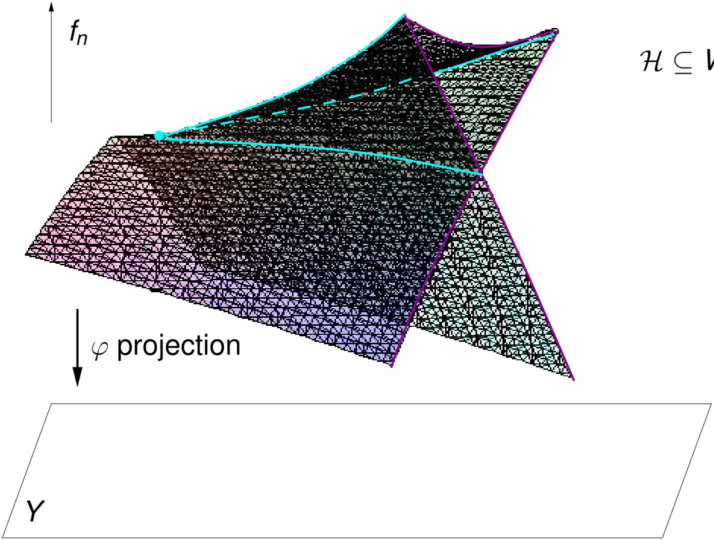

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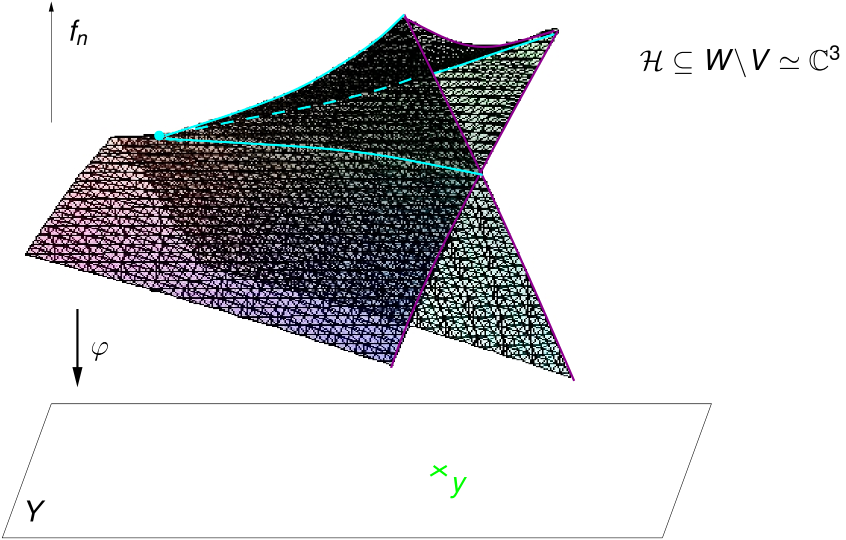

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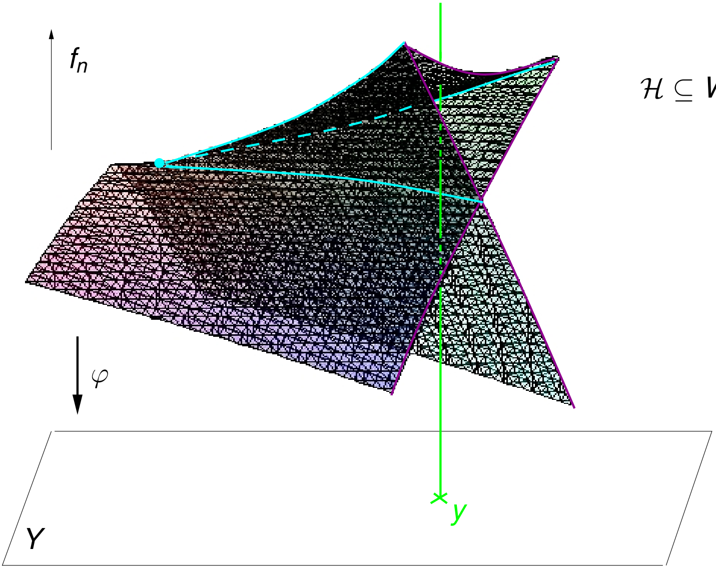

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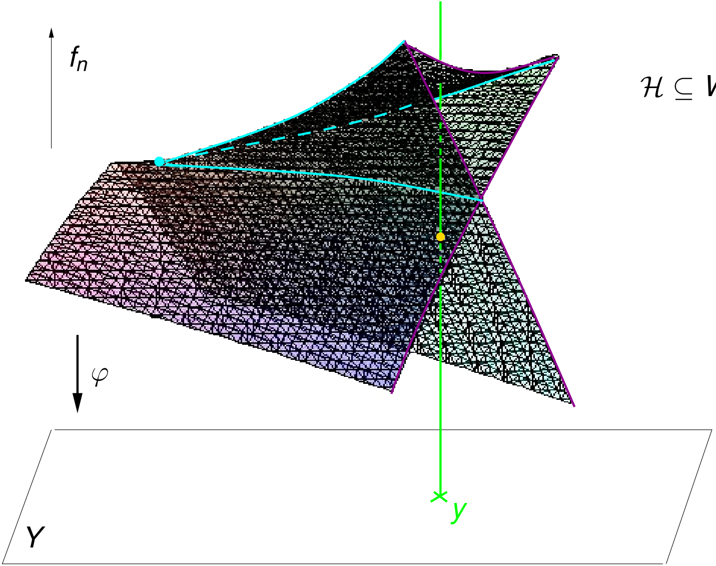

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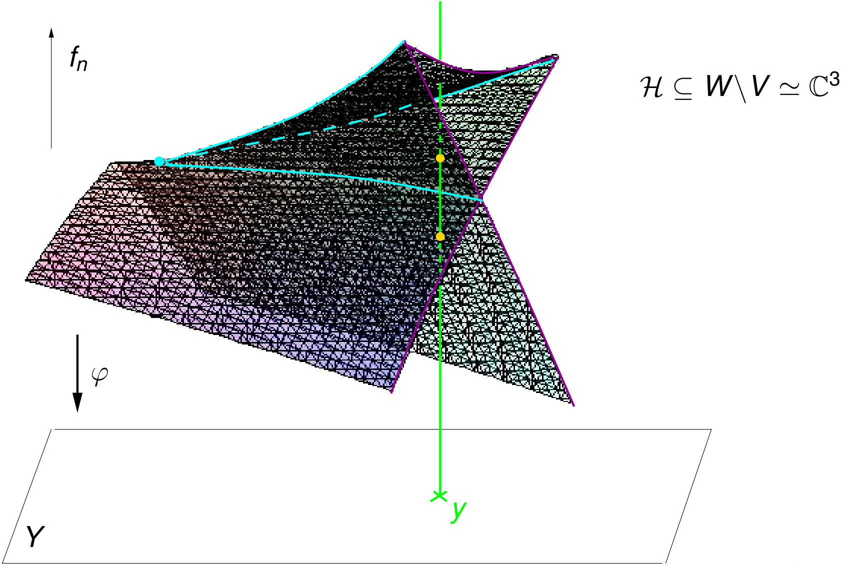

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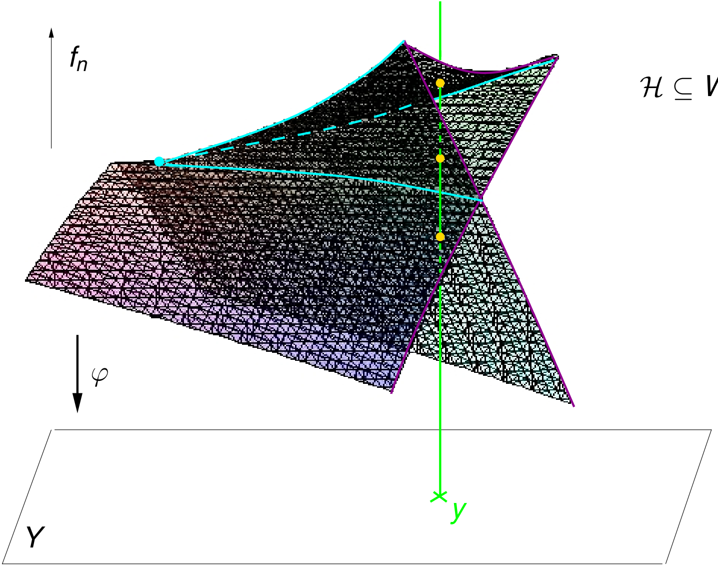

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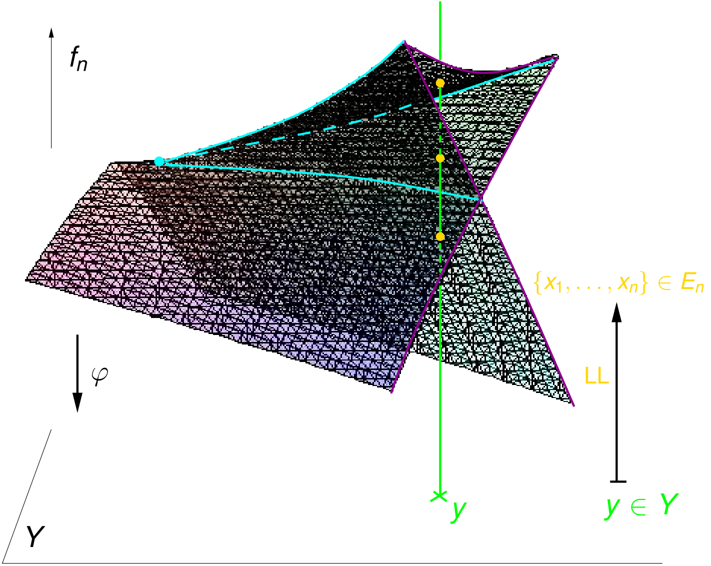

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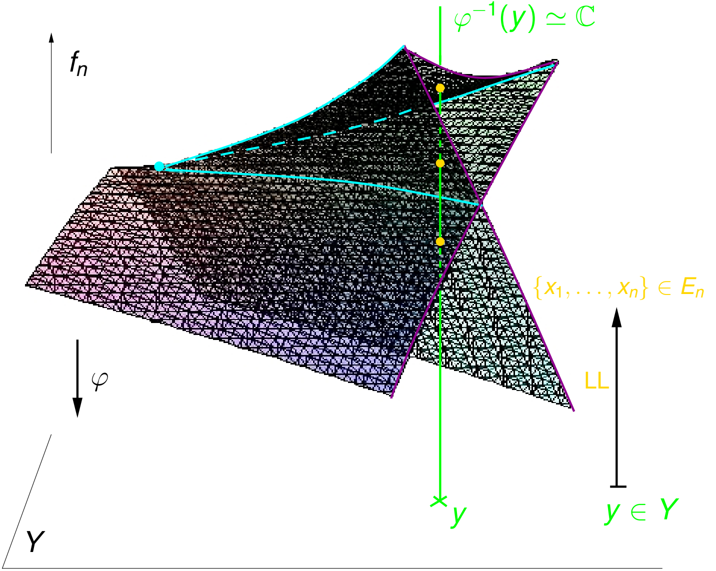

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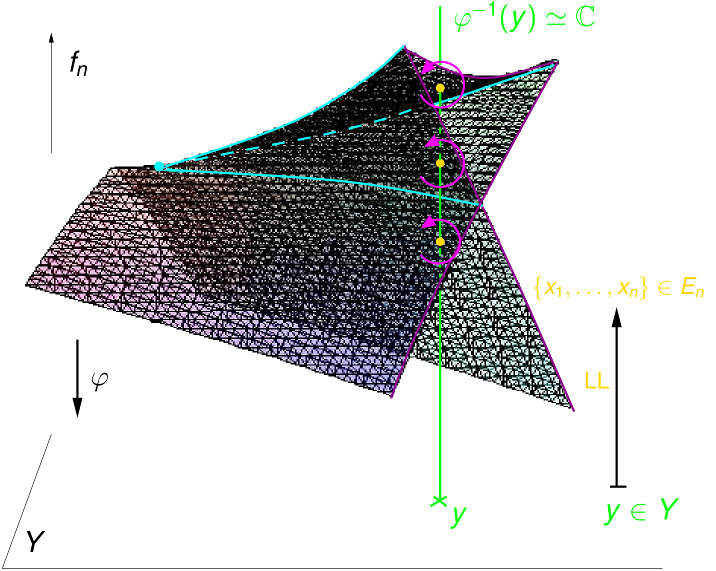

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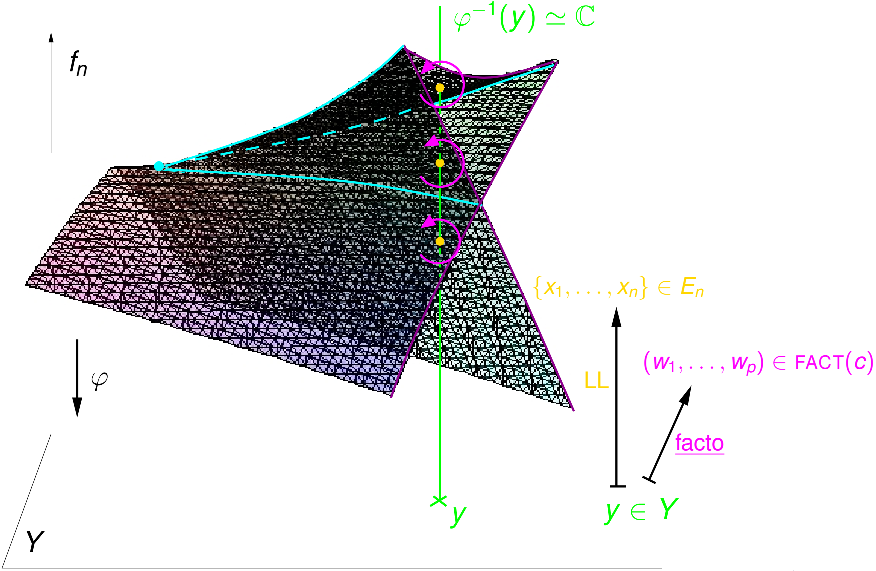

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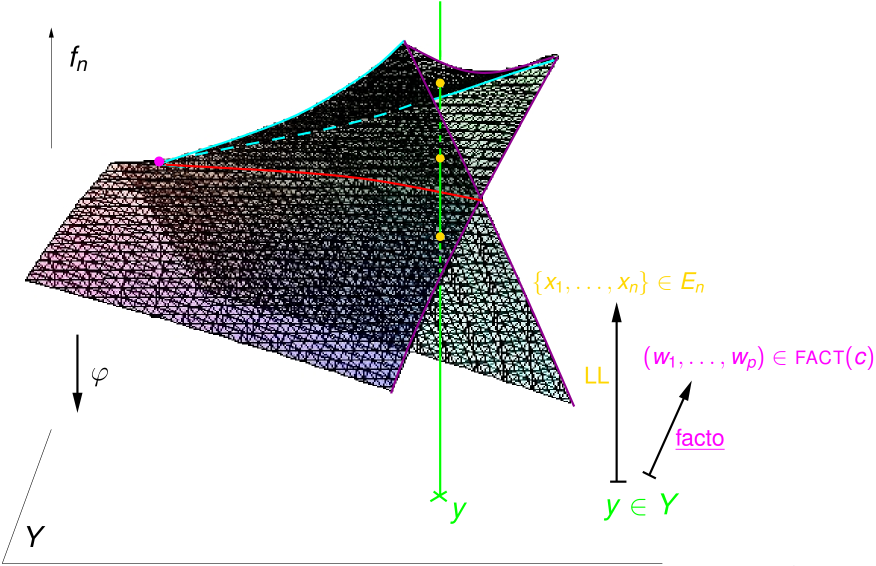

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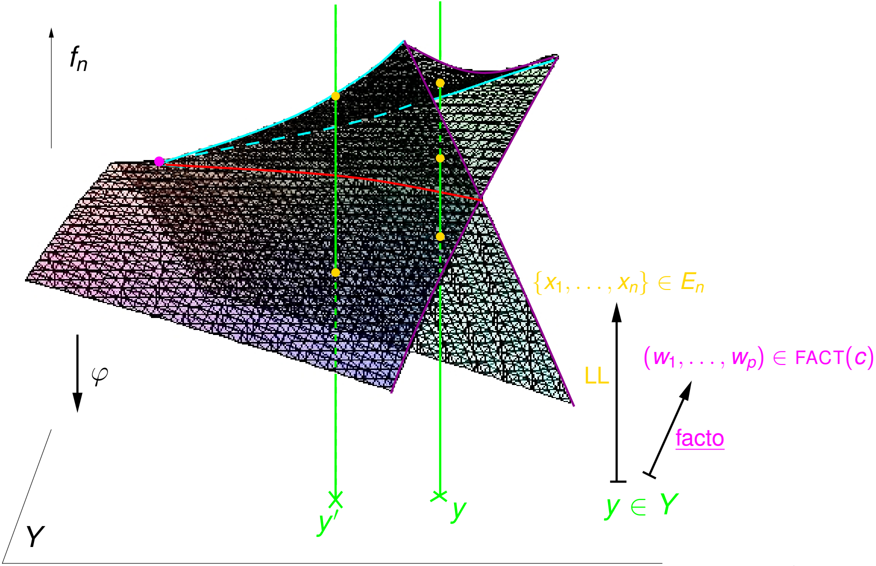

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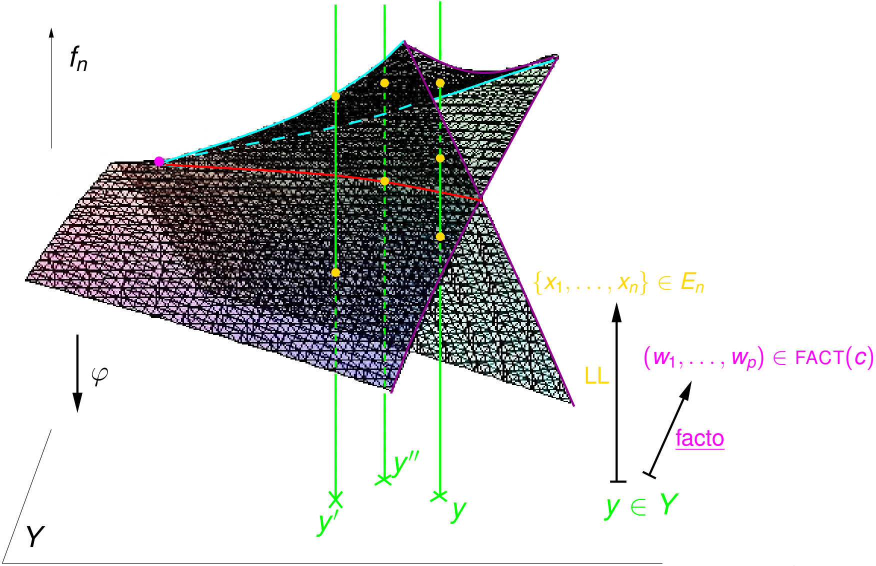

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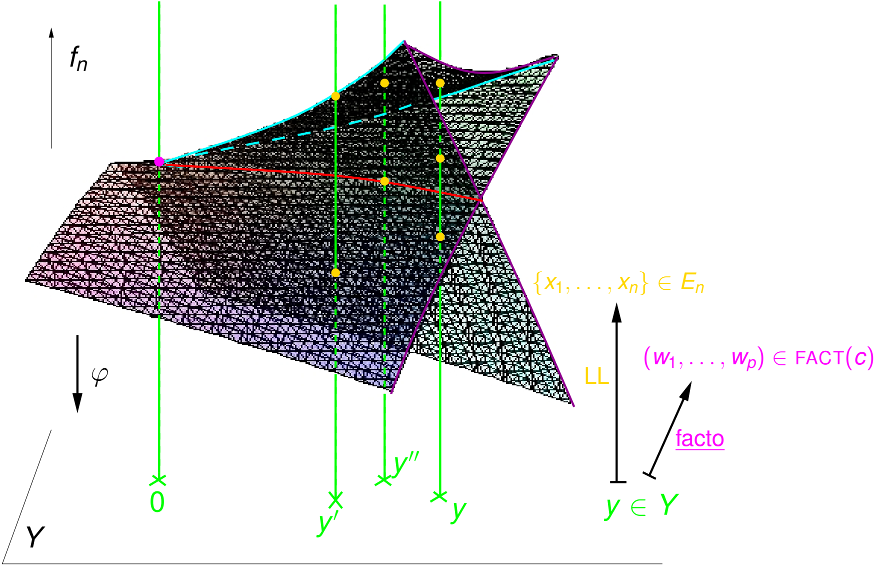

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Geometrical compatibilities:

- length of the factors ($\leftrightarrow$ multiplicities in the multiset LL(y));
- conjugacy classes of the factors ($\leftrightarrow$ parabolic strata in $\mathcal{H}$).

▸ Details

Fibers of LL and strict factorisations of c

Let ω be a multiset in E_n.

Compatibility $\Rightarrow$ $\forall y \in \mathrm{LL}^{-1}(\omega)$, the distribution of lengths of factors of $\underline{\mathrm{facto}}(y)$ is the same (composition of n).

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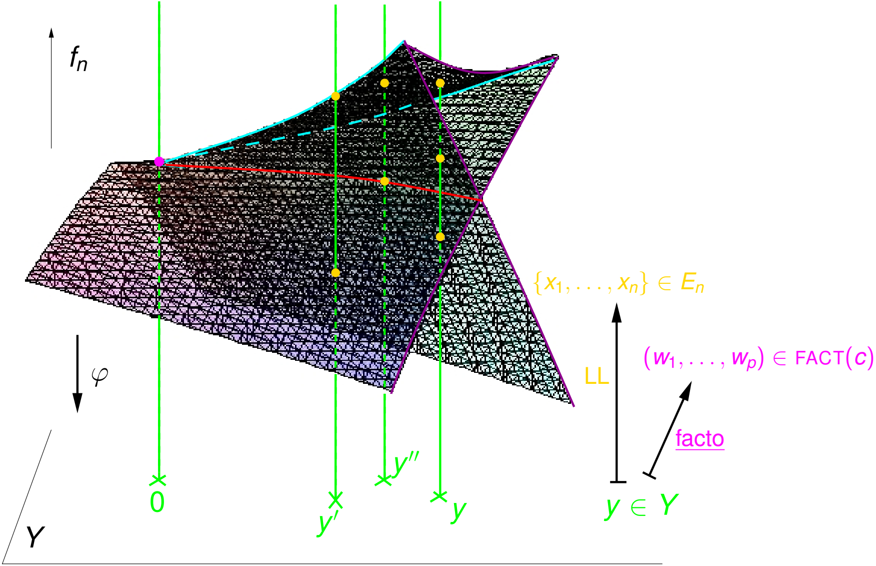
Equivalently, the product map:

$$Y \xrightarrow{\mathrm{LL} \times \underline{\text{facto}}} E_n \times \mathrm{FACT}(c)$$

is injective, and its image is the set of "compatible" pairs.

Outline

Bifurcation locus $(\mathcal{K})$ of LL



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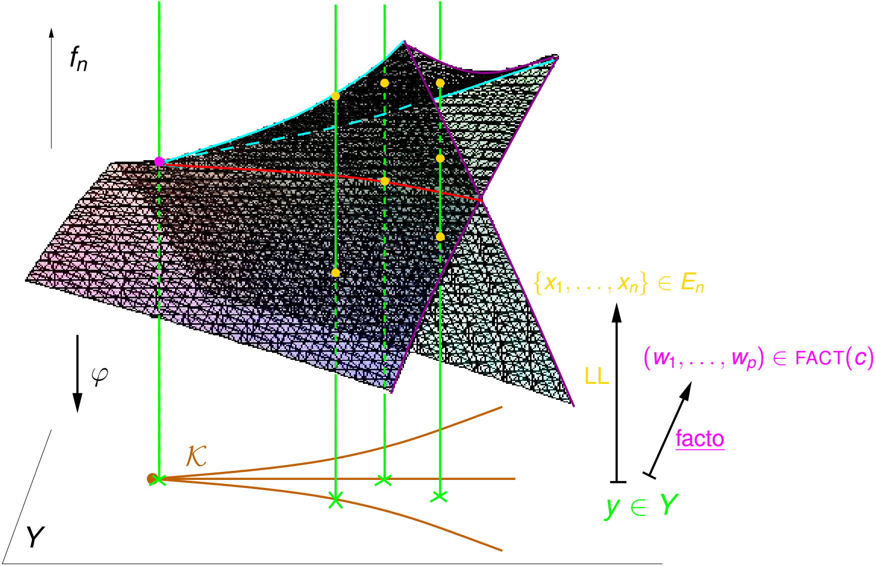

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Denote "this" set by $\bar{\mathcal{L}}_2$. Thus: $D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$

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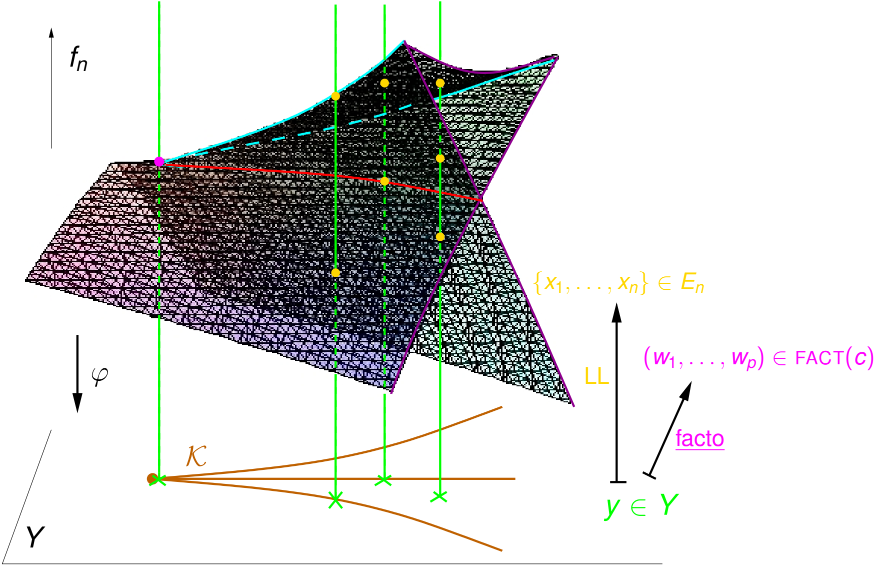

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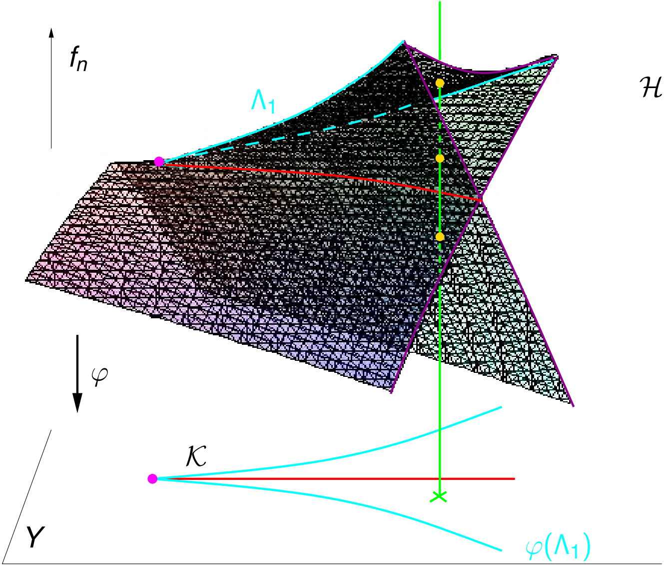

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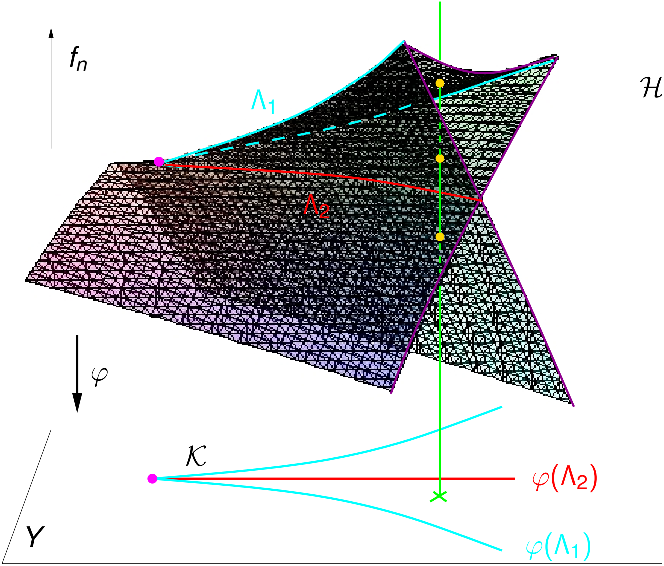

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Submaximal factorisations of type Λ

$\mathrm{FACT}^{\Lambda}_{n-1}(c)$:= set of factorisations of c in $n-1$ factors, with:

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So, $\sum \deg D_\Lambda = \deg D_{\mathrm{LL}} - \deg J_{\mathrm{LL}} = \ldots$

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- We recover what is predicted by Chapoton's formula;
- but the proof is more satisfactory and enlightening: we travelled from the numerology of $\mathrm{FACT}_n(c)$ (non-ramified part of LL) to that of $\mathrm{FACT}_{n-1}(c)$, without adding any case-by-case analysis.

Conclusion, questions

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Outline

4 Appendix
- Stratifications
- Comparison reflection groups / LL extensions

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Stratification of V with the "flats" (**intersection lattice**):
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Example of $W = A_3$: stratification of the discriminant

Return to facto

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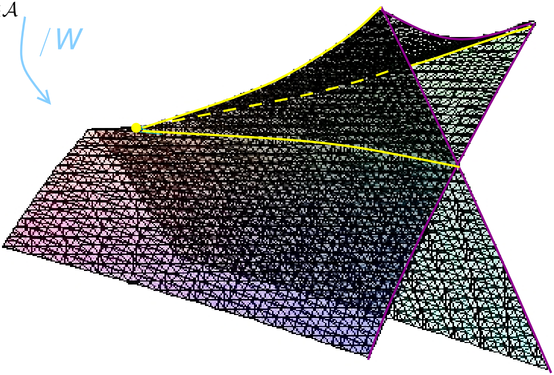

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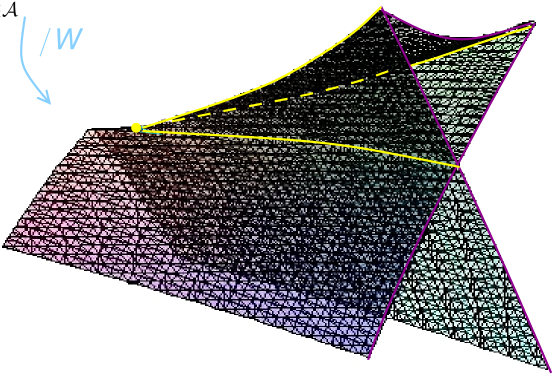

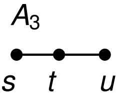

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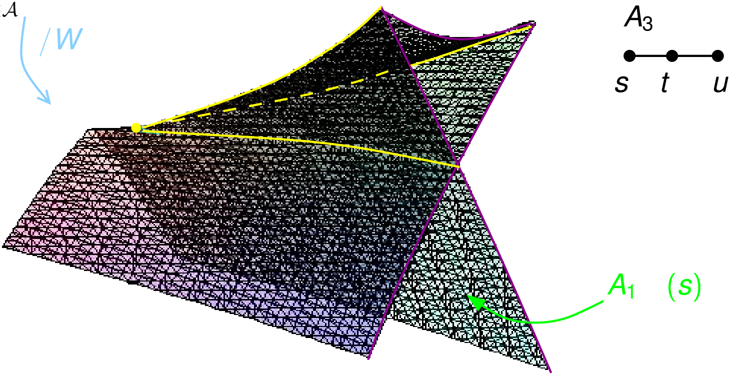

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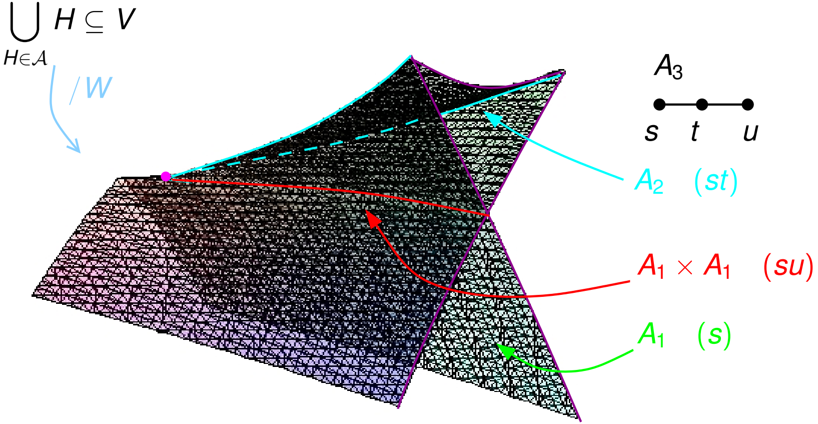
Return to facto



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$$\Delta_W(f_1, f_2, f_3) = \mathrm{Disc}(T^4 + f_1 T^2 - f_2 T + f_3 \; ; \; T)$$

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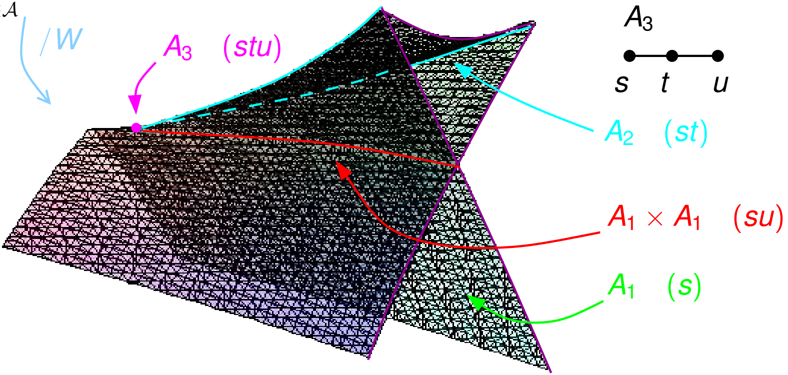
$$\bigcup_{H \in \mathcal{A}} H \subseteq V$$



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Irreducible components of $\mathcal{K}$, details

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Proposition

The $\varphi(\Lambda)$, for $\Lambda \in \bar{\mathcal{L}}_2$, are the irreducible components of $\mathcal{K}$ (where φ is the projection $V/W \twoheadrightarrow Y$).

◂ Return to Irreducible components of $\mathcal{K}$

Reflection group *vs.* Lyashko-Looijenga extension

Reflection group W	Extension LL
$V \to V/W$ $\mathbb{C}[f_1, \dots, f_n] = \mathbb{C}[V]^W \subseteq \mathbb{C}[V]$ degree $\lvert W \rvert$	$Y \to \mathbb{C}^{n-1}$ $\mathbb{C}[a_2, \dots, a_n] \subseteq \mathbb{C}[f_1, \dots, f_{n-1}]$ degree $n!\ h^n / \lvert W \rvert$
$V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$ Generic fiber $\simeq W$	$Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}}$ $\simeq \mathrm{Red}_R(c)$
ramified on $\bigcup_{H \in \mathcal{A}} H \twoheadrightarrow \mathcal{H}$	$\mathcal{K} = \bigcup_{\Lambda \in \bar{\mathcal{L}}_2} \varphi(\Lambda) \twoheadrightarrow E_n - E_n^{\mathrm{reg}}$
$\Delta_W = \prod_{H \in \mathcal{A}} \alpha_H^{e_H}$	$D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$
$J_W = \prod \alpha_H^{e_H - 1}$ $e_H = \lvert W_H \rvert$	$J_{\mathrm{LL}} = \prod D_\Lambda^{r_\Lambda - 1}$ $r_\Lambda =$ pseudo-order of elements of NCP_W of type Λ

Return